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Almost Periodicity in Linear Topological Spaces - Revisited

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Abstract

Almost periodic functions with values in Banach spaces, and even more generally in Fréchet spaces and p-Fréchet spaces, have been investigated by many authors. The purpose of this paper is to investigate the extent to which some results of these authors also hold in the setting of topological vector spaces, not necessarily locally convex or p-locally convex. In fact, most of our results (Theorems 3.3, 3.5–3.8) do not require even completeness or metrizability of the range space. We thus extend and unify several known results.

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1 Introduction

The theory of almost periodic functions was mainly created and published during 1924-1926 by the Danish mathematician Harold Bohr; Bohr's work was preceded by the important investigations of P. Bohl and E. Esclangon (see references in [4, 8]). The theory

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attracted the interest of a number of researchers because it has been effectively applied to solutions of diverse problems, initially in the theory of harmonic analysis and differential equations. In 1933, S. Bochner [2] published an important article devoted to extension of the theory of almost periodic functions on the real line $\mathbb{R}$ with values in a Banach space E. His results were further developed by several mathematicians (see the monographs by L. Amerio and G. Prouse [1], C. Corduneanu [4], B.M. Levitan and V.V. Zhikov [8], and S. Zaidman [15]). The theory of almost periodic functions taking values in a complete metrizable locally convex space E has been initiated by G.M. N'Guérékata [9] and further developed in [3, 5, 10, 11]. A survey paper by A. I. Shtern [14] also considers almost periodic functions and representations in locally convex spaces.

In this paper, we have considered the concept of almost periodicity of functions having values in a topological vector space E, not necessarily locally convex. We are mainly concerned with studying the topological properties of almost periodic functions and demonstrate the validity of several known results, including some from [3, 5, 9, 10], to this general setting.

2 Preliminaries

Definition 2.1. Let (E, τ) be a Hausdorff topological vector space (in short, TVS) over the field $\mathbb{K}$ (= $\mathbb{R}$ or $\mathbb{C}$) with a base $\mathcal{W} = \mathcal{W}_E$ of balanced neighbourhoods of 0. A subset A of E is called:

(1) *totally bounded* if, for each $W \in \mathcal{W}$, there exists a finite set $\{x_1, x_2, ..., x_n\} \subseteq A$ such that

$$A \subseteq \cup_{i=1}^{n} (x_i + W);$$

(2) *sequentially complete* if every Cauchy sequence in A converges to a point in A;

(3) *sequentially compact* if every sequence in A has a convergent subsequence with limit in A.

Remark 2.2. [6, 13] Clearly, A is compact $\Rightarrow$ A is relatively compct $\Rightarrow$ A is totally bounded $\Rightarrow$ A is bounded; also, A is complete $\Rightarrow$ A is sequentially complete. Further:

(a) A subset A of a TVS E is compact iff it is totally bounded and complete.
(b) If E is complete, then every totally bounded subset of E is relatively compact.
(c) If E is metrizable, then a subset A of E is:
(i) compact iff it is sequentially compact,
(ii) complete iff it is sequentially complete.

Definition 2.3. A complete metrizable TVS is called an *F-space*. A locally convex F-space is called a *Fréchet space*.

Definition 2.4. ([6, 7]). For any Hausdorff topological space X, let $C(X, E)$ (resp. $C_b(X, E)$) be the set of all continuous (resp. continuous and bounded) functions $f : X \to E$. Clearly, $C_b(X, E) \subseteq C(X, E)$ and both $C(X, E)$, $C_b(X, E)$ are vector spaces over $\mathbb{K}$ with the pointwise operations of addition and scalar multiplication. The *uniform topology* u on $C_b(X, E)$ is defined as the linear topology which has a base of neighbourhoods of 0 consisting of all sets of the form

$$N(X,W) = \{f \in C_b(X,E) : f(X) \subseteq W\},$$

where W varies over $\mathcal{W}$. A sequence $\{f_n\} \subseteq C_b(X,E)$ is said to be *u-Cauchy* if, for any $W \in \mathcal{W}$, there exists $n_o \in \mathbb{N}$ such that

