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On Discrete Favard's and Berwald's Inequalities

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Abstract

In this paper, we obtain an extensions of majorization type results and extensions of weighted Favard's and Berwald's inequality. We prove positive semi-definiteness of matrices generated by differences deduced from majorization type results and differences deduced from weighted Favard's and Berwald's inequality. This implies a surprising property of exponentially convexity and log-convexity of this differences which allows us to deduce Lyapunov's inequalities for the differences, which are improvements of majorization type results and weighted Favard's and Berwald's inequalities. Analogous Cauchy's type means, as equivalent forms of exponentially convexity and log-convexity, are also studied and the monotonicity properties are proved.

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1 Introduction

Favard (1933) proved the following result ([14, p.212]).

Theorem 1.1. *Let f be a non-negative continuous concave function on $[a,b]$, not identically zero, and ϕ be a convex function on $[0,2\widetilde{f}]$, where*

$$\widetilde{f} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Then

$$\frac{1}{2\widetilde{f}}\int_0^{2\widetilde{f}} \phi(y)\,dy \geq \frac{1}{b-a}\int_a^b \phi(f(x))\,dx.$$

The following theorem can be obtained from Theorem 1.1.

Theorem 1.2. *Let f be a non-negative concave function on $[a,b] \subset \mathbb{R}$. If $q > 1$, then*

$$\frac{2^q}{q+1}\left(\frac{1}{b-a}\int_a^b f(x)dx\right)^q \geq \frac{1}{b-a}\int_a^b f^q(x)dx. \tag{1.1}$$

If $0 < q < 1$, then the reverse inequality holds in (1.1).

An important generalization of Favard's inequality is given by Berwald (1947) ([14, p.214]).

Theorem 1.3. *Let f be a non-negative, continuous concave function, not identically zero on $[a,b]$, and ψ be a continuous and strictly monotonic function on $[0,y_0]$, where y_0 is sufficiently large. If $\overline{z}$ is the unique positive root of the equation*

$$\frac{1}{\overline{z}}\int_0^{\overline{z}} \psi(y)\,dy = \frac{1}{b-a}\int_a^b \psi(f(x))\,dx,$$

then for every function $\phi : [0,y_0] \to \mathbb{R}$ which is convex with respect to ψ, we have

$$\frac{1}{\overline{z}}\int_0^{\overline{z}} \phi(y)\,dy \geq \frac{1}{b-a}\int_a^b \phi(f(x))\,dx.$$

The following theorem can be obtained from Theorem 1.3.

Theorem 1.4. *Let f be a non-negative concave function on $[a,b] \subset \mathbb{R}$. If $s > q > 0$, we have*

$$\left(\frac{q+1}{b-a}\int_a^b f^q(x)\,dx\right)^{\frac{1}{q}} \geq \left(\frac{s+1}{b-a}\int_a^b f^s(x)\,dx\right)^{\frac{1}{s}}. \tag{1.2}$$

The following theorem is given by Marshall, Olkin and Proschan (1967) [10].

Theorem 1.5. *Let $\mathbf{x}$ and $\mathbf{y}$ be positive n-tuples and $\mathbf{x}/\mathbf{y} = (x_1/y_1, x_2/y_2, ..., x_n/y_n)$. For $r \in \mathbb{R}$,*

$$F(r) := \begin{cases} \left(\frac{\sum_{i=1}^n x_i^r}{\sum_{i=1}^n y_i^r}\right)^{\frac{1}{r}}, & r \neq 0; \\ \left(\frac{\Pi_{i=1}^n x_i}{\Pi_{i=1}^n y_i}\right)^{\frac{1}{n}}, & r = 0. \end{cases}$$

If $\mathbf{y}$ and $\mathbf{x}/\mathbf{y}$ are similarly ordered, then $F(r)$ is increasing on $\mathbb{R}$. If $\mathbf{y}$ and $\mathbf{x}/\mathbf{y}$ are oppositely ordered, then $F(r)$ is decreasing on $\mathbb{R}$.

