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On a Heterogeneous Viscoelastic Fluid Heated from Below in Porous Medium

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Abstract

An attempt has been made to investigate the thermal convection of a heterogeneous Walters B' viscoelastic fluid layer through porous medium under linear stability theory. It is found that medium permeability and viscoelastic parameter have destabilizing effect while density distribution and Prandtl number have stabilizing effect on the fluid layer. The sufficient conditions depending upon the monotonic behaviour of $f(z)\left[\frac{df}{dz} \rangle 0 \; or \; < 0\right]$ for the non-existence of overstability are also derived. It is shown that the sufficient conditions for the validity of principle of exchange of stabilities for the present problem are $\frac{df}{dz} < 0$ and $F < \frac{P_1}{\varepsilon}$.Further, it has been found that the complex growth rate of an arbitrary oscillatory modes lies inside a circle in the $\sigma_r - \sigma_i$ plane, whose centre is at the origin and radius is $\frac{a}{p_1}\sqrt{\frac{|R_2|P_1}{E}}$.

AMS Subject Classification: 76A10; 76S99.

Keywords: Thermal convection; Heterogeneous Walters B' viscoelastic fluid; Porous medium; Prandtl number

1 Introduction

The derivation of the basic equations of a layer of fluid heated from below in porous medium, using Boussinesq approximation, has been given by Joseph [4]. The study of a layer of fluid heated from below in porous media is motivated both theoretically and by its practical applications in engineering disciplines. Among the applications in engineering

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disciplines one can find the food process industry, chemical process industry, solidification and centrifugal casting of metals. The development of geothermal power resources has increased general interest in the properties of convection in porous medium. The thermal convection in a layer of Newtonian fluid heated from below has been discussed in detail by Chandrasekhar [2]. Lapwood [7] has studied the stability of convective flow in a porous medium using Rayleigh's procedure. Tadie [14] has considered the oscillation criteria for bounded solutions for some nonlinear diffusion equations via Picone-type formulae. The Rayleigh instability of a thermal boundary layer in flow through a porous medium has been considered by Wooding [1]. When the fluid slowly percolates through the pores of the rock, the gross effect is represented by the well-known Darcy's law. As a result, the usual viscous term in the equations of motion of Walters B' viscoelastic fluid is replaced by the resistance term $\left[-\frac{1}{k_1}\left(\mu-\mu'\frac{\partial}{\partial t}\right)\vec{q}\right]$, where μ and μ' are the viscosity and viscoelasticity of the fluid, k_1 is the medium permeability and $\vec{q}$ is the Darcian (filter) velocity of the fluid. Generally, it is accepted that comets consist of a dusty 'snowball' of a mixture of frozen gases which, in the process of their journey, changes from solid to gas and vice-versa. The physical properties of comets, meteorites and interplanetary dust strongly suggest the importance of porosity in astrophysical context [McDonnel, 8].

In all the above studies, the fluid has been considered to be Newtonian. Since viscoelastic fluids play an important role in polymers and electrochemical industry, the studies of waves and stability in different viscoelastic fluid dynamical configuration has been carried out by several researchers in the past. The stability of a horizontal layer of Maxwell's viscoelastic fluid heated from below has been investigated by Vest and Arpaci [15]. The nature of instability and some factors may have different effects, on viscoelastic fluids as compared to the Newtonian fluids. For example, Bhatia and Steiner [1] have considered the effect of a uniform rotation on the thermal instability of a Maxwell fluid and have found that rotation has a destabilizing effect in contrast to the stabilizing effect on Newtonian fluid. In another study, Sharma and Sharma [11] have considered the thermal instability of a rotating Maxwell fluid through porous medium and found that, for stationary convection, the rotation has stabilizing effect whereas the permeability of the medium has both stabilizing as well as destabilizing effect, depending on the magnitude of rotation. In another study, Sharma [10] has studied the stability of a layer of an electrically conducting Oldroyd fluid [9] in the presence of a magnetic field and has found that the magnetic field has a stabilizing influence.

