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# Applications of the Summability Theory to the Solvability of Certain Sequence Spaces Equations with Operators of the Form $B(r,s)$

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**Abstract**

In this paper we deal with *sequence spaces inclusion equations (SSIE),* which are determined by an inclusion where each term is a *sum or a sum of products of sets of the form* $\chi_a(T)$ *and* $\chi_{f(x)}(T)$ where $f$ map $U^+$ to itself, and $\chi \in \left\{\mathbf{s}, \mathbf{s}^0, \mathbf{s}^{(c)}\right\}$, the sequence $x$ is the unknown and $T$ is a given triangle. Here we give characterizations of the *(SSIE)* $\chi_x(B(r,s)) \subset \chi_x(B(r',s'))$ and of the (SSE) $\chi_x(B(r,s)) = \chi_x(B(r',s'))$, where $\chi = s$, $s^0$, or $s^{(c)}$ and $B(r,s)$ is the generalized operator of first difference defined by $B(r,s)_n y = ry_n + sy_{n-1}$ for all $n \geq 2$ and $B(r,s)_1 y_1 = ry_1$. We give an application to the spectrum of $B(r,s)$ considered as an operator from $\chi_x$ to itself, where $\chi = \mathbf{s}$, or $\mathbf{s}^0$. Then we apply these results to the solvability of the sequence spaces equation $\chi_a + \mathbf{s}_x^{(c)}(B(r,s)) = \mathbf{s}_x^{(c)}$ where $\chi = \mathbf{s}$, $\mathbf{s}^0$, or $\mathbf{s}^{(c)}$ and $x$ is the unknown.

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**Keywords**: Matrix transformations; BK space; the spaces $\mathbf{s}_a$, $\mathbf{s}_a^0$ and $\mathbf{s}_a^{(c)}$; multiplier of sets of sequences; (SSIE); (SSE) with operator; band matrix $B(r,s)$.

## 1 Introduction

In the book entitled *Summability through Functional Analysis,* [15] Wilansky introduced sets of the form $a^{-1} * E$ where $E$ is a BK space, and $a = (a_n)_{n \geq 1}$ is a sequence satisfying $a_n \neq 0$ for all $n$. Recall that $a^{-1} * E$ is the set of all sequences $y = (y_n)_{n \geq 1}$ such that $ay \in E$. In [4] the sets $\mathbf{s}_a$, $\mathbf{s}_a^0$ and $\mathbf{s}_a^{(c)}$ were introduced by $(1/a)^{-1} * E$ with $a_n > 0$ for all $n$ and $E \in \{\ell_\infty, c_0, c\}$. In [5, 6] the sum $\chi_a + \chi'_b$ and the product $\chi_a * \chi'_b$ were defined, where $\chi, \chi'$ are any

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of the symbols $\mathbf{s}$, $\mathbf{s}^0$, or $\mathbf{s}^{(c)}$, among other things characterizations of matrix transformations mapping in the sets $\mathbf{s}_a + \mathbf{s}_b^0(\Delta^q)$ and $\mathbf{s}_a + \mathbf{s}_b^{(c)}(\Delta^q)$ were given, where $\Delta$ is the *operator of the first difference*. In [9] de Malafosse and Malkowsky gave among other things properties of the spectrum of the matrix of weighted means $\overline{N}_q$ considered as operator in the set $\mathbf{s}_a$. In [10] characterizations can be found of the sets $(\mathbf{s}_a(\Delta^q), \chi_b)$ where $\chi$ is any of the symbols $\mathbf{s}$, $\mathbf{s}^0$, or $\mathbf{s}^{(c)}$. Using spectral properties of the operator of first difference in the sets $\mathbf{s}_\alpha^0$ and $\mathbf{s}_\beta^{(c)}$ it can be found in [6] simplifications of the set $\mathbf{s}_\alpha^0\left((\Delta - \lambda I)^h\right) + \mathbf{s}_\beta^{(c)}\left((\Delta - \mu I)^l\right)$ where $h$, $l$ are complex numbers, $\alpha$, $\beta$ given sequences, and characterization of matrix transformations in this set. In [11] de Malafosse and Rakočević gave applications of the measure of noncompactness to operators on the spaces $\mathbf{s}_\alpha$, $\mathbf{s}_\alpha^0$, $\mathbf{s}_\alpha^{(c)}$ and $l_\alpha^p$.

In this paper among other things we determine the set of all sequences $x \in U^+$ such that $ry_n + sy_{n-1} = O(x_n)$ implies $r'y_n + s'y_{n-1} = O(x_n)$ $(n \to \infty)$ for all $y$ and for given reals $r$, $s$, $r'$, $s'$. This statement consists in determining the set of all $x$ such that $\mathbf{s}_x(B(r,s)) \subset \mathbf{s}_x(B(r',s'))$ where $B(r,s)$ and $B(r',s')$ are band matrix. So we are led to deal with special *sequence spaces inclusion equations (SSIE), (resp. sequence spaces equations (SSE)),* which are determined by an inclusion, (resp. identity), where each term is a *sum or a sum of products of sets of the form* $\chi_a(T)$ *and* $\chi_{f(x)}(T)$ where $f$ maps $U^+$ to itself, $\chi$ is any of the symbols $\mathbf{s}$, $\mathbf{s}^0$, or $\mathbf{s}^{(c)}$, $x$ is the unknown and $T$ is a triangle, (cf. [2, 7, 8]). In [2] some results were given on the (SSE) $\mathbf{s}_a + \mathbf{s}_x = \mathbf{s}_b$ and on the (SSE) $\mathbf{s}_{\varphi(x)} = \mathbf{s}_b$ where $\varphi$ maps from the set $U^+$ of all positive sequences to itself. In [7] it can be found a study of the (SSIE) $\chi_b'' \subset \chi_a + \chi_x'$ when $a/b \in c_0$ in the cases when $\chi$, $\chi'$, $\chi''$ are any of the symbols $\mathbf{s}$, $\mathbf{s}^0$, or $\mathbf{s}^{(c)}$, there is also a resolution of the (SSIE) with operator $\chi_a + \chi_x(\Delta) \subset \chi_x$ where $\chi$ is any of the symbols $\mathbf{s}$, $\mathbf{s}^0$, or $\mathbf{s}^{(c)}$.

This paper is organized as follows. In Section 2 we recall some well-known results on sequence spaces and matrix transformations. In Section 3 some results are recalled on the sum, the multiplier and the product of some special spaces of sequences. In Section 4 are recalled some results on the solvability of (SSIE) of the form $\chi_b'' \subset \chi_a + \chi_x'$ with $\chi$, $\chi'$, $\chi'' \in \left\{\mathbf{s}^0, \mathbf{s}^{(c)}, \mathbf{s}\right\}$. The main results are given in Sections 5 and 6. In Section 5 we explicitly describe the set of all $x \in U^+$ satisfying the inclusion $\chi_x(B(r,s)) \subset \chi_x(B(r',s'))$, or the identity $\chi_x(B(r,s)) = \chi_x(B(r',s'))$ where $\chi = \mathbf{s}$, $\mathbf{s}^0$, or $\mathbf{s}^{(c)}$ and deal with the spectrum of $B(r,s)$ considered as operator from $\chi_x$ to itself, where $\chi = \mathbf{s}$, or $\mathbf{s}^0$. In Section 6 we deal with the (SSIE) $\mathbf{s}_x^{(c)}(B(r,s)) \subset \mathbf{s}_x^{(c)}(B(r',s'))$ and focus on the special case when $rs < 0$. Finally in Section 7 we apply these results to the solvability of the (SSE) $\chi_a + \mathbf{s}_x^{(c)}(B(r,s)) = \mathbf{s}_x^{(c)}$ where $\chi = \mathbf{s}$, $\mathbf{s}^0$, or $\mathbf{s}^{(c)}$.

## 2 Notations and preliminary results

For a given infinite matrix $\Lambda = (\lambda_{nk})_{n,k \geq 1}$ we define the operators $\Lambda_n$ for any integer $n \geq 1$, by

$$\Lambda_n(y) = \sum_{k=1}^{\infty} \lambda_{nk} y_k \tag{2.1}$$

where $y = (y_k)_{k \geq 1}$, and the series are assumed convergent for all $n$. So we are led to the study of the operator $\Lambda$ defined by $\Lambda y = (\Lambda_n(y))_{n \geq 1}$ mapping between sequence spaces.

A Banach space $E$ of complex sequences with the norm $\|\|_E$ is a *BK space if each projection* $P_n$:$E \to \mathbb{C}$ defined by $y \to P_n y = y_n$ *is continuous*. A BK space $E$ is *said to have AK if every sequence* $y = (y_k)_{k\geq 1} \in E$ has a *unique representation* $y = \sum_{k=1}^{\infty} y_k e^{(k)}$ where $e^{(k)}$ is the sequence with 1 in the $k$-th position and 0 otherwise.