We will consider discrete results of Favard's and Berwald's inequalities. Berwald's result (1947) [4] and Thunsdroff's result (1932) [19], as well as the Gauss-Winckler inequality are related to a well-known result of Marshall, Olkin and Proschan for the monotonicity of ratio of means, and their result was proved by using the theory of majorization ([10]). Such results will be considered in this paper. Note that this result was previously proved in Izumi, Kobayashi and Takahashi (1934)[6] and later given in Sunouchi (1938)[18]. A simple proof of their result with weights is given by Vasić and Milovanović (1977)[20], and it can also be proved using a generalization of majorization theorem by Pečarić (1984)[17]. Moreover, by using an idea in Vasić and Milovanović's paper a more general result can be obtained ([14, p.218]).
The subject of majorization is treated extensively, see for instance, [1], [9], [11], [13] and [14] and their references. Pečarić and Abramovich (1997) [15] gave this result with positive weights.

Theorem 1.6. *Let g be a strictly increasing function from (a,b) onto (c,d), and let $f \circ g^{-1}$ be a concave function on (c,d). Let the vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ with elements from (a,b) satisfy*

$$\sum_{i=1}^{k} w_i\, g(y_i) \leq \sum_{i=1}^{k} w_i\, g(x_i), \quad k = 1, ..., n-1,$$

and

$$\sum_{i=1}^{n} w_i\, g(y_i) = \sum_{i=1}^{n} w_i\, g(x_i).$$

If $\boldsymbol{y}$ is decreasing, then

$$\sum_{i=1}^{n} w_i\, f(x_i) \leq \sum_{i=1}^{n} w_i\, f(y_i).$$

If $\boldsymbol{x}$ is increasing, then

$$\sum_{i=1}^{n} w_i\, f(y_i) \leq \sum_{i=1}^{n} w_i\, f(x_i).$$

They also gave extensions of Favard's and Berwald's theorems in [8].
Let $\boldsymbol{a}$ and $\boldsymbol{w}$ be positive n-tuples. For $p,q \in \mathbb{R}$ define the Gini mean of $\boldsymbol{a}$ with weight $\boldsymbol{w}$ by ([5, p.248])

$$\eta_n^{p,q}(\boldsymbol{a},\boldsymbol{w}) := \begin{cases} \left(\frac{\sum_{i=1}^n w_i a_i^p}{\sum_{i=1}^n w_i a_i^q}\right)^{\frac{1}{p-q}}, & p \neq q; \\ \left(\prod_{i=1}^n a_i^{w_i a_i^p}\right)^{1/\sum_{i=1}^n w_i a_i^p}, & p = q, \end{cases}$$

Some properties of Gini means are given in the next theorem ([5, p.249]).

Theorem 1.7.

$$\lim_{p \to q} \eta_n^{p,q}(\boldsymbol{a};\boldsymbol{w}) = \eta_n^{q,q}(\boldsymbol{a};\boldsymbol{w}); \lim_{p \to \infty} \eta_n^{p,q}(\boldsymbol{a};\boldsymbol{w}) = \max \boldsymbol{a}; \lim_{q \to -\infty} \eta_n^{p,q}(\boldsymbol{a};\boldsymbol{w}) = \min \boldsymbol{a}.$$

If $p_1 \leq p_2$, $q_1 \leq q_2$, then

$$\eta_n^{p_1,q_1}(\boldsymbol{a};\boldsymbol{w}) \leq \eta_n^{p_2,q_2}(\boldsymbol{a};\boldsymbol{w}); \tag{1.3}$$

If either $p_1 \neq p_2$ or $q_1 \neq q_2$, then inequality (1.3) is strict unless $\boldsymbol{a}$ is constant.

If $p \geq 1 \geq q \geq 0$, then

$$\eta_n^{p,q}(\boldsymbol{a}+\boldsymbol{b};\boldsymbol{w}) \leq \eta_n^{p,q}(\boldsymbol{a};\boldsymbol{w}) + \eta_n^{p,q}(\boldsymbol{b};\boldsymbol{w}).$$

Inequality (1.3) is known as Dresher's inequality.
Positive semi-definite matrices have a number of interesting properties. One of these is that all the eigenvalues of a positive semi-definite matrix are real and nonnegative. Positive semi-definite matrices are very important in theory of inequalities. So in classical book [2] one of the five chapters (second chapter) is devoted to them. Of course as was noted in [2, p.59-61] a very important positive semi-definite matrix is Gram matrix. The corresponding determinantal inequality is well known as Gram's inequality. In this paper we show that we can use majorization type results and weighted Favard's and Berwald's inequalities to obtain positive semi-definite matrices that is we can give determinantal form of these inequalities. Very specific form of these determinantal forms enable us to interpret our results in a form of exponentially convex functions. This is a sub-class of convex functions introduced by Bernstein in [3] (see also [11] and [12], p. 373):

Definition 1.8. A function $h:(a,b)\to\mathbb{R}$ is exponentially convex function if it is continuous and

$$\sum_{i,j=1}^{n} \xi_i\xi_j h\left(x_i + x_j\right) \geq 0$$

for all $n\in\mathbb{N}$ and all choices $\xi_i\in\mathbb{R}$, $i=1,...,n$ such that $x_i + x_j \in (a,b)$, $1\leq i,j\leq n$.