$$f_n - f_m \in N(X,W) \text{ for all } n,m \geq n_o.$$

If E is metrizable with metric d, then $(C_b(X,E),u)$ is also metrizable with respect to the metric ρ given by:

$$\rho(f,g) = \sup_{x \in X} d(f(x),g(x)), \quad f,g \in C_b(X,E).$$

The *pointwise topology* p on $C(X,E)$ is defined as the linear topology which has a base of neighbourhoods of 0 consisting of all sets of the form

$$N(D,W) = \{f \in C(X,E) : f(D) \subseteq W\},$$

where D varies over finite subsets of X and W varies over $\mathcal{W}$.Clearly, $p \leq u$ on $C_b(X,E)$.

We state the following known result for reference purpose.

Theorem 2.5. *([6], p. 71-73) Let X be a Hausdorff topological space and E a Hausdorff TVS.*

(a) *If a sequence $\{f_n\} \subseteq C_b(X,E)$ is u-convergent to a function $f : X \to E$, then $f \in C_b(X,E)$.*

(b) *If $\{f_n\}$ is a u-Cauchy sequence in $C_b(X,E)$ and if there is a function $f : X \to E$ such that $f_n(x) \to f(x)$ for each $x \in X$, then $f_n \xrightarrow{u} f$* and $f \in C_b(X,E)$.

(c) *If E is an F-space, then so is $(C_b(X,E),u)$.*

3 Main Results

In this section, we consider the concept of almost periodicity in the non-locally convex setting. In fact, we demonstrate the validity of several known results, including some from [9, 3, 5] to this general setting. We include complete proofs of results for the benefit of readers and further investigations.

Definition 3.1. A subset P of $\mathbb{R}$ is called *relatively dense* in $\mathbb{R}$ if there exists a number $\ell > 0$ such that every interval of length ℓ in $\mathbb{R}$ contains at least one point of P.

Definition 3.2. Let (E,τ) be TVS with a base $\mathcal{W} = \mathcal{W}_E$ of balanced neighbourhoods of 0. A function $f : \mathbb{R} \to E$ is called *almost periodic* if it is continuous and, for each $W \in \mathcal{W}$, there exists a number $\ell_W > 0$ such that each interval of length ℓ_W in $\mathbb{R}$ contains a point τ_W such that

$$f(t+\tau_W) - f(t) \in W \text{ for all } t \in \mathbb{R}. \qquad (*)$$

A number $\tau_W \in \mathbb{R}$ for which $(*)$ holds is called *W-translation number* of f. The above property says that, for each $W \in \mathcal{W}$, the function f has a set of W-translation numbers $P_{W,f}$

which is relatively dense in $\mathbb{R}$. The set of all almost periodic functions $f:\mathbb{R}\to E$ is denoted by $AP(\mathbb{R},E)$. For any $f:\mathbb{R}\to E$ and a fixed $h\in\mathbb{R}$, the *h-translate* of f is defined as the function $f_h:\mathbb{R}\to E$ defined by

$$f_h(t) = f(t+h),\ t\in\mathbb{R}.$$

We shall denote $H(f)=\{f_h : h\in\mathbb{R}\}$, the set of all translates of f.

Theorem 3.3. *Let E be a TVS. Let $f\in AP(\mathbb{R},E)$. Then:*

(a) *f has totally bounded range $f(\mathbb{R})$; hence f is bounded.*
(b) *f is uniformly continuous on $\mathbb{R}$.*

