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Dolbeault Cohomology Along the Vertical Liouville Distribution on Complex Finsler Bundles

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Abstract

In this paper we define a vertical Liouville distribution in the vertical foliated distribution on a complex Finsler bundle and we prove that it is an integrable one. Next, some new operators on foliated forms along the vertical Liouville distribution are defined, a Dolbeault type lemma is proved and new cohomology groups are studied.

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1 Introduction

The idea of decomposing the exterior derivative for real smooth or complex analytic foliated manifolds and the study of their cohomology is due to I. Vaisman, see [14, 15]. There are proved some Poincaré type Lemmas with respect to some differential operators corresponding to the foliated type $(0,1)$ or to the mixed type $(0,1)$ for the analytic case, respectively. Different from [14, 15], recently, A. El Kacimi Alaoui in [6, 7], study a Dolbeault cohomology along the leaves of complex foliations and states a foliated Grothendieck-Dolbeault Lemma, see [6] p. 889.

Recently, in [11] is studied a new cohomology with respect to a Liouville foliation on the tangent bundle of a real Finsler manifold and a de Rham type theorem is obtained. The main goal of the present paper is to obtain a complex analogue of these results as a Dolbeault cohomology along the vertical Liouville distribution on complex Finsler bundles. Firstly, we consider $\mathcal{V}$ the vertical foliation of a holomorphic vector bundle and following [6, 7], we make a short review about Cauchy-Riemann operators and Dolbeault cohomology groups along the leaves of the foliation $\mathcal{V}$. Next, by analogy with the real case [3, 4], we define a

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vertical complex Liouville distribution on the total space of a complex Finsler bundle (E,L) and we get an adapted basis on the holomorphic vertical distribution with respect to the orthogonal splitting $T^{1,0}\mathcal{V} = \mathcal{L}^{1,0}\mathcal{V} \oplus \{\xi\}$, where $\{\xi\}$ is the complex line bundle spanned by the vertical complex Liouville vector field ξ over (E,L) and $\mathcal{L}^{1,0}\mathcal{V}$ is the vertical Liouville distribution on (E,L). We also prove that the distribution $\mathcal{L}^{1,0}\mathcal{V}$ is an integrable one. In the last two sections, by analogy with [11], we consider new type of foliated forms of type $(0;q,0)$ and $(0;q-1,1)$, respectively, with respect to conjugated Liouville distribution $\mathcal{L}^{0,1}\mathcal{V}$ and we obtain a decomposition of the conjugated foliated differential operator $\overline{\partial}_{\mathcal{V}} = \overline{\partial}_{\mathcal{V}}^{1,0} + \overline{\partial}_{\mathcal{V}}^{0,1}$ for foliated forms of type $(0;q,0)$. Finally, by applying some results from [6, 7] concerning to the operator $\overline{\partial}_{\mathcal{V}}$ we prove a Grothendieck-Dolbeault type Lemma with respect to the operator $\overline{\partial}_{\mathcal{V}}^{1,0}$ and new cohomology groups are obtained and studied.

2 Preliminaries

Let $\pi : E \to M$ be a holomorphic vector bundle of rank m over an n-dimensional complex manifold M. Consider $\mathcal{U} = \{U_\alpha\}$ an open covering set of M, (z^k), $k = 1,\ldots,n$, local complex coordinates in chart (U,φ) and $s_U = \{s_a\}$, $a = 1,\ldots,m$, a local frame for the sections of E over U. The covering $\{U, s_U\}$, $U \in \mathcal{U}$ induces the complex coordinates system $u = (z^k,\eta^a)$ on $\pi^{-1}(U)$, where $s = \eta^a s_a$ is a section on $E_z = \pi^{-1}(z)$. In $z \in U \cap U'$, the transition functions $g_{UU'} : U \cap U' \to GL(m,\mathbb{C})$ has a local representation by the complex matrix $M^a_b(z)$ and then if (z'^k,η'^a) are the complex coordinates in $\pi^{-1}(U')$ the transition laws of these coordinates are

$$z'^k = z'^k(z)\,,\, \eta'^a = M^a_b(z)\eta^b, \tag{2.1}$$

where z'^k, M^a_b are holomorphic functions on z^j variables and $\det M^a_b \neq 0$.

As we already know, the total space of E has a natural structure of $n+m$-dimensional complex manifold because the transition functions $M^a_b(z)$ are holomorphic. Let J be the natural complex structure of the manifold E and $T^{1,0}E$ and $T^{0,1}E = \overline{T^{1,0}E}$ be its holomorphic and antiholomorphic subbundles, respectively. Let $T_{\mathbb{C}}E = T^{1,0}E \oplus T^{0,1}E$ be the complexified tangent bundle of the real tangent bundle $T_{\mathbb{R}}E$. From (2.1) it results the following changes for the natural local frames on $T^{1,0}_uE$:

$$\frac{\partial}{\partial z^j} = \frac{\partial z'^k}{\partial z^j}\frac{\partial}{\partial z'^k} + \frac{\partial M^a_b}{\partial z^j}\eta^b\frac{\partial}{\partial \eta'^a}\,,\, \frac{\partial}{\partial \eta^b} = M^a_b\frac{\partial}{\partial \eta'^a}. \tag{2.2}$$

By conjugation over all in (2.2) we obtain the change rules of the local frame on $T^{0,1}_uE$, and then the behaviour of the J complex structure is

$$J(\frac{\partial}{\partial z^k}) = i\frac{\partial}{\partial z^k}\,,\, J(\frac{\partial}{\partial \eta^a}) = i\frac{\partial}{\partial \eta^a}\,,\, J(\frac{\partial}{\partial \overline{z}^k}) = -i\frac{\partial}{\partial \overline{z}^k}\,,\, J(\frac{\partial}{\partial \overline{\eta}^a}) = -i\frac{\partial}{\partial \overline{\eta}^a}. \tag{2.3}$$

Let $\mathcal{V}$ be the vertical foliation on $E_0 = E - \{\text{zero section}\}$, i.e. the simply foliation defined by C^∞ submersion $\pi : E_0 \to M$, and characterized by $z^k = const.$ on the leaves.

The relations (2.2) show that $T^{1,0}\mathcal{V} = \mathrm{span}\{\frac{\partial}{\partial \eta^a}\} \subset T^{1,0}E$ is a foliated holomorphic vector subbundle, called the vertical distribution, which is an integrable one. In particular, $J_{\mathcal{V}}$:

$T_{\mathbb{C}}\mathcal{V} \to T_{\mathbb{C}}\mathcal{V}$, defined by

$$J_{\mathcal{V}}(\frac{\partial}{\partial\eta^a}) = i\frac{\partial}{\partial\eta^a}\,,\; J_{\mathcal{V}}(\frac{\partial}{\partial\overline{\eta}^a}) = -i\frac{\partial}{\partial\overline{\eta}^a} \tag{2.4}$$

is called the *complex structure along the leaves*, where $T_{\mathbb{C}}\mathcal{V} = T^{1,0}\mathcal{V}\oplus T^{0,1}\mathcal{V}$. We also notice that the *Nijenhuis tensor along the leaves* associated to $J_{\mathcal{V}}$ vanish, namely

$$N_{\mathcal{V}}(X,Y) = 2\{[J_{\mathcal{V}}X, J_{\mathcal{V}}Y] - [X,Y] - J_{\mathcal{V}}[J_{\mathcal{V}}X,Y] - J_{\mathcal{V}}[X, J_{\mathcal{V}}Y]\} = 0$$

for every $X,Y \in \Gamma(T_{\mathbb{C}}\mathcal{V})$.