Proposition 1.9. *Let $h:(a,b)\to\mathbb{R}$. The following propositions are equivalent.*

(i) h is exponentially convex.

(ii) h is continuous and

$$\sum_{i,j=1}^{n} \xi_i\xi_j h\left(\frac{x_i + x_j}{2}\right) \geq 0,$$

for every $n\in\mathbb{N}$, for every $\xi_i\in\mathbb{R}$ and every $x_i\in(a,b)$, $1\leq i\leq n$.

Corollary 1.10. *If ϕ is exponentially convex function, then*

$$\det\left[\phi\left(\frac{x_k + x_l}{2}\right)\right]_{k,l=1}^{n} \geq 0$$

for every $n\in\mathbb{N}$, $x_k\in I$, $k=1,2,..,n$.

Corollary 1.11. *If $h:(a,b)\to\mathbb{R}^+$ is exponentially convex function, then h is a* log-*convex function.*

As an analogy to J-convex functions, one defines convex sequences as follows [14, p.6].

Definition 1.12. A finite sequence $\{a_k\}_{k=1}^{n}$ of real numbers is said to be a convex sequence if

$$2a_k \leq a_{k-1} + a_{k+1} \text{ for all } k = 2, 3, ..., n-1.$$

In this paper, we give majorization type results in the case when only one sequence is monotonic. We also give generalization of Favard's inequality, generalization of Berwald's inequality and related results in discrete case. The paper is organized in the following way: In Section 2 we give extensions of majorization type results, generalizations of Favard's and Berwald's inequalities and related results in discrete case. In Section 3 we prove positive semi-definiteness of matrices generated by differences deduced from majorization type results and differences deduced from weighted Favard's and Berwald's inequality. This implies a surprising property of exponentially convexity and log-convexity of this differences which allows us to deduce Lyapunov's inequalities for the differences, which are improvements of majorization type results and weighted Favard's and Berwald's inequalities. In Section 4 we introduce new Cauchy's means as equivalent form of exponential convexity and log-convexity.
The results in Section 2, Section 3 and Section 4 are the discrete version of the results in [7].

2 Main Results

The following theorem is valid ([13], p.32).

Theorem 2.1. *Let φ be a convex function on an interval $I \subseteq \mathbb{R}$, $\boldsymbol{w}$, $\boldsymbol{a}$ and $\boldsymbol{b}$ be positive n-tuples and satisfy*

$$\sum_{i=1}^{k} w_i b_i \leq \sum_{i=1}^{k} w_i a_i, \quad k = 1, ..., n-1, \tag{2.1}$$

and

$$\sum_{i=1}^{n} w_i b_i = \sum_{i=1}^{n} w_i a_i. \tag{2.2}$$

If $\boldsymbol{b}$ is decreasing n-tuple, then

$$\sum_{i=1}^{n} w_i \varphi(b_i) \leq \sum_{i=1}^{n} w_i \varphi(a_i). \tag{2.3}$$

If $\boldsymbol{a}$ is increasing n-tuple, then

$$\sum_{i=1}^{n} w_i \varphi(a_i) \leq \sum_{i=1}^{n} w_i \varphi(b_i). \tag{2.4}$$

If φ is strictly convex and $\boldsymbol{a} \neq \boldsymbol{b}$, then (2.3) *and* (2.4) *are strict.*

Proof. The proof of part (3) and (4) are similar to the proof in [15].
If φ is strictly convex and $\boldsymbol{a} \neq \boldsymbol{b}$, then

$$\varphi(a_i) - \varphi(b_i) > \varphi'_+(b_i)(a_i - b_i),$$

for at least one $i = 1,...,n$. This gives strict inequality in (2.3) and (2.4). □

The following lemma is a discrete case of Lemma 1 in [8] and it can be proved by simple calculations.