We will denote by $\omega$, $c_0$, $c$, $\ell_\infty$ the sets of all sequences, the set of sequences that converge to zero, that are convergent and that are bounded respectively. If $u$ and $v$ are sequences and $E$ and $F$ are two subsets of $\omega$, then we write $uv = (u_n v_n)_n$ and

$$M(E,F) = \{u = (u_n)_{n\geq 1} \in \omega : \ uv \in F \text{ for all } v \in E\},$$

$M(E,F)$ is called the *multiplier space of E and F*. We shall use the set $U^+$ of all sequences $u = (u_n)_{n\geq 1} \in \omega$ such that $u_n > 0$ for all $n$. Using Wilansky's notations [15], we define for any sequence $a = (a_n)_{n\geq 1} \in U^+$ and for any set of sequences $E$, the set $(1/a)^{-1} * E = \{(y_n)_{n\geq 1} \in \omega \ : (y_n/a_n)_n \in E\}$. To simplify, we use the diagonal matrix $D_a$ defined by $[D_a]_{nn} = a_n$ for all $n$ and write

$$D_a * E = (1/a)^{-1} * E$$

and define $\mathbf{s}_a = D_a * \ell_\infty$, $\mathbf{s}_a^0 = D_a * c_0$ and $\mathbf{s}_a^{(c)} = D_a * c$, see for instance [5, 4, 11]. Each of the spaces $D_\alpha * \chi$, where $\chi \in \{\ell_\infty, c_0, c\}$, is a BK space normed by $\|\xi\|_{\mathbf{s}_a} = \sup_{n\geq 1} (|\xi_n|/a_n)$ and $\mathbf{s}_a^0$ has AK.

Now let $a = (a_n)_{n\geq 1}$, $b = (b_n)_{n\geq 1} \in U^+$. By $S_{a,b}$ we denote the set of infinite matrices $\Lambda = (\lambda_{nk})_{n,k\geq 1}$ such that $\|\Lambda\|_{S_{a,b}} = \sup_{n\geq 1} \left[(1/b_n) \sum_{k=1}^{\infty} |\lambda_{nk}| a_k\right] < \infty$. The set $S_{a,b}$ is a Banach space with the norm $\|\|_{S_{a,b}}$. Let $E$ and $F$ be any subsets of $\omega$. When $\Lambda$ maps $E$ into $F$ we write $\Lambda \in (E,F)$, see [3]. So we have $\Lambda y \in F$ for all $y \in E$, ($\Lambda y \in F$ means that for each $n \geq 1$ the series defined by $\Lambda_n(y) = \sum_{k=1}^{\infty} \lambda_{nk} y_k$ is convergent and $(\Lambda_n(y))_{n\geq 1} \in F$). It is well known that $\Lambda \in (\mathbf{s}_a, \mathbf{s}_b)$ if and only if $\Lambda \in S_{a,b}$. So we can write $(\mathbf{s}_a, \mathbf{s}_b) = S_{a,b}$.

When $\mathbf{s}_a = \mathbf{s}_b$ we obtain the *Banach algebra with identity* $S_{a,b} = S_a$, (see [4]) normed by $\|\Lambda\|_{S_a} = \|\Lambda\|_{S_{a,a}}$. We also have $\Lambda \in (\mathbf{s}_a, \mathbf{s}_a)$ if and only if $\Lambda \in S_a$.

If $a = (r^n)_{n\geq 1}$, the sets $S_a$, $\mathbf{s}_a$, $\mathbf{s}_a^0$ and $\mathbf{s}_a^{(c)}$ are denoted by $S_r$, $\mathbf{s}_r$, $\mathbf{s}_r^0$ and $\mathbf{s}_r^{(c)}$ respectively (see [5]). When $r = 1$, we obtain $\mathbf{s}_1 = \ell_\infty$, $\mathbf{s}_1^0 = c_0$ and $\mathbf{s}_1^{(c)} = c$, and putting $e = (1,1,...)$ we have $S_1 = S_e$. It is well known, see [3] that $(\mathbf{s}_1, \mathbf{s}_1) = (c_0, \mathbf{s}_1) = (c, \mathbf{s}_1) = S_1$. We have $\Lambda \in (\mathbf{s}_1, \mathbf{s}_1)$ if and only if

$$\Lambda \in S_1; \tag{2.2}$$

$\Lambda \in (c_0, c_0)$ if and only if (2.2) holds and $\lim_{n\to\infty} \lambda_{nk} = 0$ for all $k \geq 1$. Then $\Lambda \in (c, c)$ if and only if (2.2) holds, $\lim_{n\to\infty} \lambda_{nk} = l_k$ for all $k$ and for some scalars $l_k$, and $\lim_{n\to\infty} \sum_{k=1}^{\infty} \lambda_{nk} = l$ for some $l$.

In the following we will frequently use the fact that $\Lambda \in \left(\chi_a, \chi'_b\right)$ if and only if $D_{1/b} \Lambda D_a \in (\chi_e, \chi'_e)$ where $\chi, \chi'$ are any of the symbols $\mathbf{s}^0$, $\mathbf{s}^{(c)}$, or $\mathbf{s}$.

For any subset $E$ of $\omega$, we put $\Lambda E = \{\eta \in \omega \ : \eta = \Lambda y \text{ for some } y \in E\}$. If $F$ is a subset of $\omega$, we will write $F(\Lambda) = F_\Lambda = \{y \in \omega \ : \ \Lambda y \in F\}$.

# 3 Sum, multiplier and product of special sets sequences.

In this section we recall some properties of the *sum* $\chi_a + \chi'_b$ where $\chi$, $\chi'$ are any of the symbols $\mathbf{s}^0$, $\mathbf{s}^{(c)}$, or $\mathbf{s}$.

### 3.1 Sum of sets of the form $\chi_a$, where $\chi$ is any of the symbols $\mathbf{s}^0$, $\mathbf{s}^{(c)}$, or $\mathbf{s}$.

We state some results concerning the *sum* of particular interesting sequence spaces.

Let $E$, $F \subset \omega$ be two linear spaces. The set $E+F$ is defined by

$$E+F = \{y \in \omega : y = u+v \text{ for some } u \in E \text{ and } v \in F\}.$$

It can easily be seen that $E+F = F$ if and only if $E \subset F$. This permits us to show some of the next results.

**Theorem 3.1.** *[5] Let $a, b \in U^+$.*

*(i) a) $\mathbf{s}_a \subset \mathbf{s}_b$ if and only if $a/b \in \ell_\infty$;*

*b) $\mathbf{s}_a = \mathbf{s}_b$ if and only if there are $K_1$, $K_2 > 0$ such that*

$$K_1 \leq \frac{b_n}{a_n} \leq K_2 \text{ for all } n,$$

*c) $\mathbf{s}_a + \mathbf{s}_b = \mathbf{s}_{a+b} = \mathbf{s}_{\max(a,b)}$, where $[\max(a,b)]_n = \max(a_n, b_n)$;*

*d) $\mathbf{s}_a + \mathbf{s}_b = \mathbf{s}_a$ if and only if $b/a \in \ell_\infty$.*

*(ii) a) $\mathbf{s}_a^0 \subset \mathbf{s}_b^0$ if and only if $a/b \in \ell_\infty$;*

*b) $\mathbf{s}_a^0 = \mathbf{s}_b^0$ if and only if $\mathbf{s}_a = \mathbf{s}_b$;*

*c) $\mathbf{s}_a^0 + \mathbf{s}_b^0 = s_{a+b}^0$;*

*d) $\mathbf{s}_a^0 + \mathbf{s}_b^0 = \mathbf{s}_a^0$ if and only if $b/a \in l_\infty$;*

*e) $\mathbf{s}_a^{(c)} \subset \mathbf{s}_b^{(c)}$ if and only if $a/b \in c$;*

*f) Consider the condition*

$$a_n/b_n \to l \ (n \to \infty) \text{ for some } l > 0. \tag{3.1}$$

*Then condition (3.1) is equivalent to $\mathbf{s}_a^{(c)} = \mathbf{s}_b^{(c)}$, and (3.1) implies $\mathbf{s}_a = \mathbf{s}_b$, $\mathbf{s}_a^0 = \mathbf{s}_b^0$ and $\mathbf{s}_a^{(c)} = \mathbf{s}_b^{(c)}$.*

*(iii) a) $\mathbf{s}_{a+b}^{(c)} \subset \mathbf{s}_a^{(c)} + \mathbf{s}_b^{(c)}$;*

*b) The condition $a/(a+b) \in c$ is equivalent to $\mathbf{s}_a^{(c)} + \mathbf{s}_b^{(c)} = \mathbf{s}_{a+b}^{(c)}$;*

*c) The condition $b/a \in c$ is equivalent to $\mathbf{s}_a^{(c)} + \mathbf{s}_b^{(c)} = \mathbf{s}_{a+b}^{(c)} = \mathbf{s}_a^{(c)}$;*

*(iv) $\mathbf{s}_a^0 + \mathbf{s}_b^{(c)} = \mathbf{s}_b^{(c)}$ is equivalent to $a/b \in \ell_\infty$, and the condition $b/a \in c_0$ implies $\mathbf{s}_a^0 + \mathbf{s}_b^{(c)} = \mathbf{s}_a^0$.*

*(v) a) $\mathbf{s}_a + \mathbf{s}_b^0 = \mathbf{s}_a^0$ is equivalent to $b/a \in \ell_\infty$;*

*b) $\mathbf{s}_a + \mathbf{s}_b^0 = \mathbf{s}_b^0$ is equivalent to $a/b \in c_0$.*

*(vi) a) $\mathbf{s}_a^{(c)} + \mathbf{s}_b = \mathbf{s}_a^{(c)}$ is equivalent to $b/a \in c_0$;*

*b) $\mathbf{s}_a^{(c)} + \mathbf{s}_b = \mathbf{s}_b$ is equivalent to $a/b \in \ell_\infty$;*

*c) $\mathbf{s}_a^{(c)} + \mathbf{s}_b^{(c)} = \mathbf{s}_a^{(c)}$ is equivalent to $b/a \in c$.*

As a direct consequence of the preceding we obtain the next result.

