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MULTIPLICITY OF NODAL SOLUTIONS FOR A CLASS OF p-LAPLACIAN EQUATIONS IN $\mathbb{R}^N$

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Abstract

We consider a class of p-Laplacian equations in $\mathbb{R}^N$. By carefully analyzing the compactness of the Palais-Smale sequences and constructing Nehari manifolds, we prove that for every positive integer $m \geq 2$, there exists a nodal solution with at least $2m$ nodal domains.

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1 Introduction

In this article, we consider the following p-Laplacian equation in the entire space

$$\begin{cases} -\Delta_p u + (\lambda a(x) + 1)|u|^{p-2}u = f(x,u), & x \in \mathbb{R}^N, \\ u(x) \to 0, & |x| \to \infty, \end{cases} \tag{P_λ}$$

where $\Delta_p u := div(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator with $p \geq 2$. We assume $\lambda \geq 0$, $N > p$, moreover, a and f satisfy the following conditions:

(a_1) $a \in C(\mathbb{R}^N, \mathbb{R})$, $a(x) \geq 0$, $\Omega := int\ a^{-1}(0)$ is non-empty and has smooth boundary, $\bar{\Omega} = a^{-1}(0)$.

(a_2) There exists $M_0 > 0$ such that

$$mes(\{x \in \mathbb{R}^N : a(x) \leq M_0\}) < \infty,$$

here $mes(\cdot)$ denotes the Lebesgue measure on $\mathbb{R}^N$.

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(a_3) a is radially symmetric with respect to the first two coordinates, that is to say, if $(x_1, x_2, z_3, \cdots, z_N) \in \mathbb{R}^N$, $(y_1, y_2, z_3, \cdots, z_N) \in \mathbb{R}^N$ and $|(x_1, x_2)| = |(y_1, y_2)|$, then

$$a(x_1, x_2, z_3, \cdots, z_N) = a(y_1, y_2, z_3, \cdots, z_N).$$

(f_1) $f \in C^1(\mathbb{R}^N, \mathbb{R})$ and when $t \to 0$, $f(x,t) = o(|t|^{p-1})$ uniformly in x.

(f_2) There are constants $a_1 > 0$, $a_2 > 0$ and $p < q < p^* := \frac{Np}{N-p}$ such that

$$\left|f(x,t)\right| \le a_1(1 + |t|^{q-1}), \quad \left|f_t(x,t)\right| \le a_2(1 + |t|^{q-2})$$

for every $x \in \mathbb{R}^N$, $t \in \mathbb{R}$.

(f_3) There exists $\mu > p$ such that for every $x \in \mathbb{R}^N$, $t \in \mathbb{R}\backslash\{0\}$,

$$0 < \mu F(x,t) := \mu \int_0^t f(x,s)ds \le t f(x,t).$$

(f_4) f is radially symmetric with respect to the first two coordinates, that is to say, if $(x_1, x_2, z_3, \cdots, z_N) \in \mathbb{R}^N$, $(y_1, y_2, z_3, \cdots, z_N) \in \mathbb{R}^N$ and $|(x_1, x_2)| = |(y_1, y_2)|$, then

$$f(x_1, x_2, z_3, \cdots, z_N) = f(y_1, y_2, z_3, \cdots, z_N).$$

(f_5) $f(x,t) = -f(x,-t)$ for every $x \in \mathbb{R}^N$, $t \in \mathbb{R}$.

Under these assumptions, we have the following theorem.

Theorem 1.1. *Suppose (a_1)-(a_3) and (f_1)-(f_5) hold. For any given integer $m > 0$, there is $\Lambda_m > 0$ such that problem (P_λ) has a nodal solution with at least $2m$ nodal domains for all $\lambda \ge \Lambda_m$.*

For $p = 2$, (P_λ) turns into a Schrödinger equation of the form

$$-\Delta u + (\lambda a(x) + 1)u = f(x,u), \quad u \in H^1(\mathbb{R}^N), \qquad (S_\lambda)$$

which has been studied extensively. In [3], Bartsch and Wang showed that (S_λ) has a positive and a negative solution. If f is odd, they proved that (S_λ) possesses $k (k \in \mathbb{N})$ pairs of nontrivial solutions. Moreover, Bartsch and Wang studied the general problem

$$-\Delta u + b(x)u = f(x,u), \quad x \in \mathbb{R}^N.$$

When f is odd, they got some existence and multiplicity results.

If $f(x,u) = |u|^{q-2}u$, Bartsch and Wang showed that (S_λ) possesses multiple positive solutions in [4]. In [8], Furtado proved the existence and multiplicity of solutions with exactly two nodal domains for (P_λ), he also studied the concentration behavior of these solutions as $\lambda \to \infty$.

To prove Theorem 1.1, we will use the Nehari manifold technique. By a group constructing method from [12], we consider a minimizing problem on a group-action invariant Nehari manifold and get a nodal solution with at least $2m$ nodal domains when λ is large enough.

The paper is organized as follows. In Section 2, we give some preparation and analyze the compactness of Palais-Smale sequences. In Section 3, we prove Theorem 1.1. In the following, C will denote different constants in different places and $\|\cdot\|_q$ is the usual norm in $L^q(\mathbb{R}^N)$.

2 Preliminaries and compactness of Palais-Smale sequences

Let $W^{1,p}(\mathbb{R}^N)$ be the usual space endowed with the norm

$$\| u \|_{W^{1,p}(\mathbb{R}^N)}= \left(\int_{\mathbb{R}^N} (|\nabla u|^p + |u|^p) dx \right)^{\frac{1}{p}} .$$

In the rest of this paper, we will use E_λ denote the space

$$E := \{u \in W^{1,p}(\mathbb{R}^N) : \int_{\mathbb{R}^N} a(x)|u|^p dx < \infty\}$$

with norm

$$\| u \|_\lambda= \left(\int_{\mathbb{R}^N} (|\nabla u|^p + (\lambda a(x) + 1)|u|^p) dx \right)^{\frac{1}{p}} , \lambda \geq 0.$$

Condition (a$_1$) and the Sobolev theorem imply that the embedding $E_\lambda \hookrightarrow L^q_{loc}(\mathbb{R}^N)$ is compact for all $p \leq q < p^*$. Define a functional $\Phi_\lambda : E_\lambda \to \mathbb{R}$ as follow

$$\Phi_\lambda(u) = \frac{1}{p} \int_{\mathbb{R}^N} (|\nabla u|^p + (\lambda a(x) + 1)|u|^p) dx - \int_{\mathbb{R}^N} F(x, u) dx = \frac{1}{p} \| u \|^p_\lambda - \int_{\mathbb{R}^N} F(x, u) dx.$$

It is obvious that critical points of Φ_λ correspond to solutions of (P_λ). By (f$_1$) and (f$_2$), $\Phi_\lambda \in C^1(E_\lambda, \mathbb{R})$ for all $\lambda \geq 0$.