Let $\Omega^{p,q}(\mathcal{V})$ be the space of all foliated differential forms of type (p,q) that is, differential forms on E which can be written in local coordinates $u = (z^k, \eta^a)$, adapted to the foliation by

$$\varphi = \sum \varphi_{A_p\overline{B}_q}(z,\eta)d\eta^{A_p}\wedge d\overline{\eta}^{B_q}, \tag{2.5}$$

where $A_p = (a_1 \dots a_p)$, $B_q = (b_1 \dots b_q)$, and the sum is after the indices $a_1 < \dots < a_p$ and $b_1 < \dots < b_q$, respectively. We also notice that the coeficient functions $\varphi_{a_1\dots a_p\overline{b}_1\dots\overline{b}_q}$ are C^∞-functions on (z,η) and are skew symmetric in the indices $(a_1,\dots,a_p)$ and $(b_1,\dots,b_q)$, respectively.

Then, the set of all foliated r-differential forms on E admits the decomposition $\Omega^r(\mathcal{V}) = \oplus^r_{p+q}\Omega^{p,q}(\mathcal{V})$, $r = 0,1,\dots,2m$ and the exterior derivative along the leaves $d_{\mathcal{V}}$, admits the decomposition

$$d_{\mathcal{V}} = \partial_{\mathcal{V}} + \overline{\partial}_{\mathcal{V}}, \tag{2.6}$$

where the terms denote $(1,0)$ and $(0,1)$ foliated type, respectively. The Cauchy-Riemann operators along the leaves, are locally defined by

$$\partial_{\mathcal{V}}\varphi = \sum_{a=1}^{m}\frac{\partial\varphi_{A_p\overline{B}_q}}{\partial\eta^a}d\eta^a\wedge d\eta^{A_p}\wedge d\overline{\eta}^{B_q}\,,\; \overline{\partial}_{\mathcal{V}}\varphi = \sum_{a=1}^{m}\frac{\partial\varphi_{A_p\overline{B}_q}}{\partial\overline{\eta}^a}d\overline{\eta}^a\wedge d\eta^{A_p}\wedge d\overline{\eta}^{B_q}. \tag{2.7}$$

These operators have the properties $\partial^2_{\mathcal{V}} = \overline{\partial}^2_{\mathcal{V}} = 0$ and $\partial_{\mathcal{V}}\overline{\partial}_{\mathcal{V}} + \overline{\partial}_{\mathcal{V}}\partial_{\mathcal{V}} = 0$, respectively. The differential complex

$$0 \longrightarrow \Omega^0(\mathcal{V}) \xrightarrow{d_{\mathcal{V}}} \Omega^1(\mathcal{V}) \xrightarrow{d_{\mathcal{V}}} \dots \xrightarrow{d_{\mathcal{V}}} \Omega^{2m}(\mathcal{V}) \longrightarrow 0$$

is called the *$d_{\mathcal{V}}$-complex* of $(E,\mathcal{V})$; its homology $H^p_{\mathcal{V}}(E)$ is called the *foliated de Rham cohomology* of the holomorphic foliation $(E,\mathcal{V})$. The differential complex

$$0 \longrightarrow \Omega^{p,0}(\mathcal{V}) \xrightarrow{\overline{\partial}_{\mathcal{V}}} \Omega^{p,1}(\mathcal{V}) \xrightarrow{\overline{\partial}_{\mathcal{V}}} \dots \xrightarrow{\overline{\partial}_{\mathcal{V}}} \Omega^{p,m}(\mathcal{V}) \longrightarrow 0$$

is called the *$\overline{\partial}_{\mathcal{V}}$-complex* of $(E,\mathcal{V})$; its homology $H^{p,q}_{\mathcal{V}}(E)$ is called the *foliated Dolbeault cohomology* of the holomorphic foliation $(E,\mathcal{V})$.

Locally, the operator $\overline{\partial}_{\mathcal{V}}$ satisfies a Grothendieck-Dolbeault Lemma, namely

Theorem 2.1. *([6]). Let φ be a $\overline{\partial}_{\mathcal{V}}$ -closed foliated differential form of type (p,q) defined on an open $U \subset E$. Then, there exists a foliated differential form ψ of type $(p,q-1)$ defined on $U' \subset U$ and such that $\varphi = \overline{\partial}_{\mathcal{V}}\psi$.*

One can describe the cohomology $H^{p,\bullet}_{\mathcal{V}}(E)$ by using a sheaf which is analogous to the sheaf of germs of holomorphic p-forms on a complex manifold.

Definition 2.2. A p-form φ is said to be $\mathcal{V}$-holomorphic, if it is foliated, of type $(p,0)$ and satisfies $\overline{\partial}_{\mathcal{V}}\varphi = 0$.

Locally, a $\mathcal{V}$-holomorphic p-form can be written: $\varphi = \varphi_{A_p}(z,\eta)d\eta^{A_p}$ with $\varphi_{A_p}(z,\eta)$ holomorphic on η.

Let $\Phi^p_{\mathcal{V}}$ be the sheaf of germs of $\mathcal{V}$-holomorphic p-forms on E and $\mathcal{F}^{p,q}(\mathcal{V})$ the sheaf of germs of foliated forms of type (p,q); $\mathcal{F}^{p,q}(\mathcal{V})$ is a fine sheaf. Theorem 2.1 implies the:

Proposition 2.3. *The sequence of sheaves:*

$$0 \longrightarrow \Phi^p_{\mathcal{V}} \xrightarrow{i} \mathcal{F}^{p,0}(\mathcal{V}) \xrightarrow{\overline{\partial}_{\mathcal{V}}} \ldots \xrightarrow{\overline{\partial}_{\mathcal{V}}} \mathcal{F}^{p,m}(\mathcal{V}) \longrightarrow 0$$

is a fine resolution of $\Phi^p_{\mathcal{V}}$. *So we have* $H^q(E,\Phi^p_{\mathcal{V}}) \approx H^{p,q}_{\mathcal{V}}(E)$, *for* $p,q = 0,1,\ldots,m$.

We notice that $H^\bullet(E,\Phi^p_{\mathcal{V}})$ is not finite dimensional because E is not compact.

3 Vertical Liouville distribution on a complex Finsler bundle

Let $\pi^*E \to E_0$ be the pullback of E by π. Given a global section $s : M \to E$ its natural lift is the section

$$\widetilde{s} : E_0 \to \pi^*E\,,\, \widetilde{s}(u) = (u, s(\pi(u))),\, u = (z,\eta) \in E_0. \tag{3.1}$$

Given a local frame $\{s_1,\ldots,s_m\}$ of E on the open set $U \subseteq M$, then $\{\widetilde{s}_1,\ldots,\widetilde{s}_m\}$ is a local frame of π^*E on $\pi^{-1}(U) \subseteq E_0$.