Proof. (a) Let $W\in\mathcal{W}$. Choose a balanced $V\in\mathcal{W}$ such that $V+V\subseteq W$. Since f is almost periodic, there exists a number $\ell=\ell_V>0$ such that each interval of length ℓ in $\mathbb{R}$ contains a point τ_V such that

$$f(t+\tau_V)-f(t)\in V \text{ for all } t\in\mathbb{R}. \tag{1}$$

By continuity of f, the set $f[0,\ell]$ is compact in E and hence totally bounded. So there exists a finite set $\{x_1,...,x_n\}\subseteq f[0,\ell]$ such that

$$f[0,\ell]\subseteq \cup_{i=1}^n (x_i+V). \tag{2}$$

We claim that $f(\mathbb{R})\subseteq \cup_{i=1}^n (x_i+W)$. [Take an arbitrary $t\in\mathbb{R}$. By (1), there exists $\tau\in[-t,-t+\ell]$ such that $f(t+\tau)-f(t)\in V$. By (2), choose $x_k\in\{x_1,...,x_n\}$ such that $f(t+\tau)-x_k\in V$. Then

$$f(t)-x_k = [f(t)-f(t+\tau)]+[f(t+\tau)-x_k]\in -V+V\subseteq W,$$

and therefore $f(t)\in x_k+W$.] Thus $f(\mathbb{R})$ is totally bounded in E.

(b) Let $W\in\mathcal{W}$. Choose a balanced $V\in\mathcal{W}$ such that $V+V+V\subseteq W$. There exists a number $\ell=\ell_V>0$ such that each interval of length ℓ in $\mathbb{R}$ contains a point τ_V such that

$$f(t+\tau_V)-f(t)\in V \text{ for all } t\in\mathbb{R}. \tag{3}$$

Now f, being almost periodic, is continuous on $\mathbb{R}$. Then f is uniformly continuous on the compact set $[-1,1+\ell_V]$, so there exists $\delta=\delta_V>0$ (we may assume $0<\delta<1$ without loss of generality) such that

$$f(s)-f(t)\in V \text{ for all } s,t\in[-1,1+\ell_V] \text{ with } |s-t|<\delta. \tag{4}$$

Let $a,b\in\mathbb{R}$ with $|a-b|<\delta$ and assume $a<b$. We claim that $f(a)-f(b)\in W$. [We may assume that $a<b$. Choose a $\tau_V\in[-a,-a+\ell_V]$ satisfying (3). Then

$$a+\tau_V\in a+[-a,-a+\ell_V]=[0,0+\ell_V]\subseteq[-1,1+\ell_V];$$

since $0<b-a<\delta<1$,

$$b+\tau_V\in b+[-a,-a+\ell_V]=[b-a,b-a+\ell_V]=[0,1+\ell_V]\subseteq[-1,1+\ell_V].$$

Also, $|(a+\tau_V)-(b+\tau_V)|=|a-b|<\delta$, so by (4),

$$f(a+\tau_V)-f(b+\tau_V)\in V. \tag{5}$$

Therefore, by (3) and (5),

$$\begin{aligned} f(a)-f(b) &= [f(a)-f(a+\tau_V)]+[f(a+\tau_V)-f(b+\tau_V)] \\ &\quad +[f(b+\tau_V)-f(b)] \\ &\in -V+V+V \subseteq W.] \end{aligned}$$

Thus f is uniformly continuous on $\mathbb{R}$. □

Remark 3.4. Clearly, by Theorem 3.3(a), $AP(\mathbb{R},E) \subseteq C_b(\mathbb{R},E)$.

Theorem 3.5. *Let E be a TVS. If $\{f_n\}$ is a sequence in $AP(\mathbb{R},E)$ such that $f_n \xrightarrow{u} f$, then $f \in AP(\mathbb{R},E)$.*

Proof. Clearly, by Theorem 2.5, f is continuous on $\mathbb{R}$. Let $W \in \mathcal{W}$. Choose a balanced $V \in \mathcal{W}$ such that $V+V+V \subseteq W$. Since $f_n \xrightarrow{u} f$, there exists an integer $N \geq 1$ such that

$$f_n(t)-f(t) \in V \text{ for all } t \in \mathbb{R} \text{ and } n \geq N. \tag{6}$$

Since f_N is almost periodic, there exists a number $\ell = \ell_V > 0$ such that each interval of length ℓ in $\mathbb{R}$ contains a point $\tau = \tau_V$ such that

$$f_N(t+\tau)-f_N(t) \in V \text{ for all } t \in \mathbb{R}. \tag{7}$$

Then, by (6) and (7), for any $t \in \mathbb{R}$

$$\begin{aligned} f(t+\tau)-f(t) &= [f(t+\tau)-f_N(t+\tau)]+[f_N(t+\tau)-f_N(t)]+[f_N(t)-f(t)] \\ &\in -V+V+V \subseteq W. \end{aligned}$$