If a sequence $(u_n) \subset E_\lambda$ satisfies that $\Phi_\lambda(u_n) \to c$ for some $c \in \mathbb{R}$ and $\Phi'_\lambda(u_n) \to 0$ as $n \to \infty$, then (u_n) is a $(PS)_c$-sequence of Φ_λ. We say Φ_λ satisfies the $(PS)_c$-condition if any $(PS)_c$-sequence of Φ_λ has a convergent subsequence.

For the space E_λ, we have the following proposition.

Proposition 2.1. *E_λ is a reflexive Banach space.*

Proof. Condition (a$_1$) and $\lambda \geq 0$ imply that the function

$$\lambda a + 1 : \mathbb{R}^N \to \mathbb{R} : x \mapsto \lambda a(x) + 1$$

is positive and measurable. According to Theorem 1.29 ([11]), it holds that

$$\varphi(X) = \int_X (\lambda a + 1) dx, \quad X \in \mathfrak{B}$$

is a measure on $\mathfrak{B}$ which is the family of Borel sets in $\mathbb{R}^N$ and

$$\int_{\mathbb{R}^N} g d\varphi = \int_{\mathbb{R}^N} g(\lambda a + 1) dx$$

for every measurable g on $\mathbb{R}^N$ with range in $[0, \infty]$.

For the measure φ, we define a space

$$L^p(\varphi) := \{u \mid u \text{ is a measurable function on } \mathbb{R}^N \text{ and } \int_{\mathbb{R}^N} |u|^p d\varphi < \infty\}$$

with norm

$$\| u \|_{L^p(\varphi)} = \left(\int_{\mathbb{R}^N} |u|^p d\varphi \right)^{\frac{1}{p}} = \left(\int_{\mathbb{R}^N} (\lambda a + 1)|u|^p dx \right)^{\frac{1}{p}}.$$

By Theorem 3.11 in [11], $L^p(\varphi)$ is a Banach space. Moreover, by Example 11.3 in [7], $L^p(\varphi)$ is reflexive for all $1 < p < \infty$.

Assume $(u_n) \subset E_\lambda$ is a Cauchy sequence, that is to say $\| u_n - u_m \|_\lambda \to 0$ as $m, n \to \infty$. Then $\| \nabla u_n - \nabla u_m \|_{L^p(\mathbb{R}^N)} \to 0$ and $\| u_n - u_m \|_{L^p(\varphi)} \to 0$. Since $L^p(\mathbb{R}^N)$ and $L^p(\varphi)$ are complete, there exist u and v such that

$$\nabla u_n \to u \in L^p(\mathbb{R}^N),$$

$$u_n \to v \in L^p(\varphi).$$

Since $L^p(\varphi) \hookrightarrow L^p(\mathbb{R}^N)$,

$$u_n \to v \in L^p(\mathbb{R}^N).$$

From the proof of the fact that $W^{1,p}(\mathbb{R}^N)$ is Banach space, we have $u = \nabla v$. So $v \in E_\lambda$ and $\| u_n - v \|_\lambda \to 0$. This proves that E_λ is complete.

Define

$$T : E_\lambda \to L^p(\mathbb{R}^N) \times L^p(\varphi) : u \mapsto (|\nabla u|, u),$$

here $\| \cdot \|_{L^p(\mathbb{R}^N) \times L^p(\varphi)} := \| \cdot \|_{L^p(\mathbb{R}^N)} + \| \cdot \|_{L^p(\varphi)}$. Then $\| \cdot \|_{E_\lambda}$ and $\| \cdot \|_{L^p(\mathbb{R}^N) \times L^p(\varphi)}$ are equivalent norms, so E_λ is equivalent to $T(E_\lambda)$ which is a closed subspace of $L^p(\mathbb{R}^N) \times L^p(\varphi)$. From Pettis theorem, $T(E_\lambda)$ is reflexive and so E_λ is reflexive. □

The following proposition is the main conclusion of this section.

Proposition 2.2. *Suppose* (a_1)-(a_2) *and* (f_1)-(f_3) *hold. Then for any* $c \neq 0$ *there exists* $\Lambda_c > 0$ *such that* Φ_λ *satisfies the* $(PS)_c$*-condition for all* $\lambda \geq \Lambda_c$.

The proof of Proposition 2.2 consists of a series of lemmas which occupy the rest of this section. The thoughts of proof for these lemmas are inspired by Lemma 2.3-2.5 in [4].

Lemma 2.3. *Let* K_λ *be the set of critical points of* Φ_λ. *Then there exists* $\sigma > 0$*(independent of* $\lambda \geq 0$*) such that* $\| u \|_\lambda \geq \| u \|_{W^{1,p}(\mathbb{R}^N)} \geq \sigma$ *for all* $u \in K_\lambda \setminus \{0\}$.

Proof. For any $\epsilon > 0$, by (f_1), there exists $t \in [0,1]$, if $|u| < t$, then $|f(x,u)| < \epsilon |u|^{p-1}$, if $t < |u| < 1$, by (f_2),

$$f(x,u) < a_1(1 + |u|^{q-1}) < 2a_1 = t^{q-1} \frac{2a_1}{t^{q-1}} < A_\epsilon |u|^{q-1},$$

if $|u| \geq 1$, by (f_2),

$$f(x,u) < a_1(1 + |u|^{q-1}) < 2a_1 |u|^{q-1}.$$

Thus, for any $\epsilon > 0$, there exists $A_\epsilon > 0$ such that

$$f(x,u) \leq \epsilon |u|^{p-1} + A_\epsilon |u|^{q-1}, \quad \forall x \in \mathbb{R}^{\mathbb{N}}, u \in \mathbb{R}. \tag{2.1}$$

Choose $\epsilon = 1/2$, then for $u \in K_\lambda \backslash \{0\}$,

$$\begin{aligned} 0 &= \langle \Phi'_\lambda(u), u\rangle \\ &= \int_{\mathbb{R}^N} (|\nabla u|^p + (\lambda a(x)+1)|u|^p)dx - \int_{\mathbb{R}^N} f(x,u)udx \\ &\geq \| u \|_\lambda^p - \frac{1}{2}\int_{\mathbb{R}^N} |u|^p dx - C\int_{\mathbb{R}^N} |u|^q dx \\ &\geq \frac{1}{2}\| u \|_\lambda^p - C \| u \|_q^q \\ &\geq \frac{1}{2}\| u \|_{W^{1,p}(\mathbb{R}^N)}^p - C\| u \|_{W^{1,p}(\mathbb{R}^N)}^q \end{aligned}$$

where $C > 0$ is independent of λ. Hence there exists $\sigma > 0$ such that $\| u \|_{W^{1,p}(\mathbb{R}^N)} \geq \sigma$. □

Lemma 2.4. *There exists $c_0 > 0$ (independent of λ) such that if (u_n) is a $(PS)_c$-sequence of Φ_λ then*

$$\limsup_{n\to\infty} \| u_n \|_\lambda^p \leq \frac{\mu p c}{\mu - p}$$

and if $c \neq 0$, then $c \geq c_0$.