**Corollary 3.2.** *[5, 6] The following properties are equivalent,*

*(i) $b/a \in \ell_\infty$,*

*(ii) $\mathbf{s}_a^0 + \mathbf{s}_b^0 = s_a^0$,*

*(iii) $\mathbf{s}_a + \mathbf{s}_b = \mathbf{s}_a$,*

*(iv) $\mathbf{s}_a^{(c)} + \mathbf{s}_b^0 = \mathbf{s}_a^{(c)}$,*

*(v) $\mathbf{s}_a + \mathbf{s}_b^{(c)} = \mathbf{s}_a$.*

## 3.2 The multiplier of certain sets of sequences

First we need to recall some well-known results. We have

**Lemma 3.3.** *Let $E$ and $F$ be arbitrary subsets of $\omega$, $u = (u_n)_{n\geq 1}$ with $u_n \neq 0$. Then*
*(i) $M(E,F) \subset M\left(\widetilde{E},F\right)$ for all $\widetilde{E} \subset E$,*
*(ii) $M(E,F) \subset M\left(E,\widetilde{F}\right)$ for all $F \subset \widetilde{F}$.*

We also have, see [5, Lemma 3.1, p. 648] and [5, Example 1.28, p. 157],

**Lemma 3.4.** *(i) $M(c_0,\chi) = \ell_\infty$ for $\chi = c_0$, $c$, or $\ell_\infty$,*
*(ii) $M(\ell_\infty, c_0) = c_0$ and $M(\ell_\infty,\ell_\infty) = \ell_\infty$,*
*(iii) $M(c,c) = c$ and $M(\ell_\infty, c) = c_0$.*

*Remark* 3.5. Since $c_0 \subset c \subset \ell_\infty$, it can easily be deduced from Lemma 3.3 and Lemma 3.4 that $\ell_\infty = M(\ell_\infty,\ell_\infty) \subset M(c,\ell_\infty) \subset M(c_0,\ell_\infty) = \ell_\infty$ and $M(c,\ell_\infty) = \ell_\infty$. We also have

$$M(c,c_0) = M(\ell_\infty,c_0) = c_0 \text{ and } M(c_0,c_0) = \ell_\infty.$$

We deduce the next corollary from the preceding one.

**Corollary 3.6.** *[6] (i) $M\left(\mathbf{s}_a^0,\chi_b'\right) = \mathbf{s}_{b/a}$ where $\chi'$ is any of the symbols $\mathbf{s}^0$, $\mathbf{s}^{(c)}$, or $\mathbf{s}$,*
*(ii) $M(\chi_a,\mathbf{s}_b) = s_{b/a}$ where $\chi$ is any of the symbols $\mathbf{s}^{(c)}$, or $\mathbf{s}$,*
*(iii) $M\left(\mathbf{s}_a,\mathbf{s}_b^{(c)}\right) = \mathbf{s}_{b/a}^0$ and $M\left(\mathbf{s}_a^{(c)},\mathbf{s}_b^{(c)}\right) = \mathbf{s}_{b/a}^{(c)}$.*

## 3.3 Product of sets of the form $\chi_\xi$ where $\chi$ is any of the symbols s, $\mathbf{s}^0$, or $\mathbf{s}^{(c)}$.

In this subsection we will deal with some properties of the *product* $E * F$ of particular subsets $E$ and $F$ of $\omega$. These results generalize some of those given in [5, 10].

For any given sets of sequences $E$ and $F$, we will write

$$E * F = \{uv \in \omega : \ u \in E \text{ and } v \in F\}.$$

We immediately have the following results,

**Proposition 3.7.** *Let $a$, $b$, $\gamma \in U^+$. Then*
*(i) $\mathbf{s}_a * \mathbf{s}_b = \mathbf{s}_a * \mathbf{s}_b^{(c)} = \mathbf{s}_{ab}$,*
*(ii) $s_a * \mathbf{s}_b^0 = \mathbf{s}_a^0 * \mathbf{s}_b^0 = \mathbf{s}_a^{(c)} * \mathbf{s}_b^0 = \mathbf{s}_{ab}^0$,*
*(iii) $s_a^{(c)} * \mathbf{s}_b^{(c)} = \mathbf{s}_{ab}^{(c)}$.*
*(iv) $\mathbf{s}_a^{(c)} * \mathbf{s}_b = \mathbf{s}_{ab}$,*
*(v) Let $\chi$ be any of the symbols $\mathbf{s}$, $\mathbf{s}^0$, or $\mathbf{s}^{(c)}$.*
*Then $\chi_a * \mathbf{s}_b^0 = \mathbf{s}_\gamma^0$ if and only if there are $K_1$, $K_2 > 0$ such that*

$$K_1\gamma_n \leq a_n b_n \leq K_2\gamma_n \text{ for all } n.$$

*(vi) a) $\mathbf{s}_a * \mathbf{s}_b = \mathbf{s}_a * \mathbf{s}_\gamma$ if and only if $\mathbf{s}_b = \mathbf{s}_\gamma$,*
*b) $\mathbf{s}_a^0 * \mathbf{s}_b^0 = \mathbf{s}_a^0 * \mathbf{s}_\gamma^0$ if and only if $\mathbf{s}_b^0 = \mathbf{s}_\gamma^0$.*

## 4 Solvability of some sequence spaces inclusion equations

In this section among other we determine for given sequences $a$ and $b$ the set of all sequences $x = (x_n)_{n\geq 1} \in U^+$ such that $\mathbf{s}_b^{(c)} \subset \mathbf{s}_a^{(c)} + \mathbf{s}_x$. The last (SSIE) means that for every $y \in \omega$ the condition

$$\frac{y_n}{b_n} \to l \ (n \to \infty)$$

implies there are $u$ and $v \in \omega$ such that $y = u + v$ and $u_n/a_n \to l_1$ and $v_n/x_n = O(1)$ $(n \to \infty)$ for some scalars $l$ and $l_1$.

### 4.1 (SSIE) of the form $\chi''_b \subset \chi_a + \chi'_x$ with $\chi, \chi', \chi'' \in \left\{\mathbf{s}^0, \mathbf{s}^{(c)}, \mathbf{s}\right\}$.

**Theorem 4.1.** *[7] Let $a$ and $b \in U^+$.*

*(i) Assume $a/b \in c_0$. Each one of the next (SSIE) where $\chi$, $\chi'$ are any of the symbols* $\mathbf{s}$, $\mathbf{s}^0$, *or* $\mathbf{s}^{(c)}$

*a)* $\mathbf{s}_b^0 \subset \chi_a + \chi'_x$,

*b)* $\chi''_b \subset \chi_a + s_x$ *where* $\chi'' \in \left\{\mathbf{s}, \mathbf{s}^{(c)}\right\}$,

*is equivalent to* $\mathbf{s}_x \supset \mathbf{s}_b$, *that is,* $x_n \geq K b_n$ *for all n and for some* $K > 0$.

*(ii) Let $a/b \in \ell_\infty$. Then the (SSIE)*

$\chi''_b \subset \mathbf{s}_a^0 + \mathbf{s}_x^0$ *where* $\chi''$ *is* $\mathbf{s}^{(c)}$, *or* $\mathbf{s}$

*is equivalent to* $\mathbf{s}_b \subset \mathbf{s}_x^0$, *that is,* $\lim_{n\to\infty} x_n/b_n = \infty$.

*(iii) If $b/a \in \ell_\infty$ then each of the next (SSIE) holds for all $x \in U^+$ where*

*a)* $s_b^0 \subset \chi_a + \chi'_x$ *where* $\chi, \chi'$ *are any of the symbols* $\mathbf{s}$, $\mathbf{s}^0$, *or* $\mathbf{s}^{(c)}$;

*b)* $\chi''_b \subset \mathbf{s}_a + \mathbf{s}_x$ *where* $\chi'' \in \left\{\mathbf{s}, \mathbf{s}^{(c)}\right\}$.

We immediately deduce the following.