Lemma 2.2. *Let $\boldsymbol{v}$ be a positive n-tuple. If $\boldsymbol{x}$ is an increasing real n-tuple, then*

$$\sum_{i=1}^{k} x_i v_i \sum_{i=1}^{n} v_i \leq \sum_{i=1}^{n} x_i v_i \sum_{i=1}^{k} v_i, \ \ k = 1,...,n. \tag{2.5}$$

If $\boldsymbol{x}$ is a decreasing real n-tuple, then the reverse inequality holds in (2.5).

The following theorem is a generalization of discrete weighted Favard's inequality.

Theorem 2.3. *Let $\varphi : (0,1) \to \mathbb{R}$ be a convex function and also let $\boldsymbol{w}$, $\boldsymbol{a}$ and $\boldsymbol{b}$ be positive n-tuples.*

Let $\boldsymbol{a}/\boldsymbol{b}$ be a decreasing n-tuple. If $\boldsymbol{a}$ is an increasing n-tuple, then

$$\sum_{i=1}^{n} w_i \varphi\left(\frac{a_i}{\sum_{i=1}^{n} a_i w_i}\right) \leq \sum_{i=1}^{n} w_i \varphi\left(\frac{b_i}{\sum_{i=1}^{n} b_i w_i}\right). \tag{2.6}$$

If $\boldsymbol{b}$ is a decreasing n-tuple, then the reverse inequality holds in (2.6).

Let $\boldsymbol{a}/\boldsymbol{b}$ be an increasing n-tuple. If $\boldsymbol{b}$ is an increasing n-tuple, then

$$\sum_{i=1}^{n} w_i \varphi\left(\frac{b_i}{\sum_{i=1}^{n} b_i w_i}\right) \leq \sum_{i=1}^{n} w_i \varphi\left(\frac{a_i}{\sum_{i=1}^{n} a_i w_i}\right). \tag{2.7}$$

If $\boldsymbol{a}$ is a decreasing n-tuple, then the reverse inequality holds in (2.7).

If φ is strictly convex function and $\boldsymbol{a} \neq \boldsymbol{b}$, then the strict inequalities hold in (2.6) *and* (2.7) *and their reverse cases.*

Proof. Using Lemma 2.2 with

$$\boldsymbol{v} = \boldsymbol{b}\,\boldsymbol{w}, \quad \boldsymbol{x} = \boldsymbol{a}/\boldsymbol{b},$$

we obtain

$$\sum_{i=1}^{n} a_i w_i \sum_{i=1}^{k} b_i w_i \leq \sum_{i=1}^{k} a_i w_i \sum_{i=1}^{n} b_i w_i, \ \ k = 1,...,n,$$

implies

$$\sum_{i=1}^{k} w_i \left(\frac{b_i}{\sum_{i=1}^{n} b_i w_i}\right) \leq \sum_{i=1}^{k} w_i \left(\frac{a_i}{\sum_{i=1}^{n} a_i w_i}\right). \tag{2.8}$$

By using Theorem 2.1 and $\boldsymbol{a}$ is increasing, we have

$$\sum_{i=1}^{n} w_i \varphi\left(\frac{a_i}{\sum_{i=1}^{n} a_i w_i}\right) \leq \sum_{i=1}^{n} w_i \varphi\left(\frac{b_i}{\sum_{i=1}^{n} b_i w_i}\right).$$

Similarly, we can prove the case when $\boldsymbol{b}$ is decreasing n-tuple.

The remaining cases can be reduced to the first case switching the role of $\boldsymbol{a}$ and $\boldsymbol{b}$.

Similarly as in Theorem 2.1 for strict inequality, we can get strict inequality in (2.6), reverse inequality in (2.6), (2.7) and reverse inequality in (2.7). □

Corollary 2.4. *Let $\varphi : [0,\infty) \to \mathbb{R}$ be a convex function and $\mathbf{w}$ be a positive n-tuple.*

If $\boldsymbol{a}$ is a positive increasing concave n-tuple, then

$$\sum_{i=1}^{n} w_i \varphi\left(\frac{a_i}{\sum_{j=1}^{n} w_j a_j}\right) \leq \sum_{i=1}^{n} w_i \varphi\left(\frac{i-1}{\sum_{j=1}^{n} (j-1)\, w_j}\right). \tag{2.9}$$

If $\boldsymbol{a}$ is an increasing convex real n-tuple and $a_1 = 0$, then the reverse inequality holds in (2.9).