Proof. First we claim that if (u_n) is a $(PS)_c$-sequence of Φ_λ then (u_n) is bounded. In fact,

$$\begin{aligned} & c + o(1) + \| u_n \|_\lambda \cdot o(1) \\ &= \Phi_\lambda(u_n) - \frac{1}{\mu}\Phi'_\lambda(u_n)u_n \\ &= (\frac{1}{p} - \frac{1}{\mu})\int_{\mathbb{R}^N} (|\nabla u_n|^p + (\lambda a(x)+1)|u_n|^p)dx - \int_{\mathbb{R}^N} (F(x,u_n) - \frac{1}{\mu} f(x,u_n)u_n)dx \\ &\geq (\frac{1}{p} - \frac{1}{\mu})\int_{\mathbb{R}^N} (|\nabla u_n|^p + (\lambda a(x)+1)|u_n|^p)dx \\ &= \frac{\mu - p}{\mu p}\| u_n \|_\lambda^p \end{aligned}$$

which implies that (u_n) is bounded. By (f_3) we have

$$\begin{aligned} c &= \limsup_{n\to\infty} (\Phi_\lambda(u_n) - \frac{1}{\mu}\Phi'_\lambda(u_n)u_n) \\ &= \limsup_{n\to\infty} \left((\frac{1}{p} - \frac{1}{\mu})\int_{\mathbb{R}^N} (|\nabla u_n|^p + (\lambda a(x)+1)|u_n|^p)dx - \int_{\mathbb{R}^N} (F(x,u_n) - \frac{1}{\mu} f(x,u_n)u_n)dx \right) \\ &\geq \limsup_{n\to\infty} (\frac{1}{p} - \frac{1}{\mu})\int_{\mathbb{R}^N} (|\nabla u_n|^p + (\lambda a(x)+1)|u_n|^p)dx \\ &= \frac{\mu - p}{\mu p}\limsup_{n\to\infty} \| u_n \|_\lambda^p . \end{aligned}$$

According to (2.1), choosing $\epsilon = 1/2$, then

$$\begin{aligned} \langle \Phi'_\lambda(u), u\rangle &= \int_{\mathbb{R}^N} (|\nabla u|^p + (\lambda a(x)+1)|u|^p)dx - \int_{\mathbb{R}^N} f(x,u)udx \\ &\geq \frac{1}{2}\| u \|_\lambda^p - C\| u \|_\lambda^q . \end{aligned}$$

So there exists $\sigma_1 > 0$ such that for all $\| u \|_\lambda < \sigma_1$

$$\frac{1}{4} \| u \|_\lambda^p < \langle \Phi'_\lambda(u), u \rangle. \tag{2.2}$$

Set $c_0 = \sigma_1^p(\mu - p)/\mu p$. If $c < c_0$, then

$$\limsup_{n\to\infty} \| u_n \|_\lambda^p \leq \frac{\mu p c}{\mu - p} < \sigma_1^p.$$

Thus $\| u_n \|_\lambda < \sigma_1$ for n large enough. By (2.2),

$$\frac{1}{4} \| u_n \|_\lambda^p < \langle \Phi'_\lambda(u_n), u_n \rangle = o(1) \| u_n \|_\lambda .$$

Then $\| u_n \|_\lambda \to 0$ as $n \to \infty$. Hence $\Phi_\lambda(u_n) \to 0$, i.e., $c = 0$ □

Lemma 2.5. *There exists $\delta_0 > 0$ such that any $(PS)_c$-sequence (u_n) of Φ_λ satisfies*

$$\liminf_{n\to\infty} \| u_n \|_q^q \geq \delta_0 c.$$

Proof. The proof is similar to Lemma 5.1 of [3]. For any u, by (f_3) and (2.1), we have

$$\begin{aligned} \frac{1}{p} f(x,u)u - F(x,u) &\leq \frac{1}{p} f(x,u)u \\ &\leq \frac{\epsilon}{p}|u|^p + \frac{A_\epsilon}{p}|u|^q. \end{aligned}$$

If (u_n) is a $(PS)_c$-sequence of Φ_λ, then

$$\begin{aligned} c &= \lim_{n\to\infty} (\Phi_\lambda(u_n) - \frac{1}{p}\Phi'_\lambda(u_n)u_n) \\ &= \lim_{n\to\infty} \int_{\mathbb{R}^N} (\frac{1}{p} f(x,u_n)u_n - F(x,u_n))dx \\ &\leq \lim_{n\to\infty} \int_{\mathbb{R}^N} (\frac{\epsilon}{p}|u_n|^p + \frac{A_\epsilon}{p}|u_n|^q)dx \\ &\leq \lim_{n\to\infty} \left(\frac{\epsilon}{p} \| u_n \|_\lambda^p + \frac{A_\epsilon}{p} \int_{\mathbb{R}^N} |u_n|^q dx \right). \end{aligned}$$

By Lemma 2.4 it holds that

$$\begin{aligned} c &\leq \frac{\epsilon}{p} \cdot \frac{\mu p c}{\mu - p} + \frac{A_\epsilon}{p} \lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^q dx \\ &\leq \frac{\mu \epsilon c}{\mu - p} + \frac{A_\epsilon}{p} \lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^q dx. \end{aligned}$$

That is to say,

$$c - \frac{\mu \epsilon c}{\mu - p} \leq \frac{A_\epsilon}{p} \lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^q dx.$$

Then $\delta_0 = (1 - \frac{\mu\epsilon}{\mu - p}) \cdot \frac{p}{A_\epsilon}$ is the required constant. □

Lemma 2.6. *For any $\epsilon > 0$ there exists $\Lambda_\epsilon > 0$, $R_\epsilon > 0$ such that if (u_n) is a $(PS)_c$-sequence of Φ_λ and $\lambda \geq \Lambda_\epsilon$ then*

$$\limsup_{n\to\infty} \| u_n |_{B^c_{R_\epsilon}} \|_q^q \leq \epsilon$$

where $B^c_{R_\epsilon} = \{x \in \mathbb{R}^N : |x| \geq R_\epsilon\}$.

