Communications in **M**athematical **A**nalysis
Volume 12, Number 2, pp. 85–101 (2012) www.math-res-pub.org/cma
ISSN 1938-9787

Functional Differential Equations with State-Dependent Delay on Unbounded Domains in a Banach Space

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(Communicated by Michal Fečkan)

Abstract

In this paper we study the existence of solutions for differential equations with state dependent delay on an unbounded domain. Our results are based on the properties of the Kuratowski measure of noncompactness and Darbo's fixed point theorem.

AMS Subject Classification: 34B15, 34G20, 34K10.

Keywords: Differential equation, state-dependent delay, fixed point, infinite delay, measure of noncompactness, unbounded domain.

1 Introduction

This paper deals with the existence of solutions to the boundary value problem (BVP for short) for the differential equation with state-dependent delay on an unbounded domain of the form:

$$y''(t) = f(t, y_{\rho(t,y_t)}) \text{ a.e. } t \in J := [0,\infty), \tag{1.1}$$

$$y(t) = \phi(t), \quad t \in (-\infty, 0], \; y'(\infty) = y_\infty, \tag{1.2}$$

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where $f: J\times\mathcal{B}\to E$, $\rho: J\times\mathcal{B}\to\mathbb{R}$, $\phi\in\mathcal{B}$ are given functions, $\mathcal{B}$ is an abstract *phase space*, to be specified later, and $(E,|\cdot|)$ is a Banach space. For any function y and any $t\in[0,\infty)$, we denote by y_t the element of $\mathcal{B}$ defined by $y_t(\theta)=y(t+\theta)$ for $\theta\in(-\infty,0]$. For any $t\in J$, y_t is the history of the state y up to t.

Boundary value problems on infinite intervals frequently occur in mathematical modelling of various applied problems. For example, in the study of unsteady flow of a gas through a semi-infinite porous medium [3, 31], analysis of the mass transfer on a rotating disk in a non-Newtonian fluid [4], heat transfer in the radial flow between parallel circular disks [37], investigation of the temperature distribution in the problem of phase change of solids with temperature dependent thermal conductivity [37], as well as numerous problems arising in the study of circular membranes [5, 16, 17], plasma physics [4], nonlinear mechanics, and non-Newtonian fluid flows [6].

Differential equations with infinite delay appear frequently in applications; for instance, in physics, aeronautic, economics, engineering, populations dynamics. There exists an extensive literature for ordinary and partial differential equations with state-dependent delay, see for instance Aiello *et al* [7], Arino *et al* [8], Cao *et al* [13], Domoshnitsky *et al* [18], Hartung *et al* [23, 24, 25, 26, 27], Hernandez *et al* [28], Schumakher [39], Qesmi and Walther [38], and Walther [40]. Delay Differential equations have been used in modeling scientific phenomena for many years. Often, it has been assumed that the delay is either a fixed constant or is given as an integral in which case it is called a distributed delay, see for instance the books [22, 29, 32, 34, 41], and the papers [14, 21]. For existence results for differential equations on infinite intervals we refer the reader to [1, 2, 35, 36, 44]. In the literature devoted to equations with finite delay, the phase space is often chosen to be the space of all continuous functions on $[-r,0]$, endowed with the uniform norm topology. When the delay is infinite, the notion of phase space $\mathcal{B}$ plays an important role in the study of both qualitative and quantitative theory. A usual choice for the phase space is a normed space satisfying suitable axioms which was introduced by Hale and Kato [21], see also [30]. Other results for problems involving infinite delay can be found in the papers [9, 12, 11]. For a detailed discussion on this topic we refer the reader to the book by Hino *et al.* [29]. Some papers [36, 42, 43] deal with the existence of solutions to boundary value problems of differential equations on unbounded domains.

In this paper we will establish an existence result for a class of boundary value problems for differential equation with state-dependent delay on an unbounded domain. The technique relies on the Kuratowski measure of noncompactness and Darbo's fixed point theorem [10, 15, 33]. To our best knowledge, no papers exist in the literature that are devoted to boundary value problems for functional differential equations with state-dependent delay on a unbounded interval in the infinite dimensional Banach space. This paper initiates the study of such problems.

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper.

Consider the space $C_\infty(J,E)$ defined by

$$C_\infty(J,E) = \{y \in C(J,E) : \lim_{t\to\infty} \frac{y(t)}{1+t} \text{ exists}\},$$

equipped with the norm

$$\|y\|_\infty = \sup_{t\in J} \frac{|y(t)|}{1+t}.$$

It is easy to verify that C_∞ is a Banach space [36]. We use α, α_C and α_∞ to denote the Kuratowski measure of noncompactness of bounded sets in the spaces E, $C(I,E)$ and C_∞ respectively, I is a compact interval of J. As for the definition of the Kuratowski measure of noncompactness, we refer to the reference [20].

In this paper, we will employ an axiomatic definition of the phase space $\mathcal{B}$ introduced by Hale and Kato in [21] and follow the terminology used in [29]. Thus, $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ will be a seminormed linear space of functions mapping $(-\infty,0]$ into E. For $\psi \in \mathcal{B}$ the norm of ψ is defined by

$$\|\psi\|_{\mathcal{B}} = \sup\{|\psi(\theta)| : \theta \in (-\infty,0]\}.$$

For the definition of the phase space $\mathcal{B}$ we introduce the following axioms.

(A_1) If $y : (-\infty,+\infty) \to E$, $y_0 \in \mathcal{B}$, then for every $t \in J$ the following conditions hold:
(i) $y_t \in \mathcal{B}$;
(ii) There exists a positive constant H such that $|y(t)| \leq H\|y_t\|_{\mathcal{B}}$;
(iii) There exist two functions $K(\cdot), M(\cdot) : J \to J$, independent of y, with K continuous, M locally bounded and $\sup_{t\in J} K(t) < +\infty$, $\sup_{t\in J} M(t) < +\infty$ such that:

$$\|y_t\|_{\mathcal{B}} \leq K(t)\sup\{\,|y(s)| : 0 \leq s \leq t\} + M(t)\|y_0\|_{\mathcal{B}}.$$

(A_2) The space $\mathcal{B}$ is complete.