Let $L = F^2 : E \to \mathbb{R}_+ \cup \{0\}$ be a complex Finsler structure on E (for necessary definitions see for instance [1, 2, 8, 13]), and we set

$$H_a = \frac{\partial L}{\partial \eta^a}\,,\, H_{\bar b} = \frac{\partial L}{\partial \overline{\eta}^b}\,,\, H_{ab} = \frac{\partial^2 L}{\partial \eta^a \partial \eta^b}\,,\, H_{a\bar b} = \frac{\partial^2 L}{\partial \eta^a \partial \overline{\eta}^b} \text{ etc.}$$

Let us put $H(Z,\overline{W}) = H_{a\bar b}(u)Z^a\overline{W}^b$, where $Z = Z^a\widetilde{s}_a(u)$, $W = W^b\widetilde{s}_b(u) \in \Gamma_u(\pi^*E)$, $u \in \pi^{-1}(U)$. Then H is globally defined. We say that L is convex if H is positive definite. If L is convex, H is a Hermitian metric in $\pi^*E \to E_0$. Also, by the homogeneity condition of a complex Finsler structure, namely $L(z,\lambda\eta) = |\lambda|^2 L(z,\eta)$ for any $\lambda \in \mathbb{C} - \{0\}$, we have, see [8], the following properties:

$$H_{ab}\eta^a = 0\,,\, H_{\bar a\bar b}\overline{\eta}^a = 0\,,\, H_a\eta^a = L\,,\, H_{\bar b}\overline{\eta}^b = L, \tag{3.2}$$

$$H_{a\bar b c}\eta^a = 0\,,\, H_{a\bar b\bar c}\overline{\eta}^b = 0\,,\, H_{a\bar b c}\overline{\eta}^b = H_{ac}, \tag{3.3}$$

$$H_{a\bar b}\eta^a = H_{\bar b}\,,\, H_{a\bar b}\overline{\eta}^b = H_a\,,\, H_{a\bar b}\eta^a\overline{\eta}^b = L. \tag{3.4}$$

The (globally defined) bundle isomorphism [5],

$$\gamma : \pi^*E \to T^{1,0}\mathcal{V}\,,\, \gamma(\widetilde{s}_a) = \frac{\partial}{\partial \eta^a}, a = 1,\ldots,m, \tag{3.5}$$

induces a Hermitian metric structure on $T^{1,0}\mathcal{V}$, denoted again by H, and defined by

$$H(Z,\overline{W}) = H_{a\bar{b}}Z^a\overline{W}^b, \text{ for any } Z = Z^a\frac{\partial}{\partial\eta^a},\ W = W^b\frac{\partial}{\partial\eta^b} \in \Gamma(T^{1,0}\mathcal{V}). \tag{3.6}$$

An important global vertical vector field is defined by $\xi = \eta^a\frac{\partial}{\partial\eta^a}$ and it is called *the vertical complex Liouville vector field* (or vertical radial complex vector field). We notice that the third equation of (3.4) says that

$$L = H(\xi,\bar{\xi}) > 0, \tag{3.7}$$

so ξ is an embedding of E into $T^{1,0}\mathcal{V}$.

Let $\{\xi\}$ be the complex line bundle over E spanned by ξ and we define the vertical Liouville distribution on (E,L) as the complementary orthogonal distribution, denoted by $\mathcal{L}^{1,0}\mathcal{V}$, to $\{\xi\}$ in $T^{1,0}\mathcal{V}$ with respect to H, namely $T^{1,0}\mathcal{V} = \mathcal{L}^{1,0}\mathcal{V}\oplus\{\xi\}$. Hence, $\mathcal{L}^{1,0}\mathcal{V}$ is defined by

$$\Gamma(\mathcal{L}^{1,0}\mathcal{V}) = \{Z\in\Gamma(T^{1,0}\mathcal{V})\,;\, H(Z,\bar{\xi}) = 0\}. \tag{3.8}$$

Consequently, let us consider the vertical vector fields

$$Z_a = \frac{\partial}{\partial\eta^a} - t_a\xi\,,\ a = 1,\dots,m, \tag{3.9}$$

where the functions $t_a(z,\eta)$ are defined by the conditions

$$H(Z_a,\bar{\xi}) = 0\,,\ a = 1,\dots,m. \tag{3.10}$$

Thus, the above conditions become $H(\frac{\partial}{\partial\eta^a},\bar{\xi}) - t_aH(\xi,\bar{\xi}) = 0$ for every $a = 1,\dots,m$, so, taking into account (3.4) and (3.7), we obtain the local expression of the functions t_a in a local chart $(U,(z^k,\eta^a))$ as

$$t_a = \frac{H_a}{L}\,,\ a = 1,\dots,m. \tag{3.11}$$

If $(U',(z'^i,\eta'^b))$ is another local chart on E, then on $U\cap U'\neq\phi$, we have

$$t'_b = \frac{H'_{b\bar{d}}\bar{\eta}'^d}{L} = \frac{1}{L}M^{\bar{d}}_{\bar{a}}\bar{\eta}^a M^c_b M^{\bar{a}}_{\bar{d}}H_{c\bar{a}} = M^c_b t_c,$$

so we obtain the following changing rule for the vector fields from (3.9)

$$Z'_b = M^a_b Z_a\,,\ b = 1,\dots,m. \tag{3.12}$$

By conjugation we obtain the decomposition

$$T_{\mathbb{C}}\mathcal{V} = \mathcal{L}^{1,0}\mathcal{V}\oplus\{\xi\}\oplus\mathcal{L}^{0,1}\mathcal{V}\oplus\{\bar{\xi}\}.$$

Proposition 3.1. *The functions* $\{t_a\}$, $a = 1,\dots,m$, *satisfies*

$$t_a\eta^a = t_{\bar{a}}\bar{\eta}^a = 1\ ,\ Z_a\eta^a = Z_{\bar{a}}\bar{\eta}^a = 0, \tag{3.13}$$

$$\frac{\partial t_a}{\partial\eta^b} = \frac{H_{ab}}{L} - t_at_b\ ,\ \frac{\partial t_a}{\partial\eta^{\bar{b}}} = \frac{H_{a\bar{b}}}{L} - t_at_{\bar{b}}, \tag{3.14}$$

$$\xi t_a = -t_a\,,\ \bar{\xi}t_a = 0\,,\ \eta^a\frac{\partial t_a}{\partial\eta^b} = -t_b\,;\ \eta^a(\xi t_a) = -1. \tag{3.15}$$

Proof. We have that $t_a\eta^a = \frac{H_a}{L}\eta^a = 1$ and similarly $t_{\bar{a}}\bar{\eta}^a = \frac{H_{\bar{a}}}{L}\bar{\eta}^a = 1$, where we used (3.11) and the last two equalities from (3.2). Now, $Z_a\eta^a = (\frac{\partial}{\partial\eta^a} - t_a\xi)\eta^a = 1 - t_a\eta^a = 0$ and similarly for conjugated. Thus, the relations (3.13) are proved. Similarly, by direct calculations using (3.2), (3.4) and (3.11), one gets (3.14) and (3.15). □

Theorem 3.2. *The vertical Liouville distribution $\mathcal{L}^{1,0}\mathcal{V}$ is integrable.*

Proof. The proof of this theorem is based on the ideas of Theorem 3.1. from [3]. Let $Z, W \in \Gamma(\mathcal{L}^{1,0}\mathcal{V})$. As $T^{1,0}\mathcal{V}$ is an integrable distribution on E, it is sufficient to show that $[Z, W]$ has no component with respect to ξ.