Proof. For $R > 0$, we set

$$A(R) := \{x \in \mathbb{R}^N : |x| > R, a(x) \geq M_0\},$$

$$B(R) := \{x \in \mathbb{R}^N : |x| > R, a(x) < M_0\}.$$

According to Lemma 2.4,

$$\begin{aligned}
\int_{A(R)} |u_n|^p dx &\leq \frac{1}{\lambda M_0 + 1} \int_{\mathbb{R}^N} (\lambda a(x) + 1)|u_n|^p dx \\
&\leq \frac{1}{\lambda M_0 + 1} \int_{\mathbb{R}^N} (|\nabla u_n|^p + (\lambda a(x) + 1)|u_n|^p) dx \\
&\leq \frac{1}{\lambda M_0 + 1} (\frac{\mu pc}{\mu - p}) \to 0, \quad as\ \lambda \to \infty.
\end{aligned}$$

Choosing s, s' such that $ps < p^*, 1/s + 1/s' = 1$. Applying Hölder inequality and (a_2), we have

$$\begin{aligned}
\int_{B(R)} |u_n|^p dx &\leq \left(\int_{\mathbb{R}^N} |u_n|^{ps} dx\right)^{1/s} \left(\int_{B(R)} dx\right)^{1/s'} \\
&\leq C \| u_n \|_\lambda^p \cdot (mes(B(R)))^{1/s'} \to 0, \quad as\ R \to \infty.
\end{aligned}$$

Setting $\theta = \frac{N(q-p)}{pq}$, the Gagliardo-Nirenberg inequality yields

$$\begin{aligned}
\int_{B^c_R} |u_n|^q dx &\leq C \| \nabla u_n|_{B^c_R} \|_p^{\theta q} \cdot \| u_n|_{B^c_R} \|_p^{(1-\theta)q} \\
&\leq C \| u_n \|_\lambda^{\theta q} \left(\int_{A(R)} |u_n|^p dx + \int_{B(R)} |u_n|^p dx\right)^{(1-\theta)q/p} \\
&\leq C (\frac{\mu pc}{\mu - p})^{\theta q/p} \left(\int_{A(R)} |u_n|^p dx + \int_{B(R)} |u_n|^p dx\right)^{(1-\theta)q/p}.
\end{aligned}$$

The first summand on the right can be arbitrarily small if λ is large. The second summand on the right will be arbitrarily small if R is large by (a_2). This completes the proof. □

The next two results will overcome the lack of Hilbertian structure.

Lemma 2.7. *(Lemma 3 of [1]) Set $M \geq 1$, $p \geq 2$ and $A(y) = |y|^{p-2}y$, $y \in \mathbb{R}^M$. Consider a sequence of vector functions $\eta_n : \mathbb{R}^N \to \mathbb{R}^M$ such that $(\eta_n) \subset (L^p(\mathbb{R}^N))^M$ and $\eta_n(x) \to 0$ for a.e. $x \in \mathbb{R}^N$. Then, if there exists $M > 0$ such that*

$$\int_{\mathbb{R}^N} |\eta_n|^p dx \leq M \quad for\ all\ n \in \mathbb{N},$$

then we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\left|A(\eta_n)+A(\vartheta)-A(\eta_n+\vartheta)\right|^{\frac{p}{p-1}}dx=0$$

for each $\vartheta\in(L^p(\mathbb{R}^N))^M$.

Remark 2.8. From the proof of the Lemma 2.7, we can conclude that if

$$\int_{\mathbb{R}^N}(\lambda a(x)+1)|\eta_n|^p dx\le M\quad for\ all\ n\in\mathbb{N},$$

then for each $\vartheta\in(L^p(\varphi))^M$,

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}(\lambda a(x)+1)\left|A(\eta_n)+A(\vartheta)-A(\eta_n+\vartheta)\right|^{\frac{p}{p-1}}dx=0.$$

Lemma 2.9. *Let* (u_n) *be a* $(PS)_c$*-sequence of* Φ_λ*, then, up to a sequence,* $u_n\rightharpoonup u$ *in* E_λ *and* u *is a weak solution of* (P_λ)*. Moreover,* $u_n^1=u_n-u$ *is a* $(PS)_{c'}$*-sequence of* Φ_λ*, here* $c'=c-\Phi_\lambda(u)$*.*

Proof. First, (u_n) is bounded in E_λ by Lemma 2.4, hence there is a subsequence of (u_n) such that

$$\begin{array}{l}u_n\rightharpoonup u\in E_\lambda,\quad as\ n\to\infty,\\ u_n\to u\in L^q_{loc}(\mathbb{R}^N),\quad p\le q<p^*,\\ u_n(x)\to u(x)\quad a.e.\ x\in\mathbb{R}^N.\end{array}\tag{2.3}$$

We claim that

$$\nabla u_n(x)\to\nabla u(x)\quad a.e.\ x\in\mathbb{R}^N.\tag{2.4}$$

In fact, define $P_n:\mathbb{R}^N\to R$ as follow

$$P_n(x)=(|\nabla u_n(x)|^{p-2}\nabla u_n(x)-|\nabla u(x)|^{p-2}\nabla u(x))\nabla(u_n(x)-u(x))\tag{2.5}$$

and $K\subset\mathbb{R}^N$ is a compact subset. For any given $\epsilon>0$, set

$$K_\epsilon=\{x\in\mathbb{R}^N: dist(x,K)\le\epsilon\}.$$

Choose a cut-off function $\psi\in C^\infty(\mathbb{R}^N)$ such that $0\le\psi\le1$, $\psi\equiv1$ in K and $\psi\equiv0$ in $\mathbb{R}^N\setminus K_\epsilon$, then by the definition of P_n we have

$$\begin{aligned}0\le\int_K P_n dx &\le \int_{\mathbb{R}^N}P_n\psi dx\\ &= \int_{\mathbb{R}^N}|\nabla u_n|^p\psi dx-\int_{\mathbb{R}^N}|\nabla u_n|^{p-2}(\nabla u_n\cdot\nabla u)\psi dx\\ &\quad+\int_{\mathbb{R}^N}|\nabla u|^{p-2}(\nabla u\cdot\nabla(u-u_n))\psi dx.\end{aligned}\tag{2.6}$$

Since (ψu_n) is bounded in E_λ and $\Phi'_\lambda(u_n)\to0$, it holds that

$$\lim_{n\to\infty}\langle\Phi'_\lambda(u_n),\psi u_n\rangle=\lim_{n\to\infty}\langle\Phi'_\lambda(u_n),\psi u\rangle=0.$$