Denote

$$K_\infty = \sup\{K(s) : s \in J\}$$

and

$$M_\infty = \sup\{M(s) : s \in J\}.$$

Example 2.1. Define for a positive constant γ the following standard space

$$C_\gamma := \{\phi : (-\infty,0] \to E \text{ continuous such that } \lim_{\theta\to-\infty} e^{\gamma\theta}\phi(\theta) \text{ exists in } E\}.$$

It is known from [29] that C_γ with the norm $\|\phi\|_\gamma = \sup_{\theta\leq 0} e^{\gamma\theta}|\phi(\theta)|$, $\phi \in C_\gamma$, satisfies the axioms $(A1)$, $(A2)$ with $H = 1$, $K(t) = e^{-\gamma t}$ and $M(t) = e^{-\gamma t}$ for all $t \geq 0$. It follows that

$$K_\infty = 1, \quad M_\infty = 1.$$

Set

$$\mathcal{B}_\infty = \{y : (-\infty, +\infty) \to E : \; y|_{(-\infty,0]} \in \mathcal{B} \text{ and } y|_J \in C_\infty(J,E)\},$$

and let $\|.\|_\infty$ the seminorm in $\mathcal{B}_\infty$ be defined by

$$\|y\|_\infty = \|y_0\|_{\mathcal{B}} + \sup_{t\in[0,+\infty)} \frac{|y(t)|}{1+t}.$$

Let $L^1(J,E)$ denote the Banach space of measurable functions $y : J \longrightarrow E$ which are Bochner integrable and normed by

$$\|y\|_{L^1} = \int_0^\infty |y(t)|dt.$$

Definition 2.2. The map $f : J \times \mathcal{B} \to E$ is said to be Carathéodory if:

(i) The function $t \longmapsto f(t,u)$ is measurable for each $u \in \mathcal{B}$;

(ii) The function $u \longmapsto f(t,u)$ is continuous for almost all $t \in J$.

$AC^1(J,E)$ denotes the space of differentiable functions $y : J \to E$ whose first derivative y' is absolutely continuous.

The following properties of the Kuratowski measure of noncompactness and the Darbo's fixed point theorem are needed for our discussion.

Lemma 2.3. *[20] If $H \subset C(I,E)$ is bounded and equicontinuous, then the function $t \mapsto \alpha(H(t))$ is continuous on I and*

$$\alpha_C(H) = \max_{t\in I} \alpha(H(t)),$$

where $H(t) = \{x(t), x \in H\}$, $t \in I$, I is a compact interval of J.

Definition 2.4. Let D be a bounded, closed and convex subset of the Banach space C_∞. A bounded and continuous operator $T : D \to D$ is called a strict set contraction if there is a constant $0 \le k < 1$ such that $\alpha(T(S)) \le k\alpha(S)$ for any bounded set $S \subset D$.

Lemma 2.5. *[15] Let D be a bounded, closed and convex subset of the Banach space C_∞. If the operator $T : D \to D$ is a strict set contraction, then T has a fixed point in D.*

3 Existence of Solutions

Let $h : J \to E$ be an integrable function and consider the differential equation

$$y''(t) = h(t), \text{ a.e, } t \in J. \tag{3.1}$$

We shall refer to (3.1), (1.2) as (LP). Let

$$\Omega = \{y : (-\infty, +\infty) \to E : y|_{(-\infty,0]} \in \mathcal{B} \text{ and } y|_J \in AC^1(J,E)\}.$$

By a solution to (LP) we mean a function $y \in \Omega$ which satisfies (3.1),(1.2). We need the following auxiliary result:

Lemma 3.1. *[1] Let h be an integrable function. A function $y \in \Omega$ is a solution of (LP) if and only if y is a solution of the integral equation*

$$y(t) = \begin{cases} \phi(t) & \text{if } t \in (-\infty, 0], \\ \\ \phi(0) + t y_\infty + \int_0^{+\infty} G(t,s)h(s)ds & \text{if } t \in J, \end{cases} \tag{3.2}$$

where $G(t,s)$ is the Green function defined by

$$G(t,s) = \begin{cases} -s, & 0 \le s \le t < \infty \\ \\ -t, & 0 \le t \le s < \infty. \end{cases} \tag{3.3}$$

We will need to introduce the following hypotheses

(H_ϕ) The function $t \to \phi_t$ is continuous from $\mathcal{R}(\rho^-) = \{\rho(s,\varphi) : (s,\varphi) \in J \times \mathcal{B}, \rho(s,\varphi) \le 0\}$ into $\mathcal{B}$ and there exists a continuous and bounded function $L^\phi : \mathcal{R}(\rho^-) \to (0,\infty)$ such that $\|\phi_t\|_{\mathcal{B}} \le L^\phi(t)\|\phi\|_{\mathcal{B}}$ for every $t \in \mathcal{R}(\rho^-)$.

$(H1)$ There exist $p \in L^1(J, \mathbb{R}_+)$ and $\psi : [0,\infty) \to (0,\infty)$ continuous and nondecreasing such that

$$|f(t,u)| \le p(t)\psi(\|u\|_{\mathcal{B}}) \text{ for each } t \in J \text{ and all } u \in \mathcal{B},$$

with

$$\int_0^\infty p(s)ds < \infty.$$

$(H2)$ For any $r > 0$ and compact interval I with $I \subset J$, $f(t,y)$ is uniformly continuous on $I \times B(0,r)$, where 0 is the zero element of $\mathcal{B}$.

$(H3)$ The function $f : J \times \mathcal{B} \to E$ is Carathéodory.

$(H4)$ There exists a nonnegative constant R, such that

$$\psi(K_\infty R + K_\infty|\phi(0)| + M_\infty\|\phi\|_{\mathcal{B}}) \int_0^{+\infty} p(s)ds \le R.$$

$(H5)$ There exists a nonnegative function $l \in L^1(J, \mathbb{R}_+)$ such that

$$\alpha(f(t,B)) \le l(t)\alpha(B), \quad t \in J,$$

where B is any bounded subset of $\mathcal{B}$ and $\int_0^{+\infty}(1+t)l(t)dt < 1$.

The next result is a consequence of the phase space axioms.