If $\mathbf{a}$ is a positive decreasing concave n-tuple, then

$$\sum_{i=1}^{n} w_i \varphi\left(\frac{a_i}{\sum_{j=1}^{n} w_j a_j}\right) \leq \sum_{i=1}^{n} w_i \varphi\left(\frac{n-i}{\sum_{j=1}^{n} (n-j)\, w_j}\right). \tag{2.10}$$

If $\boldsymbol{a}$ is a decreasing convex real n-tuple and $a_n = 0$, then the reverse inequality holds in (2.10).

Proof. (1) Take $b_1 = \varepsilon < a_1/a_2$, $b_i = i-1$ $(2 \leq i \leq n)$. So, a_i/b_i $(1 \leq i \leq n)$ is a decreasing n-tuple. Using Theorem 2.3 (2.6), we have

$$\begin{aligned} &\sum_{i=1}^{n} w_i \varphi\left(\frac{a_i}{\sum_{j=1}^{n} w_j a_j}\right) \\ &\quad \leq w_1 \varphi\left(\frac{\varepsilon}{\varepsilon w_1 + \sum_{j=2}^{n} (j-1)\, w_j}\right) \\ &\quad + \sum_{i=2}^{n} w_i \varphi\left(\frac{i-1}{\varepsilon w_1 + \sum_{j=2}^{n} (j-1)\, w_j}\right). \end{aligned}$$

When $\varepsilon \to 0$, then

$$\begin{aligned} &\sum_{i=1}^{n} w_i \varphi\left(\frac{a_i}{\sum_{j=1}^{n} w_j a_j}\right) \\ &\quad \leq w_1 \varphi(0) + \sum_{i=2}^{n} w_i \varphi\left(\frac{i-1}{\sum_{j=2}^{n} (j-1)\, w_j}\right) \\ &\quad = \sum_{i=1}^{n} w_i \varphi\left(\frac{i-1}{\sum_{j=1}^{n} (j-1)\, w_j}\right). \end{aligned}$$

Since $\mathbf{a}$ is an increasing convex n-tuple and $a_1 = 0$, then $a_i/(i-1)$ $(2 \leq i \leq n)$ is an increasing n-tuple. Using Theorem 2.3 (2.7), we have

$$\sum_{i=2}^{n} w_i \varphi\left(\frac{i-1}{\sum_{j=2}^{n} (j-1)\, w_j}\right) \leq \sum_{i=2}^{n} w_i \varphi\left(\frac{a_i}{\sum_{j=2}^{n} w_j a_j}\right),$$

or equivalently

$$w_1\,\varphi\left(\frac{0}{\sum_{j=1}^{n}(j-1)\,w_j}\right)+\sum_{i=2}^{n}w_i\,\varphi\left(\frac{i-1}{\sum_{j=1}^{n}(j-1)\,w_j}\right)$$
$$\leq w_1\,\varphi\left(\frac{0}{\sum_{j=1}^{n}w_j\,a_j}\right)+\sum_{i=2}^{n}w_i\,\varphi\left(\frac{a_i}{\sum_{j=1}^{n}w_j\,a_j}\right),$$

implies

$$\sum_{i=1}^{n}w_i\,\varphi\left(\frac{i-1}{\sum_{j=1}^{n}(j-1)\,w_j}\right)\leq\sum_{i=1}^{n}w_i\,\varphi\left(\frac{a_i}{\sum_{j=1}^{n}w_j\,a_j}\right).$$

The remaining cases can be proved by using the similar procedure as in the first case. □

The following corollary is an application of Theorem 2.3.

Corollary 2.5. *Let* $\mathbf{w}$*,* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *be positive n-tuples and* $\varphi(x)=x^p$*, where* $p>1$ *or* $p<0$*.*
Let $\boldsymbol{a}/\boldsymbol{b}$ *be a decreasing n-tuple. If* $\boldsymbol{a}$ *is an increasing n-tuple, then*

$$\frac{\sum_{i=1}^{n}a_i^p\,w_i}{\sum_{i=1}^{n}b_i^p\,w_i}\leq\left(\frac{\sum_{i=1}^{n}a_iw_i}{\sum_{i=1}^{n}b_i\,w_i}\right)^p. \tag{2.11}$$

If $\boldsymbol{b}$ *is a decreasing n-tuple, then the reverse inequality holds in* (2.11)*.*
Let $\boldsymbol{a}/\boldsymbol{b}$ *be an increasing n-tuple. If* $\boldsymbol{b}$ *is an increasing n-tuple, then*

$$\left(\frac{\sum_{i=1}^{n}a_iw_i}{\sum_{i=1}^{n}b_i\,w_i}\right)^p\leq\frac{\sum_{i=1}^{n}a_i^p\,w_i}{\sum_{i=1}^{n}b_i^p\,w_i}. \tag{2.12}$$