That is to say,

$$\begin{aligned} o(1) &= \int_{\mathbb{R}^N} (|\nabla u_n|^p \psi + |\nabla u_n|^{p-2}(\nabla u_n \cdot \nabla \psi) u_n + (\lambda a(x)+1)|u_n|^p \psi) dx \\ &\quad - \int_{\mathbb{R}^N} f(x,u_n)\psi u_n dx, \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} o(1) &= \int_{\mathbb{R}^N} (|\nabla u_n|^{p-2}(\nabla u_n \cdot \nabla \psi) u + |\nabla u_n|^{p-2}(\nabla u_n \cdot \nabla u)\psi) dx \\ &\quad + \int_{\mathbb{R}^N} (\lambda a(x)+1)|u_n|^{p-2} u_n u \psi dx - \int_{\mathbb{R}^N} f(x,u_n)\psi u dx. \end{aligned} \tag{2.8}$$

Up to a subsequence, we can assume that $\psi u_n \rightharpoonup \psi u$ in E_λ, so

$$\lim_{n\to\infty} \langle \Phi_\lambda'(u), \psi u - \psi u_n \rangle = 0.$$

That is

$$\begin{aligned} o(1) &= \int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \cdot \nabla (u-u_n)\psi dx + \int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \cdot \nabla \psi (u-u_n) dx \\ &\quad + \int_{\mathbb{R}^N} (\lambda a(x)+1)|u|^{p-2} u(u-u_n)\psi dx - \int_{\mathbb{R}^N} f(x,u)\psi(u-u_n) dx. \end{aligned} \tag{2.9}$$

By (2.6)-(2.9) and the fact that $\psi \equiv 0$ in $\mathbb{R}^N \setminus K_\epsilon$, we have

$$\begin{aligned} 0 &\leq \int_K P_n dx \\ &\leq \int_{K_\epsilon} f(x,u_n)\psi u_n dx - \int_{K_\epsilon} |\nabla u_n|^{p-2}(\nabla u_n \cdot \nabla \psi) u_n dx - \int_{K_\epsilon} (\lambda a(x)+1)|u_n|^p \psi dx \\ &\quad + \int_{K_\epsilon} |\nabla u_n|^{p-2}(\nabla u_n \cdot \nabla \psi) u dx + \int_{K_\epsilon} (\lambda a(x)+1)|u_n|^{p-2} u_n u \psi dx - \int_{K_\epsilon} f(x,u_n)\psi u dx \\ &\quad - \int_{\mathbb{R}^N} |\nabla u|^{p-2}(\nabla u \cdot \nabla \psi)(u-u_n) dx - \int_{K_\epsilon} (\lambda a(x)+1)|u|^{p-2} u(u-u_n)\psi dx \\ &\quad + \int_{K_\epsilon} f(x,u)\psi(u-u_n) dx + o(1) \\ &= \int_{K_\epsilon} |\nabla u_n|^{p-2}(\nabla u_n \cdot \nabla \psi)(u-u_n) dx - \int_{\mathbb{R}^N} |\nabla u|^{p-2}(\nabla u \cdot \nabla \psi)(u-u_n) dx \\ &\quad + \int_{K_\epsilon} (\lambda a(x)+1)(|u_n|^{p-2} u_n u - |u_n|^p)\psi dx + \int_{K_\epsilon} (\lambda a(x)+1)(|u|^{p-2} u u_n - |u|^p)\psi dx \\ &\quad + \int_{K_\epsilon} f(x,u_n)\psi(u_n-u) dx + \int_{K_\epsilon} f(x,u)\psi(u-u_n) dx + o(1) \\ &:= A_1 + A_2 + A_3 + A_4 + A_5 + A_6 + o(1), \quad as\ n \to \infty, \end{aligned} \tag{2.10}$$

where

$$\begin{aligned} A_1 &= \int_{K_\epsilon} |\nabla u_n|^{p-2}(\nabla u_n \cdot \nabla\psi)(u-u_n)dx, \\ A_2 &= \int_{K_\epsilon} |\nabla u|^{p-2}(\nabla u \cdot \nabla\psi)(u-u_n)dx, \\ A_3 &= \int_{K_\epsilon} (\lambda a(x)+1)(|u_n|^{p-2}u_n u - |u_n|^p)\psi dx, \\ A_4 &= \int_{K_\epsilon} (\lambda a(x)+1)(|u|^{p-2}u u_n - |u|^p)\psi dx, \\ A_5 &= \int_{K_\epsilon} f(x,u_n)\psi(u_n-u)dx, \\ A_6 &= \int_{K_\epsilon} f(x,u)\psi(u-u_n)dx. \end{aligned}$$

Since (u_n) is bounded in E_λ, thus $u_n \to u \in L^p(K_\epsilon)$. So we have

$$\begin{aligned} |A_1| &\le |\nabla\psi|_\infty \int_{K_\epsilon} |\nabla u_n|^{p-1}|u_n-u|dx \\ &\le |\nabla\psi|_\infty \| u_n \|_\lambda^{p-1} \| u_n - u \|_{p,K_\epsilon} \\ &= o(1), \quad as\, n \to \infty. \end{aligned}$$

In the same way, $\lim_{n\to\infty} A_2 = 0$. The Hölder inequality, $a(x)\psi$ is bounded in K_ϵ and $u_n \to u$ in $L^p(K_\epsilon)$ imply that

$$\begin{aligned} |A_3| &= \left| \int_{K_\epsilon} (\lambda a(x)+1)\psi|u_n|^{p-2}u_n(u-u_n)dx \right| \\ &\le C\left(\int_{K_\epsilon} |u_n|^p dx\right)^{(p-1)/p} \left(\int_{K_\epsilon} |u-u_n|^p dx\right)^{1/p} \\ &\le C \| u_n \|_\lambda^{p-1} \| u_n - u \|_{p,K_\epsilon} \\ &= o(1), \quad as\, n \to \infty. \end{aligned}$$

Similarly, $\lim_{n\to\infty} A_4 = 0$. As for A_5,

$$\begin{aligned} |A_5| &\le \left(\int_{K_\epsilon} |f(x,u_n)|^{q/(q-1)}dx\right)^{(q-1)/q} \left(\int_{K_\epsilon} |u_n-u|^q dx\right)^{1/q} \\ &\le C\left(\int_{K_\epsilon} (1+u_n^{q-1})^{q/(q-1)}dx\right)^{(q-1)/q} \left(\int_{K_\epsilon} |u_n-u|^q dx\right)^{1/q} \\ &\le C\left(\int_{K_\epsilon} (1+|u_n|^q)dx\right)^{(q-1)/q} \left(\int_{K_\epsilon} |u_n-u|^q dx\right)^{1/q} \\ &\le \left(C + C \| u_n \|_\lambda^{q-1}\right) \| u_n - u \|_{q,K_\epsilon} \\ &= o(1), \quad as\, n \to \infty. \end{aligned}$$

Similarly, $\lim_{n\to\infty} A_6 = 0$. Therefore, we can rewrite (2.10) as

$$0 \le \int_K (|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u)\cdot\nabla(u_n - u)dx \to 0, \quad as\, n \to \infty.$$

Using the fact that $(|a|^{p-2}a - |b|^{p-2}b)(a-b) \ge C_p|a-b|^p$ for every $a, b \in \mathbb{R}^N$([13], p.210), we obtain

$$\lim_{n\to\infty}\int_K |\nabla u_n - \nabla u|^p dx = 0. \tag{2.11}$$

Since K is arbitrary, (2.4) holds.