Lemma 3.2. *([[28], Lemma 2.1]) If $y : (-\infty,\infty) \to E$ is a function such that $y_0 = \phi$ and $y|_J \in C(J,E)$, then*

$$\|y_s\|_{\mathcal{B}} \le (M_\infty + L^\phi)\|\phi\|_{\mathcal{B}} + K_\infty \sup\{\|y(\theta)\|; \theta \in [0, \max\{0,s\}]\}, \quad s \in \mathcal{R}(\rho^-) \cup J,$$

where

$$L^\phi = \sup_{t \in \mathcal{R}(\rho^-)} L^\phi(t).$$

Remark 3.3. We remark that condition (H_ϕ) is satisfied by functions which are continuous and bounded. In fact, if the space $\mathcal{B}$ satisfies axiom C_2 in [29] then there exists a constant $L > 0$ such that $\|\phi\|_{\mathcal{B}} \leq L\sup\{\|\phi(\theta)\| : \theta \in (-\infty, 0]\}$ for every $\phi \in \mathcal{B}$ that is continuous and bounded (see [29] Proposition 7.1.1) for details. Consequently,

$$\|\phi_t\|_{\mathcal{B}} \leq L\frac{\sup_{\theta\leq 0}\|\phi(\theta)\|}{\|\phi\|_{\mathcal{B}}}\|\phi\|_{\mathcal{B}}, \text{ for every } \phi \in \mathcal{B}\backslash\{0\}.$$

Theorem 3.4. *Assume that hypotheses* $(H1)-(H5)$ *and* (H_ϕ) *hold. Then the problem (1.1)-(1.2) has at least one solution on* $(-\infty, +\infty)$*.*

Proof. Define the operator $T : \mathcal{B}_\infty \to \mathcal{B}_\infty$ by:

$$T(y)(t) = \begin{cases} \phi(t), & \text{if } \ t \in (-\infty, 0], \\ \phi(0) + ty_\infty + \int_0^{+\infty} G(t,s)f(s, y_{\rho(s,y_s)})ds & \text{if } \ t \in [0, \infty) \end{cases} \tag{3.4}$$

where the Green's function $G(t,s)$ is given by (3.3). Clearly, from Lemma 3.1, the fixed points of T are solutions to (1.1)–(1.2).
Let $x(\cdot) : (-\infty, +\infty) \to E$ be the function defined by

$$x(t) = \begin{cases} \phi(t), & \text{if } \ t \in (-\infty, 0], \\ \phi(0) + ty_\infty, & \text{if } \ t \in [0, \infty). \end{cases}$$

Then $x_0 = \phi$. For each $z \in \mathcal{B}_\infty$ with $z_0 = 0$, we denote by $\bar{z}$ the function defined by

$$\bar{z}(t) = \begin{cases} 0, & \text{if } \ t \in (-\infty, 0], \\ z(t), & \text{if } \ t \in [0, \infty). \end{cases}$$

If $y(\cdot)$ satisfies the integral equation

$$y(t) = \phi(0) + ty_\infty + \int_0^\infty G(t,s)f(s, y_{\rho(s,y_s)})ds,$$

we can decompose $y(.)$ into $y(t) = \bar{z}(t) + x(t)$, $t \geq 0$, which implies $y_t = \bar{z}_t + x_t$, for every $t \in J$, and the function $z(\cdot)$ satisfies

$$z(t) = \int_0^\infty G(t,s)f(s, \bar{z}_{\rho(s,\bar{z}_s+x_s)} + x_{\rho(s,\bar{z}_s+x_s)})ds.$$

Let

$$C = \{z \in \mathcal{B}_\infty : \ z_0 = 0\}.$$

Let $\|\cdot\|_0$ be the norm in C defined by

$$\|z\|_0 = \|z_0\|_{\mathcal{B}} + \sup_{t\in J}\frac{|z(t)|}{1+t} = \sup_{t\in J}\frac{|z(t)|}{1+t} = \|z\|_\infty.$$

We define the operator $P : C \to C$ by

$$P(z)(t) = \int_0^\infty G(t,s)f(s, \bar{z}_{\rho(s,\bar{z}_s+x_s)} + x_{\rho(s,\bar{z}_s+x_s)})ds.$$

Obviously the operator T has a fixed point if and only if P has one, so we need to prove that P has a fixed point. We shall show that P satisfies the properties of the Kuratowski measure of noncompactness and Darbo's fixed point theorem assumptions. The proof will be given in four Steps.

Step 1: *P is continuous*

Let $\{y_n\}_{n=1}^{\infty} \subset C$ and $y \in C$ such that $y_n \to y$ as $n \to \infty$. Then $\{y_n\}_{n=1}^{\infty}$ is a bounded set of C, i.e. there exists $m > 0$ such that $\|y_n\|_\infty \le m$ for $n \ge 1$. We also have by taking the limit that $\|y\|_\infty \le m$. Now by $(H1)$, for any $\varepsilon > 0$, there exists $l > 0$ such that

$$\int_l^{\infty} p(s)ds < \frac{\varepsilon}{3\psi(m)}. \tag{3.5}$$

Condition $(H2)$ yields that there exists $N > 0$ such that for $n > N$ and $t \in [0,l]$,

$$\left| f(t, y^n_{\rho(t,y^n_t)}) - f(t, y_{\rho(t,y_t)}) \right| < \frac{\varepsilon}{3l}. \tag{3.6}$$

Therefore, for $t \in [0,l]$ and $n > N$, we can obtain from (3.4), (3.5), (3.6)

$$\begin{aligned}
&\frac{|P(z_n)(t) - P(z)(t)|}{1+t} \le \\
&\int_0^t \frac{s}{1+t} |f(s, \overline{z_n}_{\rho(s,\overline{z_n}_s + x_s)} + x_{\rho(s,\overline{z_n}_s + x_s)}) - f(s, \bar{z}_{\rho(s,\bar{z}_s + x_s)} + x_{\rho(s,\bar{z}_s + x_s)})| ds \\
+\ & \int_t^{+\infty} \frac{t}{1+t} |f(s, \overline{z_n}_{\rho(s,\overline{z_n}_s + x_s)} + x_{\rho(s,\overline{z_n}_s + x_s)}) - f(s, \bar{z}_{\rho(s,\bar{z}_s + x_s)} + x_{\rho(s,\bar{z}_s + x_s)})| ds \\
\le\ & \int_0^t |f(s, \overline{z_n}_{\rho(s,\overline{z_n}_s + x_s)} + x_{\rho(s,\overline{z_n}_s + x_s)}) - f(s, \bar{z}_{\rho(s,\bar{z}_s + x_s)} + x_{\rho(s,\bar{z}_s + x_s)})| ds \\
+\ & \int_t^{\infty} |f(s, \overline{z_n}_{\rho(s,\overline{z_n}_s + x_s)} + x_{\rho(s,\overline{z_n}_s + x_s)}) - f(s, \bar{z}_{\rho(s,\bar{z}_s + x_s)} + x_{\rho(s,\bar{z}_s + x_s)})| ds \\
\le\ & \int_0^l |f(s, \overline{z_n}_{\rho(s,\overline{z_n}_s + x_s)} + x_{\rho(s,\overline{z_n}_s + x_s)}) - f(s, \bar{z}_{\rho(s,\bar{z}_s + x_s)} + x_{\rho(s,\bar{z}_s + x_s)})| ds \\
+\ & 2\psi(m) \int_l^{\infty} p(s)ds \\
\le\ & \frac{\varepsilon}{3} + \frac{2\varepsilon}{3} = \varepsilon.
\end{aligned}$$