**Corollary 4.2.** *Let $a \in c_0$ and let $\chi$, $\chi'$ be any of the symbols* $\mathbf{s}$, $\mathbf{s}^0$, *or* $\mathbf{s}^{(c)}$. *Then*

*(i) each of the inclusion equations*

$$c_0 \subset \chi_a + \chi'_x,$$

$$\chi''_1 \subset \chi_a + \mathbf{s}_x,$$

*where $\chi''$ is either $s^{(c)}$, or $s$ is equivalent to $x_n \geq K$ for all n and for some $K > 0$.*

*(ii) If $a \in \ell_\infty$, then the inclusion equation*

$$\chi''_1 \subset \mathbf{s}_a^0 + \mathbf{s}_x^0,$$

*where* $\chi''$ *is either* $\mathbf{s}^{(c)}$, *or* $\mathbf{s}$ *is equivalent to* $x_n \to \infty$ $(n \to \infty)$.

### 4.2 The operators $C(\xi)$, $\Delta(\xi)$ and the sets $\widehat{\Gamma}$, $\widehat{C}$, $\Gamma$ and $\widehat{C_1}$

An infinite matrix $T = (t_{nk})_{n,k\geq 1}$ is said to be a triangle if $t_{nk} = 0$ for $k > n$ and $t_{nn} \neq 0$ for all $n$. Now let $U$ be the set of all sequences $(u_n)_{n\geq 1} \in \omega$ with $u_n \neq 0$ for all $n$. The next operators are used for many applications, see for instance [4, 13, 12, 14]. The triangle $C(\xi)$ for $\xi = (\xi_n)_{n\geq 1} \in U$, is defined by $[C(\xi)]_{nk} = 1/\xi_n$ for $k \leq n$. The infinite matrix $\Delta(\xi)$ is the triangle whose the non-zero entries are given by $[\Delta(\xi)]_{nn} = \xi_n$, and $[\Delta(\xi)]_{n,n-1} = -\xi_{n-1}$, for all $n$, with the convention $\xi_0 = 0$. It can be shown that the triangle $\Delta(\xi)$ is the inverse of $C(\xi)$, that is, $C(\xi)(\Delta(\xi)y) = \Delta(\xi)(C(\xi)y) = y$ for all $y \in \omega$. If $\xi = e$ we obtain the well-known operator of the *first difference* denoted by $\Delta(e) = \Delta$. We then have $\Delta y_n = y_n - y_{n-1}$ for all $n \geq 1$, with the convention $y_0 = 0$. It is usually written $\Sigma = C(e)$. Note that $\Delta = \Sigma^{-1}$ and $\Delta, \Sigma \in S_R$ for any $R > 1$.

Consider the sets

$$\widehat{C} = \left\{\xi \in U^+ : [C(\xi)\xi]_n = \frac{1}{\xi_n}\sum_{k=1}^{n}\xi_k \to l \ (n \to \infty) \text{ for some scalar } l\right\}$$

$$\widehat{C_1} = \left\{\xi \in U^+ : \quad [C(\xi)\xi]_n = \frac{1}{\xi_n}\sum_{k=1}^{n}\xi_k = O(1) \ \ (n \to \infty)\right\},$$

$$\widehat{\Gamma} = \left\{\xi \in U^+ : \lim_{n\to\infty}\left(\frac{\xi_{n-1}}{\xi_n}\right) < 1\right\},$$

$$\Gamma = \left\{\xi \in U^+ : \limsup_{n\to\infty}\left(\frac{\xi_{n-1}}{\xi_n}\right) < 1\right\}$$

and

$$G_1 = \{\xi \in U^+ : \xi_n \geq C\gamma^n \text{ for all } n \text{ and for some } C > 0 \text{ and } \gamma > 1\}.$$

By [4, Proposition 2.1, p. 1786] and [9, Proposition 2.2 p. 88] we obtain the next lemma.

**Lemma 4.3.** *We have* $\widehat{C} = \widehat{\Gamma} \subset \Gamma \subsetneq \widehat{C_1} \subset G_1$.

We also need the next results.

**Lemma 4.4.** *[5, Proposition 9, p. 300] Let* $a, b \in U^+$. *Then*
*(i) following statements are equivalent*
*(a)* $\chi_a(\Delta) = \chi_b$ *where* $\chi$ *is any of the symbols* $\mathbf{s}$, *or* $\mathbf{s}^0$,
*(b)* $a \in \widehat{C_1}$ *and* $\mathbf{s}_a = \mathbf{s}_b$.
*(ii)* $a \in \widehat{\Gamma}$ *if and only if* $\mathbf{s}_a^{(c)}(\Delta) = \mathbf{s}_a^{(c)}$.

We then have the following examples.

**Example 4.5.** Assume $b \in \widehat{\Gamma}$ and $a/b \in c_0$ and consider the (SSIE)

$$\mathbf{s}_b^{(c)}(\Delta) \subset \mathbf{s}_a^0 + \mathbf{s}_x. \tag{4.1}$$

We have $\mathbf{s}_b^{(c)}(\Delta) = \mathbf{s}_b^{(c)}$ since $b \in \widehat{\Gamma}$, and (4.1) is equivalent to $\mathbf{s}_b^{(c)} \subset \mathbf{s}_a^0 + \mathbf{s}_x$. Then by Theorem 4.1, inclusion equation (4.1) holds if and only if $s_x \supset s_b$, that is, $x_n \geq Kb_n$ for all $n$.

We are led to state the next application which is a direct application of the preceding.

**Example 4.6.** Let $r, u > 0$ with $r > 1$ and $u < r$. Consider the set $\Upsilon$ of all $x \in U^+$ such that

$$\frac{\Delta y_n}{r^n} \to l \text{ implies } y_n = o(u^n) + O(x_n) \ \ (n \to \infty) \text{ for some } l \in \mathbb{C} \text{ and for all } y \in \omega.$$

$\Upsilon$ is determined by the solutions of the inclusion equation $\mathbf{s}_r^{(c)}(\Delta) \subset \mathbf{s}_u^0 + \mathbf{s}_x$, where $x \in U^+$ is the unknown. By Lemma 4.4 we have $\mathbf{s}_r^{(c)}(\Delta) = \mathbf{s}_r^{(c)}$ and as we have just seen we easily deduce that $\Upsilon$ is the set of all sequences $x$ such that $x_n \geq Kr^n$ for all $n$.

# 5 On some (SSIE) and (SSE) with operators of the form $B(r,s)$

## 5.1 The (SSIE) $\chi_x(B(r,s)) \subset \chi_x(B(r',s'))$ where $\chi = \mathbf{s}$, or $\mathbf{s}^0$.

In this subsection among other things, we are interested in the study of the equivalence

$$ry_n + sy_{n-1} = O(x_n) \text{ if and only if } r'y_n + s'y_{n-1} = O(x_n) \ (n \to \infty) \text{ for all } y \in \omega,$$

for $r$, $s$, $r'$, $s'$ reals, which consists in determining the set of all $x \in U^+$ such that $s_x(B(r,s)) = \mathbf{s}_x(B(r',s'))$, similarly we easily see that the equivalence

$$\frac{ry_n + sy_{n-1}}{x_n} \to 0 \text{ if and only if } \frac{r'y_n + s'y_{n-1}}{x_n} \to 0 \ (n \to \infty), \text{ for all } y \in \omega,$$

reduces to the study of the (SSE) with operators $\mathbf{s}_x^0(B(r,s)) = \mathbf{s}_x^0(B(r',s'))$.

Let $B(r,s)$ where $r$, $s$ are reals be the lower triangular matrix

$$B(r,s) = \begin{pmatrix} r & & & & \\ s & r & & 0 & \\ & s & r & & \\ 0 & & . & . & \\ & & & . & . \end{pmatrix}.$$

For $r$, $s \neq 0$, the matrix $B(r,s)$ was introduced by Altay and Basar [1] and was called the *generalized operator of first difference*. When $r = -s = 1$, the matrix $B(r,s)$ reduces to the operator of first difference $\Delta$. Here we deal with the (SSIE) with operators

$$\mathbf{s}_x(B(r,s)) \subset \mathbf{s}_x(B(r',s')). \tag{5.1}$$

In the following we use the notations $\alpha = s/r$, $\alpha' = s'/r'$ for $r, r' \neq 0$, $\delta = \begin{vmatrix} r & r' \\ s & s' \end{vmatrix} = rs' - r's$ and we write $\widetilde{B} = B(r',s')B^{-1}(r,s)$ for $r \neq 0$. Now we can state the next result.