There are many elastico-viscous fluids that cannot be characterized by Maxwell's or Oldroyd's constitutive relations. One such class of viscoelastic fluids is Walters B' fluid [16] having relevance and importance in geophysical fluid dynamics, chemical technology, and petroleum industry. Walters' [17] reported that the mixture of polymethyl methacrylate and pyridine at 25^0C containing 30.5g of polymer per litre with density 0.98g per litre behaves very nearly as the Walters B' viscoelastic fluid. Polymers are used in the manufacture of spacecrafts, aeroplanes, tyres, belt conveyers, ropes, cushions, seats, foams, plastics engineering equipments, contact lens, etc. Walters B' viscoelastic fluid forms the basis for the manufacture of many such important and useful products. Chakraborty and Sengupta [3] have studied the flow of unsteady viscoelastic (Walters B' liquid) conducting fluid through two porous concentric non-conducting infinite circular cylinders rotating with different angular velocities in the presence of uniform axial magnetic field. Sharma and Kumar [12] studied the stability of the plane interface separating two viscoelastic (Walters B') superposed fluids of uniform densities. In another study, Sharma and Kumar [13] studied Rayleigh-Taylor instability of superposed conducting Walters B' viscoelastic fluids in hydromagnetics. Kumar [5] has considered the thermal instability of a layer of Walters B' viscoelastic fluid acted on by a uniform rotation and found that for stationary convection, rotation has a stabilizing effect. Kumar et al. [6] have considered the stability of plane interface separating the Walters B' viscoelastic superposed fluids of uniform densities in the presence of suspended particles.

Keeping in mind the importance in various fields particularly in the soil sciences, groundwater hydrology, geophysical, astrophysical and biometrics, the thermal convection of a viscoelastic (Walters B') incompressible and heterogeneous fluid layer saturated with porous medium, where density is $\rho_0 f(z)$, ρ_0 being a positive constant having the dimension of density, and $f(z)$ is a monotonic function of the vertical coordinate z, with $f(0) = 1$ has been considered in the present paper.

2 Formulation of the Problem and Perturbation Equations

Let us consider an infinite horizontal layer of incompressible and heterogeneous Walters B' viscoelastic fluid of thickness 'd ', in porous medium of porosity ε and medium permeability k_1, bounded by the planes $z = 0$ and $z = d$. Let z-axis be vertically upwards. The interstitial fluid of variable density is viscous and incompressible. The initial inhomogenenity in the fluid is assumed to be of the form$\rho_0 f(z)$, where ρ_0 is the density at the lower boundary and $f(z)$ be the function of vertical co-ordinate z such that $f(0) = 1$. The fluid layer is infinite in horizontal direction and is heated from below leading to an adverse temperature gradient $\beta = {}^{(T_0 - T_1)}/_{d}$, where T_0 and T_1 are the constant temperatures of the lower and upper boundaries. The effective density is the superposition of the inhomogeneity described by (a) $\rho = \rho_0 f(z)$, and (b) $\rho = \rho_0[1+\alpha(T_0 - T)]$ which is caused by temperature gradient. This leads to the effective density

$$\rho = \rho_0 [f(z) + \alpha(T_0 - T)], \tag{2.1}$$

where α is the coefficient of thermal expansion.

Let $\vec{q}(u, v, w)$, p, μ, μ' *and* κ be the filter velocity, pressure, coefficient of viscosity, viscoelasticity and thermal diffusivity of the fluid, respectively.