Thus we conclude that $\|Py_n - Py\|_\infty \le \varepsilon$ as $n > N$, namely, P is continuous.

Step 2: *P maps bounded sets into bounded sets in C.*

Indeed, it is enough to show that for any $\eta > 0$, there exists a positive constant ℓ such that for each $z \in B_\eta = \{z \in C : \|z\|_\infty \leq \eta\}$, we have $\|Pz\|_\infty \leq l$. By $(H1)$ we have for each $t \in J$,

$$\begin{aligned}
\frac{|P(z)(t)|}{1+t} &\leq \int_0^t \frac{s}{1+t}|f(s,\overline{z}_{\rho(s,\overline{z}_s+x_s)}+x_{\rho(s,\overline{z}_s+x_s)})|ds \\
&+ \int_t^{+\infty} \frac{t}{1+t}|f(s,\overline{z}_{\rho(s,\overline{z}_s+x_s)}+x_{\rho(s,\overline{z}_s+x_s)})|ds \\
&\leq \int_0^{+\infty} p(s)\psi(\|\overline{z}_{\rho(s,\overline{z}_s+x_s)}+x_{\rho(s,\overline{z}_s+x_s)}\|)ds \\
&\leq \psi(K_\infty\eta + K_\infty|\phi(0)| + M\|\phi\|_{\mathcal{B}}) \int_0^{+\infty} p(s)ds := \ell.
\end{aligned}$$

Step 3:

(a) *The* $\{\frac{PB(t)}{1+t}\}$ *is equicontinuous on any compact interval I of J, and for any bounded set B of C.*

We note that $P(z)(t)$ can be written as

$$P(z)(t) = \int_0^t (t-s)f(s,\overline{z}_{\rho(s,\overline{z}_s+x_s)}+x_{\rho(s,\overline{z}_s+x_s)})ds - \int_0^\infty tf(s,\overline{z}_{\rho(s,\overline{z}_s+x_s)}+x_{\rho(s,\overline{z}_s+x_s)})ds.$$

In view of the condition $(H1)$ and the boundedness of B there exists $m > 0$ such that

$$\int_0^\infty \left|f(s,y_{\rho(s,y_s)})\right|ds \leq m \text{ for any } y \in B. \tag{3.7}$$

In order to prove (**a**) let $t_1,\ t_2 \in [a,b]$ where $[a,b] \subset J,\ t_1 < t_2$, let the constant η be such that $\|z\|_\infty \leq \eta$ for any $z \in B$. Then,

$$\begin{aligned}
\left|\tfrac{P(z)(t_2)}{1+t_2} - \tfrac{P(z)(t_1)}{1+t_1}\right| &\leq \left|\tfrac{t_2}{1+t_2} - \tfrac{t_1}{1+t_1}\right| \int_0^\infty |f(s,\overline{z}_{\rho(s,\overline{z}_s+x_s)}+x_{\rho(s,\overline{z}_s+x_s)})|ds \\
&\quad + \int_0^{t_2} \tfrac{t_2-s}{1+t_2}|f(s,\overline{z}_{\rho(s,\overline{z}_s+x_s)}+x_{\rho(s,\overline{z}_s+x_s)})|ds \\
&- \int_0^{t_1} \tfrac{t_2-s}{1+t_2}|f(s,\overline{z}_{\rho(s,\overline{z}_s+x_s)}+x_{\rho(s,\overline{z}_s+x_s)})|ds \\
&+ \int_0^{t_1} \tfrac{t_2-s}{1+t_2}|f(s,\overline{z}_{\rho(s,\overline{z}_s+x_s)}+x_{\rho(s,\overline{z}_s+x_s)})|ds \\
&- \int_0^{t_1} \tfrac{t_1-s}{1+t_1}|f(s,\overline{z}_{\rho(s,\overline{z}_s+x_s)}+x_{\rho(s,\overline{z}_s+x_s)})|ds \\
&\leq m\left|\tfrac{t_2}{1+t_2} - \tfrac{t_1}{1+t_1}\right| + \int_{t_1}^{t_2} |f(s,\overline{z}_{\rho(s,\overline{z}_s+x_s)}+x_{\rho(s,\overline{z}_s+x_s)})|ds \\
&+ \int_0^{t_1} \left|\tfrac{t_2-s}{1+t_2} - \tfrac{t_1-s}{1+t_1}\right| |f(s,\overline{z}_{\rho(s,\overline{z}_s+x_s)}+x_{\rho(s,\overline{z}_s+x_s)})|ds \\
&\leq m\left|\tfrac{t_2}{1+t_2} - \tfrac{t_1}{1+t_1}\right| + \psi(K_\infty\eta + K_\infty|\phi(0)| + M\|\phi\|_{\mathcal{B}}) \int_{t_1}^{t_2} p(s)ds \\
&+ \psi(K_\infty\eta + K_\infty|\phi(0)| + M_\infty\|\phi\|_{\mathcal{B}}) \int_0^{t_1} \left|\tfrac{t_2-s}{1+t_2} - \tfrac{t_1-s}{1+t_1}\right| p(s)ds.
\end{aligned}$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero.