Since the set P_{V,f_N} of almost periods of f_N is relatively dense, we take $P_{V,f} = P_{V,f_N}$. Hence f is also almost periodic. □

Theorem 3.6. *Let E be a TVS. If $f:\mathbb{R}\to E$ is almost periodic, then the functions (i) λf ($\lambda \in \mathbb{K}$), (ii) $\overline{f}(t) \equiv f(-t)$ and (iii) $f_h(t) = f(t+h)$ ($h \in \mathbb{R}$) are also almost periodic.*

Proof. (i) This is trivial if $\lambda = 0$. Suppose $\lambda \neq 0$. Let $W \in \mathcal{W}$ be balanced. Then $V = \lambda^{-1}W \in \mathcal{W}$. Since f is almost periodic, there exists a number $\ell = \ell_V > 0$ such that each interval of length ℓ in $\mathbb{R}$ contains a point τ_V such that

$$f(t+\tau_V)-f(t) \in V \text{ for all } t \in \mathbb{R}.$$

Then

$$(\lambda f)(t+\tau_V)-(\lambda f)(t) = \lambda[f(t+\tau_V)-f(t)] \in \lambda V = W \text{ for all } t \in \mathbb{R}.$$

Hence λf is almost periodic.

(ii) Let $W \in \mathcal{W}$ be balanced. There exists a number $\ell = \ell_W > 0$ such that each interval of length ℓ in $\mathbb{R}$ contains a point τ such that

$$f(t+\tau)-f(t) \in W \text{ for all } t \in \mathbb{R}.$$

Put $s = -t$; we get:

$$\overline{f}(s-\tau) - \overline{f}(s) = f(-s+\tau) - f(-s) = f(t+\tau) - f(t) \in W.$$

Therefore $\overline{f}(s-\tau) - \overline{f}(s) \in W$ for every $s \in \mathbb{R}$. This shows that $\overline{f}$ is almost periodic with $-\tau$ as a W-translation number.

(iii) Let $W \in \mathcal{W}$ be balanced, and let $h \in \mathbb{R}$. There exists a number $\ell = \ell_W > 0$ such that each interval of length ℓ in $\mathbb{R}$ contains a point τ such that

$$f(t+\tau) - f(t) \in W \text{ for all } t \in \mathbb{R}.$$

Replacing t by $t+h$, we have

$$f(t+h+\tau) - f(t+h) \in W \text{ for all } t \in \mathbb{R},$$

or

$$f_h(t+\tau) - f_h(t) \in W \text{ for all } t \in \mathbb{R}.$$

□

Theorem 3.7. *Let E and F be a TVSs, and let $f \in AP(\mathbb{R}, E)$. If $g : \overline{f(\mathbb{R})} \to F$ is any continuous function, then the composed function $g \circ f \in AP(\mathbb{R}, F)$.*

Proof. Let $U \in \mathcal{W}_F$ be balanced. Since $\overline{f(\mathbb{R})}$ is compact, g is uniformly continuous on $\overline{f(\mathbb{R})}$ and so there exists a $W \in \mathcal{W}_E$ such that

$$g(x) - g(y) \in U \text{ for all } x, y \in \overline{f(\mathbb{R})} \text{ with } x - y \in W.$$

Since f is almost periodic, there exists a number $\ell = \ell_W > 0$ such that each interval of length ℓ in $\mathbb{R}$ contains a point τ such that

$$f(t+\tau) - f(t) \in W \text{ for all } t \in \mathbb{R}.$$

Consequently,

$$g[f(t+\tau)] - g[f(t)] \in U \text{ for all } t \in \mathbb{R}.$$

Thus $g \circ f \in AP(\mathbb{R}, F)$. □

We now proceed to establish the Bochner's criteria for almost periodicity. As a first step, we obtain:

Theorem 3.8. *Let E be a TVS and $f : \mathbb{R} \to E$ a continuous function. If the set of translates $H(f) = \{f_h : h \in \mathbb{R}\}$ is u-sequentially compact in $C_b(\mathbb{R}, E)$, then f is almost periodic.*

Proof. Suppose f is not almost periodic Then there exists a $W \in \mathcal{W}$ such that for every $\ell > 0$, there exists an interval of length ℓ, $[-a, -a+\ell]$ (say) which contains no W-translation number of f. Consequently, for every $h \in [-a, -a+\ell]$, there exists $t_h \in \mathbb{R}$ such that

$$f(t_h + h) - f(t_h) \notin W.$$

Let us consider $h_1 \in \mathbb{R}$ and an interval (a_1, b_1) with $b_1 - a_1 > 2|h_1|$ which contains no W-translation number of f. Now let $h_2 = \frac{a_1+b_1}{2}$. Since

$$-\frac{b_1 - a_1}{2} < h_1 < \frac{b_1 - a_1}{2},$$

$h_2 - h_1 \in (a_1, b_1)$ and therefore $h_2 - h_1$ cannot be a W-translation number of f. Let us consider another interval (a_2, b_2) with $b_2 - a_2 > 2(|h_1| + |h_2|)$, which contains no W-translation number of f. Let $h_3 = \frac{a_2+b_2}{2}$; then $h_3 - h_1$, $h_3 - h_2 \in (a_2, b_2)$ and therefore $h_3 - h_1$and $h_3 - h_2$ cannot be W-translation number of f. We proceed and get a sequence $\{h_n\}$ such that no $h_m - h_n$ $(m > n)$ is a W-translation number of f; that is, there exists $t_{mn} \in \mathbb{R}$ with

$$f(t_{mn} + h_m - h_n) - f(t_{mn}) \notin W. \tag{8}$$

Put $s_{mn} = t_{mn} - h_n$. Then (8) becomes:

$$f(s_{mn} + h_m) - f(s_{mn} + h_n) \notin W. \tag{9}$$

Since $H(f)$ is sequentially compact in $C_b(\mathbb{R}, E)$, there exists a subsequence $\{k_n\}$ of $\{h_n\}$ such that $\{f\{t + k_n)\}$ is u-convergent on $\mathbb{R}$. Then, for $W \in \mathcal{W}$, there exists an $N = N_W \geq 1$ such that if m, $n > N$ (we may take $m > n$), we have:

$$f(t + k_m) - f(t + k_n) \in W \text{ for every } t \in \mathbb{R}. \tag{10}$$

Taking $t = s_{mn}$ in (10), we get a contradiction to (9). Thus f is almost periodic. □

As a converse of the above theorem, we obtain:

Theorem 3.9. *Let E be an F-space and $f : \mathbb{R} \to E$ an almost periodic function. Then the set of translates $H(f) = \{f_h : h \in \mathbb{R}\}$ is u-compact in $C_b(\mathbb{R}, E)$.*

Proof. Since E is an F-space, by Theorem 2.5, $(C_b(\mathbb{R}, E), u)$ is also an F-space. Therefore, it suffices to show that any sequence $\{f_{h_n}\}$ in $H(f)$ has a u-Cauchy subsequence.

Let $S = \{a_n\}$ be a dense sequence in $\mathbb{R}$. Since $f(\mathbb{R})$ is totally bounded and hence relatively compact in the F-space E, we can extract from $\{f_{h_n}(a_1)\}$ a convergent subsequence. Let $\{f_{h_{1,n}}\}$ be the subsequence of $\{f_{h_n}\}$ which converges at a_1. We apply the same argument as above to the sequence $\{f_{h_{1,n}}\}$ to choose a subsequence $\{f_{h_{2,n}}\}$ which converges at a_2. We continue the process and consider the diagonal sequence $\{f_{h_{n,n}}\}$ which converges at each a_n in S. Call this last sequence by $\{f_{k_n}\}$. We claim that this sequence is u-Cauchy on $\mathbb{R}$.

