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Sinai-Ruelle-Bowen Measure Leaks

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Abstract

In this article, we study about the escaping rate of the Sinai-Ruelle-Bowen (SRB) measure through holes of positive measure constructed in the Julia set of hyperbolic rational maps. An explicit formula for the escaping rate through holes of different sizes is obtained and the dependence of this rate on the position of the hole is discussed. Later, the Hausdorff dimension of the survivor set is also computed. For an easier and better understanding, the simple quadratic polynomial restricted on the unit circle is studied.

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1 Introduction

Let $T : X \longrightarrow X$ be a transformation that preserves an ergodic probability measure μ. We create a hole H in the phase space with positive measure and study the dynamics of T restricted on $X \backslash H$, i.e., we keep track of orbits as long as they do not enter H and ignore them once they enter H. The study of such systems called *open dynamical systems* were introduced by Pianigiani and Yorke in [19]. For a measure supported on X, the escape velocity quantifies the rate at which it leaks through the hole in an open dynamical system. In recent papers, Bunimovich and Yurchenko [1] studied the doubling map on the unit interval and the Haar measure escaping through dyadic intervals, Keller and Liverani [15] studied expanding maps on the unit interval and absolutely continuous invariant probability measures and Ferguson and Pollicott [8] studied expanding maps on Riemannian manifold and Gibbs measures.

In this paper, we prove analogous results in the setting of complex dynamics; hyperbolic rational maps restricted on their Julia sets and the Sinai-Ruelle-Bowen (SRB) measure. As

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an example, we consider the quadratic polynomial map $z \longmapsto z^2$ restricted on the unit circle in the complex plane and work with the Lyubich's measure that equidistributes periodic points. As and when necessary, we work on appropriate full shifts in symbolic dynamics and translate the results in our setting via a conjugacy.

In section 2, we write all necessary definitions and state the main results of this paper in theorems (2.1), (2.2) and (2.3). In section 3, a brief description of symbolic dynamics is given. In section 4, we define a quantity called autocorrelation (as studied by Guibas and Odlyzko in [9], Eriksson in [7] and Cakir, Chryssaphinou and Månsson in [2]). We give a proof of theorem (2.1) in section 5 based on ideas developed by Bunimovich and Yurchenko in [1]. Later, we explain theorem (2.1) by considering the example of the polynomial map written above and state an analogous theorem for this map in theorem (5.1). In sections 6 and 7, we define the Ruelle operator and the perturbed Ruelle operator and study their spectral properties; the important ones being the spectral gap of the operators, convergence of the spectral radii and the quasi-compactness of the perturbed Ruelle operator. Theorem (2.2) is proved in section 8 using results from spectral theory of the Ruelle operator and the perturbed Ruelle operator stated in the previous sections and using ideas from Ferguson and Pollicott as in [8]. We conclude this section by stating an analogous theorem for the polynomial map in theorem (8.2). In section 9, we prove theorem (2.3) along the lines of the proof of theorem (2.2). We conclude the paper by looking at an analogous result for our favourite quadratic polynomial map, as stated in theorem (9.1).

2 Escape velocity of a measure

Let $\widehat{\mathbb{C}}$ denote the Riemann sphere and let T be a rational map defined on the Riemann sphere. By degree of the rational map, we mean the number of inverse images for a typical point $z \in \widehat{\mathbb{C}}$ counted with multiplicity. In other words, the maximum among the degrees of the two relatively prime polynomials whose quotient yields the rational map is defined to be its degree denoted by d. For our purpose of study in this paper, we shall only consider those rational maps whose degree is at least 2. One of the several possible definitions of the *Julia set* $\mathcal{J} \subset \widehat{\mathbb{C}}$ of T states that it is the closure of the set of all repelling periodic points, i.e.,

$$\mathcal{J} := \overline{\left\{z \in \widehat{\mathbb{C}} : T^p z = z \text{ for some } p \in \mathbb{Z}^+ \text{ and } |(T^p)'(z)| > 1\right\}}. \tag{2.1}$$

Elementary observations reveal that the rational map remains completely invariant on its Julia set, i.e., $T^{-1}(\mathcal{J}) = \mathcal{J}$. For more properties of Julia sets of rational maps, please refer [16]. We focus on *hyperbolic* rational maps in this paper, i.e., there exists $C > 0$ and $\lambda > 1$ such that for all $z \in \mathcal{J}$ and $n \geq 1$ we have $|(T^n)'(z)| \geq C\lambda^n$.

Let $f : \mathcal{J} \longrightarrow \mathbb{R}$ be a Hölder continuous function. We define a quantity called *pressure*, in accordance with thermodynamic formalism.

$$\mathfrak{P}(f) := \sup\left\{h_\mu(T) + \int f d\mu : \mu \text{ is a } T\text{-invariant probability measure}\right\}, \tag{2.2}$$

where $h_\mu(T)$ is the entropy of T with respect to the measure μ, see [23] for more details. Then, by a result due to Denker and Urbanski in [6], there exists a unique equilibrium Sinai-Ruelle-Bowen (SRB) measure, denoted by μ_f realising the supremum in the definition of pressure. It is quite obvious that the support of the non-atomic SRB measure is the Julia set.

Fix $z \in \mathcal{J}$ and for any $\epsilon > 0$, consider the neighbourhood $H_{z,\epsilon}$ in $\mathcal{J}$ centered at z of radius ϵ. We shall call this a *hole* in $\mathcal{J}$, provided $\mu_f(H_{z,\epsilon}) > 0$. By the Poincaré recurrence theorem, we have that the orbit of μ_f-a.e. $\zeta \in H_{z,\epsilon}$ returns to $H_{z,\epsilon}$ infinitely often. We define the *Poincaré recurrence time* of a hole $H_{z,\epsilon}$ to be

$$\tau(H_{z,\epsilon}) \;:=\; \inf\left\{n \in \mathbb{Z}^+ : \mu_f(T^n H_{z,\epsilon} \cap H_{z,\epsilon}) > 0\right\}. \tag{2.3}$$

It is clear from the definition that all measurable subsets of $H_{z,\epsilon}$ have a recurrence time of at least $\tau(H_{z,\epsilon})$ while all measurable supersets of $H_{z,\epsilon}$ have a recurrence time of utmost $\tau(H_{z,\epsilon})$.

We now focus on T acting on $\mathcal{J}\backslash H_{z,\epsilon}$, i.e., we keep track of the orbit of $\zeta \in \mathcal{J}\backslash H_{z,\epsilon}$ as long as it remains there and no longer bother after it enters $H_{z,\epsilon}$. Owing to density of periodic points of T in $\mathcal{J}$, any hole $H_{z,\epsilon}$ in $\mathcal{J}$ with positive measure contains periodic points. Let $\vartheta(\zeta)$ denote the *escape time* of any point $\zeta \in \mathcal{J}$ into the hole $H_{z,\epsilon}$, i.e.,

$$\vartheta(\zeta) \;:=\; \inf\{n \in \mathbb{Z}^+ : T^n\zeta \in H_{z,\epsilon}\}. \tag{2.4}$$

Let $\Phi_n(H_{z,\epsilon})$ denote the set of points in $\mathcal{J}$ whose escape time into $H_{z,\epsilon}$ is utmost n while $\Psi_n(H_{z,\epsilon})$ denote the set of points in $\mathcal{J}$ whose escape time into $H_{z,\epsilon}$ is at least $n+1$, i.e.,

$$\Phi_n(H_{z,\epsilon}) \;:=\; \{\zeta \in \mathcal{J} : \vartheta(\zeta) \le n\} \quad ; \quad \Psi_n(H_{z,\epsilon}) \;:=\; \{\zeta \in \mathcal{J} : \vartheta(\zeta) > n\}. \tag{2.5}$$

We set $\Phi_0(H_{z,\epsilon}) := H_{z,\epsilon}$ and $\Psi_0(H_{z,\epsilon}) := \mathcal{J}\backslash H_{z,\epsilon}$ and observe that

$$\Phi_0(H_{z,\epsilon}) \subset \Phi_1(H_{z,\epsilon}) \subset \Phi_2(H_{z,\epsilon}) \subset \cdots$$

while

$$\Psi_0(H_{z,\epsilon}) \supset \Psi_1(H_{z,\epsilon}) \supset \Psi_2(H_{z,\epsilon}) \supset \cdots.$$

Moreover, $\Phi_n(H_{z,\epsilon}) \cup \Psi_n(H_{z,\epsilon}) = \mathcal{J}$ for all n.

The *upper and lower exponential escape velocities* of the measure μ_f through the hole $H_{z,\epsilon}$ denoted by $-\log r_{\mu_f}(H_{z,\epsilon})$ and $-\log r^{\mu_f}(H_{z,\epsilon})$ respectively are defined as

$$\log r_{\mu_f}(H_{z,\epsilon}) \;=\; \liminf_{n\to\infty} \frac{1}{n} \log \mu_f\left(\Psi_n(H_{z,\epsilon})\right) \tag{2.6}$$

$$\log r^{\mu_f}(H_{z,\epsilon}) \;=\; \limsup_{n\to\infty} \frac{1}{n} \log \mu_f\left(\Psi_n(H_{z,\epsilon})\right). \tag{2.7}$$

If $r_{\mu_f}(H_{z,\epsilon}) = r^{\mu_f}(H_{z,\epsilon})$ we say that the *exponential escape velocity* of the measure μ_f through the hole $H_{z,\epsilon}$ is well-defined and is equal to $-\log r(\mu_f(H_{z,\epsilon}))$. Clearly,

$$0 \le r_{\mu_f}(H_{z,\epsilon}) \le r^{\mu_f}(H_{z,\epsilon}) \le 1,$$

thereby the escape velocity lies between 0 and ∞ where we allow the notation $\log 0 = -\infty$. The larger the escape velocity, the faster the measure leaks through the hole.

Being equipped with all necessary definitions, we now state the main results of this paper. The first compares the escape velocity through two different holes of the same size.