(b) *Equiconvergence at infinity.*

We shall show that for a given $\varepsilon > 0$, there exists a constant $N > 0$ such that $\left|\frac{Pz(t_1)}{1+t_1} - \frac{Pz(t_2)}{1+t_2}\right| < \varepsilon$

for any $t_1,\ t_2 \geq N$ and $z \in B$.

$$\begin{aligned}\left|\frac{P(z)(t_2)}{1+t_2} - \frac{P(z)(t_1)}{1+t_1}\right| &\leq \left|\frac{t_2}{1+t_2} - \frac{t_1}{1+t_1}\right|\left|\int_0^\infty f(s,\overline{z}_{\rho(s,\overline{z}_s+x_s)} + x_{\rho(s,\overline{z}_s+x_s)})ds\right| \\ &+ \left|\int_0^{t_2} \frac{t_2-s}{1+t_2} f(s,\overline{z}_{\rho(s,\overline{z}_s+x_s)} + x_{\rho(s,\overline{z}_s+x_s)})ds\right. \\ &- \left.\int_0^{t_1} \frac{t_1-s}{1+t_1} f(s,\overline{z}_{\rho(s,\overline{z}_s+x_s)} + x_{\rho(s,\overline{z}_s+x_s)})ds\right|.\end{aligned}$$

It is sufficient to prove that

$$\left|\int_0^{t_2} \frac{s}{1+t_2} f(s,\overline{z}_{\rho(s,\overline{z}_s+x_s)} + x_{\rho(s,\overline{z}_s+x_s)})ds - \int_0^{t_1} \frac{s}{1+t_1} f(s,\overline{z}_{\rho(s,\overline{z}_s+x_s)} + x_{\rho(s,\overline{z}_s+x_s)})ds\right| \leq \varepsilon.$$

The relation (3.7) yields that there exists $N_1 > 0$ such that

$$\left|\int_{N_1}^\infty f(s,\overline{z}_{\rho(s,\overline{z}_s+x_s)} + x_{\rho(s,\overline{z}_s+x_s)})ds\right| \leq \frac{\varepsilon}{3} \text{ uniformly with respect to } z \in B. \tag{3.8}$$

On the other hand since $\lim_{t\to\infty} \frac{t-N_1}{1+t} = 1$ there exists $N \geq N_1$, such that for every $t_1,\ t_2 \geq N$, and $s \in [0, N_1]$ we have :

$$\begin{aligned}\left|\frac{t_1-s}{1+t_1} - \frac{t_2-s}{1+t_2}\right| &\leq \left(1-\frac{t_1-s}{1+t_1}\right)+\left(1-\frac{t_2-s}{1+t_2}\right) \\ &\leq \left(1-\frac{t_1-N_1}{1+t_1}\right)+\left(1-\frac{t_2-N_1}{1+t_2}\right) \\ &\leq \frac{\varepsilon}{3m}.\end{aligned}$$

Thus we have

$$\left|\frac{t_1-s}{1+t_1} - \frac{t_2-s}{1+t_2}\right| \leq \frac{\varepsilon}{3m}. \tag{3.9}$$

Now take $t_1,\ t_2 \geq N$, then from (3.7), (3.8), (3.9) we arrive at

$$\begin{aligned}
& |\int_0^{t_2} \frac{t_2 - s}{1+t_2} f(s, \overline{z}_{\rho(s,\overline{z}_s+x_s)} + x_{\rho(s,\overline{z}_s+x_s)})ds \\
& - \int_0^{t_1} \frac{t_1 - s}{1+t_1} f(s, \overline{z}_{\rho(s,\overline{z}_s+x_s)} + x_{\rho(s,\overline{z}_s+x_s)})ds| \\
\leq \ & \int_0^{N_1} \left| \frac{t_2 - s}{1+t_2} - \frac{t_1 - s}{1+t_1} \right| |f(s, \overline{z}_{\rho(s,\overline{z}_s+x_s)} + x_{\rho(s,\overline{z}_s+x_s)})|ds \\
& + \int_{N_1}^{t_1} \frac{t_1 - s}{1+t_1} |f(s, \overline{z}_{\rho(s,\overline{z}_s+x_s)} + x_{\rho(s,\overline{z}_s+x_s)})|ds \\
& + \int_{N_1}^{t_2} \frac{t_2 - s}{1+t_2} |f(s, \overline{z}_{\rho(s,\overline{z}_s+x_s)} + x_{\rho(s,\overline{z}_s+x_s)})|ds \\
& < \frac{\varepsilon}{3m} \int_0^{\infty} |f(s, \overline{z}_{\rho(s,\overline{z}_s+x_s)} + x_{\rho(s,\overline{z}_s+x_s)})|ds \\
& +2 \int_{N_1}^{\infty} |f(s, \overline{z}_{\rho(s,\overline{z}_s+x_s)} + x_{\rho(s,\overline{z}_s+x_s)})|ds \\
& < \varepsilon.
\end{aligned}$$

Step 4: Let

$$B = \{z \in C : \|z\|_\infty \leq R\},$$

where R is the constant defined by $(H4)$. We will show now that P maps B into itself. Indeed, for any $z \in B$, by the condition $(H1)$ we get

$$\begin{aligned}
\frac{|Pz(t)|}{1+t} \quad & \leq \quad \int_0^t \frac{s}{1+t} |f(s, \overline{z_n}_{\rho(s,\overline{z_n}_s+x_s)} + x_{\rho(s,\overline{z_n}_s+x_s)})|ds \\
& + \quad \int_t^{\infty} \frac{t}{1+t} |f(s, \overline{z}_{\rho(s,\overline{z}_s+x_s)} + x_{\rho(s,\overline{z}_s+x_s)})|ds \\
& \leq \quad \psi(K_\infty R + K_\infty |\phi(0)| + M_\infty \|\phi\|_{\mathcal{B}}) \int_0^{+\infty} p(s)ds \\
& \leq \quad R.
\end{aligned}$$

Hence, $\|Pz\|_\infty \leq R$ and we conclude that $P : B \to B$. In what follows we show that the operator $P : B \to B$ is a strict set contraction. From Steps 1 and 2 we know that $P : B \to B$ is bounded and continuous. Finally we need to prove that there exists a constant $0 \leq k < 1$ such that

$$\alpha_\infty(PS) \leq k\alpha_\infty(S)$$

for $S \subset B$. Furthermore, from Step 3 it is enough to verify

$$\sup_{t\in J} \alpha\left(\frac{PS(t)}{1+t}\right) \leq k\alpha_\infty(S). \tag{3.10}$$