Now, the relevant equations for our problem through porous medium are given by

$$\frac{\rho}{\varepsilon}\left[\frac{\partial \vec{q}}{\partial t} + \frac{1}{\varepsilon}(\vec{q}.\nabla)\vec{q}\right] = -\nabla p + \rho\vec{g} - \frac{1}{k_1}\left(\mu - \mu'\frac{\partial}{\partial t}\right)\vec{q}, \tag{2.2}$$

$$\nabla.\vec{q} = 0 \ , \tag{2.3}$$

$$\varepsilon\frac{\partial \rho}{\partial t} + (\vec{q}.\nabla)\rho = 0 \ , \tag{2.4}$$

$$E\frac{\partial T}{\partial t}+(\vec{q}.\nabla)T=\kappa\nabla^2 T \quad , \tag{2.5}$$

where $E=\varepsilon+(1-\varepsilon)\frac{\rho_S C_S}{\rho_0 C_f}$; ρ_0, C_f; ρ_S, C_S stands for the density and heat capacity of fluid and solid matrix, respectively.

Now the initial state whose stability is to be examined is characterized by

$$\vec{q}=0,\ T=T_0-\beta z,\ \rho=\rho_0\left[f(z)+\alpha\beta z\right],\ p=p_0-\int_0^z g\rho\, dz \ ,$$

where p_0 is the pressure at $\rho=\rho_0$.

Let $\delta\rho$, δp, θ *and* $\vec{q}(u,\ v,\ w)$ denotes respectively the perturbations in density ρ, pressure p, temperature T and velocity $\vec{q}$ (initially zero). Then the linearized perturbations equations describing the system are written as

$$\frac{1}{\varepsilon}\frac{\partial\vec{q}}{\partial t}=-\frac{1}{\rho_0}\nabla\delta p+\vec{g}\frac{\delta\rho}{\rho_0}-\frac{1}{k_1}\left(\nu-\nu'\frac{\partial}{\partial t}\right)\vec{q}\ , \tag{2.6}$$

$$\nabla.\,\vec{q}=0\ , \tag{2.7}$$

$$\varepsilon\frac{\partial\delta\rho}{\partial t}+\rho_0 w\frac{df}{dz}=0\ , \tag{2.8}$$

$$E\frac{\partial\theta}{\partial t}=\beta w+\kappa\nabla^2\theta\ . \tag{2.9}$$

3 Analysis in Terms of Normal Modes

The analysis of an arbitrary disturbance is carried out in terms of normal modes following Chandrasekhar [2]. The stability of each of the modes is discussed separately. Assuming that the perturbed quantities are of the form

$$[w,\ \theta]=[W(z),\ \Theta(z)]\ \exp\left(ik_x x+ik_y y+nt\right)\ , \tag{3.1}$$

where k_x and k_y are the wave numbers along the x- and y- directions.
$k=\left(k_x^2+k_y^2\right)^{1/2}$ is the resultant wave number and n is the growth rate of disturbances.

Using expression (3.1), equations (2.6)-(2.9), on simplification, give

$$n\left[\frac{1}{\varepsilon}n+\frac{1}{k_1}(\nu-\nu' n)\right]\left(\frac{d^2}{dz^2}-k^2\right)W=-gk^2\left(\frac{1}{\varepsilon}\frac{df}{dz}\right)W-gk^2\alpha\Theta\ , \tag{3.2}$$

$$E\frac{\partial\Theta}{\partial t}=\beta W+\kappa\left(\frac{d^2}{dz^2}-k^2\right)\Theta. \tag{3.3}$$

Equations (3.2) and (3.3) in non-dimensional form can be written as

$$\sigma p_1\left[\frac{\sigma}{\varepsilon}+\frac{(1-F\sigma)}{P_1}\right]\left(D^2-a^2\right)W=-\sigma p_1\frac{g\alpha a^2 d^2\Theta}{\nu}-\frac{g\alpha a^2 d^4}{\kappa\nu}\left(\frac{df}{dz}\right)W\,, \tag{3.4}$$

$$\left(D^2 - a^2 - E\sigma\, p_1\right)\Theta = -\left(\frac{\beta d^2}{\kappa}\right)W, \tag{3.5}$$

here we have put $x = x*d,\ y = y*d,\ z = z*d,\ D = {}^{d}\!/_{dz*}$ and thereafter dropping stars for simplicity. Also we have put

$a = kd,\ \sigma = \frac{nd^2}{\nu},\ p_1 = \frac{\nu}{\kappa}$ (Prandtl number), $P_1 = \frac{k_1}{d^2}$ (dimensionless medium permeability) and $F = \frac{\nu'}{d^2}$.