For any $\omega \in C_0^\infty(\mathbb{R}^N)$, we set $K = supp(\omega)$. From the proof of (2.11), it holds that $\nabla u_n \to \nabla u$ and $u_n \to u$ in $L^p(K)$. Thus

$$\begin{aligned}\lim_{n\to\infty}\int_{\mathbb{R}^N} |\nabla u_n|^{p-2}\nabla u_n \omega dx &= \lim_{n\to\infty}\int_K |\nabla u_n|^{p-2}\nabla u_n \omega dx\\ &= \int_K |\nabla u|^{p-2}\nabla u \omega dx\\ &= \int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u \omega dx,\end{aligned}$$

and

$$\begin{aligned}\lim_{n\to\infty}\int_{\mathbb{R}^N} (\lambda a(x)+1)|u_n|^{p-2}u_n \omega dx &= \lim_{n\to\infty}\int_K (\lambda a(x)+1)|u_n|^{p-2}u_n \omega dx\\ &= \int_K (\lambda a(x)+1)|u|^{p-2}u \omega dx\\ &= \int_{\mathbb{R}^N} (\lambda a(x)+1)|u|^{p-2}u \omega dx.\end{aligned}$$

By (f_2) we have

$$\begin{aligned}\lim_{n\to\infty}\int_{\mathbb{R}^N} f(x,u_n)\omega dx &= \lim_{n\to\infty}\int_K f(x,u_n)\omega dx\\ &= \int_K f(x,u)\omega dx\\ &= \int_{\mathbb{R}^N} f(x,u)\omega dx.\end{aligned}$$

Hence

$$\langle \Phi'_\lambda(u), \omega\rangle = \lim_{n\to\infty}\langle \Phi'_\lambda(u_n), \omega\rangle, \quad \forall \omega \in C_0^\infty(\mathbb{R}^N).$$

Since $C_0^\infty(\mathbb{R}^N)$ is dense in E_λ, for any $\omega \in E_\lambda$, we have

$$\langle \Phi'_\lambda(u), \omega\rangle = \lim_{n\to\infty}\langle \Phi'_\lambda(u_n), \omega\rangle = 0, \tag{2.12}$$

i.e., $\Phi'_\lambda(u) = 0$. Therefore u is a weak solution of (P_λ).

Next we consider the new sequence $u_n^1 = u_n - u$ and we will show that

$$\Phi_\lambda(u_n^1) \to c - \Phi_\lambda(u), \quad as\, n \to \infty, \tag{2.13}$$

and

$$\Phi'_\lambda(u_n^1) \to 0, \quad as\, n \to \infty. \tag{2.14}$$

We observe that

$$\begin{aligned}\Phi_\lambda(u_n^1) &= \Phi_\lambda(u_n) - \Phi_\lambda(u) + \frac{1}{p}\int_{\mathbb{R}^N}(|\nabla u|^p - |\nabla u_n|^p + |\nabla u_n^1|^p)dx \\ &\quad + \frac{1}{p}\int_{\mathbb{R}^N}(\lambda a(x)+1)(|u|^p - |u_n|^p + |u_n^1|^p)dx \\ &\quad + \int_{\mathbb{R}^N}(F(x,u_n^1+u) - F(x,u_n^1) - F(x,u))dx.\end{aligned} \tag{2.15}$$

According to Brézis-Lieb Lemma ([14], Lemma 1.32), we can rewrite (2.15) as

$$\Phi_\lambda(u_n^1) = \Phi_\lambda(u_n) - \Phi_\lambda(u) + \int_{\mathbb{R}^N}(F(x,u_n^1+u) - F(x,u_n^1) - F(x,u))dx + o(1). \tag{2.16}$$

For any $\epsilon > 0$, choose $R(\epsilon) > 0$ such that

$$\int_{B^c_{R(\epsilon)}} |u|^p dx \le \epsilon, \int_{B^c_{R(\epsilon)}} |u|^q dx \le \epsilon, \tag{2.17}$$

where $B^c_{R(\epsilon)} = \{x \in \mathbb{R}^N : |x| \ge R(\epsilon)\}$. By ($f_1$)-($f_3$), we have

$$\begin{aligned}&\int_{B^c_{R(\epsilon)}} \left|F(x,u_n^1+u) - F(x,u_n^1)\right| dx \\ &\le \int_{B^c_{R(\epsilon)}} \left|f(x,u_n^1+\xi u)\right| \cdot |u| dx \\ &\le C\int_{B^c_{R(\epsilon)}} \left((|u_n^1|+|u|)^{p-1} + (|u_n^1|+|u|)^{q-1}\right)\cdot |u| dx \\ &\le C\int_{B^c_{R(\epsilon)}} \left(|u_n^1|^{p-1} + |u|^{p-1} + (|u_n^1|+|u|)^{q-1}\right)\cdot |u| dx \\ &\le C \parallel u_n^1 \parallel^{p-1}_{L^p(B^c_{R(\epsilon)})} \cdot \parallel u \parallel_{L^p(B^c_{R(\epsilon)})} + C \parallel u \parallel^p_{L^p(B^c_{R(\epsilon)})} \\ &\quad + C\left(\int_{B^c_{R(\epsilon)}} (|u_n^1|+|u|)^q dx\right)^{(q-1)/q}\left(\int_{B^c_{R(\epsilon)}} |u|^q dx\right)^{1/q} \\ &= O(\epsilon).\end{aligned}$$

By (2.1) and (f_3),

$$\int_{B^c_{R(\epsilon)}} F(x,u)dx \le C\int_{B^c_{R(\epsilon)}} (|u|^p + |u|^q)dx = O(\epsilon).$$

Since ϵ is arbitrary, we obtain (2.13).