Theorem 2.1. *Let H_{z_1,ϵ_1} and H_{z_2,ϵ_2} be any two holes of the same size in $\mathcal{J}$ i.e., $\mu_f(H_{z_1,\epsilon_1}) = \mu_f(H_{z_2,\epsilon_2})$. Then,*

$$\tau(H_{z_1,\epsilon_1}) > \tau(H_{z_2,\epsilon_2}) \quad \textit{implies} \quad \log r\Big(\mu_f(H_{z_1,\epsilon_1})\Big) < \log r\Big(\mu_f(H_{z_2,\epsilon_2})\Big). \tag{2.8}$$

In fact,

$$\left.\begin{aligned} \mu_f(\Psi_n(H_{z_1,\epsilon_1})) &< \mu_f(\Psi_n(H_{z_2,\epsilon_2})) \quad \forall\ n \geq \tau(H_{z_2,\epsilon_2})\ ; \\ \mu_f(\Phi_n(H_{z_1,\epsilon_1})) &> \mu_f(\Phi_n(H_{z_2,\epsilon_2})) \quad \forall\ n \geq \tau(H_{z_2,\epsilon_2}). \end{aligned}\right\} \tag{2.9}$$

The next result describes the escape velocity of the measure μ_f through a sequence of holes $\{H^n_{z,\epsilon}\}_{n\geq 1}$ asymptotically as we decrease the size of the hole to zero. In other words, we consider a real sequence $\{\epsilon_n\}_{n\geq 1}$ that converges to 0 and look at the sequence of holes with centre z whose radius is given by ϵ_n.

Theorem 2.2. *Let $\{H^n_{z,\epsilon}\}_{n\geq 1}$ be a sequence of holes with centre z in $\mathcal{J}$ such that $\mu_f(H^n_{z,\epsilon}) > 0$ for all n. Then,*

$$\begin{aligned} \lim_{n\nearrow\infty} \frac{-\log r\Big(\mu_f(H^n_{z,\epsilon})\Big)}{\mu_f(H^n_{z,\epsilon})} &= \lim_{\epsilon_n\searrow 0} \frac{-\log r\Big(\mu_f(H_{z,\epsilon_n})\Big)}{\mu_f(H_{z,\epsilon_n})} \\ &= \begin{cases} 1 & \textit{if}\ \ T^p z \neq z\ \textit{for any}\ p \in \mathbb{Z}^+\ ; \\ 1 - \exp[f^p(z) - p\mathfrak{P}(f)] & \textit{if}\ \ T^p z = z\ \textit{for some}\ p \in \mathbb{Z}^+, \end{cases} \end{aligned}$$

where $f^p(z) = f(z) + f(Tz) + \cdots + f(T^{p-1}z)$.

The next theorem is about the Hausdorff dimension of the survivor set, $\Psi_\infty(H_{z,\epsilon})$ defined as

$$\Psi_\infty(H_{z,\epsilon}) := \Big\{\zeta \in \mathcal{J} : T^k\zeta \notin H_{z,\epsilon},\ \ \forall\ k \geq 0\Big\}.$$

In other words, $\Psi_\infty(H_{z,\epsilon})$ is the set of points in $\mathcal{J}$ whose orbits are bounded away from z at least by a distance of ϵ forever. Let $f = -s\log|T'|$ where s is the Hausdorff dimension of $\mathcal{J}$. We denote by $\mu_s = \mu_{-s\log|T'|}$ the associated equilibrium state. Let s_ϵ be the Hausdorff dimension of $\Psi_\infty(H_{z,\epsilon})$.

Theorem 2.3. *Let $\{H^n_{z,\epsilon}\}_{n\geq 1}$ be a sequence of holes with centre z in $\mathcal{J}$ such that $\mu_f(H^n_{z,\epsilon}) > 0$ for all n. Then,*

$$\begin{aligned} \lim_{n\nearrow\infty} \frac{s - s_{\epsilon_n}}{\mu_s(H^n_{z,\epsilon})} &= \lim_{\epsilon_n\searrow 0} \frac{s - s_{\epsilon_n}}{\mu_s(H_{z,\epsilon_n})} \\ &= \frac{1}{\int \log|T'| d\mu_s} \begin{cases} 1 & \textit{if}\ \ T^p z \neq z\ \textit{for any}\ p \in \mathbb{Z}^+\ ; \\ 1 - \exp[f^p(z)] & \textit{if}\ \ T^p z = z\ \textit{for some}\ p \in \mathbb{Z}^+, \end{cases} \end{aligned}$$

where $f^p(z) = f(z) + f(Tz) + \cdots + f(T^{p-1}z)$ with $f = -s\log|T'|$.

3 Symbolic dynamics

In this section, we set the stage for symbolic dynamics. Consider the *full symbolic space* Σ that consists of words of infinite length and Σ^* that consists of words of finite length on d symbols, i.e.,

$$\begin{aligned} \Sigma &:= \{1,2,\cdots,d\}^{\mathbb{Z}^+} &= \{x=(x_0,x_1,x_2,\cdots): x_i\in\{1,2,\cdots,d\}\ \forall\ i\in\mathbb{Z}^+\}\ ; \\ \Sigma^* &:= \{1,2,\cdots,d\}^n &= \{x=(x_0,x_1,x_2,\cdots x_{n-1}): x_i\in\{1,2,\cdots,d\}\ \forall\ 1\le i\le n-1<\infty\}. \end{aligned}$$

Being a full symbolic space on d symbols, we know that the $d\times d$ transition matrix M corresponding to the space is aperiodic and has all entries 1. We now define the *shift map* $\sigma:\Sigma\longrightarrow\Sigma$ that shifts all infinite sequences one place to the left with the first term being deleted; $(\sigma x)_i = x_{i+1};\ i\ge 0$ and $\varsigma:\Sigma^*\longrightarrow\Sigma^*$ that shifts all finite sequences one place to the right with the last term being deleted and making a blank space at the left-most position; for a n-lettered word, $x=(x_0,x_1,\cdots,x_{n-1})\in\Sigma^*$, $(\varsigma x)_i = x_{i-1};\ 1\le i\le n-1$ while $(\varsigma x)_0$ is now a blank space that has no letter. For $k<n$, $(\varsigma^k x)_i = x_{i-k}$ meaning that x_0 occurs at the k-th position of $\varsigma^k x$ and that the $k-1$ positions before remain empty.

We say that two distinct points in the set of symbols have distance 1 and a point is at distance 0 from itself; $d(i,j) = 1-\delta_{i,j}$ where $\delta_{i,j}$ is the Kronecker delta function. Then each point is closed and open. In the discrete topology defined by the metric, the set of symbols is compact. It is now only natural to endow the corresponding Tychonov product topology on Σ thereby making it a compact space. In this topology, the *cylinder sets* i.e., any subset $C_{w,t}$ of Σ for $w=(w_0,w_1,\cdots,w_k)\in\Sigma^*$ of the form

$$C_{w,t} \ = \ [w]_t \ := \{x\in\Sigma : x_t = w_0, x_{t+1}=w_1,\cdots,x_{t+k}=w_k\},$$

are both closed and open. Moreover, any open and closed subset of Σ is a finite union of cylinder sets $C_{w,t}$ for $w\in\Sigma^*$. These cylinder sets form a basis for the topology: any open set is a (finite or countable) union of cylinder sets. As a consequence, Σ has topological dimension 0. The product metric defined on Σ is given by

$$d(x,y) \ := \ \sum_{i=0}^{\infty}\frac{1-\delta_{x_i,y_i}}{2^i}. \tag{3.1}$$

In fact, for any $\xi\in(0,1)$, one can define a metric on Σ as

$$d(x,y) \ := \ \xi^{n(x,y)} \quad \text{where} \quad n(x,y) = \sup\{n : x_i = y_i \text{ for } 0\le i\le n\}. \tag{3.2}$$

If $x_i = y_i$ for all i, we define $n(x,y)=\infty$, so that $d(x,y)=0$. Though there are other possible metrics that could be defined on this space, we remark that no particular one is more important than the other. In fact, the topology dictated by either of these metrics agrees with the product topology mentioned above. Observe that σ is a local homeomorphism with $\#\{y:\sigma(y)=x\}\le d$ for all $x\in\Sigma$.

We now define a family of measures on the cylinder sets of Σ by taking $p=(p_1,p_2,\cdots,p_d)$ to be any positive probability vector, i.e., $p_i>0$ for all i and $\sum p_i = 1$. Probability measures of this kind are called *Bernoulli measures*. For a cylinder set $C_{w,t}\subset\Sigma$, we define

$$\mu_B(C_{w,t}) \ = \ \mu_B([w]_t) \ = \ \mu_B([w_0,w_1,\cdots w_k]_t) \ := \ p_{w_0}p_{w_1}\cdots p_{w_k}. \tag{3.3}$$

Though we note that the measure of the cylinder set $C_{w,t}$ is independent of its position t, we shall still include the position variable while denoting a cylinder set for a different purpose that will become clear later. We will further assume that the probability vector p is equidistributed; i.e., $p_i = 1/d$ for all $1 \leq i \leq d$.

The transition matrix M being aperiodic is equivalent to the shift map σ being topologically mixing. By the Perron-Frobenius theory, M has a maximal positive eigenvalue $\rho > 1$, and $\log\rho = h(\sigma)$, the topological entropy of σ. Observe in this case that $\rho = d$. Moreover, we know from [16] that any hyperbolic rational map $T : \mathcal{J} \longrightarrow \mathcal{J}$ is a quotient of this full shift, i.e., there exists a conjugacy map $\pi : \Sigma \longrightarrow \mathcal{J}$ such that $T \circ \pi = \pi \circ \sigma$. It is then clear that the periodic points and the eventually periodic points in $\mathcal{J}$ correspond to the periodic points and the eventually periodic points in Σ.