Eliminating Θbetween equations (3.4) and (3.5), we obtain

$$\sigma p_1\left[\frac{\sigma}{\varepsilon} + \frac{(1 - F\sigma)}{P_1}\right]\left(D^2 - a^2\right)\left(D^2 - a^2 - E\sigma p_1\right)W$$
$$= \sigma p_1 R a^2 W - a^2 R_2\left(D^2 - a^2 - E\sigma p_1\right)W\,, \tag{3.6}$$

where $R = \frac{g\alpha\beta d^4}{\nu\kappa}$ is the Rayleigh number and $R_2 = \frac{gd^4}{\kappa\nu\varepsilon}\left(\frac{df}{dz}\right)$.

Consider the case where both the boundaries are free and the medium adjoining the fluid is non-conducting. The appropriate boundary conditions for this case are (Chandrasekhar [2])

$$W = D^2 W = 0,\ \ \Theta = 0\ \ at\ \ z = 0,\ 1. \tag{3.7}$$

Using the boundary conditions (3.7), one can show that all even-order derivatives of W must vanish for $z = 0$ *and* $z = 1$ and hence the proper solution of (3.6) characterizing the lowest mode is

$$W = W_0\ \sin \pi z, \tag{3.8}$$

where W_0 is constant.

Substituting (3.8) in equation (3.6) and letting $a^2 = \pi^2 x,\ R = R_1\pi^4,\ R_2 = R_3\pi^4,\ P = P_1\pi^2$ and $\sigma = i\sigma_1\pi^2$, we obtain the dispersion relation

$$R_1 = \frac{1}{x}\left[\frac{i\sigma_1}{\varepsilon} + \frac{1 - i\sigma_1 F\pi^2}{P}\right](1 + x)(1 + x + i\sigma_1 p_1 E)$$
$$+ \frac{iR_3}{\sigma_1 p_1}(1 + x + i\sigma_1 p_1 E). \tag{3.9}$$

It is clear from equation (3.9) that for an arbitrary value of σ_1, R_1 is complex. But from the physical consideration R_1 is a real. Therefore, the condition that R_1 will be real imply a relation between real and imaginary parts of σ_1. But as we are interested in specifying the critical Rayleigh number for the onset of instability via a state of pure oscillation, we shall suppose that σ_1 is real in the above equations and try to obtain the conditions for such solution to exist.

Assuming σ_1 is real in the equation (3.9) and separating the real and imaginary parts, both of which must vanish separately, which led to the following equations

$$R_1 = \left(\frac{1+x}{x}\right)\left[\frac{1+x}{P} - \frac{\sigma_1^2 E p_1}{\varepsilon} + \frac{\sigma_1^2 E p_1 F \pi^2}{P}\right] - R_3 E, \tag{3.10}$$

and

$$\frac{(1+x)^2}{x\varepsilon} - \frac{(1+x)^2 F\pi^2}{xP} + \left(\frac{1+x}{x}\right)\frac{Ep_1}{P} + \frac{R_3(1+x)}{\sigma_1^2 p_1} = 0. \tag{3.11}$$

Equations (3.10) and (3.11) must satisfy simultaneously if the overstability is to be occurring. For the numerical calculations, we set the fixed value for p_1, E, P, F, ε *and* R_3 in equation (3.11) and consequently σ_1^2 is found for different value of x. σ_1 is assumed to be real, negative and complex value of σ_1^2 are rejected. Then R_1 is calculated from equation (3.10) keeping the same value of p_1, E, P, F, ε *and* R_3 for different values of σ_1^2 obtained from equation (3.10). Repeating the same calculations for the different values of the R_3 (density distribution), P (medium permeability), F (viscoelasticity parameter) and p_1 (Prandtl number), we arrive at the conclusion that the value of R_1 decreases with increase in the value of P and F. This implies that the medium permeability and viscoelastic parameter has a destabilizing effect on fluid layer. Also it is found that value of R_1 increases with increase in the value of density distribution R_3 and p_1(Prandtl number), showing the stabilizing effect of density distribution and Prandtl number on the fluid layer as shown in the table.