**Theorem 5.1.** *Let $r$, $s$, $r'$ and $s'$ be reals.*

*(i) Let $r$, $s \neq 0$. Then*

*(a) if $\delta = 0$, then (SSIE) (5.1) holds for all $x$, and we have $s_x(B(r,s)) = s_x(B(r',s'))$;*

*(b) if $\delta \neq 0$, then (5.1) holds if and only if*

$$x \in D_{(|\alpha|^n)_n} * \widehat{C_1},$$

*(that is $(x_n/|\alpha|^n)_n \in \widehat{C_1}$). So if*

$$\overline{\lim_{n\to\infty}} \left(\frac{x_{n-1}}{x_n}\right) < \left|\frac{r}{s}\right|, \tag{5.2}$$

*then (5.1) holds.*

*(ii) Assume $r \neq 0$ and $s = 0$. Then*

*a) if $s' \neq 0$ then (SSIE) (5.1) holds if and only if*

$$\sup_n \left(\frac{x_{n-1}}{x_n}\right) < \infty \text{ ;}$$

*b) if* $s' = 0$ *then (5.1) holds for all x.*
*(iii) Assume* $r = 0$ *and* $s \neq 0$. *Then*
*a) if* $r' = 0$ *then (5.1) holds for all x;*
*b) if* $r' \neq 0$ *then (5.1) holds if and only if*

$$\sup_n \left( \frac{x_n}{x_{n-1}} \right) < \infty. \tag{5.3}$$

*(iv) If* $r = s = 0$, *then*
*a) if* $r'$, *or* $s' \neq 0$ *then (5.1) has no solution,*
*b) if* $r' = s' = 0$ *then (5.1) holds for all x.*

*Proof.* (i) First for $r \neq 0$ the band matrix $B(r,s)$ is invertible, its inverse is a triangle and elementary calculations give

$$\left[B^{-1}(r,s)\right]_{nk} = \frac{1}{r}(-\alpha)^{n-k} \text{ for } 1 \leq k \leq n.$$

Inclusion (5.1) is equivalent to $I \in (\mathbf{s}_x(B(r,s)), \mathbf{s}_x(B(r',s')))$, that is, to

$$\widetilde{B} = B(r',s')B^{-1}(r,s) \in (\mathbf{s}_x, \mathbf{s}_x).$$

This means

$$D_{1/x}\widetilde{B}D_x \in (\ell_\infty, \ell_\infty). \tag{5.4}$$

For $k = n$ we obtain $\widetilde{B}_{nn} = r'/r$, and for $k \leq n-1$ we have

$$\begin{aligned} \widetilde{B}_{nk} &= s'\left[B^{-1}(r,s)\right]_{n-1,k} + r'\left[B^{-1}(r,s)\right]_{nk} \\ &= s'\frac{1}{r}(-\alpha)^{n-k-1} + \frac{r'}{r}(-\alpha)^{n-k} \\ &= (-\alpha)^{n-k-1}\left[\frac{s'}{r} + \frac{r'}{r}\left(-\frac{s}{r}\right)\right] \\ &= (-\alpha)^{n-k-1}\frac{\delta}{r^2}. \end{aligned}$$

From the characterization of $(\ell_\infty, \ell_\infty)$ we deduce that (5.4) holds if and only if

$$\sum_{k=1}^{n} \left| \left[D_{1/x}\widetilde{B}D_x\right]_{nk} \right| = \left|\frac{r'}{r}\right| + \left|\frac{\delta}{rs}\right| \left( \frac{1}{\frac{x_n}{|\alpha|^n}} \sum_{k=1}^{n-1} \frac{x_k}{|\alpha|^k} \right) \leq K$$

for all $n$ and for some $K$.

(a) If $\delta = 0$ the previous sum reduces to $|r'/r|$ and the inclusion (5.1) holds for all $x$.

(b) If $\delta \neq 0$ it can easily be deduced that (5.1) holds if and only if $(x_n/|\alpha|^n)_n \in \widehat{C_1}$. Then we have $(x_n/|\alpha|^n)_n \in \Gamma$ if and only if (5.2) holds, and since $\widehat{C_1} \supset \Gamma$ by Lemma 4.3, we conclude that (5.2) implies (5.1). This concludes the proof of (i) b).

(ii) Case $r \neq 0$ and $s = 0$. Since $B(r,s) = rI$ we have

$$\mathbf{s}_x(B(r,s)) = \mathbf{s}_x.$$

So inclusion (5.1) is equivalent to $D_{1/x}B(r',s')D_x \in (\ell_\infty,\ell_\infty)$, that is,

$$\left|r'\right| + \left|s'\right| \frac{x_{n-1}}{x_n} \leq K' \text{ for all } n. \tag{5.5}$$

If $s' \neq 0$ then

$$\frac{x_{n-1}}{x_n} \leq \frac{K' - |r'|}{|s'|} \text{ for all } n$$

and if $s' = 0$ condition (5.5) holds for all $x$.

(iii) Case $r = 0$, $s \neq 0$. We have

$$\mathbf{s}_x(B(0,s)) = \left\{ y \in \omega : \frac{y_n}{x_{n+1}} = O(1) \ \ (n \to \infty) \right\} = \mathbf{s}_{x^+}$$

where $x^+ = (x_{n+1})_n$. Then (5.1) is successively equivalent to $\mathbf{s}_{x^+} \subset \mathbf{s}_x(B(r',s'))$, $B(r',s') \in (\mathbf{s}_{x^+},\mathbf{s}_x)$ and to

$$\left|r'\right| \frac{x_{n+1}}{x_n} + \left|s'\right| \leq K'' \text{ for all } n \text{ and for some } K'' > 0. \tag{5.6}$$

(a) If $r' = 0$, then (5.6) trivially holds for all $x$.

(b) If $r' \neq 0$, we then have $x_{n+1}/x_n \leq (K'' - |s'|)/|r'|$ for all $n$ and inclusion equation (5.1) holds for all $x \in U^+$.

(iv) a) Assume $r = s = 0$. Then $s_x(B(0,0)) = \omega$ and the inclusion $\omega \subset s_x(B(r',s'))$ implies $r' = s' = 0$. Indeed assume either $r'$, or $s'$ is different from zero. Let $r' \neq 0$ and consider the cases $s'/r' \geq 0$ and $s'/r' < 0$. If $s'/r' \geq 0$ take $y = (R^n x_n)_n \in \omega$ with $R > 1$, we then have

$$\left|\frac{B(r',s')y_n}{x_n}\right| = \frac{|r'|}{x_n}\left|y_n + \frac{s'}{r'}y_{n-1}\right| \geq \left|r'\right| R^n \text{ for all } n.$$

Then

$$\left|\frac{B(r',s')y_n}{x_n}\right| \to \infty \ \ (n \to \infty)$$

and $\omega \subset s_x(B(r',s'))$ is impossible. If $s'/r' < 0$ taking $y_n = (-R)^n x_n$ with $R > 1$ we then have

$$\left|\frac{B(r',s')y_n}{x_n}\right| = \left|\frac{r'}{x_n}\left(y_n + \frac{s'}{r'}y_{n-1}\right)\right| \geq \left|r'\right| R^{n-1} \text{ for all } n$$

and we conclude as above. The case $s' \neq 0$ can be treated similarly.

b) is trivial. □

**Proposition 5.2.** *Theorem 5.1 holds when $s_x$ is replaced by $s_x^0$.*

*Proof.* The proof follows the same lines as above. Note that here in part (i) b) we have $D_{1/x}\widetilde{B}D_x \in (c_0,c_0)$ equivalent to (5.4) and $\left[D_{1/x}\widetilde{B}D_x\right]_{nk} \to 0$ $(n \to \infty)$. But for $\delta \neq 0$, condition (5.4) implies $(x_n/|\alpha|^n)_n \in \widehat{C_1}$ and $x_n/|\alpha|^n \to \infty$ $(n \to \infty)$ by Lemma 4.3, then for $n > k$ we have

$$\left[D_{1/x}\widetilde{B}D_x\right]_{nk} = \frac{1}{x_n}(-\alpha)^{n-k-1}\frac{\delta}{r^2}x_k = \frac{(-\alpha)^n}{x_n}\left[(-\alpha)^{-k-1}\frac{\delta}{r^2}x_k\right] = o(1) \ \ (n \to \infty) \text{ for all } k.$$

□

## 5.2 On some (SSE) with operators $B(r,s)$ and $B(r',s')$

### 5.2.1 The (SSE) $\mathbf{s}_x(B(r,s)) = \mathbf{s}_x(B(r',s'))$

Consider now the next sequence spaces equations with operators

$$\mathbf{s}_x(B(r,s)) = \mathbf{s}_x \tag{5.7}$$

and

$$\mathbf{s}_x(B(r,s)) = \mathbf{s}_x(B(r',s')). \tag{5.8}$$

We immediately deduce from Theorem 5.1 the following corollaries.

**Corollary 5.3.** *(i) Let $r,\ s \neq 0$. Then (SSE) (5.7) holds if and only if $x \in D_{(|\alpha|^n)_n} * \widehat{C_1}$.*

*(ii) If $r \neq 0,\ s = 0$ then (5.7) holds for all $x$.*

*(iii) If $r = 0,\ s \neq 0$ then (5.7) holds if and only if there are $K_1,\ K_2 > 0$ such that*

$$K_1 \leq \frac{x_n}{x_{n-1}} \leq K_2 \text{ for all } n.$$

*(iv) If $r = s = 0$ then (5.7) has no solution.*

*Proof.* Proof of (i). First we have $B(r,s) \in (\mathbf{s}_x, \mathbf{s}_x)$ if and only if

$$\frac{x_{n-1}}{x_n} \leq K \text{ for all } n\text{, and for some } K > 0, \tag{5.9}$$

and from the expression of $B^{-1}(r,s)$, we have $B^{-1}(r,s) \in (\mathbf{s}_x, \mathbf{s}_x)$ if and only if

$$\frac{1}{x_n} \sum_{k=1}^{n} |\alpha|^{n-k} x_k \leq K' \text{ for all } n,$$

that is, $x \in D_{(|\alpha|^n)_n} * \widehat{C_1}$. Now $x \in D_{(|\alpha|^n)_n} * \widehat{C_1}$ implies

$$\frac{1}{x_n} |\alpha|^{n-(n-1)} x_{n-1} \leq K' \text{ for all } n,$$

that is, (5.9). So (5.9) holds if and only if $x \in D_{(|\alpha|^n)_n} * \widehat{C_1}$. This concludes the proof of part (i).