4 Autocorrelation

Let $\mathcal{J}$ be divided into d^N holes of the same size, i.e., we consider the holes $\mathcal{P}_{(N)} = \{H_{z_i,\epsilon_i}\}_{i=1}^{d^N}$ such that

$$H_{z_i,\epsilon_i} \cap H_{z_j,\epsilon_j} = \phi \quad \text{for} \quad i \neq j \quad ; \quad \bigcup_{i=1}^{d^N} \overline{H_{z_i,\epsilon_i}} = \mathcal{J} \quad \text{and} \quad \mu_f(H_{z_i,\epsilon_i}) = \mu_f\left(H_{z_j,\epsilon_j}\right).$$

In fact, we consider probability measures that are equidistributed. Hence, $\mu_f(H_{z_i,\epsilon_i}) = d^{-N}$ for all i. To each element $H_{z_i,\epsilon_i} \in \mathcal{P}_{(N)}$, there is a corresponding cylinder set $C_{w,1} \subset \Sigma$ where $w \in \Sigma^*$ is a N-lettered word. Moreover, the set of points in $\mathcal{J}$ that do not enter H_{z_i,ϵ_i} in the first $n+1$ iterations of T correspond to the set of points in Σ that do not enter $C_{w,1}$ in the first $n+1$ iterations of σ. In fact, this is the same as the set of points in Σ that do not have the N-lettered word w in its first $N+n+1$ positions. In other words,

$$\begin{aligned}
\mathcal{J}\backslash\Phi_n(H_{z_i,\epsilon_i}) &= \Psi_n(H_{z_i,\epsilon_i}) \\
&= \{\zeta \in \mathcal{J} \ : \ \vartheta(\zeta) > n\} \\
&= \left\{\zeta \in \mathcal{J} \ : \ T^k\zeta \notin H_{z_i,\epsilon_i} \text{ for } 0 \leq k \leq n\right\} \\
&\equiv \left\{x \in \Sigma \ : \ \sigma^k x \notin C_{w,1} \text{ for } 0 \leq k \leq n\right\} \\
&= \{x \in \Sigma \ : \ x \notin C_{w,k} \text{ for } 0 \leq k \leq n\} \\
&= \{x \in \Sigma \ : \ \vartheta(x) > n\} \\
&= \Psi_n(C_{w,1}) \\
&= \Sigma\backslash\Phi_n(C_{w,1}).
\end{aligned}$$

Here, by a slight abuse of notation, we have defined τ, ϑ, Φ_n and Ψ_n for cylinder sets in Σ with the same meaning as earlier. Therefore, the escape velocity of μ_f into the hole $H_{z_i,\epsilon_i} \in \mathcal{P}_{(N)}$ is the same as the escape velocity of μ_B into the appropriate cylinder set $C_{w,1}$,

$$\log r\left(\mu_f(H_{z_i,\epsilon_i})\right) \equiv \log r(\mu_B(C_{w,1})) = \lim_{n\to\infty} \frac{1}{n} \log\left(\frac{\#\Psi_n(C_{w,1})}{d^{n+N}}\right), \tag{4.1}$$

provided the limit exists. However, an asymptotic result as obtained by Guibas and Odlyzko in [9], states $\#\Psi_n(C_{w,1}) \sim c(\kappa_w)^n$ for some constants c and $\kappa_w < d$ depending on w. A little necessary work on this asymptotic result yields the following lemma.

Lemma 4.1. *The escape velocity* $-\log r(\mu_B(C_{w,1}))$ *is well-defined and depends only on* $w \in \Sigma^*$. *Moreover,*

$$-\log r(\mu_B(C_{w,1})) = -\lim_{n\to\infty} \frac{1}{n} \log\left(\frac{\#\Psi_n(C_{w,1})}{d^n}\right) = h(\sigma) - \log \kappa_w. \tag{4.2}$$

The following autocorrelation function of finite words was studied by Guibas and Odlyzko in [9]. Let $w \in \Sigma^*$ be a N-lettered word. According to their definition, the *autocorrelation* of w is the value of the polynomial $\wp_w(z) = a_1 z^{N-1} + a_2 z^{N-2} + \cdots + a_N$ evaluated at $z = d$. Here, the coefficients a_k are given by

$$a_k := \begin{cases} 1 & \text{if } (w)_i = (\varsigma^{k-1} w)_i \text{ for } k-1 \le i \le N-1 ; \\ 0 & \text{otherwise.} \end{cases} \tag{4.3}$$

The following lemma gives the relation between the number of words in Σ that do not contain a particular sub-word $w \in \Sigma^*$ and the autocorrelation of w, as studied by Guibas and Odlyzko in [9], Eriksson in [7] and Cakir, Chryssaphinou and Månsson in [2].

Lemma 4.2. *Let* v *and* w *be two N-lettered words such that* $\wp_v(d) < \wp_w(d)$. *Then*

$$\lim_{n\to\infty} \frac{\log(\#\Psi_n(C_{v,1}))}{n} < \lim_{n\to\infty} \frac{\log(\#\Psi_n(C_{w,1}))}{n}. \tag{4.4}$$

In fact,

$$\#\Psi_n(C_{v,1}) < \#\Psi_n(C_{w,1}) \text{ for all } n > \inf\left\{k : a_k^{(v)} \neq a_k^{(w)}\right\} - 1, \tag{4.5}$$

where $a_k^{(v)}$ *and* $a_k^{(w)}$ *are coefficients as defined above in the polynomial expression of* $\wp_v(z)$ *and* $\wp_w(z)$ *respectively.*

5 Proof of Theorem (2.1)

Let H_{z_1,ϵ_1} and H_{z_2,ϵ_2} be any two holes in $\mathcal{J}$ such that $\mu_f(H_{z_1,\epsilon_1}) = \mu_f(H_{z_2,\epsilon_2})$. Then as earlier, we can look at the two associated cylinder sets $C_{v,1}$ and $C_{w,1}$ respectively in Σ. The condition that the measure of the two holes in $\mathcal{J}$ are equal implies $\mu_B(C_{v,1}) = \mu_B(C_{w,1})$. But the measure μ_B is dictated by the probability vector p that is equidistributed amongst its entries. Hence, v and w must be words that have the same number of letters, say N. This further implies that H_{z_1,ϵ_1} and H_{z_2,ϵ_2} belongs to $\mathcal{P}_{(N)}$.

Let $C_{v,1}$ contain a periodic point of prime period $M < N$. Then $a_k^{(v)} = a_{M+k}^{(v)}$ in the valid range of k. In particular, $a_1^{(v)} = a_{M+1}^{(v)} = 1$ while $a_2^{(v)} = \cdots = a_M^{(v)} = 0$. In other words, the first non-zero coefficient in the polynomial expression $\wp_v(z)$ after $a_1^{(v)}$ is $a_{\tau(C_{v,1})+1}^{(v)}$. Hence, if $\tau(C_{v,1}) > \tau(C_{w,1})$, it is clear that $\wp_v(d) < \wp_w(d)$. Then, by lemma (4.2), we have

$$\#\Psi_n(C_{v,1}) < \#\Psi_n(C_{w,1}) \text{ for all } n > \inf\left\{k : a_k^{(v)} \neq a_k^{(w)}\right\} - 1. \tag{5.1}$$

Observe that $\inf\left\{k : a_k^{(v)} \neq a_k^{(w)}\right\}$ is nothing but $\tau(C_{w,1}) + 1$. Hence,

$$\begin{aligned} \mu_B(\Psi_{n-N}(C_{v,1})) &= \frac{\#\Psi_n(C_{v,1})}{d^{n+N}} \\ &< \frac{\#\Psi_n(C_{w,1})}{d^{n+N}} \qquad \text{for all } n > \tau(C_{w,1}) \\ &= \mu_B(\Psi_{n-N}(C_{w,1})). \end{aligned} \tag{5.2}$$

Moreover, since Φ_n and Ψ_n form a dichotomy of the probability space, we have for all $n > \tau(C_{w,1})$,

$$\mu_B(\Phi_{n-N}(C_{v,1})) = 1-\mu_B(\Psi_{n-N}(C_{v,1})) > 1-\mu_B(\Psi_{n-N}(C_{w,1})) = \mu_B(\Phi_{n-N}(C_{w,1})). \quad (5.3)$$

We invoke lemma (4.2) again to prove the comparison result on the escape velocities of μ_B through these cylinder sets when $\tau(C_{v,1}) > \tau(C_{w,1})$. From equation (5.1), we have for all $n > \tau(C_{w,1})$,

$$\#\Psi_n(C_{v,1}) < \#\Psi_n(C_{w,1}) \implies \lim_{n\to\infty} \frac{1}{n} \log\left(\frac{\#\Psi_n(C_{v,1})}{d^n}\right) < \lim_{n\to\infty} \frac{1}{n} \log\left(\frac{\#\Psi_n(C_{w,1})}{d^n}\right).$$

Hence,

$$\log r(\mu_B(C_{v,1})) < \log r(\mu_B(C_{w,1})). \quad (5.4)$$

The inequalities (5.2), (5.3) and (5.4) when stated in the language of appropriate holes in the Julia set yield,

$$\left.\begin{aligned} \mu_f(\Psi_n(H_{z_1,\epsilon_1})) &< \mu_f(\Psi_n(H_{z_2,\epsilon_2})) \ \forall\ n \geq \tau(H_{z_2,\epsilon_2})\ ; \\ \mu_f(\Phi_n(H_{z_1,\epsilon_1})) &> \mu_f(\Phi_n(H_{z_2,\epsilon_2})) \ \forall\ n \geq \tau(H_{z_2,\epsilon_2})\ ; \end{aligned}\right\} \quad (5.5)$$

$$\tau(H_{z_1,\epsilon_1}) > \tau(H_{z_2,\epsilon_2}) \text{ implies } \log r\Big(\mu_f(H_{z_1,\epsilon_1})\Big) < \log r\Big(\mu_f(H_{z_2,\epsilon_2})\Big), \quad (5.6)$$

thereby proving theorem (2.1).