If $\boldsymbol{a}$ *is a decreasing n-tuple, then the reverse inequality holds in* (2.12)*.*
If $\varphi(x)=x^p$*,* $0<p<1$*, then the reverse inequality holds in* (2.11)*, reverse inequality in* (2.11)*,* (2.12) *and reverse inequality in* (2.12)*.*

The following result is an application of Corollary 2.4.

Corollary 2.6. *Let* $\mathbf{w}$ *be a positive n-tuple and* $\varphi(x)=x^p$*, where* $p>1$*.*
If $\boldsymbol{a}$ *is a positive increasing concave n-tuple, then*

$$\frac{\sum_{i=1}^{n}a_i^p\,w_i}{\sum_{i=1}^{n}(i-1)^p\,w_i}\leq\left(\frac{\sum_{i=1}^{n}a_i\,w_i}{\sum_{i=1}^{n}(i-1)\,w_i}\right)^p. \tag{2.13}$$

If $\boldsymbol{a}$ *is an increasing convex n-tuple and* $a_1=0$*, then the reverse inequality holds in* (2.13)*.*
If $\boldsymbol{a}$ *is a positive decreasing concave n-tuple, then*

$$\frac{\sum_{i=1}^{n}a_i^p\,w_i}{\sum_{i=1}^{n}(n-i)^p\,w_i}\leq\left(\frac{\sum_{i=1}^{n}a_i\,w_i}{\sum_{i=1}^{n}(n-i)\,w_i}\right)^p. \tag{2.14}$$

If $\boldsymbol{a}$ *is a decreasing convex n-tuple and* $a_n=0$*, then the reverse inequality holds in* (2.14)*.*
If $\varphi(x)=x^p$*,* $0<p<1$*, then the reverse inequality holds in* (2.13)*, reverse inequality in* (2.13)*,* (2.14) *and reverse inequality in* (2.14)*.*

The following theorem is a slight extension of Theorem 1.6:

Theorem 2.7. *Let* $\mathbf{w}$*,* $\mathbf{a}$ *and* $\mathbf{b}$ *be an positive n-tuples. Suppose* $\psi, \varphi : [0,\infty) \to \mathbb{R}$ *are such that* ψ *is a strictly increasing function and* φ *is a convex function with respect to* ψ *i.e.,* $\varphi \circ \psi^{-1}$ *is convex. Suppose also that*

$$\sum_{i=1}^{k} w_i \psi(b_i) \leq \sum_{i=1}^{k} w_i \psi(a_i), \quad k = 1, ..., n-1, \tag{2.15}$$

and

$$\sum_{i=1}^{n} w_i \psi(b_i) = \sum_{i=1}^{n} w_i \psi(a_i). \tag{2.16}$$

If $\mathbf{b}$ *is a decreasing n-tuple, then*

$$\sum_{i=1}^{n} w_i \varphi(b_i) \leq \sum_{i=1}^{n} w_i \varphi(a_i). \tag{2.17}$$

If $\mathbf{a}$ *is an increasing n-tuple, then*

$$\sum_{i=1}^{n} w_i \varphi(a_i) \leq \sum_{i=1}^{n} w_i \varphi(b_i). \tag{2.18}$$

If $\varphi \circ \psi^{-1}$ *is strictly convex and* $\mathbf{a} \neq \mathbf{b}$*, then* (2.17) *and* (2.18) *are strict.*

Proof. Without loss of generality, it is sufficient to prove the case when $\psi(t) = t$, but this case is proved in Theorem 2.1. □

The following theorem is a generalization of discrete weighted Berwald's inequality.

Theorem 2.8. *Let* $\mathbf{w}$*,* $\mathbf{a}$ *and* $\mathbf{b}$ *be positive n-tuples. Suppose* $\psi, \varphi : [0,\infty) \to \mathbb{R}$ *are such that* ψ *is a continuous and strictly increasing function and* φ *is a convex function with respect to* ψ *i.e.,* $\varphi \circ \psi^{-1}$ *is convex. Let* z_1 *be such that*

$$\sum_{i=1}^{n} w_i \psi(z_1 b_i) = \sum_{i=1}^{n} w_i \psi(a_i). \tag{2.19}$$

Let $\mathbf{a}/\mathbf{b}$ *be a decreasing n-tuple. If* $\mathbf{a}$ *is an increasing n-tuple, then*

$$\sum_{i=1}^{n} w_i \varphi(a_i) \leq \sum_{i=1}^{n} w_i \varphi(z_1 b_i). \tag{2.20}$$

If $\mathbf{b}$ *is a decreasing n-tuple, then the reverse inequality holds in* (2.20).