For any $\omega \in E_\lambda$, it holds that

$$\langle \Phi'_\lambda(u_n^1), \omega\rangle = \langle \Phi'_\lambda(u_n), \omega\rangle - \langle \Phi'_\lambda(u), \omega\rangle - \int_{\mathbb{R}^N}(f(x,u_n^1) - f(x,u_n) + f(x,u))\omega dx + A + B$$

where

$$\begin{aligned} A &:= \int_{\mathbb{R}^N} (\left|\nabla u_n^1\right|^{p-2}\nabla u_n^1 + |\nabla u|^{p-2}\nabla u - |\nabla u_n|^{p-2}\nabla u_n)\nabla\omega dx, \\ B &:= \int_{\mathbb{R}^N} (\lambda a(x)+1)(\left|u_n^1\right|^{p-2}u_n^1 + |u|^{p-2}u - |u_n|^{p-2}u_n)\omega dx. \end{aligned}$$

By Hölder inequality and Lemma 2.7, set $\eta_n = \nabla u_n^1$ and $\vartheta = \nabla u$, we have

$$\begin{aligned} |A| &\leq \left(\int_{\mathbb{R}^N} (\left|\nabla u_n^1\right|^{p-2}\nabla u_n^1 + |\nabla u|^{p-2}\nabla u - |\nabla u_n|^{p-2}\nabla u_n)^{\frac{p}{p-1}} dx\right)^{\frac{p-1}{p}} \parallel \nabla\omega \parallel_p \\ &\leq o(1) \parallel \omega \parallel_\lambda, \quad as\, n\to\infty. \end{aligned}$$

Choose $\eta_n = u_n^1$ and $\vartheta = u$, by Hölder inequality and Remark 2.8, it holds that

$$\begin{aligned} |B| &\leq \left(\int_{\mathbb{R}^N} (\lambda a(x)+1)(\left|u_n^1\right|^{p-2}u_n^1 + |u|^{p-2}u - |u_n|^{p-2}u_n)^{\frac{p}{p-1}} dx\right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^N} (\lambda a(x)+1)|\omega|^p dx\right)^{\frac{1}{p}} \\ &\leq o(1) \parallel \omega \parallel_\lambda, \quad as\, n\to\infty. \end{aligned}$$

Therefore, in order to obtain (2.14), by (2.12) we only need to show

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} (f(x,u_n^1) - f(x,u_n) + f(x,u))\omega dx = 0. \tag{2.18}$$

For any $\epsilon > 0$, choose $R(\epsilon) > 0$ such that

$$\left(\int_{B^c_{R(\epsilon)}} |u|^p dx\right)^{1/p} \leq \epsilon, \qquad \left(\int_{B^c_{R(\epsilon)}} |u|^q dx\right)^{1/q} \leq \epsilon.$$

Thus

$$\begin{aligned} \int_{B^c_{R(\epsilon)}} \left|f(x,u)\omega\right| dx &\leq C\int_{B^c_{R(\epsilon)}} (|u|^{p-1} + |u|^{q-1})|\omega| dx \\ &\leq C\cdot\epsilon^{p-1}\cdot \parallel \omega \parallel_\lambda + C\cdot\epsilon^{q-1}\cdot \parallel \omega \parallel_\lambda, \end{aligned}$$

and

$$\begin{aligned} &\int_{B^c_{R(\epsilon)}} \left|f(x,u_n^1) - f(x,u_n^1+u)\right|\cdot|\omega| dx \\ &\leq \int_{B^c_{R(\epsilon)}} \left|f_t(x,u_n^1+\xi u)\right|\cdot|u|\cdot|\omega| dx \\ &\leq C\int_{B^c_{R(\epsilon)}} \left((\left|u_n^1\right| + |u|)^{p-2} + (\left|u_n^1\right| + |u|)^{q-2}\right)\cdot|u|\cdot|\omega| dx \\ &\leq C\int_{B^c_{R(\epsilon)}} \left(\left|u_n^1\right|^{p-2} + |u|^{p-2} + \left|u_n^1\right|^{q-2} + |u|^{q-2}\right)\cdot|u|\cdot|\omega| dx \\ &\leq C \parallel u_n^1 \parallel^{p-2}_{L^p(B^c_{R(\epsilon)})} \cdot \parallel u \parallel_{L^p(B^c_{R(\epsilon)})} \cdot \parallel \omega \parallel_\lambda + C \parallel u \parallel^{p-1}_{L^p(B^c_{R(\epsilon)})} \cdot \parallel \omega \parallel_\lambda \\ &\quad + C \parallel u_n^1 \parallel^{q-2}_{L^q(B^c_{R(\epsilon)})} \cdot \parallel u \parallel_{L^q(B^c_{R(\epsilon)})} \cdot \parallel \omega \parallel_\lambda + C \parallel u \parallel^{q-1}_{L^q(B^c_{R(\epsilon)})} \cdot \parallel \omega \parallel_\lambda \\ &\leq C\cdot\epsilon\cdot \parallel \omega \parallel_\lambda + C\cdot\epsilon^{p-1}\cdot \parallel \omega \parallel_\lambda + C\cdot\epsilon^{q-1}\cdot \parallel \omega \parallel_\lambda . \end{aligned}$$

By Lebesgue's Dominated Convergence Theorem, it holds that

$$\lim_{n\to\infty}\int_{B_{R(\epsilon)}}(f(x,u_n^1)-f(x,u_n)+f(x,u))\omega dx=0.$$

Since ϵ is arbitrary, we obtain (2.18). This completes the proof. □

Proof of Proposition 2.2 Choose $0<\epsilon<\delta_0 c_0/2$, here $c_0>0$ is given by Lemma 2.4 and $\delta_0>0$ is given by Lemma 2.5. According to Lemma 2.6, we choose $\Lambda_\epsilon>0$ and $R_\epsilon>0$, then $\Lambda_c=\Lambda_\epsilon$ is required. Considering a $(PS)_c$-sequence (u_n) of Φ_λ where $\lambda\geq\Lambda_c$ and $c\neq 0$. By Lemma 2.9, $u_n^1=u_n-u$ is a $(PS)_{c'}$-sequence of Φ_λ where $c'=c-\Phi_\lambda(u)$.