Let $t_0 \in \mathbb{R}$, and let $W \in \mathcal{W}$. Choose a balanced $V \in \mathcal{W}$ such that $V + V + V + V + V \subset W$. By almost periodicity of f, let $\ell = \ell(V) > 0$ be such that each interval of length ℓ in $\mathbb{R}$ contains a point $\tau = \tau_V$ such that

$$f(t + \tau) - f(t) \in V \text{ for all } t \in \mathbb{R}. \tag{11}$$

By uniform continuity of f over $\mathbb{R}$, there exists $\delta = \delta_V > 0$ such that

$$f(t) - f(t') \in V \text{ for every } t, t' \in \mathbb{R} \text{ with } \left|t - t'\right| < \delta. \tag{12}$$

We divide the interval $[0,\ell]$ into subintervals $I_1,...,I_r$, each of length smaller than δ. Since S is dense in $\mathbb{R}$, for each $1 \le j \le r$, I_j contains a point b_j of S and set $S_0 = \{b_1,...,b_r\}$. Since $\{f_{k_n}\}$ is p-convergent on S, it follows that $\{f_{k_n}\}$ is u-convergent on the finite set S_0. Then there exists an interger $N = N_V \ge 1$ such that

$$f(b_i + k_n) - f(b_i + k_m) \in V \text{ for every } i = 1,...,J \text{ and } n,m > N. \tag{13}$$

Now, choose $\tau_0 \in [-t_0, -t_0 + \ell]$ satisfying (11). Then $t_0 + \tau \in [0,\ell]$, so choose $b_j \in S_0$ such that $|t_0 + \tau_0 - b_j| < \delta$. So, by (12),

$$f(t_0 + \tau_0 + k_n) - f(b_j + k_n) \in V \text{ for every } n \ge 1. \tag{14}$$

Therefore, if $n,m > N$, by applying (11), (13), (14) we get

$$\begin{aligned}
f_{k_n}(t_0) - f_{k_m}(t_0) &= f(t_0 + k_n) - f(t_0 + k_m)\\
&= [f(t_0 + k_n) - f(t_0 + k_n + \tau_0)]\\
&\quad +[f(t_0 + k_n + \tau_0) - f(b_j + k_n)]\\
&\quad +[f(b_j + k_n) - f(b_j + k_m)]\\
&\quad +[f(b_j + k_m) - f(t_0 + \tau_0 + k_m)]\\
&\quad +[f(t_0 + k_m + \tau_0) - f(t_0 + k_m)]\\
&\in V + V + V + V + V \subset W.
\end{aligned}$$

Therefore the subsequence $\{f_{k_n}\}$ of $\{f_{h_n}\}$ is u-Cauchy on $\mathbb{R}$. Consequently, $\{f_{k_n}\}$ *is u-convergent on* $\mathbb{R}$. □

Combining Theorems 3.8 and 3.9, we have:

Corollary 3.10. *(Bochner's Criterion) Let E be an F-space. Then a continuous function $f : \mathbb{R} \to E$ is almost periodic function iff the set $H(f) = \{f_h : h \in \mathbb{R}\}$ is u-compact in $C_b(\mathbb{R}, E)$.*

Theorem 3.11. *Let E be an F-space, and let $f_1,..., f_m \in AP(\mathbb{R}, E)$. Define $F : \mathbb{R} \to E^m$ by*

$$F(t) = (f_1(t),..., f_m(t)),\ t \in \mathbb{R}.$$

Then $F \in AP(\mathbb{R}, E^m)$. In particular, for any $W \in \mathcal{W}$, $f_1,..., f_m$ have common W-translation numbers.