Variation of Rayleigh Number R_1 with respect to R_3 (density distribution) for the onset of stability for fixed value of $p_1 = 0.1, E = 1, P = 0.5, \varepsilon = 0.5, x = 0.5, F = 0.2$		Variation of Rayleigh Number R_1 with respect to P (medium permeability) for the onset of stability for fixed value of $p_1 = 0.1, E = 1, R_3 = 10, \varepsilon = 0.5, x = 0.5, F = 0.2$		Variation of Rayleigh Number R_1 with respect to p_1 (Prandtl number)for the onset of stability for fixed value of $E = 1, P = 0.5, R_3 = 10, \varepsilon = 0.5, x = 0.5, F = 0.2$		Variation of Rayleigh Number R_1 with respect to F (viscoelastic parameter) for the onset of stability for fixed value of $p_1 = 0.1, E = 1, P = 0.5, \varepsilon = 0.5, x = 0.5, R_3 = 10$	
R_3	R_1	P	R_1	p_1	R_1	F	R_1
5	9.37	0.1	45.38	0.1	9.74	0.2	9.96
0.1	9.74	0.2	22.97	0.2	10.58	0.2	9.69
.1	9.96	0.3	15.47	0.3	11.57	0.3	9.51
20	10.42	0.4	11.87	0.4	12.74	0.4	9.42
25	10.79	0.5	9.74	0.5	14.18	0.5	9.36

4 Non-existence of Overstability

From equation (3.11), we have

$$\sigma_1^2 = -\frac{R_3}{p_1\left[\frac{(1+x)}{x\varepsilon} - \frac{(1+x)\pi^2 F}{xP} + \frac{Ep_1}{xP}\right]}. \tag{4.1}$$

Case I: If $R_3 > 0$ i.e. $\frac{df}{dz} > 0$ and $\frac{1}{\varepsilon} > \frac{\pi^2 F}{P}$ i.e. $F < \frac{P}{\varepsilon\pi^2}$, then σ_1^2 is negative in equation (4.1) which contradicts the given hypothesis that σ_1 is real and thus overstability cannot exist. Thus $\frac{df}{dz} > 0$ and $F < \frac{P}{\varepsilon\pi^2}$ are the sufficient conditions for the non-existence of overstability.

Case II: If $R_3 < 0$ i.e. $\frac{df}{dz} < 0$ and $F > \left[\frac{P}{\varepsilon} + \frac{EP_1}{1+x}\right]$ then σ_1^2 is negative in equation (4.1) which contradicts the given hypothesis that σ_1 is real and thus overstability cannot exist. Thus $\frac{df}{dz} < 0$ and $F > \left[\frac{P}{\varepsilon} + \frac{EP_1}{1+x}\right]$ are the sufficient conditions for the non-existence of overstability in this case.

5 Princple of Exchange of Stabilities

Theorem: The principle of exchange of stabilities is not valid for the problem.

Proof: If possible, let the principle of exchange of stabilities be valid. Then for $\sigma = 0$, equation (3.6) reduces to

$$\left(D^2 - a^2\right) W = 0. \tag{5.1}$$

Now multiplying the equation (5.1) with $W*$ (complex conjugate of W) and integrating over the range of z and making use of boundary conditions (3.7), we obtain

$$W = 0,$$

which is the only solution.

Again for $\sigma = 0$, equation (3.5) yield

$$\left(D^2 - a^2\right)\Theta = -\left(\frac{\beta d^2}{\kappa}\right) W. \tag{5.2}$$

Substituting $W = 0$ and using boundary conditions (3.7), we get

$$\Theta = 0,$$

which is the only solution.