For a better understanding and a simpler statement of the result, we consider the example of the polynomial map; $T : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ defined by $z \longmapsto z^2$. Then it is clear that the Julia set $\mathcal{J}$ of this polynomial map is the unit circle in the complex sphere, $\mathbb{S}^1 = \{\exp(i\alpha) : 0 \leq \alpha \leq 2\pi\}$. The *Lyubich's measure* μ that equidistributes the periodic points in the Julia set is defined as,

$$\mu := \lim_{n\to\infty} \frac{1}{2^n+1} \sum_{T^n z = z} \delta_z, \quad (5.7)$$

where δ_z is the Dirac delta measure at the point z and the convergence is in the weak* topology. It is easy to verify that μ is a probability measure whose support is $\mathcal{J}$, no point carries mass, i.e., $\mu(\{\zeta\}) = 0$ and that T preserves μ. In fact, μ is the Parry measure that maximises the entropy of the polynomial map T i.e., $\mu = \mu_0$ with $f \equiv 0$.

Any hole in $\mathcal{J}$ has to be an arc from $\exp(i\alpha)$ to $\exp(i(\alpha+\epsilon))$ on $\mathbb{S}^1$, for some $\epsilon > 0$. Now let $\mathcal{J}$ be divided into 2^N parts; by H_j we mean the arc on $\mathbb{S}^1$ from $\exp[2\pi i(j-1)/2^N]$ to $\exp[2\pi i(j)/2^N]$. Then a restatement of theorem (2.1) for this polynomial map reads as follows.

Theorem 5.1. *Let H_i and H_j be any two holes in $\mathcal{J} = \mathbb{S}^1$ such that $\tau(H_i) > \tau(H_j)$. Then,*

$$\log r(\mu(H_i)) < \log r\Big(\mu(H_j)\Big). \quad (5.8)$$

In fact,

$$\left.\begin{aligned} \mu(\Psi_n(H_i)) &< \mu\Big(\Psi_n(H_j)\Big) \ \forall\ n \geq \tau(H_j)\ ; \\ \mu(\Phi_n(H_i)) &> \mu\Big(\Phi_n(H_j)\Big) \ \forall\ n \geq \tau(H_j). \end{aligned}\right\} \quad (5.9)$$

6 Ruelle operator

We return to the general setting of a hyperbolic rational map T restricted on its Julia set $\mathcal{J}$. As earlier, fix $z \in \mathcal{J}$ and let $H_{z,\epsilon}$ be a hole in $\mathcal{J}$. By a sequence of holes $\{H^n_{z,\epsilon}\}_{n\geq 1}$ whose size decreases to zero asymptotically, we mean to consider a real sequence $\{\epsilon_n\}_{n\geq 1}$ that converges to 0 and look at the sequence of holes $\{H_{z,\epsilon_n}\}_{n\geq 1}$ where every element of the sequence $H_{z,\epsilon_n} \in \mathcal{P}_{(m)}$ for some $m \in \mathbb{Z}^+$. Observe that $\{H_{z,\epsilon_n}\}$ is then a nested sequence of holes in $\mathcal{J}$ such that $\bigcap_{n\geq 1} H_{z,\epsilon_n} = \{z\}$. We remark that all the extra assumptions that Ferguson and Pollicott need in [8] are naturally satisfied in our setting.

To proceed in our analysis, we need two Banach spaces of real-valued functions defined on $\mathcal{J}$. In order to meet these demands, we first define an *oscillation function*, as studied by Keller in [13]. For $u : \mathcal{J} \longrightarrow \mathbb{R}$, the oscillation function with respect to the hole $H_{z,\epsilon}$ denoted by $\operatorname{osc}(u, H_{z,\epsilon})$ is defined as

$$\operatorname{osc}(u, H_{z,\epsilon}) \ := \ \operatorname{ess\,sup}\{|u(x) - u(y)| : x, y \in H_{z,\epsilon}\}. \tag{6.1}$$

As $\operatorname{osc}(u, H_{\cdot,\epsilon})$ is lower semi-continuous and therefore measurable, it is possible to define a p-norm for $1 \leq p \leq \infty$, $\operatorname{osc}_p(u, H_\epsilon) = \|\operatorname{osc}(u, H_{\cdot,\epsilon})\|_p$, where this is only a semi-norm. The following lemma assimilated from Parry and Pollicott [18] and Keller [13] gives our two Banach spaces.

Lemma 6.1. *For an integrable real-valued function u defined on $\mathcal{J}$, consider*

$$\mathcal{B}_1 \ := \left\{u \in L^1(\mu_f) : \|u\|_{\mathcal{B}_1} < \infty\right\} \quad \textit{where} \quad \|u\|_{\mathcal{B}_1} \ := \ \sup_{n\geq 1}\left(\frac{osc_1(u, H_{\epsilon_n})}{\epsilon_n}\right) \ + \ \|u\|_1, \tag{6.2}$$

while for an arbitrary real-valued continuous function u defined on $\mathcal{J}$, consider

$$\mathcal{B}_\infty \ := \ \{u : \mathcal{J} \longrightarrow \mathbb{R} : \|u\|_{\mathcal{B}_\infty} < \infty\} \quad \textit{where} \quad \|u\|_{\mathcal{B}_\infty} \ := \ \sup_{n\geq 1}\left(\frac{osc_\infty(u, H_{\epsilon_n})}{\epsilon_n}\right) \ + \ \|u\|_\infty. \tag{6.3}$$

Then, $\mathcal{B}_1$ and $\mathcal{B}_\infty$ are Banach spaces equipped with the norms $\|\cdot\|_{\mathcal{B}_1}$ and $\|\cdot\|_{\mathcal{B}_\infty}$ respectively. Moreover, the set $\{u \in \mathcal{B}_1 : \|u\|_{\mathcal{B}_1} \leq c\}$ is L^1-compact for all $c > 0$.

We now define the Ruelle (transfer) operator $\mathcal{L}_f : \mathcal{B}_i \longrightarrow \mathcal{B}_i$ for $i \in \{1, \infty\}$ as

$$\begin{aligned} \left(\mathcal{L}_f u\right)(x) \ &:= \ \sum_{y\in T^{-1}x} \exp[f(y)]\, u(y) \ ; \\ \text{so that} \quad \left(\mathcal{L}^n_f u\right)(x) \ &= \ \sum_{y\in T^{-n}x} \exp[f^n(y)]\, u(y), \end{aligned} \tag{6.4}$$

where $f^n(y) = f(y) + f(Ty) + \cdots + f(T^{n-1}y)$. The following standard result from [18] describes the spectral gap of the Ruelle operator $\mathcal{L}_f$ defined on $\mathcal{B}_\infty$ and gives an alternate characterisation of pressure.

Proposition 6.2. *Consider the Ruelle operator $\mathcal{L}_f : \mathcal{B}_\infty \longrightarrow \mathcal{B}_\infty$ as defined in equation (6.4). Then,*

1. *There is a simple maximal positive eigenvalue* λ *of* $\mathcal{L}_f$ *with corresponding eigenfunction* $g \in \mathcal{B}_\infty$ *being strictly positive and bounded. In fact,*

$$\log \lambda \;=\; \mathfrak{P}(f) \;=\; \lim_{n\to\infty} \frac{1}{n} \log\left(\mathcal{L}_f^n 1\right)(x). \tag{6.5}$$

2. *The remainder of the spectrum of* $\mathcal{L}_f : \mathcal{B}_\infty \longrightarrow \mathcal{B}_\infty$ *is contained in a disc of radius strictly smaller than* λ.

3. $\lambda^{-n}\mathcal{L}_f^n u \to g \int u d\mu_f$ *uniformly for all* $u \in \mathcal{B}_\infty$*, where* g *is the strictly positive eigenfunction corresponding to* λ *that satisfies* $\int g d\mu_f = 1$.

Making use of an estimate on the $\mathcal{B}_\infty$-norm of the iterates of the Ruelle operator due to Denker, Przytycki and Urbanski in [5] that states $\|\mathcal{L}_f^n u\|_{\mathcal{B}_\infty} \le c\|u\|_{\mathcal{B}_\infty}$ for all $n \in \mathbb{Z}^+$, one can normalise the transfer operator as done by Haydn in [10] such that the simple maximal eigenvalue $\lambda = 1$ with corresponding eigenfunction being the constant, i.e., $\mathcal{L}_f 1 = 1$. Our next aim is to obtain a result (spectral gap) similar to proposition (6.2) for the Ruelle operator $\mathcal{L}_f$ defined on $\mathcal{B}_1$. We include a short proof for the convenience of the reader.

Proposition 6.3. *Consider the Ruelle operator* $\mathcal{L}_f : \mathcal{B}_1 \longrightarrow \mathcal{B}_1$ *as defined in equation (6.4). Then*

1. *There is a simple maximal positive eigenvalue* λ *of* $\mathcal{L}_f$ *with corresponding eigenfunction* $g \in \mathcal{B}_1$ *being strictly positive and bounded. In fact,* $\lambda = 1$ *and* $g = 1$*, i.e.,* $\mathcal{L}_f 1 = 1$.

2. *The remainder of the spectrum of* $\mathcal{L}_f : \mathcal{B}_1 \longrightarrow \mathcal{B}_1$ *is contained in a disc of radius strictly smaller than* $\lambda = 1$.

3. $\mathcal{L}_f^n u \to \int u d\mu_f$ *uniformly in* L^1*-norm for all* $u \in \mathcal{B}_1$.