Proof. Let $\{h_n\}$ be a sequence in $\mathbb{R}$. Consider the sequence $\{f_{1,h_n}\}$ of translates of function f_1 corresponding to $\{h_n\}$. Since f_1 is almost periodic, by using Bochner's criteria, we can extract from $\{f_{1,h_n}\}$ a uniformly convergent subsequence, denoting again by $\{f_{1,h_n}\}$. Continuing this process, we extract from $\{f_{m,h_n}\}$ a uniformly convergent subsequence, denoted also by $\{f_{m,h_n}\}$. Then the sequence $\{(f_{1,h_n}..., f_{m,h_n})\}$ has a subsequence which is easily seen to be u-convergent on $\mathbb{R}$. Hence F is almost periodic. Next, for any $W \in \mathcal{W}$, there exists a number $\ell = \ell_W > 0$ such that each interval of length ℓ in $\mathbb{R}$ contains a point τ_W such that

$$F(t + \tau_W) - F(t) \in W^m \text{ for all } t \in \mathbb{R}.$$

Consequently,

$$f_i(t + \tau_W) - f_i(t) \in W \text{ for all } t \in \mathbb{R} \text{ and } i = 1,...,m.$$

□

Theorem 3.12. *Let E be a F-space, and let $f,g \in AP(\mathbb{R},E)$. Then $f+g \in AP(\mathbb{R},E)$.*

Proof. In view of the Bochner's criteria, we need to show that $H(f+g) = \{f_h + g_h : h \in \mathbb{R}\}$ is u-compact in $C_b(\mathbb{R},E)$. Define $F : \mathbb{R} \to E^2$ by

$$F(t) = (f(t), g(t)), t \in \mathbb{R}.$$

By Theorem 3.11, $F \in AP(\mathbb{R},E^2)$ and hence, by Bochner's criteria, $H(F) = \{(f_h, g_h) : h \in \mathbb{R}\}$ is u-compact in $C_b(\mathbb{R},E^2)$. Now define $S : C_b(\mathbb{R},E) \times C_b(\mathbb{R},E) \to C_b(\mathbb{R},E)$ by

$$S(u,v) = u+v, \ u,v \in C_b(\mathbb{R},E).$$

This is a continuous function, hence $S(H(F)) = H(f+g))$ is u-compact in $C_b(\mathbb{R},E)$. □

We next consider completeness of the space $(AP(\mathbb{R},E),u)$.

Theorem 3.13. *Let E be an F-space. Then the vector space $AP(\mathbb{R},E)$ is u-complete.*

Proof. By Theorems 3.6 and 3.12, $AP(\mathbb{R},E)$ is a vector space. Further, since each $f \in AP(\mathbb{R},E)$ is bounded, $AP(\mathbb{R},E)$ is a vector subspace of $C_b(\mathbb{R},E)$. Since E is complete, by Theorem 2.5, $C_b(\mathbb{R},E)$ is u-complete. So we need only to see that $AP(\mathbb{R},E)$ is u-closed in $C_b(\mathbb{R},E)$.

Let $f \in C_b(\mathbb{R},E)$ with $f \in u\text{-}cl[AP(\mathbb{R},E)]$. Then there exists a sequence $\{f_n\} \subseteq AP(\mathbb{R},E)$ such that $f_n \xrightarrow{u} f$. Since each f_n is almost periodic and $f_n \xrightarrow{u} f$, by Theorem 3.5, f is almost periodic. Hence $f \in AP(\mathbb{R},E)$. Thus $AP(\mathbb{R},E)$ is u-closed in $C_b(\mathbb{R},E)$. □

Remark 3.14. Our results include extensions of following results:

(a) Theorems 1-6 and Corollary 1 of [9], where E is a Fréchet space.

(b) Theorem 3.1 and Propositin 3.3. of [3], where E is again a Fréchet space

(c) Theorems 3.2-3.8 of [5], where E is a p-Fréchet space, $0 < p < 1$.

(d) The corresponding results in the monographs [1, 4, 8, 15], where E is a Banach space.

Remark 3.15. We mention that a continuous functions $f : [a;b] \to E$ need not be integrable in the Riemann sense, if E is not locally convex ([12], p. 123). However, if E is a locally pseudoconvex F-space, then all analytic functions $f : [a;b] \to E$ are integrable in the Bochner-Lebesgue sense ([12], Theorem 3.5.2). Using a similar approach, almost periodicity of functions with values in p-Fréchet spaces, $0 < p < 1$, has been considered in [5]. This paper also contains some applications to differential equations and to dynamical systems.

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