Hence $W = \Theta = 0$, which contradicts the given hypothesis that the initial stationary state solution are perturbed.

Therefore, the principle of exchange of stabilities is not valid for the problem.

6 Sufficient Conditions for Stability

Now multiplying the equation (3.4) with $W*$ (complex conjugate of W) and integrating over the range of z and making use of boundary conditions (3.7), we obtain

$$\sigma p_1\left[\frac{\sigma}{\varepsilon}+\frac{(1-F\sigma)}{P_1}\right]\int_0^1\left(D^2-a^2\right)WW*\,dz=-\frac{\sigma p_1 g\alpha a^2 d^2}{\nu}\int_0^1\Theta W*\,dz$$
$$-a^2R_2\int_0^1 WW*\,dz. \tag{6.1}$$

With the help of equation (3.5), equation (6.1) written as

$$p_1\left[\frac{\sigma}{\varepsilon}+\frac{(1-F\sigma)}{P_1}\right]\int_0^1\left(|DW|^2+a^2|W|^2\right)dz+\frac{p_1 g a^2\kappa\alpha}{\beta\nu}\int_0^1\left(|D\Theta|^2+|\Theta|^2\right)dz$$
$$+\frac{\sigma*Ep_1^2 g a^2\alpha\kappa}{\beta\nu}\int_0^1\left(|\Theta|^2\right)dz-\frac{a^2R_2\sigma*}{|\sigma|^2}\int_0^1|W|^2\,dz=0. \tag{6.2}$$

Letting $\sigma=\sigma_r+i\sigma_i$ in equation (6.2) and equating the real part, we have

$$\sigma_r\left[p_1\left\{\frac{1}{\varepsilon}-\frac{F}{P_1}\right\}\int_0^1\left(|DW|^2+a^2|W|^2\right)dz\right.$$
$$\left.+\frac{Ep_1^2 g a^2\kappa\alpha}{\beta\nu}\int_0^1|\Theta|^2\,dz-\frac{a^2}{|\sigma|^2}R_2\int_0^1|W|^2\,dz\right]=$$
$$-\left[\frac{p_1}{P_1}\int_0^1\left(|DW|^2+a^2|W|^2\right)dz+\frac{p_1 g a^2\kappa\alpha}{\beta\nu}\int_0^1\left(|D\Theta|^2+a^2|\Theta|^2\right)dz\right]. \tag{6.3}$$

Equation (6.3) imply that $\sigma_r<0$ if $\frac{df}{dz}<0$ and $\frac{1}{\varepsilon}>\frac{F}{P_1}$ i.e. $F<\frac{P_1}{\varepsilon}$.

Thus, $\frac{df}{dz}<0$ and $F<\frac{P_1}{\varepsilon}$are the sufficient conditions for the stability of the system.

7 Necessary Condition for the Validity of Principle of Exchange of Stabilities

Put $\sigma=i\sigma_i$ in the equation (6.2), where σ_i is real, and on equating imaginary part, we get

$$\sigma_i\left[p_1\left(\frac{1}{\varepsilon}-\frac{F}{P_1}\right)\int_0^1\left(|DW|^2+a^2|W|^2\right)dz-\frac{Ep_1^2 g a^2\alpha\kappa}{\nu\beta}\int_0^1|\Theta|^2\,dz\right.$$
$$\left.+\frac{a^2R_2}{|\sigma_i|^2}\int_0^1|W|\,dz\right]=0\,. \tag{7.1}$$

Equation (7.1) show that $\sigma=0$ or $\sigma\neq0$, which mean that modes may be non-oscillatory or oscillatory. Equation (7.1) must be satisfied at the marginal state. Since $\sigma_r=0$, now for the principle of exchange of stabilities is to be valid at the marginal state, we must have $\sigma_i=0$, which implies that terms inside the bracket must be positive.