By using (3.8), we obtain that $Z \in \Gamma(\mathcal{L}^{1,0}\mathcal{V})$ if and only if

$$H_{a\bar{b}}Z^a\bar{\eta}^b = 0, \tag{3.16}$$

where $Z^a(z,\eta)$ are the components of Z. Differentiate (3.16) with respect to η^c we get

$$H_{a\bar{b}c}Z^a\bar{\eta}^b + H_{a\bar{b}}\frac{\partial Z^a}{\partial\eta^c}\bar{\eta}^b = 0, \text{ for any } c = 1,\ldots,m \tag{3.17}$$

and taking into account the last equation of (3.3) we have

$$H_{ac}Z^a + H_{a\bar{b}}\frac{\partial Z^a}{\partial\eta^c}\bar{\eta}^b = 0, \text{ for any } c = 1,\ldots,m. \tag{3.18}$$

Thus,

$$\begin{aligned} H([Z,W],\xi) &= H_{a\bar{b}}\bar{\eta}^b(\frac{\partial W^a}{\partial\eta^c}Z^c - \frac{\partial Z^a}{\partial\eta^c}W^c) \\ &= -(H_{ac}W^aZ^c - H_{ac}Z^aW^c) \\ &= 0 \end{aligned}$$

which finish the proof. □

We also notice that the above theorem it follows from the straightforward calculus of Lie brackets, namely

$$[Z_a, Z_b] = t_aZ_b - t_bZ_a\,, \; [Z_a,\xi] = Z_a \tag{3.19}$$

$$[Z_a, Z_{\bar{b}}] = 0\,, \; [Z_{\bar{a}},\xi] = 0 \tag{3.20}$$

and its conjugates.

By the conditions (3.10), $\{Z_1,\ldots,Z_m\}$ are m vectors fields orthogonal to ξ, so they belong to the $(m-1)$-dimensional distribution $\mathcal{L}^{1,0}\mathcal{V}$. It results that they are linear dependent and, from (3.13) we obtain

$$Z_m = -\frac{1}{\eta^m}\sum_{a=1}^{m-1}\eta^aZ_a, \tag{3.21}$$

since the local coordinate η^m is nonzero everywhere.

We have

Proposition 3.3. *The system of vertical vector fields $\{Z_1,\ldots,Z_{m-1},\xi\}$ is a local basis of $\Gamma(T^{1,0}\mathcal{V})$, called adapted.*

Proof. The proof is similar with the analogue result from real case, see [10], and it consist to check that the rank of the matrix of change from the natural basis $\{\frac{\partial}{\partial\eta^a}\}$, $a = 1,\ldots,m$ of $\Gamma(T^{1,0}\mathcal{V})$ to $\{Z_1,\ldots,Z_{m-1},\xi\}$ is equal to m. □

In the end of this section we notice the following concludent remark: Let $(U', (z'^i, \eta'^b))$ and $(U, (z^k, \eta^a))$ be two local charts which domains overlap, where η'^b and η^m are nonzero functions (in every local chart on E there is at least one nonzero coordinate function η^a). The adapted basis in U' is $\{Z'_1,\ldots,Z'_{b-1},Z'_{b+1},\ldots,Z'_m,\xi\}$. Now, similarly to [11], the determinant of the change matrix $\{Z_1,\ldots,Z_{m-1},\xi\} \to \{Z'_1,\ldots,Z'_{b-1},Z'_{b+1},\ldots,Z'_m,\xi\}$ on $T^{1,0}\mathcal{V}$ is equal to $(-1)^{m+b}(\frac{\eta'^b}{\eta^m})\det M^a_c \neq 0$.

4 New operators on foliated forms with respect to vertical Liouville distribution

Proposition 4.1. *The foliated* $(0,1)$*-form* $\overline{\omega}_0 = t_{\overline{a}}d\overline{\eta}^a$ *is globally defined and satisfies*

$$\overline{\omega}_0(\overline{\xi}) = 1\,,\ \overline{\omega}_0(Z_{\overline{\alpha}}) = 0\,,\ \overline{\omega}_0 = \overline{\partial}_{\mathcal{V}}(\ln L) \tag{4.1}$$

for all $\alpha = 1,\ldots,m-1$*, where* Z_α *are given by (3.9) and* L *is the complex Finsler structure.*

Proof. In $U \cap U' \neq \phi$ we have $\overline{\omega}'_0 = t'_{\overline{b}}d\overline{\eta}'^b = M^{\overline{a}}_{\overline{b}}t_{\overline{a}}M^{\overline{b}}_{\overline{c}}d\overline{\eta}^c = t_{\overline{a}}d\overline{\eta}^a = \overline{\omega}_0$. We also have $d\overline{\eta}^a(\overline{\xi}) = \overline{\eta}^a$, for all $a = 1,\ldots,m$, and taking into account the first relation of (3.13) it results

$$\overline{\omega}_0(\overline{\xi}) = 1\,,\ \overline{\omega}_0(Z_{\overline{\alpha}}) = t_{\overline{a}}d\overline{\eta}^a(\frac{\partial}{\partial\overline{\eta}^\alpha} - t_{\overline{\alpha}}\overline{\xi}) = t_{\overline{a}}\delta^{\overline{a}}_{\overline{\alpha}} - t_{\overline{\alpha}}t_{\overline{a}}\overline{\eta}^a = 0,$$

where $\delta^{\overline{a}}_{\overline{\alpha}}$ denotes the Kronecker symbols. By conjugation in the relation (3.21) it results also $\overline{\omega}_0(Z_{\overline{m}}) = 0$. Now, we have

$$\overline{\partial}_{\mathcal{V}}(\ln L) = \frac{\partial \ln L}{\partial\overline{\eta}^a}d\overline{\eta}^a = \frac{H_{\overline{a}}}{L}d\overline{\eta}^a = t_{\overline{a}}d\overline{\eta}^a = \overline{\omega}_0,$$

which ends the proof. □

We notice that the equality $\overline{\omega}_0 = \overline{\partial}_{\mathcal{V}}(\ln L)$ shows that $\overline{\omega}_0$ is an $\overline{\partial}_{\mathcal{V}}$-exact foliated $(0,1)$-form and the conjugated vertical Liouville distribution $\mathcal{L}^{0,1}\mathcal{V}$ is defined by the equation $\overline{\omega}_0 = 0$.

In the following, we will consider $\Omega^{0,q}(\mathcal{V}) \subset \Omega^{p,q}(\mathcal{V})$ the subspace of all foliated forms of type $(0,q)$ on E.

Definition 4.2. A foliated $(0,q)$-form $\varphi \in \Omega^{0,q}(\mathcal{V})$ is called a $(0;q_1,q_2)$-form iff for any vertical vector fields $Z_1,\ldots,Z_q \in \Gamma(T^{0,1}\mathcal{V})$, $q = q_1+q_2$, we have $\varphi(Z_1,\ldots,Z_q) \neq 0$ only if q_1 arguments are in $\Gamma(\mathcal{L}^{0,1}\mathcal{V})$ and q_2 arguments are in $\Gamma(\{\overline{\xi}\})$.

Since $\{\overline{\xi}\}$ is a line distribution, we can talk only about $(0;q_1,q_2)$-forms with $q_2 \in \{0,1\}$. We will denote the space of $(0;q_1,q_2)$-forms by $\Omega^{0;q_1,q_2}(\mathcal{V})$. By the above definition, we have the equivalence

$$\varphi \in \Omega^{0;q-1,1}(\mathcal{V}) \Leftrightarrow \varphi(Z_1,\ldots,Z_q) = 0\,,\ \forall Z_1,\ldots,Z_q \in \Gamma(\mathcal{L}^{0,1}\mathcal{V}). \tag{4.2}$$

Proposition 4.3. *Let φ be a nonzero foliated $(0,q)$-form on E. The following assertions are true*

(i) $\varphi \in \Omega^{0;q,0}(\mathcal{V})$ *iff* $i_{\overline{\xi}}\varphi = 0$*, where* i_X *denotes the interior product.*

(ii) The foliated $(0,q-1)$*-form* $i_{\overline{\xi}}\varphi$ *is a* $(0;q-1,0)$*-form.*

(iii) $\varphi \in \Omega^{0;q-1,1}(\mathcal{V})$ *implies* $i_{\overline{\xi}}\varphi \neq 0$.