Define

$$P_n z(t) = -\int_0^t sf(s, \overline{z_n}_{\rho(s,\overline{z_n}_s+x_s)} + x_{\rho(s,\overline{z_n}_s+x_s)})ds - \int_t^n tf(s, \overline{z_n}_{\rho(s,\overline{z_n}_s+x_s)} + x_{\rho(s,\overline{z_n}_s+x_s)}).$$

Thus

$$\left|\frac{P_n z(t)}{1+t} - \frac{Pz(t)}{1+t}\right| \le \psi(K_\infty R + K_\infty|\phi(0)| + M\|\phi\|_{\mathcal{B}})\int_n^{+\infty} p(s)ds.$$

This shows $d\left(\frac{P_n S(t)}{1+t}, \frac{PS(t)}{1+t}\right) \to 0$ as $n \to \infty$, $t \in J$, where $d(.)$ denotes the Hausdorff α metric in the space E. Thus, by the property of α we get

$$\lim_{n\to\infty} \alpha\left(\frac{P_n S(t)}{1+t}\right) = \alpha\left(\frac{PS(t)}{1+t}\right) \quad t \in J.$$

Now we estimate $\alpha(\frac{P_n S(t)}{1+t})$, Step 2, implies that $\{\frac{D}{1+t}\}$ is equicontinuous on any compact intervals of J and hence, $\frac{S(t)}{1+t}$ is equicontinuous on any compact interval of J. By condition $(H2)$, it is easy to show that $\{f(s, \overline{z}_{\rho(s,\overline{z}_s+x_s)} + x_{\rho(s,\overline{z}_s+x_s)}) : \overline{z}_s + x_s \in S\}$ is equicontinuous on $[0, N]$. Moreover, $\{f(s, \overline{z}_{\rho(s,\overline{z}_s+x_s)} + x_{\rho(s,\overline{z}_s+x_s)}) : \overline{z}_s + x_s \in S\}$ is bounded on $[0, N]$ by $(H1)$. Using Lemma 2.3, Step 4 and condition $(H5)$, we arrive at

$$\begin{aligned}
\alpha\left(\frac{P_n S(t)}{1+t}\right) &\le \int_0^t \alpha(\{f(s, \overline{z}_{\rho(s,\overline{z}_s+x_s)} + x_{\rho(s,\overline{z}_s+x_s)})ds : \overline{z}_s + x_s \in S\}) \\
&\quad + \int_t^n \alpha(\{f(s, \overline{z}_{\rho(s,\overline{z}_s+x_s)} + x_{\rho(s,\overline{z}_s+x_s)})ds : \overline{z}_s + x_s \in S\}) \\
&\le \int_0^n l(s)\alpha(S(s))ds \\
&\le \int_0^n (1+s)l(s)\alpha\left(\frac{S(s)}{1+s}\right)ds \\
&\le \int_0^n (1+s)l(s)ds\alpha_\infty(S).
\end{aligned}$$

Thus, from (3.10) and Step 4, it follows immediately that

$$\alpha_\infty(PS) \le \int_0^\infty (1+s)l(s)ds.\alpha_\infty(S) = k\alpha_\infty(S)$$

here $k := \int_0^\infty (1+s)l(s)ds$ and obviously we have $0 \le k < 1$ by $(H5)$. Therefore, Darbo's fixed point theorem asserts that BVP (1.1)-(1.2) has at least one solution in D and the proof is finished. □

4 An Example

To apply our results, we consider the functional differential equation with state-dependent delay of the form:

$$\begin{cases} u_n''(t) = \frac{u_n(t-\sigma(u_n(t)))}{2(1+t^2)} + \frac{\ln(1+u_{n+1}(t-\sigma(u_{n+1}(t))))}{ne^{\sqrt{t}}} & \text{if } t \in (0, +\infty), \\ u_n(t) = \phi_n(t), \quad u_n'(\infty) = u_{n_\infty} & \text{if } t \in (-\infty, 0] \end{cases} \tag{4.1}$$

where $\sigma \in C(\mathbb{R}, [0, \infty))$, $\phi_i \in C((-\infty, 0], \mathbb{R})$, $i = 1, \ldots, n, \ldots$.

Let

$$E = \{u = (u_1, u_2, \ldots, u_n, \ldots) : \sup_n |u_n| < \infty\}$$

with the norm $u = \sup_n |u_n|$. Then E is a Banach space. Set $\gamma > 0$. For the phase space, we choose $\mathcal{B} = \mathcal{B}_\gamma$ to be defined by

$$\mathcal{B}_\gamma = \{u \in C((-\infty, 0], E) : \lim_{\theta \to -\infty} e^{\gamma\theta} u_i(\theta) \text{ exists}, \ \ i = 1, \ldots n, \ldots\}$$

with the norm

$$\|u\|_\gamma = \sup_{i=1,\ldots n,\ldots} \ \sup_{\theta \in (-\infty,0]} e^{\gamma\theta} |u_i(\theta)|.$$

Let $u : (-\infty, +\infty) \to E$ be such that $u_0 \in \mathcal{B}_\gamma$. Then

$$\begin{aligned} \lim_{\theta \to -\infty} e^{\gamma\theta} u_{i_t}(\theta) &= \lim_{\theta \to -\infty} e^{\gamma\theta} u_i(t+\theta) \\ &= \lim_{\theta \to -\infty} e^{\gamma(\theta - t)} u_i(\theta) \\ &= e^{-\gamma\, t} \lim_{\theta \to -\infty} e^{\gamma\theta} u_{i_0}(\theta). \end{aligned}$$

Hence $u_t \in \mathcal{B}_\gamma$. Finally we prove that

$$\|u_t\|_\gamma \le K(t) \sup\{\, |u_t(s)| : s \in J\} + M(t)\|u_0\|_\gamma,$$

where $K_\infty = M_\infty = H = 1$. We have $u_{i_t}(\theta) = u_i(t+\theta)$.
If $t + \theta \le 0$ we get

$$\|u_{i_t}(\theta)\| \le \sup\{\, |u_i(t+\theta)| : -\infty \le \theta \le 0\}.$$

For $t + \theta \ge 0$ we have

$$\|u_{i_t}(\theta)\| \le \sup\{\, |u_i(t+\theta)| : \theta \in J\}.$$

Thus for all $t + \theta \in (-\infty, +\infty)$, we get

$$\|u_{t_i}(\theta)\| \le \sup\{\, |u_i(\theta)| : -\infty \le \theta \le 0\} + \sup\{\, |u_i(\theta)| : \theta \in J\}.$$