Now for $\sigma=0$, equation (3.5) reduces to

$$\left(D^2 - a^2\right)\Theta = -\left(\frac{\beta d^2}{\kappa}\right)W. \tag{7.2}$$

Multiplying both sides by $\Theta*$ (complex conjugate of Θ) of the equation (7.2), integrating by parts for sufficient number of times over the range of z and making use of boundary conditions (3.7) and separating the real parts of the equation so obtained, we have

$$\int_0^1 \left(|D\Theta|^2 + a^2 |\Theta|^2\right) dz = Re\left(\frac{\beta d^2}{\kappa}\right)\int_0^1 W\Theta * dz. \tag{7.3}$$

Now $\quad Re \int_0^1 W\Theta * dz \leq \left|\int_0^1 \Theta * W dz\right|$,

$$\leq \int_0^1 |\Theta * W| dz \ ,$$

$$\leq \int_0^1 |\Theta||W| dz \ ,$$

$$\leq \sqrt{\int_0^1 |\Theta|^2 dz}\ \sqrt{\int_0^1 |W|^2 dz}\ . \tag{7.4}$$

(Schwartz inequality)

Combining inequalities (7.3) and (7.4), we get

$$\int_0^1 \left(|D\Theta|^2 + a^2 |\Theta|^2\right) dz \leq \left(\frac{\beta d^2}{\kappa}\right)\sqrt{\int_0^1 |\Theta|^2 dz}\ \sqrt{\int_0^1 |W|^2 dz}, \tag{7.5}$$

which in turn implies that

$$\int_0^1 \left(|D\Theta|^2\right) dz \leq \left(\frac{\beta d^2}{\kappa}\right)\sqrt{\int_0^1 |\Theta|^2 dz}\ \sqrt{\int_0^1 |W|^2 dz}. \tag{7.6}$$

Whence we derive from inequality (7.6) and using Rayleigh-Ritz inequality $\int_0^1 \left(|D\Theta|^2\right) dz \geq \pi^2 \int_0^1 |\Theta|^2 dz$, we get

$$\pi^2 \sqrt{\int_0^1 |\Theta|^2 dz} \ \leq \frac{\beta d^2}{\kappa}\sqrt{\int_0^1 |W|^2 dz}\ ,$$

which implies that

$$\int_0^1 |\Theta|^2 dz \leq \left(\frac{\beta d^2}{\kappa\pi^2}\right)^2 \int_0^1 |W|^2 dz\ . \tag{7.7}$$

Using inequality (7.7) in equation (7.1), we have

$$\sigma_i\left\{p_1\left(\frac{1}{\varepsilon} - \frac{F}{P_1}\right)\int_0^1 \left(|DW|^2 + a^2 |W|^2\right) dz\right.$$

$$-\frac{Ep_1^2 g a^2 \alpha\kappa}{\nu\beta}\int_0^1 |\Theta|^2\,dz + \frac{a^2 R_2}{|\sigma_i|^2}\int_0^1 |W|^2\,dz\Bigg\} >$$

$$\sigma_i\Bigg\{p_1\left(\frac{1}{\varepsilon}-\frac{F}{P_1}\right)\int_0^1 |DW|^2\,dz + p_1 a^2\left[\left(\frac{1}{\varepsilon}-\frac{F}{P_1}\right)-\frac{Ep_1 R}{\pi^4}\right]\int_0^1 |W|^2\,dz$$
$$+\frac{a^2 R_2}{|\sigma_i|^2}\int_0^1 |W|^2\,dz\Bigg\} \tag{7.8}$$

But if $\frac{1}{\varepsilon} > \frac{F}{P_1}$ and $\left(\frac{1}{\varepsilon}-\frac{F}{P_1}\right) > \frac{Ep_1R}{\pi^4}$ i.e $\frac{1}{\varepsilon} > \frac{F}{P_1} + \frac{Ep_1R}{\pi^4}$. , then the terms inside the bracket in R.H.S. of inequality (7.8) is positive which in turn implies that L.H.S. of the inequality must be positive. Thus, if $\frac{1}{\varepsilon} > \frac{F}{P_1}$ i.e. $F < \frac{P_1}{\varepsilon}$, then $\sigma_i = 0$.