(iv) If there is a $(0;q-1,0)$*-form* α *such that* $\varphi = \overline{\omega}_0 \wedge \alpha$ *then* $\varphi \in \Omega^{0;q-1,1}(\mathcal{V})$.

Proof. It follows in a similar manner with the proof of Proposition 2.2. from [11]. □

Proposition 4.4. *For every foliated* $(0,q)$*-form* φ *there are two forms* $\varphi_1 \in \Omega^{0;q,0}(\mathcal{V})$ *and* $\varphi_2 \in \Omega^{0;q-1,1}(\mathcal{V})$ *such that* $\varphi = \varphi_1 + \varphi_2$*, uniquely.*

Proof. Let φ be a nonzero foliated $(0,q)$-form. If $i_{\overline{\xi}}\varphi = 0$, then by Proposition 4.3, we have $\varphi \in \Omega^{0;q,0}(\mathcal{V})$. So $\varphi = \varphi + 0$.

If $i_{\overline{\xi}}\varphi \neq 0$, then let φ_2 be the foliated $(0,q)$-form given by $\overline{\omega}_0 \wedge i_{\overline{\xi}}\varphi$. By Proposition 4.3 (iv), it results φ_2 is a $(0;q-1,1)$-form. Moreover, putting $\varphi_1 = \varphi - \varphi_2$, we have

$$i_{\overline{\xi}}\varphi_1 = i_{\overline{\xi}}\varphi - i_{\overline{\xi}}(\overline{\omega}_0 \wedge i_{\overline{\xi}}\varphi) = i_{\overline{\xi}}\varphi - \overline{\omega}_0(\overline{\xi}) i_{\overline{\xi}}\varphi = 0$$

since $\overline{\omega}_0(\overline{\xi}) = 1$. So, φ_1 is a $(0;q,0)$-form and φ_1 and φ_2 are unique defined by φ. Obviously $\varphi = \varphi_1 + \varphi_2$. □

We have to remark that only the zero form can be simultaneous a $(0;q,0)$- and a $(0;q-1,1)$-form, respectively. The above proposition leads to the following decomposition:

$$\Omega^{0,q}(\mathcal{V}) = \Omega^{0;q,0}(\mathcal{V}) \oplus \Omega^{0;q-1,1}(\mathcal{V}). \tag{4.3}$$

A consequence of the Propositions 4.3 and 4.4 is

Proposition 4.5. *Let* φ *be a foliated* $(0,q)$*-form. We have the equivalence*

$$\varphi \in \Omega^{0;q-1,1}(\mathcal{V}) \Leftrightarrow \exists \alpha \in \Omega^{0;q-1,0}(\mathcal{V}) \text{ such that } \varphi = \overline{\omega}_0 \wedge \alpha. \tag{4.4}$$

Moreover, the form α *is uniquely determined.*

Taking into account the characterization given in Proposition 4.3 (i) and the relation (4.4), it follows

Proposition 4.6. *The following assertions hold:*

(i) If $\varphi \in \Omega^{0;q,0}(\mathcal{V})$ *and* $\psi \in \Omega^{0;s,0}(\mathcal{V})$*, then* $\varphi \wedge \psi \in \Omega^{0;q+s,0}(\mathcal{V})$.

(ii) If $\varphi \in \Omega^{0;q,1}(\mathcal{V})$ *and* $\psi \in \Omega^{0;s,0}(\mathcal{V})$*, then* $\varphi \wedge \psi \in \Omega^{0;q+s,1}(\mathcal{V})$.

(iii) If $\varphi \in \Omega^{0;q,1}(\mathcal{V})$ *and* $\psi \in \Omega^{0;s,1}(\mathcal{V})$*, then* $\varphi \wedge \psi = 0$.

Example 4.7. (i) $\overline{\omega}_0 \in \Omega^{0;0,1}(\mathcal{V})$ since there is the $(0;0,0)$-form, the constant 1 function on E, such that $\overline{\omega}_0 = \overline{\omega}_0 \cdot 1$.

(ii) $\theta^{\overline{a}} = d\overline{\eta}^a - \overline{\eta}^a\overline{\omega}_0 \in \Omega^{0;1,0}(\mathcal{V})$, for each $a = 1,\ldots,m$. Indeed

$$\theta^{\overline{a}}(\overline{\xi}) = d\overline{\eta}^a(\overline{\xi}) - \overline{\omega}_0(\overline{\xi})\overline{\eta}^a = 0,$$

so $i_{\overline{\xi}}\theta^{\overline{a}} = 0$. We have to remark that the foliated $(0,1)$-forms $\{\theta^{\overline{a}}\}$, $a = 1,\ldots,m$ are linear dependent, since $\sum t_{\overline{a}}\theta^{\overline{a}} = 0$.

(iii) $i_{\overline{\xi}}(\theta^{\overline{a}} \wedge \theta^{\overline{b}})(Z) = \theta^{\overline{a}}(\overline{\xi})\theta^{\overline{b}}(Z) - \theta^{\overline{b}}(\overline{\xi})\theta^{\overline{a}}(Z) = 0$, for any vertical vector field $Z \in \Gamma(T^{0,1}\mathcal{V})$, hence $\theta^{\overline{a}} \wedge \theta^{\overline{b}} \in \Omega^{0;2,0}(\mathcal{V})$.

Proposition 4.8. *$\overline{\partial}_{\mathcal{V}}\varphi$ is a $(0;q,1)$-form, for any $(0;q-1,1)$-form φ.*

Proof. Let φ be a $(0;q-1,1)$-form. By (4.4), there is an unique $(0;q-1,0)$-form α such that $\varphi = \overline{\omega}_0 \wedge \alpha$. By Proposition 4.4 we also have that $\alpha = i_{\overline{\xi}}\varphi$. Taking into account that $\overline{\omega}_0$ is an $\overline{\partial}_{\mathcal{V}}$-exact form, it follows

$$\overline{\partial}_{\mathcal{V}}\varphi = \overline{\partial}_{\mathcal{V}}(\overline{\omega}_0 \wedge \alpha) = -\overline{\omega}_0 \wedge \overline{\partial}_{\mathcal{V}}\alpha = -\overline{\omega}_0 \wedge \beta_1 - \overline{\omega}_0 \wedge \beta_2,$$

where β_1 and β_2 are the $(0;q,0)$- and $(0;q-1,1)$-forms, respectively, components of the $(0,q)$-form $\overline{\partial}_{\mathcal{V}}\alpha$. By (4.4) we have $\beta_2 = \overline{\omega}_0 \wedge \gamma$ with $\gamma \in \Omega^{0;q-1,0}(\mathcal{V})$, so $\overline{\partial}_{\mathcal{V}}\varphi = -\overline{\omega}_0 \wedge \beta_1$. Then $\overline{\partial}_{\mathcal{V}}\varphi \in \Omega^{0;q,1}(\mathcal{V})$. □

We can write

$$\overline{\partial}_{\mathcal{V}}(\Omega^{0;q-1,1}(\mathcal{V})) \subset \Omega^{0;q,1}(\mathcal{V}). \tag{4.5}$$

Now, we can consider p_1 and p_2 the projections of the module $\Omega^{0,q}(\mathcal{V})$ on its direct sumands from the relation (4.3), namely

$$p_1 : \Omega^{0,q}(\mathcal{V}) \to \Omega^{0;q,0}(\mathcal{V})\,,\ p_1\varphi = \varphi - \overline{\omega}_0 \wedge i_{\overline{\xi}}\varphi \tag{4.6}$$

$$p_2 : \Omega^{0,q}(\mathcal{V}) \to \Omega^{0;q-1,1}(\mathcal{V})\,,\ p_2\varphi = \overline{\omega}_0 \wedge i_{\overline{\xi}}\varphi \tag{4.7}$$

for any $\varphi \in \Omega^{0,q}(\mathcal{V})$.