Proof. For $u \in \mathcal{B}_1$, choose a function $v \in \mathcal{B}_\infty$ such that $\|u - v\|_1 < \epsilon$. Then using the triangle inequality and statement (3) of proposition (6.2) that asserts uniform convergence, we obtain uniform convergence in this case,

$$\left\|\mathcal{L}_f^n u - \int u d\mu_f\right\|_1 \;\le\; \left\|\mathcal{L}_f^n u - \mathcal{L}_f^n v\right\|_1 \;+\; \left\|\mathcal{L}_f^n v - \int v d\mu_f\right\|_1 \;+\; \left\|\int v d\mu_f - \int u d\mu_f\right\|_1 \;<\; 3\epsilon.$$

Since μ_f is an ergodic T-invariant probability measure on $\mathcal{J}$, one can consider an integrable function u defined on $\mathcal{J}$ such that $\int u d\mu_f = 0$. In fact, due to a result by Pollicott and Sharp in [20], there exists a unique $t \in \mathbb{R}$ for which $\int f d\mu_{tf} = 0$. Similar to the estimate due to Denker, Przytycki and Urbanski in [5], Keller in [13] gives an estimate on the $\mathcal{B}_1$-norm of the iterates of the Ruelle operator that states, $\|\mathcal{L}_f^n u\|_{\mathcal{B}_1} \le c\|u\|_{\mathcal{B}_1}$. Using this estimate and the L^1-compactness of the set $\{u \in \mathcal{B}_1 : \|u\|_{\mathcal{B}_1} \le c\}$ for all $c > 0$, one then obtains the spectral gap of $\mathcal{L}_f$ defined on $\mathcal{B}_1$. □

7 Perturbed Ruelle operator

In this section, we introduce a perturbation of the Ruelle (transfer) operator dependent on a hole. As earlier, let $\{H^n_{z,\epsilon}\}_{n\geq 1}$ be a decreasing sequence of holes in $\mathcal{J}$ whose size decreases to 0 asymptotically. The perturbed Ruelle (transfer) operator $\mathcal{L}_{f,n} : \mathcal{B}_i \longrightarrow \mathcal{B}_i$ for $i \in \{1, \infty\}$ is defined as

$$\left(\mathcal{L}_{f,n}u\right)(x) \;:=\; \mathcal{L}_f\left(\chi_{H_{z,\epsilon_n}} u\right)(x) \;=\; \sum_{y\in T^{-1}x} \exp\left[f(y)\right]\left(\chi_{H_{z,\epsilon_n}}(y)\, u(y)\right). \tag{7.1}$$

Observe that the terms in the summand contribute to the sum iff $y \in H_{z,\epsilon_n}$. Hence,

$$\left(\mathcal{L}^k_{f,n}u\right)(x) \;=\; \sum_{y\in T^{-k}x} \exp\left[f^k(y)\right]\left(\prod_{i=0}^{k-1} \chi_{H_{z,\epsilon_n}}\left(T^i(y)\right)\right)u(y).$$

In other words, the summand in the k-th iterate of the perturbed Ruelle operator contributes to the sum iff all of the k points in that pre-image branch of x is in the hole H_{z,ϵ_n}.

We now study certain properties of the perturbed Ruelle operator. In particular, we ensure that the hypotheses of the following theorem assimilated from Hennion as in [11] and Keller and Liverani as in [14] is satisfied in our setting so that we may use a consequence of the assertion of the same that speaks about the convergence of the sequence $\{\lambda_n\}$ and the quasi-compactness of the perturbed Ruelle operator. This theorem forms the crux of weak perturbation theory.

Theorem 7.1. *Consider the family of perturbed Ruelle operators* $\mathcal{L}_{f,n} : \mathcal{B}_1 \longrightarrow \mathcal{B}_1$.

1. *Let* $\|\cdot\|_{\mathcal{B}_1,w}$ *be a weak norm defined on* $\mathcal{B}_1$ *that is dominated by the* $\mathcal{B}_1$*-norm, i.e., for all* $u \in \mathcal{B}_1$*, we have* $\|u\|_{\mathcal{B}_1,w} \leq \|u\|_{\mathcal{B}_1}$.

2. *Let* $\mathcal{L}_{f,n}$ *satisfy a uniform Lasota - Yorke inequality, i.e., there exists constants* $c_1, c_2 > 0$ *and* $\alpha \in (0,1)$ *such that for all* k, n *and* $u \in \mathcal{B}_1$*, we have*

$$\left\|\mathcal{L}^k_{f,n}u\right\|_{\mathcal{B}_1} \;\leq\; c_1\alpha^k\|u\|_{\mathcal{B}_1} \;+\; c_2\|u\|_{\mathcal{B}_1,w}.$$

3. *The maximal eigenvalue* λ_n *of the perturbed operator* $\mathcal{L}_{f,n}$ *is isolated.*

4. $\lim_{n\to\infty} \left|\left|\left|\mathcal{L}_f - \mathcal{L}_{f,n}\right|\right|\right| = 0$ *where* $|||\mathcal{L}||| = \sup\left\{\|\mathcal{L}u\|_{\mathcal{B}_1,w} : \|u\|_{\mathcal{B}_1} \leq 1\right\}$.

5. *The closed unit ball in* $\mathcal{B}_1$ *is compact in the weak norm,* $\|\cdot\|_{\mathcal{B}_1,w}$.

Then λ_n *converges to* λ *and* $\mathcal{L}_{f,n}$ *is quasi-compact.*

Hypothesis 1 : Define a weak norm as done by Keller and Liverani in [14] on $\mathcal{B}_1$ as

$$\|u\|_{\mathcal{B}_1,w} \;:=\; \sup_{k\geq 0}\;\sup_{n\geq 1}\; \frac{1}{\alpha^n}\int_{T^{-k}H_{z,\epsilon_n}} |u|\,d\mu_f \;+\; \|u\|_1\,, \quad \text{where} \quad \alpha \in \left(d^{-1}, 1\right). \tag{7.2}$$

Consider a hole $H_{z,\epsilon} \in \mathcal{P}_{(1)}$. Then,

$$
\begin{aligned}
|u(z)| &\leq \operatorname{osc}(u, H_{z,\epsilon}) + \frac{1}{d}\int_{H_{z,\epsilon}} |u| d\mu_f \\
&\leq \frac{1}{d}\left(\int_{H_{z,\epsilon}} \operatorname{osc}(u, H_{z,\epsilon})\, d\mu_f + \int_{H_{z,\epsilon}} |u| d\mu_f\right) \\
&\leq \frac{1}{d}\|u\|_{\mathcal{B}_1}. \qquad (7.3)
\end{aligned}
$$

Therefore, if $\alpha \in \left(d^{-1}, 1\right)$, then making use of the inequality in (7.3), it is easy to observe that the weak norm $\|\cdot\|_{\mathcal{B}_1,w}$ is dominated by the strong norm, $\|\cdot\|_{\mathcal{B}_1}$.

Hypothesis 2 : Before we prove a uniform Lasota - Yorke inequality, we state the following lemma that will be useful in the sequel. A proof is not hard to write.

Lemma 7.2. *There exists a constant $c > 0$ such that for any positive integers n and k and for all $u \in \mathcal{B}_1$, we have*

$$
\sup_{m\geq 1}\left(\frac{osc_1\left(\prod_{i=0}^{k-1}\chi_{H_{z,\epsilon_n}} u, H_{\epsilon_m}\right)}{\epsilon_m}\right) \leq \sup_{m\geq 1}\left(\frac{osc_1\left(u, H_{\epsilon_m}\right)}{\epsilon_m}\right) + \frac{c\,\|u\|_{\mathcal{B}_1,w}}{\alpha^k}. \qquad (7.4)
$$

Let $u \in L^1(\mu_f)$. Then

$$
\|\mathcal{L}_{f,n} u\|_1 \leq \int \mathcal{L}_f |\chi_{H_{z,\epsilon_n}} u| d\mu_f \leq \|u\|_1. \qquad (7.5)
$$

Moreover, for any fixed $k \geq 0$ and $m \geq 1$,

$$
\begin{aligned}
\frac{1}{\alpha^m}\int_{T^{-k}H_{z,\epsilon_m}} \left|\mathcal{L}_{f,n} u\right| d\mu_f &\leq \frac{1}{\alpha^m}\int_{T^{-k}H_{z,\epsilon_m}} \mathcal{L}_f \left|\chi_{H_{z,\epsilon_n}} u\right| d\mu_f \\
&= \frac{1}{\alpha^m}\int_{T^{-(k+1)}H_{z,\epsilon_m}} \chi_{H_{z,\epsilon_n}} |u|\, d\mu_f \\
&\leq \frac{1}{\alpha^m}\int_{T^{-(k+1)}H_{z,\epsilon_m}} |u|\, d\mu_f. \qquad (7.6)
\end{aligned}
$$

Therefore, making use of the inequalities in (7.5) and (7.6) and iterating $\mathcal{L}_{f,n}$, one obtains

$$
\left\|\mathcal{L}_{f,n}^k\right\|_{\mathcal{B}_1,w} \leq 1. \qquad (7.7)
$$

Finally, we observe that

$$
\begin{aligned}
\left\|\mathcal{L}_{f,n}^k u\right\|_{\mathcal{B}_1} &= \sup_{m\geq 1}\left(\frac{\operatorname{osc}_1\left(\mathcal{L}_{f,n}^k u, H_{\epsilon_m}\right)}{\epsilon_m}\right) + \|\mathcal{L}_{f,n}^k u\|_1 \\
&\leq \sup_{m\geq 1}\left(\frac{\operatorname{osc}_1\left(u, H_{\epsilon_m}\right)}{\epsilon_m}\right) + \frac{c\,\|u\|_{\mathcal{B}_1,w}}{\alpha^k} + \|u\|_1 \\
&\leq \|u\|_{\mathcal{B}_1} + \frac{c\,\|u\|_{\mathcal{B}_1,w}}{\alpha^k}. \qquad (7.8)
\end{aligned}
$$

Hypothesis 3 : A result due to Collet, Martínez and Schmitt as in [4] on perturbed Ruelle (transfer) operators $\mathcal{L}_{f,n} : \mathcal{B}_\infty \longrightarrow \mathcal{B}_\infty$ states that for every $n \geq 1$, there exists a simple maximal eigenvalue $\lambda_n > 0$ with corresponding strictly positive eigenfunction $g_n : \mathcal{J} \longrightarrow \mathbb{R}$. Then, making use of propositions (6.2) and (6.3) from the previous section, it is possible to deduce that $\mathcal{L}_{f,n} g_n = \lambda_n g_n$ for $\mathcal{L}_{f,n} : \mathcal{B}_1 \longrightarrow \mathcal{B}_1$ and that $g_n \in \mathcal{B}_1$.