Thus

$$\|u_t\|_\gamma \le \|u\|_0 + \sup\{\, |u(\theta)| : \theta \in J\}.$$

It is clear that $(\mathcal{B}_\gamma, \|u\|_\gamma)$ is a Banach space. We can conclude that $\mathcal{B}_\gamma$ is a phase space.
Take $f(t,u) = (f_1(t,u), \ldots\ldots, f_n(t,u), \ldots)$ with

$$f_n(t,u) = \frac{u_n(0)}{2(1+t^2)} + \frac{\ln(1 + u_{n+1}(0))}{n e^{\sqrt{t}}}.$$

Evidently $f \in C(J \times \mathcal{B}_\gamma, E)$ and

$$|f(t,u)| \le \left[\frac{1}{2(1+t^2)} + \frac{1}{e^{\sqrt{t}}}\right] |u|$$

$$\rho(t,u) = t - \sigma(u(0)), \ \ (t,u) \in [0,\infty) \times \mathcal{B}_\gamma,$$

(4.1) can be written as (1.1)-(1.2). Thus condition $(H1)$ is satisfied with

$$p(t) = \frac{1}{2(1+t^2)} + \frac{1}{e^{\sqrt{t}}}$$

$$\int_0^{+\infty} p(t)dt < \infty$$

$$\psi(\|u\|) = |u|.$$

It is easy to see that the condition $(H2)$ is also satisfied. Now let us verify condition $(H5)$. Denote $f = f^1 + f^2$ where

$$f^1 = \frac{u_n(0)}{2(1+t^2)},$$

$$f^2 = \frac{\ln(1+u_{n+1}(0))}{ne^{\sqrt{t}}}.$$

Then we can obtain that for any bounded set $B \subset \mathcal{B}_\gamma$, $\alpha(f^2(t,B)) = 0$. Indeed, let $u^m \subset \mathcal{B}_\gamma$ be bounded, so $\|u^m\| \leq M,\ \ m = 1,2,\ldots,$ where $u^m = (u_1^m,\ldots,u_n^m,\ldots)$. Then we have, for fixed $t \in J$,

$$f_n^{(2)}(t,u^{(m)}) \leq \frac{M}{n},\ \ n = 1,\ldots, \tag{4.2}$$

so $\{f_n^2(t,u^m)\}$ is bounded, and we can choose subsequence $\{u^{(m_k)}\} \subset \{u^{(m)}\}$ such that

$$f_n^{(2)}(t,u^{(m_k)}) \to v_n \text{ as } \ k \to \infty,\ \ n = 1,\ldots, \tag{4.3}$$

and from (4.2) we get

$$\|v_n\| \leq \frac{M}{n},\ \ n = 1,\ldots, \tag{4.4}$$

and thus $v = \{v_1,\ldots,v_n,\ldots\} \in \mathcal{B}_\gamma$. For given $\varepsilon > 0$, (4.2) and (4.3) imply that there exists $N > 0$ such that

$$|f_n^{(2)}(t,u^{(m_k)})| < \frac{\varepsilon}{2},\ \ |v_n| < \varepsilon,\ \ n > N,\ \ k = 1,\ldots, \tag{4.5}$$

On the other hand, from (4.3) we obtain that there exits $N > 0$ such that

$$|f_n^{(2)}(t,u^{(m_k)}) - v_n| < \varepsilon,\ \ k > N,\ \ n = 1,2,3,..., \tag{4.6}$$

It follows from (4.5) and (4.6) that $\|f^{(2)}(t,u^{(m_k)}) - v\| \leq \varepsilon$ for $k > N$. This means that $\|f^{(2)}(t,u^{(m_k)}) - v\| \to 0$ as $k \to \infty$ and we have that $f^{(2)}(t,B)$ is relatively compact for any bounded $B \subset \mathcal{B}_\gamma$, thus, $\alpha(f^{(2)}(t,B)) = 0$. Consequently, we arrive at

$$\alpha(f(t,B)) \leq (\alpha f^{(1)}(t,B)) = \frac{\alpha(B)}{2(1+t^2)}.$$

Since $\int_0^{+\infty} \frac{dt}{2(1+t^2)} < 1$, we conclude that condition $(H5)$ is satisfied with $l(t) = \frac{1}{2(1+t^2)}$. From the choice of the function ψ we can easily show that there exists a positive constant R satisfying

$$\psi(R + |\phi(0)| + \|\phi\|_{\mathcal{B}}) \int_0^{+\infty} p(s)ds \leq R.$$

Theorem 3.4 ensures that the problem (4.1) has a solution.

Acknowledgments

The authors thank the referees for their careful reading of the manuscript.

References

[1] R.P. Agarwal and D. O'Regan, *Infinite Interval Problems for Differential, Difference and Integral Equations*, Academic Publishers, Dordrecht, 2001.

[2] R.P. Agarwal and D. O'Regan, Boundary value problems of nonsingular type on the semi-infinite interval, *Tohoku. Math. J.* **51** (1999), 391-397.

[3] R.P. Agarwal and D. O'Regan, Infinite interval problems modelling the flow of a gas through a semi-infinite porous medium, *Stud. Appl. Math.* **108** (2002), 245-257.

[4] R.P. Agarwal and D. O'Regan, Infinite interval problems modeling phenomena which arise in the theory of plasma and electrical potential theory, *Stud. Appl. Math.* **111** (2003), 339-358.

[5] R.P. Agarwal and D. O'Regan, An infinite interval problem arising in circularly symmetric deformations of shallow membrane caps, *Internal. J. Nonlin. Mech.* **39** (2004), 779-784.

[6] R.P. Agarwal and D. O'Regan, Infinite interval problems arising in nonlinear mechanics and non-Newtonian fluid flows, *Internat. J. Nonlin. Mech.* **38** (2003), 1369-1376.

[7] W.G. Aiello, H.I. Freedman, J. Wu, Analysis of a model representing stage-structured population growth with state-dependent time delay. *SIAM J. Appl. Math.* **52** (1992), 855–869.

[8] O. Arino, K. Boushaba, A. Boussouar, A mathematical model of the dynamics of the phytoplankton-nutrient system. Spatial heterogeneity in ecological models (Alcal de Henares, 1998). *Nonlinear Anal. RWA.* **1** (2000), 69–87.