For an arbitrary foliated $(0,q)$-form φ, we have $\overline{\partial}_{\mathcal{V}}\varphi = \overline{\partial}_{\mathcal{V}}(p_1\varphi) + \overline{\partial}_{\mathcal{V}}(p_2\varphi)$. The relation (4.3) shows that $\overline{\partial}_{\mathcal{V}}(p_2\varphi)$ is a $(0;q,1)$-form, hence $p_1\overline{\partial}_{\mathcal{V}}(p_2\varphi) = 0$. It results

$$p_1\overline{\partial}_{\mathcal{V}}\varphi = p_1\overline{\partial}_{\mathcal{V}}(p_1\varphi)\,,\ p_2\overline{\partial}_{\mathcal{V}}\varphi = p_2\overline{\partial}_{\mathcal{V}}(p_1\varphi) + p_2\overline{\partial}_{\mathcal{V}}(p_2\varphi). \tag{4.8}$$

The above relations prove that

$$\overline{\partial}_{\mathcal{V}}(\Omega^{0;q,0}(\mathcal{V})) \subset \Omega^{0;q+1,0}(\mathcal{V}) \oplus \Omega^{0;q,1}(\mathcal{V}) \tag{4.9}$$

which allows to define the following operators:

$$\overline{\partial}^{1,0}_{\mathcal{V}} : \Omega^{0;q,0}(\mathcal{V}) \to \Omega^{0;q+1,0}(\mathcal{V}),\ \overline{\partial}^{1,0}_{\mathcal{V}}\varphi = p_1\overline{\partial}_{\mathcal{V}}\varphi, \tag{4.10}$$

$$\overline{\partial}^{0,1}_{\mathcal{V}} : \Omega^{0;q,0}(\mathcal{V}) \to \Omega^{0;q,1}(\mathcal{V}),\ \overline{\partial}^{0,1}_{\mathcal{V}}\varphi = p_2\overline{\partial}_{\mathcal{V}}\varphi, \tag{4.11}$$

so that

$$\overline{\partial}_{\mathcal{V}}|_{\Omega^{0;q,0}(\mathcal{V})} = \overline{\partial}^{1,0}_{\mathcal{V}} + \overline{\partial}^{0,1}_{\mathcal{V}}. \tag{4.12}$$

Proposition 4.9. *The operator* $\overline{\partial}^{1,0}_{\mathcal{V}}$ *satisfies*

(i) $\overline{\partial}^{1,0}_{\mathcal{V}}(\varphi\wedge\psi)=\overline{\partial}^{1,0}_{\mathcal{V}}\varphi\wedge\psi+(-1)^q\varphi\wedge\overline{\partial}^{1,0}_{\mathcal{V}}\psi$, for any $\varphi\in\Omega^{0;q,0}(\mathcal{V})$, $\psi\in\Omega^{0;s,0}(\mathcal{V})$.

(ii) $(\overline{\partial}^{1,0}_{\mathcal{V}})^2=0$.

Proof. (i) Let $\varphi\in\Omega^{0;q,0}(\mathcal{V})$ and $\psi\in\Omega^{0;s,0}(\mathcal{V})$. Since

$$\overline{\partial}_{\mathcal{V}}(\varphi\wedge\psi)=\overline{\partial}_{\mathcal{V}}\varphi\wedge\psi+(-1)^q\varphi\wedge\overline{\partial}_{\mathcal{V}}\psi$$

then by (4.12) it follows that

$$\overline{\partial}^{1,0}_{\mathcal{V}}(\varphi\wedge\psi)+\overline{\partial}^{0,1}_{\mathcal{V}}(\varphi\wedge\psi)=\overline{\partial}^{1,0}_{\mathcal{V}}\varphi\wedge\psi+\overline{\partial}^{0,1}_{\mathcal{V}}\varphi\wedge\psi+(-1)^q\varphi\wedge\overline{\partial}^{1,0}_{\mathcal{V}}\psi+(-1)^q\varphi\wedge\overline{\partial}^{0,1}_{\mathcal{V}}\psi.$$

By equating the $(0;q+s+1,0)$ components in the both members of above relation, we get the desired result.

(ii) Let φ be a $(0;q,0)$-form. By (4.6) and (4.10) we have that $\overline{\partial}^{1,0}_{\mathcal{V}}\varphi=\overline{\partial}_{\mathcal{V}}\varphi-\overline{\omega}_0\wedge i_{\overline{\xi}}(\overline{\partial}_{\mathcal{V}}\varphi)$. Thus, using $(\overline{\partial}_{\mathcal{V}})^2=0$, $\overline{\partial}_{\mathcal{V}}\overline{\omega}_0=0$ and $i_{\overline{\xi}}\overline{\omega}_0=1$, by direct calculations, one gets

$$\begin{aligned}(\overline{\partial}^{1,0}_{\mathcal{V}})^2\varphi &= \overline{\partial}^{1,0}_{\mathcal{V}}(\overline{\partial}_{\mathcal{V}}\varphi)-\overline{\partial}^{1,0}_{\mathcal{V}}(\overline{\omega}_0\wedge i_{\overline{\xi}}(\overline{\partial}_{\mathcal{V}}\varphi))\\ &= -\overline{\partial}_{\mathcal{V}}(\overline{\omega}_0\wedge i_{\overline{\xi}}(\overline{\partial}_{\mathcal{V}}\varphi))+\overline{\omega}_0\wedge i_{\overline{\xi}}(\overline{\partial}_{\mathcal{V}}(\overline{\omega}_0\wedge i_{\overline{\xi}}(\overline{\partial}_{\mathcal{V}}\varphi)))\\ &= \overline{\omega}_0\wedge\overline{\partial}_{\mathcal{V}}(i_{\overline{\xi}}(\overline{\partial}_{\mathcal{V}}\varphi))+\overline{\omega}_0\wedge i_{\overline{\xi}}(-\overline{\omega}_0\wedge\overline{\partial}_{\mathcal{V}}(i_{\overline{\xi}}(\overline{\partial}_{\mathcal{V}}\varphi)))\\ &= \overline{\omega}_0\wedge\overline{\partial}_{\mathcal{V}}(i_{\overline{\xi}}(\overline{\partial}_{\mathcal{V}}\varphi))-\overline{\omega}_0\wedge\overline{\partial}_{\mathcal{V}}(i_{\overline{\xi}}(\overline{\partial}_{\mathcal{V}}\varphi))=0\end{aligned}$$

which completes the proof. □

Definition 4.10. We say that a $(0;q,0)$-form φ is $\overline{\partial}^{1,0}_{\mathcal{V}}$-closed if $\overline{\partial}^{1,0}_{\mathcal{V}}\varphi=0$ and it is called $\overline{\partial}^{1,0}_{\mathcal{V}}$-exact if $\varphi=\overline{\partial}^{1,0}_{\mathcal{V}}\psi$ for some $\psi\in\Omega^{0;q-1,0}(\mathcal{V})$.

Example 4.11. (i) For a foliated $(0,1)$-form φ, we have $p_1\varphi=\varphi-\varphi(\overline{\xi})\overline{\omega}_0$ and $p_2\varphi=\varphi(\overline{\xi})\overline{\omega}_0$.