Hypothesis 4 : Consider $u \in \mathcal{B}_1$ such that $\|u\|_{\mathcal{B}_1} \leq 1$. Then

$$\begin{aligned} \left\| \left(\mathcal{L}_f - \mathcal{L}_{f,n} \right) u \right\|_1 &= \left\| \mathcal{L}_f \chi_{H_{z,\epsilon_n}} u \right\|_1 \\ &\leq \left\| \chi_{H_{z,\epsilon_n}} u \right\|_1 \\ &\leq \mu_f (H_{z,\epsilon_n}) \|u\|_\infty \\ &\leq \mu_f (H_{z,\epsilon_n}) \|u\|_{\mathcal{B}_1} \\ &\leq c_1 \mu_f (H_{z,\epsilon_n}). \end{aligned} \tag{7.9}$$

Therefore, for any fixed $k \geq 0$ and $m \geq 1$, we have

$$\frac{1}{\alpha^m} \int_{T^{-k} H_{z,\epsilon_m}} \left| \left(\mathcal{L}_f - \mathcal{L}_{f,n} \right) u \right| d\mu_f \leq \frac{c_1}{\alpha^m} \mu_f \left(T^{-(k+1)} H_{z,\epsilon_m} \cap H_{z,\epsilon_n} \right) \|u\|_{\mathcal{B}_1}. \tag{7.10}$$

Making use of the inequality in (7.10) and the Gibbs' property of SRB measures that prescribes a constant $c_2 > 0$ such that

$$\mu_f \left(T^{-k} H_{z,\epsilon_m} \cap H_{z,\epsilon_n} \right) \leq c_2 \mu_f (H_{z,\epsilon_m}) \mu_f (H_{z,\epsilon_n}) \quad \forall \; m, n \in \mathbb{Z}^+ \text{ and } k \geq n, \tag{7.11}$$

one infers that hypothesis 4 is satisfied.

Hypothesis 5: Let the closed unit ball of $\mathcal{B}_1$ be denoted by D,

$$D = \{ u \in \mathcal{B}_1 : \|u\|_{\mathcal{B}_1} \leq 1 \}.$$

Let $\{u_m\}_{m \geq 1}$ be a sequence in D. We shall prove that there exists a subsequence $\{u_{m_i}\}_{i \geq 1}$ of $\{u_m\}_{m \geq 1}$ in D such that

$$\|u_{m_i} - u\|_{\mathcal{B}_1, w} \rightarrow 0 \qquad \text{as} \quad i \rightarrow \infty.$$

Let $\sup_{i \geq 1} \|u - u_{m_i}\|_\infty = c < \infty$, since $u, u_{m_i} \in D$. Given $\epsilon > 0$, choose a positive integer N such that

$$\frac{1}{\alpha^n} \mu_f (H_{z,\epsilon_n}) \leq \frac{\epsilon}{c} \quad \forall \; n > N.$$

Choose a positive integer I such that

$$\|u - u_{m_i}\|_1 \leq \frac{\epsilon}{\alpha^N} \quad \forall \; i > I.$$

This is possible by lemma (6.1) since D is L^1-compact.

Therefore, for fixed n, k when $n > N$ we have

$$\frac{1}{\alpha^n} \int_{T^{-k} H_{z,\epsilon_n}} \left| u - u_{m_i} \right| d\mu_f \leq \frac{1}{\alpha^n} \mu_f (H_{z,\epsilon_n}) \left\| u - u_{m_i} \right\|_\infty \leq \epsilon. \tag{7.12}$$

However, when $n \leq N$, we have for $i \geq I$,

$$\frac{1}{\alpha^n} \int_{T^{-k}H_{z,\epsilon_n}} \left|u - u_{m_i}\right| \, d\mu_f \;\leq\; \frac{1}{\alpha^n} \int \left|u - u_{m_i}\right| \, d\mu_f \;\leq\; \epsilon. \tag{7.13}$$

Equations (7.12) and (7.13) in the definition of the weak norm as in (7.2) along-with lemma (6.1) ensures that hypothesis 5 is satisfied in our setting.

Hence, λ_n converges to λ. Moreover, the quasi-compactness of $\mathcal{L}_{f,n}$ implies that one can decompose

$$\mathcal{L}_{f,n} \;=\; \lambda_n E_{f,n} \,+\, F_{f,n}, \tag{7.14}$$

where $E_{f,n}$ is a projection onto the eigenspace $\{cg_n : c \in \mathbb{C}\}$ and $E_{f,n}F_{f,n} = F_{f,n}E_{f,n} = 0$. The spectral radius of $F_{f,n}$ is strictly smaller than λ_n. Moreover,

$$\left\|E_{f,n}1\right\|_\infty \;\leq\; c_1, \quad \text{for some } c_1 > 0, \text{ independent of } n \text{ and} \tag{7.15}$$

$$\left\|F^k_{f,n}1\right\|_\infty \;\leq\; c_2\beta^k \;\forall\, n \geq N \quad \text{where } c_2 > 0 \text{ and } 0 < \beta < 1,\ \forall\, k \geq 1. \tag{7.16}$$

We devote the rest of this section to try and establish a relationship between λ_n and the pressure of a function defined on an appropriate subset of $\mathcal{J}$. Let $H_{z,\epsilon_n} \in \mathcal{P}_{(m)}$ for some $m \in \mathbb{Z}^+$. Consider a sequence of positive integers $\{k_n\}_{n\geq 1}$ such that the hole $H_{z,\epsilon_{k_n}} \in \mathcal{P}_{(r)}$, where $r < m$ implying $H_{z,\epsilon_n} \subset H_{z,\epsilon_{k_n}}$. Then defining $\mathcal{J}_n := \bigcap_{i\geq 0} \mathcal{J} \backslash T^{-i} H_{z,\epsilon_n}$, it is possible to infer that $\mathcal{J}_{k_n} \subset \mathcal{J}_n$. Since the pre-images of any typical $x \in \mathcal{J}$ is uniformly distributed in $\mathcal{J}$, it is only fair to expect that each element $H_{z_i,\epsilon_i} \in \mathcal{P}_{(k)}$ contains a point y such that $T^k y = x$. Therefore, if

$$\lim_{k\to\infty} \frac{1}{k} \log\left(\mathcal{L}^k_f 1\right)(x) \;=\; \lim_{k\to\infty} \frac{1}{k} \log\left(\sum_{y \in H_{z_i,\epsilon_i} \in \mathcal{P}_{(k)}} \exp(f^k y) \right) \;=\; \mathfrak{P}(f), \tag{7.17}$$

it is only but natural to expect that

$$\lim_{k\to\infty} \frac{1}{k} \log\left(\mathcal{L}^k_{f,n} 1\right)(x) \;=\; \lim_{k\to\infty} \frac{1}{k} \log\left(\sum_{y \in H_{z_i,\epsilon_i} \in \mathcal{P}_{(k)} : H_{z_i,\epsilon_i} \cap \mathcal{J}_n \neq \phi} \exp(f^k y) \right) \;=\; \mathfrak{P}_{\mathcal{J}_n}(f). \tag{7.18}$$

8 Proof of theorem (2.2)

In this section, we prove theorem (2.2), that describes the escape velocity of the measure μ_f through a sequence of holes $\{H^n_{z,\epsilon}\}_{n\geq 1}$ asymptotically as we decrease the size of the hole to zero. As earlier, this means that we consider a real sequence $\{\epsilon_n\}_{n\geq 1}$ that converges to 0 and look at the sequence of holes with centre z whose radius is given by ϵ_n. Throughout this section, we shall assume that $\mathfrak{P}(f) = 0$, thereby allowing $\mathcal{L}_f$ to have a maximal eigenvalue 1. However, the reader is urged to observe that a minor variation in the proof is all that is necessary when $\mathfrak{P}(f) \neq 0$.

Consider

$$\begin{aligned}
\mu_f(\Psi_{k-1}(H_{z,\epsilon_n})) &= \mu_f(\{\zeta\in\mathcal{J}:\vartheta(\zeta)>k-1\}) \\
&= \mu_f\left(\left\{\zeta\in\mathcal{J}:T^i\zeta\notin H_{z,\epsilon_n},\ 0\le i\le k-1\right\}\right) \\
&= \int\left(\prod_{i=0}^{k-1}\chi_{H_{z,\epsilon_n}}\left(T^i\zeta\right)\right)d\mu_f \\
&= \int\mathcal{L}_f^k\left(\prod_{i=0}^{k-1}\chi_{H_{z,\epsilon_n}}\left(T^i\zeta\right)\right)d\mu_f \\
&= \int\mathcal{L}_{f,n}^k 1\,d\mu_f \\
&= \lambda_n^k\int E_{f,n}1\,d\mu_f+\int F_{f,n}^k 1\,d\mu_f \qquad (8.1)
\end{aligned}$$

Therefore, using the definition of escape velocity and the assertion of theorem (7.1) as written in inequalities (7.15) and (7.16) in equation (8.1), we obtain

$$\log r\left(\mu_f(H_{z,\epsilon})\right) = \lim_{k\to\infty}\frac{1}{k}\log\mu_f(\Psi_{k-1}(H_{z,\epsilon_n})) = \log\lambda_n. \qquad (8.2)$$

Therefore,

$$\begin{aligned}
\lim_{n\to\infty}\frac{-\log r\left(\mu_f(H_{z,\epsilon})\right)}{\mu_f(H_{z,\epsilon})} &= \lim_{n\to\infty}\frac{-\log\lambda_n}{\mu_f(H_{z,\epsilon})} \\
&= \lim_{n\to\infty}\frac{\lambda-\lambda_n}{\mu_f(H_{z,\epsilon})}\frac{\log\lambda-\log\lambda_n}{\lambda-\lambda_n} \\
&= \lim_{n\to\infty}\frac{\lambda-\lambda_n}{\mu_f(H_{z,\epsilon})}. \qquad (8.3)
\end{aligned}$$

We denote by $\mu_{f,n}$ the restriction of μ_f to the hole H_{z,ϵ_n} as a probability measure, i.e.,

$$\mu_{f,n}(\cdot) := \frac{\mu_f(\cdot\cap H_{z,\epsilon_n})}{\mu_f(H_{z,\epsilon_n})}.$$

The rest of the proof follows from the next lemma that is adapted from results due to Hirata as in [12] to our settings.