Thus the sufficient condition for the validity of principle of exchange of stabilities for the problem is $F < \frac{P_1}{\varepsilon}$.

8 Circular Exclusion Theorem for Oscillatory Modes

Now multiplying the equation (3.4) with $W*$ (complex conjugate of W) and integrating over the range of $z$and making use of equation (3.5) and boundary conditions (3.7), we obtain

$$\sigma p_1\left[\frac{\sigma}{\varepsilon}+\frac{1}{P_1}(1-F\sigma)\right]\int_0^1\left(D^2-a^2\right)\left(D^2-a^2-\sigma Ep_1\right)WW*\,dz$$
$$=\sigma p_1 a^2 R\int_0^1 WW*\,dz - a^2R_2\int_0^1\left(D^2-a^2-\sigma Ep_1\right)WW*\,dz\ . \tag{8.1}$$

Multiplying equation (8.1) by $\sigma*$ (complex conjugate of σ) and dividing by $|\sigma|^2$ $(\sigma \neq 0)$, we get

$$p_1\left[\frac{\sigma}{\varepsilon}+\frac{1}{P_1}(1-F\sigma)\right]\int_0^1\left(\left|D^2W\right|^2+2a^2|DW|^2+a^4|W|^2\right)dz$$
$$+\sigma Ep_1^2\left[\frac{\sigma}{\varepsilon}+\frac{1}{P_1}(1-F\sigma)\right]\int_0^1\left(|DW|^2+a^2|W|^2\right)dz$$
$$=p_1a^2R\int_0^1|W|^2\,dz+\frac{\sigma*a^2R_2}{|\sigma|^2}\int_0^1\left(|DW|^2+a^2|W|^2\right)dz$$
$$+a^2R_2Ep_1\int_0^1|W|^2\,dz. \tag{8.2}$$

Putting $\sigma = i\sigma$ in equation (8.2) and equating the imaginary parts, we get

$$p_1\left[\frac{1}{\varepsilon}-\frac{F}{P_1}\right]\int_0^1\left(\left|D^2W\right|^2+2a^2|DW|^2+a^4|W|^2\right)dz$$
$$+\frac{Ep_1^2}{P_1}\int_0^1\left(|DW|^2+a^2|W|^2\right)dz+\frac{a^2R_2}{|\sigma|^2}\int_0^1\left(|DW|^2+a^2|W|^2\right)dz=0. \tag{8.3}$$

Equation (8.3) can be written as

$$p_1\left[\frac{1}{\varepsilon}-\frac{F}{P_1}\right]\int_0^1\left(\left|D^2W\right|^2+2a^2\left|DW\right|^2+a^4\left|W\right|^2\right)dz$$
$$+\left[\frac{Ep_1^2}{P_1}+\frac{a^2R_2}{|\sigma|^2}\right]\int_0^1\left(|DW|^2+a^2|W|^2\right)dz=0. \tag{8.4}$$

If $R_2<0$ i.e. $\frac{df}{dz}<0$ (Take $R_2=-|R_2|$) and $\frac{1}{\varepsilon}>\frac{F}{P_1}$ i.e. $F<\frac{P_1}{\varepsilon}$, then from equation (8.4), we must have

$$\frac{Ep_1^2}{P_1}<\frac{a^2|R_2|}{|\sigma|^2},$$

or

$$|\sigma|^2<\frac{a^2|R_2|P_1}{Ep_1^2} \; . \tag{8.5}$$

Hence the complex growth rate of an arbitrary mode, must lie inside the circle whose radius depend upon the wave length of the modes, medium permeability and density distribution (but it does not depend upon the Rayleigh number of configuration).

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