(ii) Let $f\in\mathcal{F}(E)$ and $\overline{\partial}_{\mathcal{V}}f=\frac{\partial f}{\partial\overline{\eta}^a}d\overline{\eta}^a$ its conjugated foliated derivative. Locally, we have

$$\overline{\partial}^{0,1}_{\mathcal{V}}f=p_2\overline{\partial}_{\mathcal{V}}f=(\overline{\partial}_{\mathcal{V}}f)(\overline{\xi})\overline{\omega}_0=\overline{\xi}(f)\overline{\omega}_0$$

and

$$\begin{aligned}\overline{\partial}^{1,0}_{\mathcal{V}}f &= p_1\overline{\partial}_{\mathcal{V}}f=\overline{\partial}_{\mathcal{V}}f-(\overline{\partial}_{\mathcal{V}}f)(\overline{\xi})\overline{\omega}_0\\ &= \frac{\partial f}{\partial\overline{\eta}^a}d\overline{\eta}^a-\overline{\eta}^a\frac{\partial f}{\partial\overline{\eta}^a}\overline{\omega}_0=\frac{\partial f}{\partial\overline{\eta}^a}\theta^{\overline{a}},\end{aligned}$$

where $\theta^{\overline{a}}$ are the $(0;1,0)$-forms given in Example 4.7. Moreover, taking into account the relation (3.4) and the fact $\sum t_{\overline{a}}\theta^{\overline{a}}=0$, it results that locally

$$\overline{\partial}^{1,0}_{\mathcal{V}}f=(Z_{\overline{a}}f)\theta^{\overline{a}}. \tag{4.13}$$

We have
$$\overline{\partial}^{1,0}_{\mathcal{V}}\overline{\eta}^a = (Z_{\overline{a}}\overline{\eta}^b)\theta^{\overline{a}} = \delta^{\overline{b}}_{\overline{a}}\theta^{\overline{a}} - t_{\overline{a}}\overline{\xi}(\overline{\eta}^b)\theta^{\overline{a}} = \theta^{\overline{b}} - (t_{\overline{a}}\theta^{\overline{a}})\overline{\eta}^b = \theta^{\overline{b}},$$
so the $(0;1,0)$-forms $\theta^{\overline{b}}$ are exactly the $\overline{\partial}^{1,0}_{\mathcal{V}}$-derivatives of the local coordinates $\overline{\eta}^b$, for all $b = 1,\dots,m$.

(iii) The $(0;2,0)$-forms $\overline{\partial}^{1,0}_{\mathcal{V}}\overline{\eta}^b \wedge \overline{\partial}^{1,0}_{\mathcal{V}}\overline{\eta}^c$ are $\overline{\partial}^{1,0}_{\mathcal{V}}$-closed, for all $b,c = 1,\dots,m$.

Let us consider now an arbitrary foliated $(0,1)$-form on E. It is locally given in U by $\varphi = \varphi_{\overline{a}}d\overline{\eta}^a$, with $\varphi_{\overline{a}} \in \mathcal{F}(U)$ such that in $U \cap U' \neq \phi$ we have $\varphi'_{\overline{b}} = M^{\overline{a}}_{\overline{b}}\varphi_{\overline{a}}$. By the Proposition 4.3, φ is a $(0;1,0)$-form on E iff $i_{\overline{\xi}}\varphi = 0$ which is locally equivalent with $\varphi_{\overline{a}}\overline{\eta}^a = 0$. Then, locally we have
$$\varphi = \varphi_{\overline{a}}d\overline{\eta}^a = \varphi_{\overline{a}}(\overline{\partial}^{1,0}_{\mathcal{V}}\overline{\eta}^a + \overline{\eta}^a\overline{\omega}_0) = \varphi_{\overline{a}}\overline{\partial}^{1,0}_{\mathcal{V}}\overline{\eta}^a + (\varphi_{\overline{a}}\overline{\eta}^a)\overline{\omega}_0 = \varphi_{\overline{a}}\overline{\partial}^{1,0}_{\mathcal{V}}\overline{\eta}^a.$$
Conversely, the expression locally given by $\varphi_{\overline{a}}\overline{\partial}^{1,0}_{\mathcal{V}}\overline{\eta}^a$, with the functions $\varphi_{\overline{a}}$ satisfying $\varphi'_{\overline{b}} = M^{\overline{a}}_{\overline{b}}\varphi_{\overline{a}}$ is a $(0;1,0)$-form because $\overline{\partial}^{1,0}_{\mathcal{V}}\overline{\eta}^a(\overline{\xi}) = 0$, for all $a = 1,\dots,m$.

5 A $\overline{\partial}^{1,0}_{\mathcal{V}}$-cohomology

In this section we define and study some new cohomology groups of (E,L) with rspect to vertical Liouville distribution.

By the Proposition 4.9, we can consider the differential complex
$$0 \longrightarrow \Omega^{0;0,0}(\mathcal{V}) \xrightarrow{\overline{\partial}^{1,0}_{\mathcal{V}}} \Omega^{0;1,0}(\mathcal{V}) \xrightarrow{\overline{\partial}^{1,0}_{\mathcal{V}}} \dots \xrightarrow{\overline{\partial}^{1,0}_{\mathcal{V}}} \Omega^{0;m-1,0}(\mathcal{V}) \longrightarrow 0,$$
which will be called *the* $\overline{\partial}^{1,0}_{\mathcal{V}}$*-complex* of $(E,L,\mathcal{V})$; its homology $H^{0;q,0}_{\mathcal{V}}(E)$ is called *the Dolbeault cohomology along the vertical Liouville distribution on the complex Finsler bundle* (E,L).

Now, by using the Theorem 2.1 we can prove a resolution property of the operator $\overline{\partial}^{1,0}_{\mathcal{V}}$. Firstly, we prove a Grothendieck-Dolbeault type Lemma for the operator $\overline{\partial}^{1,0}_{\mathcal{V}}$, namely

Theorem 5.1. *Let $\varphi \in \Omega^{0;q,0}(\mathcal{V}|_U)$ be a $\overline{\partial}^{1,0}_{\mathcal{V}}$-closed form and $q \geq 1$. Then there exists $\psi \in \Omega^{0;q-1,0}(\mathcal{V}|_{U'})$, and such that $\varphi = \overline{\partial}^{1,0}_{\mathcal{V}}\psi$ on $U' \subset U$.*

Proof. Let $\varphi \in \Omega^{0;q,0}(\mathcal{V}|_U)$ such that $\overline{\partial}^{1,0}_{\mathcal{V}}\varphi = 0$. Then
$$\overline{\partial}_{\mathcal{V}}\varphi = \overline{\partial}^{1,0}_{\mathcal{V}}\varphi + \overline{\partial}^{0,1}_{\mathcal{V}}\varphi = \overline{\partial}^{0,1}_{\mathcal{V}}\varphi = \overline{\omega}_0 \wedge i_{\overline{\xi}}(\overline{\partial}_{\mathcal{V}}\varphi),$$
so $\overline{\partial}_{\mathcal{V}}\varphi = 0$ (modulo terms containing $\overline{\omega}_0$).