Lemma 8.1. *Let H_{z,ϵ_n} be a sequence of holes in $\mathcal{J}$. Then*

$$\lim_{n\to\infty}\frac{\int E_{f,n}\left(\mathcal{L}_f\chi_{H_{z,\epsilon_n}}\right)d\mu_{f,n}}{\mu_f(H_{z,\epsilon_n})} = \begin{cases}1 & \text{if } T^pz\neq z \text{ for any } p\in\mathbb{Z}^+;\\ 1-\exp[f^pz] & \text{if } T^pz=z \text{ for some } p\in\mathbb{Z}^+,\end{cases} \qquad (8.4)$$

where $f^pz=f(z)+f(Tz)+\cdots+f(T^{p-1}z)$.

Proof. Similar to the Poincaré recurrence time of a hole and the escape time for points in $\mathcal{J}$ into the hole, we define the first return time (if it exists) of the orbit of any $\zeta\in H_{z,\epsilon_n}$ as

$$\tau_n(\zeta) := \inf\left\{k\in\mathbb{Z}^+:T^k\zeta\in H_{z,\epsilon_n}\right\}.$$

Then,

$$
\begin{aligned}
\int \tau_n(\zeta)\, d\mu_{f,n} &= \sum_{k\geq 1} k \times \mu_{f,n}\left(\{\zeta \in H_{z,\epsilon_n} : \tau_n(\zeta) = k\}\right) \\
&= \mu_{f,n}\left(\{\zeta \in H_{z,\epsilon_n} : \tau_n(\zeta) = 1\}\right) + \sum_{k\geq 2} k \times \mu_{f,n}\left(\{\zeta \in H_{z,\epsilon_n} : \vartheta(\zeta) > k-1\}\right) \\
&=: T_1 + \sum_{k\geq 2} k \int \mathcal{L}_{f,n}^{k} 1 d\mu_{f,n} \\
&= T_1 + \sum_{k\geq 2} k \int \mathcal{L}_{f,n}^{k-1}\left(\mathcal{L}_{f,n} 1\right) d\mu_{f,n} \\
&= T_1 + \sum_{k\geq 2} k \int \mathcal{L}_{f,n}^{k-1}\left(\mathcal{L}_f \chi_{H_{z,\epsilon_n}}\right) d\mu_{f,n} \\
&= T_1 + \left(\int E_{f,n}\left(\mathcal{L}_f \chi_{H_{z,\epsilon_n}}\right) d\mu_{f,n}\right)\left(\sum_{k\geq 2} k\lambda_n^{k-1}\right) \\
&\qquad + \sum_{k\geq 2}\left(\int F_{f,n}^{k-1}\left(\mathcal{L}_f \chi_{H_{z,\epsilon_n}}\right) d\mu_{f,n}\right) \\
&=: T_1 + \left(T_{2,1}\left(\frac{1}{(1-\lambda_n)^2} - 1\right)\right) + \sum_{k\geq 2} \int F_{f,n}^{k-1}\left(\mathcal{L}_{f,n} 1\right) d\mu_{f,n} \\
&=: T_1 + (T_{2,1} \times T_{2,2}) + \sum_{k\geq 2} \int F_{f,n}^{k} 1 d\mu_{f,n} \\
&=: T_1 + (T_{2,1} \times T_{2,2}) + T_3. \qquad (8.5)
\end{aligned}
$$

However, Kac's theorem says

$$
\int \tau_n(\zeta)\, d\mu_{f,n} = \frac{1}{\mu_f(H_{z,\epsilon_n})}. \qquad (8.6)
$$

Therefore, we have from equations (8.5) and (8.6),

$$
\frac{\int E_{f,n}\left(\mathcal{L}_f \chi_{H_{z,\epsilon_n}}\right) d\mu_{f,n}}{\mu_f(H_{z,\epsilon_n})} = \underbrace{T_{2,1}}_{\to 0} \underbrace{(T_1 - T_{2,1} + T_3)}_{=\,O(1)} + \left(\frac{T_{2,1}}{1-\lambda_n}\right)^2. \qquad (8.7)
$$

Hence, it is sufficient for us to worry about the last term in equation (8.7). Consider

$$
\begin{aligned}
\frac{T_{2,1}}{1-\lambda_n} &= \frac{\int E_{f,n}\left(\mathcal{L}_f \chi_{H_{z,\epsilon_n}}\right) d\mu_{f,n}}{1-\lambda_n} \\
&= \frac{\int E_{f,n}\left(1 - \mathcal{L}_{f,n} 1\right) d\mu_{f,n}}{1-\lambda_n} \\
&= \int E_{f,n} d\mu_{f,n} \qquad (8.8)
\end{aligned}
$$

Observe that the proof of lemma (8.1) and hence the proof of theorem (2.2) will be completed once we evaluate $\lim_{n\to\infty} \int E_{f,n} d\mu_{f,n}$. We consider the two cases of z now.

Case 1 : z is a non-periodic point.

Since z is a non-periodic point and $\bigcap_{n\geq 1} H_{z,\epsilon_n} = \{z\}$, one can see that for any $k \in \mathbb{Z}^+$, there exists a number $N_k \in \mathbb{Z}^+$ such that $H_{z,\epsilon_{N_k}} \cap T^{-j} H_{z,\epsilon_{N_k}} = \phi$, for $j = 1, 2, \cdots, k$. In particular, for any $\zeta \in T^{-k} H_{z,\epsilon_n}$, we have $\prod_{i=0}^{k-1} \chi_{H_{z,\epsilon_n}}\left(T^i \zeta\right) = 1$, where $n > N_k$. Moreover,

$$\chi_{H_{z,\epsilon_n}}(\zeta)\mathcal{L}^k_{f,n}1(\zeta) \;=\; \chi_{H_{z,\epsilon_n}}(\zeta)\mathcal{L}^k_f\left(\prod_{i=0}^{k-1}\chi_{H_{z,\epsilon_n}}\left(T^i\zeta\right)\right) \;=\; \chi_{H_{z,\epsilon_n}}(\zeta)\mathcal{L}^k_f 1(\zeta) \;=\; \chi_{H_{z,\epsilon_n}}(\zeta),$$

thereby asserting

$$\int \mathcal{L}_{f,n} 1 d\mu_{f,n} \quad = \quad 1, \qquad \text{for} \quad n > N_k. \tag{8.9}$$

Hence,

$$\begin{aligned}
\left|1 - \int E_{f,n} 1 \; d\mu_{f,n}\right| \;&=\; \left|\int \mathcal{L}^k_{f,n} 1 d\mu_{f,n} - \int E_{f,n} 1 \; d\mu_{f,n}\right| \\
&=\; \left|(\lambda_n^k - 1)\int E_{f,n} 1 \; d\mu_{f,n} + \int F^k_{f,n} 1 d\mu_{f,n}\right| \\
&\leq\; c\left(\left|1 - \lambda_n^k\right| + \beta^k\right).
\end{aligned} \tag{8.10}$$

The last line in the proof uses the assertion of theorem (7.1) as written in inequalities (7.15) and (7.16).

Case 2 : z is a periodic point of period p.

Fix some large positive integer m and set $k = pm$. Now consider

$$\begin{aligned}
\chi_{H_{z,\epsilon_n}}(x) - \chi_{H_{z,\epsilon_n}}(x)\mathcal{L}^k_{f,n}1(x) \;&=\; \left(\chi_{H_{z,\epsilon_n}}(x)\right)\left(\sum_{y=T^{-k}x} \exp(f^k y)\chi_{\bigcup_{i=0}^{k-1} T^{-i} H_{z,\epsilon_n}}(y)\right) \\
&=\; \left(\chi_{H_{z,\epsilon_n}}(x)\right)\left(\sum_{y=T^{-k}x} \exp(f^k y)\chi_{T^{k-p} H_{z,\epsilon_n}}(y)\right) \\
&=\; \left(\chi_{H_{z,\epsilon_n}}(x)\right)\left(\sum_{y=T^{-pm}x} \exp(f^{pm} y)\chi_{T^{-p(m-1)} H_{z,\epsilon_n}}(y)\right) \\
&=\; \left(\chi_{H_{z,\epsilon_n}}(x)\right)\left(\mathcal{L}^{pm}_f\left(\chi_{H_{z,\epsilon_n}} \circ T^{p(m-1)}\right)(x)\right) \\
&=\; \left(\chi_{H_{z,\epsilon_n}}(x)\right)\left(\mathcal{L}^p_f\left(\chi_{H_{z,\epsilon_n}}\right)(x)\right).
\end{aligned} \tag{8.11}$$

Hence,

$$\begin{aligned}
\left|1 - \exp(f^p z) - \int \mathcal{L}^k_{f,n} 1 \; d\mu_{f,n}\right| \;&\leq\; \left|\mathcal{L}^p_f\left(\chi_{H_{z,\epsilon_n}}\right)(x) - \exp(f^p z) \; d\mu_{f,n}\right| \\
&\leq\; \sup_{y \in H_{z,\epsilon_r} \in \mathcal{P}_{(k_n+p}} \left|f^p y - f^p z\right| \\
&\leq\; \frac{\epsilon_n |f|_\alpha}{1-\alpha} \\
&\to\; 0, \quad \text{as} \;\; n \to \infty.
\end{aligned} \tag{8.12}$$

Here, $|f|_\alpha$ denotes the usual Hölder semi-norm defined by

$$|f|_\alpha \;=\; \sup\left\{\frac{|f(x)-f(y)|}{|x-y|^\alpha} \;:\; x,y\in\mathcal{J}\right\}.$$

Hence for the periodic case, the assertion analogous to equation (8.9) is that

$$\lim_{n\to\infty}\int \mathcal{L}^k_{f,n}1\,d\mu_{f,n} \;=\; 1-\exp\,[f^p z]. \tag{8.13}$$

Now, a little more work analogous to the estimate in (8.10) completes the proof. □

As promised, we conclude this section with our favourite example; $z\longmapsto z^2$ restricted on $\mathbb{S}^1$. A sequence of holes here looks like

$$\{H_n := \text{ arc on } \mathbb{S}^1 \text{ from } \exp(i\alpha) \text{ to } \exp(i(\alpha+\epsilon_n))\}_{n\geq 1}, \;\text{ where } \epsilon_n\to 0.$$

Observe in this case that $\exp(i\alpha)$ is a periodic point iff α is a rational multiple of 2π. Hence, a restatement of theorem (2.2) in this situation reads as follows.