[9] M. Ayachi, Variational methods and almost periodic solutions of second order functional differential equations with infinite delay. *Commun. Math. Anal.* **9** (2010), no. 1, 15–31.

[10] J. Banaś and K. Goebel, *Measure of Noncompactness in Banach Spaces*, Marcel Dekker, 1980.

[11] P.H. Bezandry, T. Diagana, Existence of square-mean almost periodic solutions to some stochastic hyperbolic differential equations with infinite delay. *Commun. Math. Anal.* **8** (2010), no. 2, 103–124.

[12] A. Boufala, H. Bouzahir, A. Maaden, Local existence for partial neutral functional integrodifferential equations with infinite delay. *Commun. Math. Anal.* **9** (2010), no. 2, 149–168.

[13] Y. Cao, J. Fan, T.C. Gard, The effects of state-dependent time delay on a stage-structured population growth model. *Nonlinear Anal. TMA.* **19** (1992), 95–105.

[14] C. Corduneanu and V. Lakshmikantham, Equations with unbounded delay, *Nonlinear Anal.* **4** (1980), 831-877.

[15] G. Darbo, Punti uniti in transformazioni a condominio non compacto *Rend. Sem. Mat. Univ. Padova*, **24** (1955), 8492.

[16] R.W. Dickey, Membrane caps under hydrostatic pressure, *Quart. Appl. Math.* **46** (1988), 95-104.

[17] R.W. Dickey, Rotationally symmetric solutions for shallow membrane caps, *Quart. AppL Math.* **47** (1989), 571-581.

[18] A. Domoshnitsky, M. Drakhlin, E. Litsyn, On equations with delay depending on solution. *Nonlinear Anal. TMA.* **49** (2002), 689-701.

[19] D. Guo, A class of second-order impulsive integro-differential equations on unbounded domain in a Banach space. *Appl. Math. Comput.* **125** (2002), 59-77.

[20] D. Guo, V. Lakshmikantham, X. Liu, *Nonlinear Integral Equations in Abstract Spaces*. **373**. Kluwer Academic Publishers Group, Dordrecht, 1996.

[21] J. Hale and J. Kato, Phase space for retarted equations with infinite delay, *Funkcial. Ekvac.* **21** (1978), 11-41.

[22] J. K. Hale and S. M. Verduyn Lunel, *Introduction to Functional Differential Equations*, Applied Mathematical Sciences 99, Springer-Verlag, New York, 1993.

[23] F. Hartung, Linearized stability in periodic functional differential equations with state-dependent delays. *J. Comput. Appl. Math.* **174** (2005), 201-211.

[24] F. Hartung, Parameter estimation by quasilinearization in functional differential equations with state-dependent delays: a numerical study. Proceedings of the Third World Congress of Nonlinear Analysis, Part 7 (Catania, 2000) *Nonlinear Anal. TMA.* **47** (2001), 4557–4566.

[25] F. Hartung, T.L. Herdman, J. Turi. Parameter identification in classes of neutral differential equations with state-dependent delays. *Nonlinear Anal. TMA* **39** (2000), 305–325.

[26] F. Hartung, T. Krisztin, H.-O. Walther, J. Wu, Functional differential equations with state-dependent delays: theory and applications. Handbook of differential equations: ordinary differential equations. Vol. III, 435–545, Handb. Differ. Equ., Elsevier/North-Holland, Amsterdam, 2006.

[27] F. Hartung and J. Turi. Identification of parameters in delay equations with state-dependent delays. *Nonlinear Anal. TMA* **29** (1997), no. 11, 1303–1318.

[28] E. Hernández, A. Prokopczyk and L. Ladeira, A note on partial functional differential equations with state-dependent delay. *Nonlinear Anal. Real World Appl.* **7** (2006), 510-519.

[29] Y. Hino, S. Murakami, T. Naito, *Functional Differential Equations with Unbounded Delay*, Springer-Verlag, Berlin, 1991.

[30] F. Kappel and W. Shappacher, Some considerations to the fondamental theory of infinite delay equations, *J. Differential Equations* **37** (1980), 141-183.

[31] R.E. Kidder, Unsteady flow of gas through a semi-infinite porous medium, *J. Appl. Mech.* **27** (1957), 329-332.

[32] V. Kolmanovskii and A. Myshkis, *Introduction to the Theory and Applications of Functional-Differential Equations*, Kluwer Academic Publishers, Dordrecht, 1999.

[33] C. Kuratowski, Sur les espaces complets *Fundam. Math.* **15** (1930), 301-309.

[34] V. Lakshmikantham, L. Wen, B. Zhang, *Theory of Differential Equations with Unbounded Delay*, Mathematics and its Applications, Kluwer Academic Publishers, Dordrecht, 1994.

[35] Y.S. Liu, Boundary value problem for second order differential equations on unbounded domain. *Acta Anal. Funct. Appl.* **4** (2002), 211-216.

[36] Y. Liu, Boundary value problems for second order differential equations on unbounded domains in a Banach space *Appl. Math. Comput.* **135** (2003), 569–583.

[37] T.Y. Na, *Computational Methods in Engineering Boundary Value Problems*, Academic Press, New York, 1979.

[38] R. Qesmi and H.-O. Walther, Center-stable manifolds for differential equations with state-dependent delays. *Discrete Contin. Dyn. Syst.* **23** (2009), 1009–1033.

[39] K. Schumacher, Existence and continuous dependence for differential equation with unbounded delay *Arch Rational Mech. Anal* (1978), 315-355.

[40] H.-O. Walther, Linearized stability for semiflows generated by a class of neutral equations, with applications to state-dependent delays. *J. Dynam. Differential Equations* **22** (2010), 439–462.

[41] J. Wu, *Theory and Applications of Partial Functional Differential Equations*, Springer-Verlag, New York, 1996.

[42] B. Yan and Y. Liu, Unbounded solutions of the singular boundary value problems for second order differential equations on the half-line. *Appl. Math. Comput.* **147** (2004), 629-644.

[43] H. Zhang, L. Liu, Y. Wu, A unique positive solution for nth-order nonlinear impulsive singular integro-differential equations on unbounded domains in Banach spaces. *Appl. Math. Comput.*, **203** (2008), 649-659.

[44] X. Zhao and W. Ge, Existence of at least three positive solutions for multi-point boundary value problem on infinite intervals with p-Laplacian operator. *J. Appl. Math. Comput.* **28** (2008), 391-403.