Assume $c'\neq 0$, then by Lemma 2.4, we have $c'\geq c_0>0$. By Lemma 2.5,

$$\liminf_{n\to\infty}\| u_n^1\|_q^q\geq\delta_0 c'\geq\delta_0 c_0.$$

Lemma 2.6 implies that

$$\limsup_{n\to\infty}\| u_n^1|_{B^c_{R(\epsilon)}}\|_q^q\leq\epsilon<\frac{\delta_0 c_0}{2}.$$

Assume that $u_n^1\rightharpoonup u^1\in E_\lambda$. By the definition of u_n^1, $u^1=0$. Then

$$\begin{aligned}\delta_0 c_0 &\leq \liminf_{n\to\infty}\| u_n^1\|_q^q\\ &\leq \limsup_{n\to\infty}\| u_n^1\|_q^q\\ &< \limsup_{n\to\infty}\| u_n^1|_{B^c_{R(\epsilon)}}\|_q^q+\lim_{n\to\infty}\int_{B_{R(\epsilon)}}|u_n^1|^q dx\\ &\leq \frac{\delta_0 c_0}{2},\end{aligned}$$

a contradiction. Therefore the assumption does not hold and so $c'=0$.

From the proof of Lemma 2.4, we have $u_n^1\to 0$, i.e., $u_n\to u$. This completes the proof of Proposition 2.2.

3 Proof of Theorem 1.1

In order to prove Theorem 1.1, we will consider a constrained minimizing problem on some Nehari manifold. Inspired by [12], using the symmetrical assumption on $a(x)$ and $f(x,t)$, this minimizing problem will be further constrained on a symmetrical Nehari manifold by Palais principle of symmetric criticality([10]). Set

$$N_\lambda=\{u\in E_\lambda\backslash\{0\}:\langle\Phi'_\lambda(u),u\rangle=0\}=\{u\in E_\lambda\backslash\{0\}:\| u\|_\lambda^p=\int_{\mathbb{R}^N}f(x,u)udx\}.$$

Proof of Theorem 1.1 Denote $x=(y,z)=(y_1,y_2,z_3,\cdots,z_N)\in\mathbb{R}^N$. Let $O(2)$ be the group of orthogonal transformations acting on $\mathbb{R}^2$ by $(g,y)\mapsto gy$. For any integer $m(m\geq 2)$, define a subgroup G_m of $O(2)$(see [12]) as follows. G_m is generated by α and β where α is the rotation in the y-plane by angle $\frac{2\pi}{m}$ and β is a reflection. If $m = 2, \beta$ is a reflection in the line $y_1 = 0$, otherwise, β is a reflection in the line $y_2 = y_1 \tan\frac{\pi}{m}$. Write $\omega = y_1 + iy_2$, then

$$\alpha\omega = \omega e^{\frac{2\pi}{m}i},$$

$$\beta\omega = \bar{\omega} e^{\frac{2\pi}{m}i}.$$

For all $g \in G_m, x \in \mathbb{R}^N$, denote $gx := (gy, z)$. Define the action of G_m on E_λ as

$$(gu)x := det(g)u(g^{-1}x).$$

We claim that Φ_λ is invariant under G_m. That is to say $\Phi_\lambda \circ g = \Phi_\lambda$ for all $g \in G_m$. Indeed, by $g \in O(2)$, conditions (a_3), (f_4), (f_5) and the fact that Lebesgue measure is ivariant under orthogonal transformation, we have

$$\begin{aligned}
\Phi_\lambda(gu) &= \frac{1}{p}\int_{\mathbb{R}^N} (|\nabla(gu)(x)|^p + (\lambda a(x)+1)|(gu)(x)|^p)dx - \int_{\mathbb{R}^N} F(x,(gu)(x))dx \\
&= \frac{1}{p}\int_{\mathbb{R}^N} (|\nabla u(g^{-1}x)|^p + (\lambda a(x)+1)|u(g^{-1}x)|^p)dx - \int_{\mathbb{R}^N} F(x, det(g)u(g^{-1}x))dx \\
&= \frac{1}{p}\int_{\mathbb{R}^N} (|\nabla u(g^{-1}x)|^p + (\lambda a(g^{-1}x)+1)|u(g^{-1}x)|^p)dx - \int_{\mathbb{R}^N} F(g^{-1}x, u(g^{-1}x))dx \\
&= \frac{1}{p}\int_{\mathbb{R}^N} (|\nabla u(g^{-1}x)|^p + (\lambda a(g^{-1}x)+1)|u(g^{-1}x)|^p)dg^{-1}x - \int_{\mathbb{R}^N} F(g^{-1}x, u(g^{-1}x))dg^{-1}x \\
&= \frac{1}{p}\int_{\mathbb{R}^N} (|\nabla u(x)|^p + (\lambda a(x)+1)|u(x)|^p)dx - \int_{\mathbb{R}^N} F(x,u(x))dx = \Phi_\lambda(u).
\end{aligned}$$

Set

$$V = \{u \in E_\lambda : u(gx) = det(g)u(x), \forall g \in G_m\}$$

and define

$$N_\lambda^{G_m} := \{u \in N_\lambda : gu = u, \forall g \in G_m\} = N_\lambda \cap V.$$

Then for all $u \in N_\lambda^{G_m}$, we have

$$gu(x) = det(g)u(g^{-1}x) = det(g)det(g^{-1})u(x) = u(x), \quad \forall g \in G_m.$$

By the definition of Nehari manifold N_λ, critical points of Φ_λ constrained on N_λ(see [14]) are critical points of Φ_λ. Moreover, by Palais principle of symmetric criticality([10]), we only need to find critical points of Φ_λ restricted on $N_\lambda^{G_m}$.

Therefore, consider the following minimizing problem

$$C_\lambda^{G_m} = \inf_{u \in N_\lambda^{G_m}} \Phi_\lambda(u).$$

By (f_3) and the definition of N_λ, Φ_λ bounded from below on $N_\lambda^{G_m}$, so $-\infty < C_\lambda^{G_m} < \infty$. Choose $c = C_\lambda^{G_m}$, let $\Lambda_m := \Lambda_c$ be the corresponding constant given in Proposition 2.2. Assume $\lambda \geq \Lambda_m$ and $(u_n) \subset N_\lambda^{G_m}$ is a minimizing sequence of Φ_λ. According to the Ekeland variational principle (Theorem 8.5 in [14]), we can assume (u_n) is a $(PS)_c$-sequence. By Proposition 2.2, the infimum is achieved by some $u \in N_\lambda^{G_m}$, that is to say, $\Phi_\lambda(u) = C_\lambda^{G_m}$.

From the definition of V and the fact that $det(\beta) = -1$,

$$u(\beta x) = det(\beta)u(x) = -u(x).$$

So u will change sign when (y_1, y_2) cross perpendicularly the half lines $y_2 = \pm y_1 \tan \frac{\pi j}{m}(y_1 \geq 0)$, $j = 1, 2, ..., m$. Hence u is a nodal solution with at least $2m$ nodal domains.

This completes the proof of Theorem 1.1

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