Theorem 8.2. *Let $\{H_n\}_{n\geq 1}$ be a sequence of holes in $\mathcal{J}=\mathbb{S}^1$ such that its Lyubich's measure $\mu(H_n)>0$ for all n. Then,*

$$\begin{aligned}\lim_{n\nearrow\infty}\frac{-\log r(\mu(H_n))}{\mu(H_n)} &= \lim_{\epsilon_n\searrow 0}\frac{-\log r(\mu(H_n))}{\mu(H_n)}\\ &= \begin{cases} 1 & \text{if} \quad \alpha \notin 2\pi\mathbb{Q}\,;\\ 1-2^{-p} & \text{if} \quad \alpha(2^p-1)\in 2\pi\mathbb{Z}.\end{cases}\end{aligned} \tag{8.14}$$

9 Proof of theorem (2.3)

Denote by $\Psi_\infty(H_{z,\epsilon})$ the set of points in $\mathcal{J}$ whose forward orbits remain bounded away from z at least by a distance of ϵ, i.e.,

$$\Psi_\infty(H_{z,\epsilon}) \;:=\; \left\{\zeta\in\mathcal{J} : T^k\zeta\notin H_{z,\epsilon}\;\;\forall\, k\in\mathbb{Z}^+\right\}.$$

This set is called the *survivor set*, for obvious reasons that the orbit of points in this set never enters the hole and hence ceases to die ever.

Let f_t be a family of real-valued functions parametrised by the real variable t defined on $\mathcal{J}$ by $f_t=-t\log|T'|$. At $t=1$, this gives the Lyapunov characteristic function that describes the stability of typical orbits in $\mathcal{J}$. In conjunction with sections 6 and 7, the associated Ruelle operator and the perturbed Ruelle operator are defined for $i\in\{1,\infty\}$ as

$$\begin{aligned}\mathcal{L}_t &:= \mathcal{L}_f : \mathcal{B}_i\longrightarrow\mathcal{B}_i \text{ by } (\mathcal{L}_t u)(x) = \textstyle\sum_{y\in T^{-1}x} u(y)|T'y|^{-t}\;;\\ \mathcal{L}_{t,n} &:= \mathcal{L}_{f,n} : \mathcal{B}_i\longrightarrow\mathcal{B}_i \text{ by } (\mathcal{L}_{t,n} u)(x) = \textstyle\sum_{y\in T^{-1}x}\chi_{H_{z,\epsilon_n}}(y)u(y)|T'y|^{-t}.\end{aligned}$$

In accordance with theorems (6.2) and (6.3), let λ_t and $\lambda_{t,n}$ denote the maximal eigenvalues of $\mathcal{L}_t$ and $\mathcal{L}_{t,n}$ respectively. By a result due to Ruelle in [21], there exists a unique real number s such that the maximal eigenvalue of the associated Ruelle operator is 1, i.e., $\mathfrak{P}(-s\log|T'|)=0$. Moreover, this unique real number s is the Hausdorff dimension of $\mathcal{J}$, $\dim_{\mathcal{H}}\mathcal{J}=s$. For a sequence of holes $\{H^n_{z,\epsilon}\}_{n\geq 1}$, we denote the Hausdorff dimension of the set $\Psi_\infty(H_{z,\epsilon_n})$ by s_{ϵ_n}. In other words, s_{ϵ_n} is the unique real number such that the maximal eigenvalue of the associated perturbed Ruelle operator is 1, i.e., $\mathfrak{P}_{\mathcal{J}_n}(-s_{\epsilon_n}\log|T'|)=0$.

We also know by another result due to Ruelle as can be found in the paper by Coelho and Parry [3] that on the space of Hölder continuous functions, the map $f\longmapsto\mathfrak{P}(f)$ is real analytic. Moreover, if f and g are Hölder continuous and f is not cohomologous to a constant, then the function $t\longmapsto\mathfrak{P}(g+tf)$ is strictly convex and real analytic. By the term real analytic, we mean in the former statement that given an analytic function $f_t(z)=\sum_{n\in\mathbb{Z}^+}a_n(z)t^n$ where $|t|\leq\epsilon$, $\mathfrak{P}(f_t)$ can be expressed as a summation of terms involving powers of t and in the latter statement that $\mathfrak{P}(g+tf)$ can be expressed as a summation of terms involving powers of t.

Hence, $t\longmapsto\lambda_t$ is an analytic function. Similarly $t\longmapsto\lambda_{t,n}$ is also an analytic function. Also we have by theorem (7.1) that $\lambda_{t,n}\to\lambda_t$, where the operator $\mathcal{L}_{t,n}$ is quasi-compact. Therefore, it is clear that $\lambda'_{t,n}\to\lambda'_t$, where the differentiation is with respect to the variable t. We have using Taylor's theorem,

$$\lambda_{s_{\epsilon_n},n} \;=\; 1 \;=\; \lambda_{s,1}+\lambda'_{\xi_n,n}(s_{\epsilon_n}-s), \quad \text{for some} \quad \xi_n\in(s_{\epsilon_n},s). \tag{9.1}$$

Hence, $s-s_{\epsilon_n}=O(\mu_s(H_{z,\epsilon_n}))$. Moreover, considering one more term in the Taylor's expansion, we have

$$\lambda_{s_{\epsilon_n},n} \;=\; 1 \;=\; \lambda_{s,n}+\lambda'_{s,n}(s_{\epsilon_n}-s)+\lambda''_{\xi_n,n}O(\mu_s(H_{z,\epsilon_n})^2), \quad \text{for some} \quad \xi_n\in(s_{\epsilon_n},s). \tag{9.2}$$

Then, using the proof of theorem (2.2), we have

$$\begin{aligned}\frac{s-s_{\epsilon_n}}{\mu_s(H_{z,\epsilon_n})} &= \frac{s-s_{\epsilon_n}}{\lambda_s-\lambda_{s,n}}\,\frac{\lambda_s-\lambda_{s,n}}{\mu_s(H_{z,\epsilon_n})}\\ &= \frac{1}{\lambda'_{s,n}}\,\underbrace{\left(1+\lambda''_{\xi_n,n}O(\mu_s(H_{z,\epsilon_n})^2)\right)}_{\to 1}\begin{cases}1 & \text{if } \; T^pz\neq z \; \text{ for any } p\in\mathbb{Z}^+ \,;\\ 1-\exp(f^p(z)) & \text{if } \; T^pz=z \; \text{ for some } p\in\mathbb{Z}^+.\end{cases}\end{aligned}$$

Observe that the proof of theorem (2.3) is complete once we show $\lim_{n\to\infty}\lambda'_{s,n}=\int\log|T'|d\mu_s$. But we already know that $\lambda'_{s,n}\to\lambda'_s$. Hence, we only need to show that

$$\lim_{n\to\infty}\lambda'_s \;=\; \int\log|T'|d\mu_s.$$

Consider the eigenfunction equation $\mathcal{L}_tg_t=\lambda_tg_t$ that looks in this case as

$$\sum_{y\in T^{-1}x}\frac{g_t(y)}{|T'(y)|^t} \;=\; \lambda_tg_t(x). \tag{9.3}$$

Differentiating equation (9.3) with respect to t, we obtain

$$\sum_{y\in T^{-1}x}\left(\frac{-\log|T'|g_t(y)}{|T'(y)|^t}+\frac{g'_t(y)}{|T'(y)|^t}\right) \;=\; \lambda'_tg_t+\lambda_tg'_t. \tag{9.4}$$

Now integrating equation(9.4) with respect to $\mu_{t,n}$, we obtain

$$\lambda'_t = -\lambda_t \int \log|T'| d\mu_{t,n}. \tag{9.5}$$

Equation (9.5) yields what we require at $t = s$.

Yet again, we exploit the simple example for a better grasp by considering the case of $z \longmapsto z^2$ restricted on its Julia set $\mathcal{J} = \mathbb{S}^1$. By a result due to Manning in [17], we know that $s = \dim_{\mathcal{H}}(\mathcal{J}) = 1$. Recall that a sequence of holes in $\mathbb{S}^1$ looks like

$$\{H_n := \text{ arc on } \mathbb{S}^1 \text{ from } \exp(i\alpha) \text{ to } \exp(i(\alpha + \epsilon_n))\}_{n\geq 1}, \text{ where } \epsilon_n \to 0.$$

Further, we know from a computation in [22] that $\int \log|T'| d\mu = \log 2$. We conclude the paper with a restatement of theorem (2.3) in this situation that reads as follows.

Theorem 9.1. *Let $\{H_n\}_{n\geq 1}$ be a sequence of holes in $\mathcal{J} = \mathbb{S}^1$ such that its Lyubich's measure $\mu(H_n) > 0$ for all n. Then,*

$$\begin{aligned} \lim_{n\nearrow\infty} \frac{1 - s_{\epsilon_n}}{\mu(H_n)} &= \lim_{\epsilon_n \searrow 0} \frac{1 - s_{\epsilon_n}}{\mu(H_n)} \\ &= \frac{1}{\log 2} \begin{cases} 1 & \text{if} \quad \alpha \notin 2\pi\mathbb{Q}\,; \\ 1 - 2^{-p} & \text{if} \quad \alpha(2^p - 1) \in 2\pi\mathbb{Z}. \end{cases} \end{aligned} \tag{9.6}$$

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