(ii), (iii) and (iv) are direct consequences of Theorem 5.1. □

In the next corollary we limit our study to the case when $r,\ s \neq 0$.

**Corollary 5.4.** *Let $r,\ s \neq 0$.*

*(i) Let $r',\ s' \neq 0$.*

*(a) If $\delta = 0$ then (SSE) (5.8) holds for all $x$,*

*(b) if $\delta \neq 0$ then (5.8) holds if and only if*

$$x \in \left(D_{(|\alpha|^n)_n} * \widehat{C_1}\right) \cap \left(D_{(|\alpha'|^n)_n} * \widehat{C_1}\right). \tag{5.10}$$

*(ii) Case $r'$, or $s' = 0$.*

*(a) Let $r' \neq 0$ and $s' = 0$. Then (5.8) holds if and only if $x \in D_{(|\alpha|^n)_n} * \widehat{C_1}$.*

*(b) Let $r' = 0$ and $s' \neq 0$. Then (5.8) holds if and only if $\sup_n (x_n/x_{n-1}) < \infty$ and $x \in D_{(|\alpha|^n)_n} * \widehat{C_1}$.*

*(iii) If $r' = s' = 0$, then (5.8) has no solution.*

*Proof.* (i) (a) is a direct consequence of Theorem 5.1 (i) (a). Now we show (i) (b). From Theorem 5.1 (i) (b) we have $\mathbf{s}_x(B(r,s)) \subset \mathbf{s}_x(B(r',s'))$ if and only if $x \in D_{(|\alpha|^n)_n} * \widehat{C_1}$. Since $r'$, $s' \neq 0$ we have $\delta \neq 0$ and the (SSIE) $\mathbf{s}_x(B(r',s')) \subset \mathbf{s}_x(B(r,s))$ is equivalent to $x \in D_{(|\alpha'|^n)_n} * \widehat{C_1}$ and we conclude (5.8) holds if and only if (5.10) holds. (ii) (a) we have $r' \neq 0$ and $s' = 0$ and $\mathbf{s}_x(B(r',s')) = \mathbf{s}_x(I) = \mathbf{s}_x$. By Corollary 5.3 (i), (SSE) (5.8) is equivalent to $x \in D_{(|\alpha|^n)_n} * \widehat{C_1}$. (ii) (b) By Theorem 5.1 (i) (b) the solutions of the (SSIE) $\mathbf{s}_x(B(r,s)) \subset \mathbf{s}_x(B(0,s'))$ are determined by $x \in D_{(|\alpha|^n)_n} * \widehat{C_1}$, since $\begin{vmatrix} r & 0 \\ s & s' \end{vmatrix} = rs' \neq 0$. By Theorem 5.1 (iii) (b) the inclusion $\mathbf{s}_x(B(0,s')) \subset \mathbf{s}_x(B(r,s))$ is equivalent to $\sup_n (x_n/x_{n-1}) < \infty$. Finally (iii) is a consequence of Theorem 5.1 (iv). □

Now we deal with the equivalence

$$\Delta y_n = O(x_n) \ (n \to \infty)$$

if and only if

$$r y_n + s y_{n-1} = O(x_n) \ (n \to \infty) \text{ for all } y.$$

This statement leads to study the identity

$$\mathbf{s}_x(B(r,s)) = \mathbf{s}_x(\Delta). \tag{5.11}$$

From Corollary 5.4 we obtain the next results.

**Corollary 5.5.** *(i) Let $r$, $s \neq 0$.*

*(a) If $r = -s$ then (SSE) (5.11) holds for all $x$.*

*(b) If $r \neq -s$ then (5.11) holds if and only if*

$$x \in \widehat{C_1} \cap \left(D_{(|\alpha|^n)_n} * \widehat{C_1}\right).$$

*(ii) Let $r \neq 0$ and $s = 0$. Then (5.11) holds if and only if $x \in \widehat{C_1}$.*

*(iii) Let $r = 0$ and $s \neq 0$. Then (5.11) holds if and only if*

$$\sup_n \left(\frac{x_n}{x_{n-1}}\right) < \infty \text{ and } x \in \widehat{C_1}.$$

*(iv) If $r = s = 0$ then (5.11) has no solution.*

*Remark* 5.6. Note that for $a \in \omega$ and $b \in U$, if $\sup_{k \leq n} |a_k/b_k| \leq K |a_n/b_n|$ for all $n$ and for some $K > 0$, then

$$D_{|a|} * \widehat{C_1} \subset D_{|b|} * \widehat{C_1}.$$

In this way in Corollary 5.5 we have $\widehat{C_1} \cap \left(D_{(|\alpha|^n)_n} * \widehat{C_1}\right) = \widehat{C_1}$ if $|\alpha| \leq 1$, and $\widehat{C_1} \cap \left(D_{(|\alpha|^n)_n} * \widehat{C_1}\right) = D_{(|\alpha|^n)_n} * \widehat{C_1}$ if $|\alpha| \geq 1$.

*Remark* 5.7. The results given in Corollary 5.4 and Corollary 5.5 are also true for the (SSE) $\mathbf{s}_x^0(B(r,s)) = \mathbf{s}_x^0(B(r',s'))$.

This remark leads to the next proposition which is a consequence of Theorem 5.1 and Corollary 5.3,

**Proposition 5.8.** *Let $r$, $s \neq 0$ and let $x \in U^+$. Then the next statements are equivalent, where $\chi = \mathbf{s}$, or $\mathbf{s}^0$,*

*i)* $\chi_x(B(r,s)) \subset \chi_x$,

*ii)* $\chi_x(B(r,s)) = \chi_x$,

*iii)* $B(r,s) \in (\chi_x, \chi_x)$ *is surjective,*

*iv)* $B(r,s) \in (\chi_x, \chi_x)$ *is bijective,*

*v)* $x \in D_{(|\alpha|^n)_n} * \widehat{C_1}$.

*Remark* 5.9. Note that in Proposition 5.8 $r$ and $s$ can be non-zero complex numbers. This permits us to state the results of the next subsection.

### 5.2.2 Application to the spectrum of $B(r,s)$ considered as an operator from $\chi_x$ to itself with $\chi = \mathbf{s}$, or $\mathbf{s}^0$

We obtain the next result where $\sigma(B(r,s), \chi_x)$ is the spectrum of the continuous operator $B(r,s) \in (\chi_x, \chi_x)$, where $\chi = \mathbf{s}$, or $\mathbf{s}^0$, that is,

$$\sigma(B(r,s), \chi_x) = \{\lambda \in \mathbb{C} : B(r,s) - \lambda I \text{ as operator from } \chi_x \text{ to itself is not invertible}\}.$$

We write $\rho(B(r,s), \chi_x) = \mathbb{C} \backslash \sigma(B(r,s), \chi_x)$ for the resolvent set of $B(r,s)$. So we have $\lambda \in \rho(B(r,s), \chi_x)$ if and only if $B(r,s) - \lambda I = B(r-\lambda, s)$ as operator from $\mathbf{s}_x$ to itself is invertible. For $x \in U^+$ we put

$$\mathbf{S}_x = \{\lambda \in \mathbb{C} : \chi_x(B(r,s) - \lambda I) \subset \chi_x\},$$

and for $\lambda \neq r$ we put

$$\mathbf{S}'_\lambda = \{x \in U^+ : \chi_x(B(r,s) - \lambda I) \subset \chi_x\}.$$

Then we obtain the following result.