Hence on the space $\overline{\omega}_0 = 0$ we have that φ is $\overline{\partial}_{\mathcal{V}}$-closed. Now, by Theorem 2.1 there exists a foliated $(0,q-1)$-form τ defined on $U' \subset U$ such that
$$\varphi = \overline{\partial}_{\mathcal{V}}\tau + \lambda \wedge \overline{\omega}_0\ ,\ \lambda \in \Omega^{0,q-1}(\mathcal{V}|_{U'}). \tag{5.1}$$

Following the Proposition 4.4 we have that $\tau = \tau_1 + \overline{\omega}_0 \wedge i_{\overline{\xi}}\tau$, with $\tau_1 = p_1\tau \in \Omega^{0;q-1,0}(\mathcal{V}|_{U'})$. Now, the relation (5.1) becomes

$$\varphi = \overline{\partial}_{\mathcal{V}}\tau_1 - \overline{\omega}_0 \wedge \overline{\partial}_{\mathcal{V}} i_{\overline{\xi}}\tau + \lambda \wedge \overline{\omega}_0. \tag{5.2}$$

Here $\varphi \in \Omega^{0;q,0}(\mathcal{V}|_{U'})$, $\overline{\omega}_0 \wedge (\lambda + \overline{\partial}_{\mathcal{V}} i_{\overline{\xi}}\tau) \in \Omega^{0;q-1,1}(\mathcal{V}|_{U'})$ and

$$\overline{\partial}_{\mathcal{V}}\tau_1 = \overline{\partial}_{\mathcal{V}}^{1,0}\tau_1 + \overline{\partial}_{\mathcal{V}}^{0,1}\tau_1 \in \Omega^{0;q,0}(\mathcal{V}|_{U'}) \oplus \Omega^{0;q-1,1}(\mathcal{V}|_{U'}).$$

Now, by equating the same components in the relation (5.2) it results $\varphi = \overline{\partial}_{\mathcal{V}}^{1,0}\tau_1$ on U'. □

Let $\Phi^{0;0,0}$ be the sheaf of germs of functions on E which satisfies $\overline{\partial}_{\mathcal{V}}^{1,0} f = 0$ and $\mathcal{F}^{0;q,0}$ be the sheaf of germs of $(0;q,0)$-forms on E. We denote by $i : \Phi^{0;0,0} \to \mathcal{F}^{0;0,0}$ be the natural inclusion. The sheaves $\mathcal{F}^{0;q,0}$ are fine and taking into account the Theorem 5.1, it follows that the sequence of sheaves

$$0 \longrightarrow \Phi^{0;0,0} \xrightarrow{i} \mathcal{F}^{0;0,0} \xrightarrow{\overline{\partial}_{\mathcal{V}}^{1,0}} \mathcal{F}^{0;1,0} \xrightarrow{\overline{\partial}_{\mathcal{V}}^{1,0}} \ldots \xrightarrow{\overline{\partial}_{\mathcal{V}}^{1,0}} \mathcal{F}^{0;m-1,0} \longrightarrow 0$$

is a fine resolution of $\Phi^{0;0,0}$ and we denote by $H^q(E, \Phi^{0;0,0})$ the cohomology groups of E with the coefficients in the sheaf $\Phi^{0;0,0}$. Then, we obtain a de Rham type isomorphism

$$H^q(E, \Phi^{0;0,0}) \approx H_{\mathcal{V}}^{0;q,0}(E), \text{ for any } q = 1, \ldots, m-1. \tag{5.3}$$

By (4.5), the Dolbeault complex

$$0 \to \mathcal{F}^{0,0}(\mathcal{V}) \xrightarrow{\overline{\partial}_{\mathcal{V}}} \Omega^{0,1}(\mathcal{V}) \xrightarrow{\overline{\partial}_{\mathcal{V}}} \ldots \xrightarrow{\overline{\partial}_{\mathcal{V}}} \Omega^{0,m}(\mathcal{V}) \longrightarrow 0,$$

admits the subcomplex

$$0 \longrightarrow \Phi^{0;0,0} \xrightarrow{\overline{\partial}_{\mathcal{V}}} \Omega^{0;0,1}(\mathcal{V}) \xrightarrow{\overline{\partial}_{\mathcal{V}}} \ldots \xrightarrow{\overline{\partial}_{\mathcal{V}}} \Omega^{0;m-1,1}(\mathcal{V}) \longrightarrow 0.$$

We denote by $Z_{\mathcal{V}}^{0;q,1}(E)$ and $B_{\mathcal{V}}^{0;q,1}(E)$ the spaces of the $\overline{\partial}_{\mathcal{V}}$-closed and $\overline{\partial}_{\mathcal{V}}$-exacts $(0;q,1)$-forms, respectively, and let

$$H_{\mathcal{V}}^{0;q,1}(E) = Z_{\mathcal{V}}^{0;q,1}(E)/B_{\mathcal{V}}^{0;q,1}(E). \tag{5.4}$$

be the q-cohomology group of the last complex.

Theorem 5.2. *The cohomology groups $H_{\mathcal{V}}^{0;q,1}(E)$ and $H^q(E, \Phi^{0;0,0})$ are isomorphic.*

Proof. By Proposition 4.5 we can define the map

$$\zeta : Z_{\mathcal{V}}^{0;q,1}(E) \to Z_{\mathcal{V}}^{0;q,0}(E), \quad \zeta(\varphi) = \alpha$$

for $\alpha \in \Omega^{0;q,0}(\mathcal{V})$ such that $\varphi = \alpha \wedge \overline{\omega}_0$. It is a well-defined map since the equality

$$0 = \overline{\partial}_{\mathcal{V}}\varphi = \overline{\partial}_{\mathcal{V}}\alpha \wedge \overline{\omega}_0 = \overline{\partial}_{\mathcal{V}}^{1,0}\alpha \wedge \overline{\omega}_0 + \overline{\partial}_{\mathcal{V}}^{0,1}\alpha \wedge \overline{\omega}_0 = \overline{\partial}_{\mathcal{V}}^{1,0}\alpha \wedge \overline{\omega}_0 \tag{5.5}$$

implies $\overline{\partial}^{1,0}_{\mathcal{V}}\alpha = 0$. Moreover, ζ is a bijective morphism of groups and $\zeta(B^{0;q,1}_{\mathcal{V}}(E) = B^{0;q,0}_{\mathcal{V}}(E)$. Indeed, for $\varphi \in B^{0;q,1}_{\mathcal{V}}(E)$, there is $\theta \in \Omega^{0;q-1,1}(\mathcal{V})$ such that $\varphi = \overline{\partial}_{\mathcal{V}}\theta$. By (4.4), there are $\alpha \in \Omega^{0;q,0}(\mathcal{V}), \beta \in \Omega^{0;q-1,0}(\mathcal{V})$ such that $\varphi = \alpha \wedge \overline{\omega}_0$ and $\theta = \beta \wedge \overline{\omega}_0$. Then, we have

$$\alpha \wedge \overline{\omega}_0 = \overline{\partial}_{\mathcal{V}}(\beta \wedge \overline{\omega}_0) = \overline{\partial}^{1,0}_{\mathcal{V}}\beta \wedge \overline{\omega}_0.$$

It follows $\alpha \in B^{0;q,0}(\mathcal{V})$. Conversely, $\alpha = \overline{\partial}^{1,0}_{\mathcal{V}}\beta$ implies $\alpha \wedge \overline{\omega}_0 = \overline{\partial}_{\mathcal{V}}(\beta \wedge \overline{\omega}_0)$. We obtain that $\zeta^* : H^{0;q,1}_{\mathcal{V}}(E) \to H^{0;q,0}_{\mathcal{V}}(E)$, $\zeta^*([\varphi]) = [\zeta(\varphi)]$, for $\varphi \in Z^{0;q,1}_{\mathcal{V}}(E)$, is bijective. □

Finally, by (5.3) and above theorem the isomorphism $H^{0;q,1}_{\mathcal{V}}(E) \approx H^{0;q,0}_{\mathcal{V}}(E)$ holds for any $q = 1, \ldots, m-1$.

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