**Corollary 5.10.** *Let $r$, $s \neq 0$, let $\chi = \mathbf{s}$, or $\mathbf{s}^0$. Then we have*

*i) for $\lambda \neq r$*

$$\mathbf{S}_x = \rho(B(r,s), \chi_x) \text{ for all } x \in U^+;$$

*and*

$$\mathbf{S}'_\lambda = D_{(|\alpha_\lambda|^n)_n} * \widehat{C_1}, \tag{5.12}$$

*where $\alpha_\lambda = s/(r-\lambda)$.*

*ii) For each $x \in U^+$ we have*

*a)*

$$\lambda \in \sigma(B(r,s), \chi_x) \text{ if and only if } \lambda = r, \text{ or } \left(\left|\frac{\lambda - r}{s}\right|^n x_n\right)_n \notin \widehat{C_1}; \tag{5.13}$$

*b)*

$$\lambda \in \sigma(B(r,s), \chi_x) \text{ implies } |\lambda - r| \leq |s| \overline{\lim_{n \to \infty}} \left(\frac{x_{n-1}}{x_n}\right). \tag{5.14}$$

*Proof.* We only consider $\chi = \mathbf{s}$, since the proof in the case $\chi = \mathbf{s}^0$ follows exactly the same lines. i) Let $\lambda \in \mathbf{S}_x$ with $\lambda \neq r$. We then have

$$B^{-1}(r-\lambda, s) \in (\mathbf{s}_x, \mathbf{s}_x) \tag{5.15}$$

so by Proposition 5.8 $B(r-\lambda, s) \in (\mathbf{s}_x, \mathbf{s}_x)$ is bijective and $\lambda \in \mathbf{S}_x$. Identity (5.12) is a direct consequence of the preceding and of Proposition 5.8.

ii) Let $x \in U^+$. a) We have $\lambda \in \rho(B(r,s), \chi_x)$ if and only if $B(r-\lambda, s) \in (\mathbf{s}_x, \mathbf{s}_x)$ is bijective, that is, $x \in D_{(|\alpha_\lambda|^n)_n} * \widehat{C_1}$ and $\lambda \neq r$, by Proposition 5.8. Thus we have (5.13). b) Here we show that if $\lambda$ satisfies

$$|\lambda - r| > |s| \overline{\lim_{n\to\infty}} \left( \frac{x_{n-1}}{x_n} \right), \tag{5.16}$$

then $\lambda \in \mathbf{S}_x$. Since we have $x \in D_{(|\alpha_\lambda|^n)_n} * \Gamma$ if and only if

$$\overline{\lim_{n\to\infty}} \left( \frac{|\alpha_\lambda|^n}{x_n} \frac{x_{n-1}}{|\alpha_\lambda|^{n-1}} \right) < 1, \ \overline{\lim_{n\to\infty}} \frac{x_{n-1}}{x_n} < \frac{1}{|\alpha_\lambda|} = \left| \frac{\lambda - r}{s} \right|,$$

we deduce that $\lambda$ satisfies (5.16) if and only if $x \in D_{(|\alpha_\lambda|^n)_n} * \Gamma$. But by Lemma 4.3 we have $D_{(|\alpha_\lambda|^n)_n} * \Gamma \subset D_{(|\alpha_\lambda|^n)_n} * \widehat{C_1}$, thus by Proposition 5.8 we conclude that (5.16) implies $\lambda \in \mathbf{S}_x = \rho(B(r,s), \chi_x)$. This completes the proof. □

## 6 On the (SSIE) $s_x^{(c)}(B(r,s)) \subset s_x^{(c)}(B(r',s'))$ for $r$, $s$, $r'$, $s'$ reals

In this subsection we determine the set of all $x \in U^+$ such that

$$\frac{ry_n + sy_{n-1}}{x_n} \to l \text{ implies } \frac{r'y_n + s'y_{n-1}}{x_n} \to l' \ (n \to \infty) \text{ for all } y$$

and for some scalars $l$, $l'$. For this consider the (SSIE)

$$\mathbf{s}_x^{(c)}(B(r,s)) \subset \mathbf{s}_x^{(c)}(B(r',s')). \tag{6.1}$$

We will solve (6.1) in the general case when $\alpha_1 = -s/r \neq 0$ in Remark 6.2, but in the next theorem, we develop the interesting case $\alpha_1 > 0$ when $\delta \neq 0$.

**Theorem 6.1.** *Let $r$, $s$, $r'$ and $s'$ be reals.*

*(i) Let $r$, $s \neq 0$.*

*(a) If $\delta = 0$, then (SSIE) (6.1) holds for all $x$,*

*(b) if $\delta \neq 0$ and $\alpha_1 = -s/r > 0$, then (6.1) holds if and only if*

$$\lim_{n\to\infty} \frac{x_{n-1}}{x_n} < \frac{1}{\alpha_1}.$$

*(ii) Assume $r \neq 0$ and $s = 0$. Then*

*(a) if $s' \neq 0$, then (6.1) holds if and only if*

$$\lim_{n\to\infty} \frac{x_{n-1}}{x_n} = l \text{ for some scalar } l;$$

*(b) if* $s' = 0$ *then (6.1) holds for all x.*
*(iii) Assume* $r = 0$ *and* $s \neq 0$*. Then*
*(a) if* $r' \neq 0$ *then (6.1) holds if and only if*

$$\lim_{n\to\infty} \frac{x_n}{x_{n-1}} = l' \text{ for some scalar } l';$$

*(b) if* $r' = 0$ *then (6.1) holds for all x.*
*(iv) If* $r = s = 0$*, then*
*(a) if* $r'$*, or* $s' \neq 0$ *then (6.1) has no solution,*
*(b) if* $r' = s' = 0$ *then (6.1) holds for all x.*

*Proof.* (i) The inclusion (6.1) is equivalent to

$$D_{1/x}\widetilde{B}D_x \in (c,c). \tag{6.2}$$

As we have seen in the proof of Theorem 5.1 we have

$$\widetilde{B}_{nk} = \begin{cases} \left(-\dfrac{s}{r}\right)^{n-k-1} \dfrac{\delta}{r^2} & \text{for } k \leq n-1, \\ \dfrac{r'}{r} & \text{for } k = n. \end{cases}$$

Since $\alpha_1 = -s/r > 0$ from the characterization of $(c,c)$ we deduce that (6.2) holds if and only if

$$\sum_{k=1}^{n} \left[D_{1/x}\widetilde{B}D_x\right]_{nk} = \frac{r'}{r} - \frac{1}{rs}\delta\left(\frac{1}{\dfrac{x_n}{\alpha_1^n}}\sum_{k=1}^{n-1}\frac{x_k}{\alpha_1^k}\right) \tag{6.3}$$

tends to a limit $l$ as $n$ tends to infinity. Indeed this condition implies $D_{1/x}\widetilde{B}D_x \in S_1$ and from the proof of Proposition 5.2 we have $\left[D_{1/x}\widetilde{B}D_x\right]_{nk} \to 0$ $(n \to \infty)$ for all $k$.

(a) If $\delta = 0$ the previous sum defined in (6.3) is reduced to $r'/r$ and inclusion (6.1) holds for all $x$.

(b) If $\delta \neq 0$, inclusion (6.1) means that (6.3) is convergent and

$$\frac{1}{\dfrac{x_n}{\alpha_1^n}}\sum_{k=1}^{n-1}\frac{x_k}{\alpha_1^k} \to -\frac{l-\dfrac{r'}{r}}{\dfrac{1}{rs}\delta} \quad (n\to\infty),$$

so we have $\left(x_n/\alpha_1^n\right)_n \in \widehat{C}$. By Lemma 4.3 we have $\widehat{C} = \widehat{\Gamma}$, so (6.2) is equivalent to

$$\lim_{n\to\infty}\frac{x_{n-1}}{\alpha_1^{n-1}}\frac{\alpha_1^n}{x_n} = \alpha_1 \lim_{n\to\infty}\frac{x_{n-1}}{x_n} < 1$$

and we conclude for (i) b).

(ii) Case $r \neq 0$ and $s = 0$. Since $B(r,s) = rI$ we have $\mathbf{s}_x^{(c)}(B(r,s)) = \mathbf{s}_x^{(c)}$. So inclusion (6.1) is equivalent to $D_{1/x}B(r',s')D_x \in (c,c)$, this means that there are $K \geq 0$ and $L$ such that

$$\begin{cases} |r'| + |s'|\dfrac{x_{n-1}}{x_n} \leq K \text{ for all } n, \\ r' + s'\dfrac{x_{n-1}}{x_n} \to L \ (n\to\infty). \end{cases}$$

If $s' \neq 0$ then we have

$$\frac{x_{n-1}}{x_n} \to \frac{L-r'}{s'} \ (n \to \infty)$$

and if $s' = 0$ the previous system holds for all $x$.

(iii) Case $r = 0$, $s \neq 0$. As we have seen in the proof of Theorem 5.1 we have $\mathbf{s}_x^{(c)}(B(0,s)) = s_{x^+}^{(c)}$. Then (6.1) is successively equivalent to $\mathbf{s}_{x^+}^{(c)} \subset \mathbf{s}_x^{(c)}(B(r',s'))$, $B(r',s') \in \left(\mathbf{s}_{x^+}^{(c)}, \mathbf{s}_x^{(c)}\right)$ and to

$$(S)\begin{cases} |r'|\dfrac{x_{n+1}}{x_n} + |s'| \leq K \text{ for all } n, \\ r'\dfrac{x_{n+1}}{x_n} + s' \to L \ (n \to \infty), \end{cases}$$

for some $K \geq 0$ and some scalar $L$.

(a) If $r' \neq 0$ we have

$$\frac{x_{n+1}}{x_n} \to \frac{L-s'}{r'} \ (n \to \infty)$$

and trivially (S) holds.

(b) If $r' = 0$ system $(S)$ trivially holds for all $x \in U^+$.

(iv) If $r = s = 0$ then $s_x^{(c)}(B(0,0)) = \omega$ and as we have seen in the proof of Theorem 5.1 the inclusion $\omega \subset s_x^{(c)}(B(r',s'))$ implies $r' = s' = 0$. □

In the general case when $r$, $s$, $\delta$, $\alpha \neq 0$ we have the following,

*Remark* 6.2. Condition (6.1) holds if and only if

$$\text{(i) } \frac{\alpha^n}{x_n}\sum_{k=1}^{n-1}\frac{x_k}{\alpha^k} \to l \ (n \to \infty), \text{ (ii) } \frac{|\alpha|^n}{x_n}\sum_{k=1}^{n-1}\frac{x_k}{|\alpha|^k} \leq K \text{ for all } n, \text{ and (iii) } \frac{\alpha^n}{x_n} \to l' \ (n \to \infty)$$

for some $l$, $l'$, and $K > 0$. This result is a direct consequence of condition (6.2) in the proof of Theorem 6.1.

We can also state the next remark.

*Remark* 6.3. The characterization of $\mathbf{s}_x^{(c)}(B(r,s)) = \mathbf{s}_x^{(c)}(B(r',s'))$ can be obtained combining Theorem 6.1 and a similar theorem obtained by replacing $(r,s)$ by $(r',s')$.

# 7 Applications to some (SSIE) and (SSE) with operators of the form $B(r,s)$

## 7.1 Application 1

In this part we apply the previous results to special (SSIE) or (SSE).

We deal with the next statements $P_1(y)$ and $P_2(y)$ defined by

$P_1(y)$: $y_n + 2y_{n-1} = O(x_n)$ implies $\Delta y_n = O(x_n)$ $(n \to \infty)$

and

$P_2(y)$: $y_{n-1}/x_n \to l$ implies $\Delta y_n / x_n \to l'$ $(n \to \infty)$ for some scalars $l$, $l'$.

The question is: what is the set of all $x \in U^+$ such that $P_1(y)$ and $P_2(y)$ hold for all $y$ ?

We easily see that this problem consists in determining the set

$$\mathbf{P}=\left\{x \in U^{+} : \mathbf{s}_x(B(1,2)) \subset \mathbf{s}_x(\Delta) \text{ and } \mathbf{s}_x^{(c)}(B(0,1)) \subset \mathbf{s}_x^{(c)}(\Delta)\right\}.$$

Combining Theorem 5.1 (i) (b) and Theorem 6.1 (iii) (a) we obtain

$$\mathbf{P}=\left\{x \in U^{+} : \left(\frac{x_n}{2^n}\right)_n \in \widehat{C_1} \text{ and } \frac{x_n}{x_{n-1}} \to L \ (n \to \infty) \text{ for some } L\right\}.$$

Note that any sequence of the form $x = (R^n)_n$ with $R > 2$ belongs to **P**. We may explicitly define a special simple subset of **P**. Indeed since we have $\widehat{C_1} \supset \Gamma$ we easily see that

$$\mathbf{P} \supset \left\{x \in U^{+} : \frac{x_{n-1}}{x_n} \to L \ (n \to \infty) \text{ for some } L \in \left]0, 1/2\right[\right\}.$$

### 7.2 Application 2

Here the question is: what are the sequences $x \in U^+$ such that $P_1'(y)$ and $P_2'(y)$ hold for all $y$, where

$P_1'(y)$: $y_{n-1}/x_n \to l$ implies $\Delta y_n/x_n \to l'$ $(n \to \infty)$

and

$P_2'(y)$: $\Delta y_n/x_n \to l''$ $(n \to \infty)$ implies $y_n/x_n \to l'''$ $(n \to \infty)$ for some scalars $l$, $l'$, $l''$ and $l'''$.

These statements lead to determine the set of all $x \in U^+$ such that $s_x^{(c)}(B(r,s)) \subset s_x^{(c)}(\Delta) \subset s_x^{(c)}$. Reasoning as in Application 1 and by Theorems 5.1 and 6.1 we have

$$\begin{aligned} \mathbf{P}' &= \left\{x \in U^{+} : P_1'(y) \text{ and } P_2'(y) \text{ hold for all } y\right\} \\ &= \left\{x \in U^{+} : \frac{x_{n-1}}{x_n} \to L \ (n \to \infty) \text{ for some } L \in \left]0, 1\right[\right\}. \end{aligned}$$

### 7.3 Application 3: The (SSE) $\chi_a + s_x^{(c)}(B(r,s)) = s_x^{(c)}$

In this part we apply the previous results to solve sequence spaces equations of the form

$$\chi_a + \mathbf{s}_x^{(c)}(B(r,s)) = \mathbf{s}_x^{(c)} \text{ where } \chi = \mathbf{s}, \mathbf{s}^0, \text{ or } \mathbf{s}^{(c)} \tag{7.1}$$

When $\chi = \mathbf{s}^0$ this equation consists in determining the set of all $x \in U^+$ such that

$$\frac{y_n}{x_n} \to l \ (n \to \infty)$$

if and only if there are $u$, $v$ such that $y = u + v$ and

$$\frac{u_n}{a_n} \to 0 \text{ and } \frac{r v_n + s v_{n-1}}{x_n} \to l' \ (n \to \infty) \text{ for all } y \in \omega.$$

Each of the sets $\chi_a$, $\mathbf{s}_x^{(c)}$ and $\mathbf{s}_x^{(c)}(B(r,s))$ are linear subspaces of $\omega$. We obtain the following result.

**Theorem 7.1.** *Let $r$, $s \neq 0$ and assume $\alpha_1 = -s/r > 0$. Then equation (7.1) where $\chi = \mathbf{s}$, $\mathbf{s}^0$, or $\mathbf{s}^{(c)}$ holds if and only if*

$$\lim_{n\to\infty} \frac{x_{n-1}}{x_n} < \frac{1}{\alpha_1} \tag{7.2}$$

*and*

$$\frac{a}{x} \in M(\chi_1, c). \tag{7.3}$$

*Proof.* Equation (7.1) implies

$$\chi_a + \mathbf{s}_x^{(c)}(B(r,s)) \subset \mathbf{s}_x^{(c)} \tag{7.4}$$

which implies $s_x^{(c)}(B(r,s)) \subset s_x^{(c)}$ and condition (7.2) by Theorem 6.1. Then condition (7.4) implies $\chi_a \subset s_x^{(c)}$ and (7.3). So we have shown that equation (7.1) implies conditions (7.2) and (7.3). Now show that (7.2) and (7.3) imply (7.1). For this it is enough to note that (7.2) implies $(x_{n-1}/x_n)_{n\geq 2} \in c$ and $s_x^{(c)} \subset s_x^{(c)}(B(r,s))$ by Theorem 6.1 (ii) and $s_x^{(c)}(B(r,s)) \subset s_x^{(c)}$ by Theorem 6.1 (i) (a) so $s_x^{(c)}(B(r,s)) = s_x^{(c)}$ and $\chi_a + s_x^{(c)}(B(r,s)) = \chi_a + s_x^{(c)}$. Finally condition (7.3) implies $\chi_a \subset s_x^{(c)}$ and $\chi_a + s_x^{(c)}(B(r,s)) = s_x^{(c)}$. This concludes the proof. □

**Corollary 7.2.** *We have*

$$\widehat{\Gamma} = \left\{ x \in U^+ : \chi_1 + \mathbf{s}_x^{(c)}(\Delta) = \mathbf{s}_x^{(c)} \right\} \text{ where } \chi = \mathbf{s}, \mathbf{s}^0, \text{ or } \mathbf{s}^{(c)}. \tag{7.5}$$

*Proof.* By Theorem 7.1 we have $\chi_1 + \mathbf{s}_x^{(c)}(\Delta) = \mathbf{s}_x^{(c)}$ if and only if $x \in \widehat{\Gamma}$ and $1/x \in M(\chi_1, c)$. If $\chi = \mathbf{s}$ we have $M(\chi_1, c) = c_0$ and $x_n \to \infty$ $(n \to \infty)$, but condition $x \in \widehat{\Gamma}$ implies $1/x \in c_0$, so it also implies $1/x \in M(\chi_1, c)$ and (7.5) holds for $\chi = \mathbf{s}$. The proof is similar in the other cases since we have $c_0 \subset M(\chi_1, c)$ for $\chi = \mathbf{s}^0$, or $\mathbf{s}^{(c)}$. □

As an immediate consequence of Theorem 7.1 we have

**Corollary 7.3.** *Let $a \in U^+$. Then*
*(i) $s_a + \mathbf{s}_x^{(c)}(\Delta) = \mathbf{s}_x^{(c)}$ is equivalent to $x \in \widehat{\Gamma}$ and $a/x \in c_0$;*
*(ii) $s_a^0 + \mathbf{s}_x^{(c)}(\Delta) = \mathbf{s}_x^{(c)}$ is equivalent to $x \in \widehat{\Gamma}$ and $a/x \in s_1$;*
*(iii) $s_a^{(c)} + \mathbf{s}_x^{(c)}(\Delta) = \mathbf{s}_x^{(c)}$ is equivalent to $x \in \widehat{\Gamma}$ and $a/x \in c$.*

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