Let $\mathbf{a}/\mathbf{b}$ *be an increasing n-tuple. If* $\mathbf{b}$ *is an increasing n-tuple, then*

$$\sum_{i=1}^{n} w_i \varphi(z_1 b_i) \leq \sum_{i=1}^{n} w_i \varphi(a_i). \tag{2.21}$$

If $\mathbf{a}$ *is a decreasing n-tuple, then the reverse inequality holds in* (2.21).

If $\varphi \circ \psi^{-1}$ *is strictly convex function and* $\mathbf{a} \neq z_1\mathbf{b}$*, then strict inequality holds in* (2.20)*, reverse inequality in* (2.20), (2.21) *and reverse inequality in* (2.21).

Proof. Since ψ is continuous, therefore $F(z) = \sum_{i=1}^{n} w_i \psi(z b_i)$ for $z \geq 0$ is continuous. By using $\boldsymbol{a} > 0$ and ψ is strictly increasing, we have $F(0) = \sum_{i=1}^{n} w_i\psi(0) < \sum_{i=1}^{n} w_i\psi(a_i)$. Since $\boldsymbol{a/b}$ is bounded above, we take any $z_0 > a_i/b_i$ or $a_i < z_0 b_i$ for $i = 1,...,n$. So, $F(z_0) = \sum_{i=1}^{n} w_i \psi(z_0 b_i) > \sum_{i=1}^{n} w_i\psi(a_i)$. This shows the existence of z_1.

Because $\boldsymbol{a/b}$ is decreasing and ψ is strictly increasing function, and because

$$\sum_{i=1}^{n} w_i \psi(z_1 b_i) = \sum_{i=1}^{n} w_i \psi(a_i),$$

there is an m such that

$$\frac{a_i}{b_i} \geq z_1, \quad i = 1,...,m \quad \text{and} \quad \frac{a_i}{b_i} \leq z_1, \quad i = m+1,...,n, \tag{2.22}$$

hence

$$\sum_{i=1}^{k} w_i \psi(z_1 b_i) \leq \sum_{i=1}^{k} w_i \psi(a_i), \quad k = 1,...,n. \tag{2.23}$$

We give the proof on inequality (2.23) for the convenience of a reader. If $k = 1,...,m$, then inequality (2.23) follows immediately from the first inequality in (4.8). If $k = m+1,...,n$, then by using the equality (2.19) and the second inequality in (4.8), we obtain

$$\begin{aligned}
&\sum_{i=1}^{k} w_i \psi(z_1 b_i) \\
&\quad = \sum_{i=1}^{n} w_i \psi(z_1 b_i) - \sum_{i=k+1}^{n} w_i \psi(z_1 b_i) \\
&\quad = \sum_{i=1}^{n} w_i \psi(a_i) - \sum_{i=k+1}^{n} w_i \psi(z_1 b_i) \\
&\quad \leq \sum_{i=1}^{n} w_i \psi(a_i) - \sum_{i=k+1}^{n} w_i \psi(a_i) \\
&\quad = \sum_{i=1}^{k} w_i \psi(a_i).
\end{aligned}$$

By using the inequality (2.23), the equality (2.19), the assumption that $\varphi \circ \psi^{-1}$ is convex, $\boldsymbol{a}$ is increasing and Theorem 2.7, we obtain

$$\sum_{i=1}^{n} w_i \varphi(a_i) \leq \sum_{i=1}^{n} w_i \varphi(z_1 b_i).$$

By using the inequality (2.23), the equality (2.19), the assumption that $\varphi \circ \psi^{-1}$ is convex, $\boldsymbol{b}$ is decreasing and Theorem 2.7, we obtain

$$\sum_{i=1}^{n} w_i \varphi(z_1 b_i) \leq \sum_{i=1}^{n} w_i \varphi(a_i).$$

The remaining cases can be proved analogously. □

Corollary 2.9. *Let* $\mathbf{w}$ *be a positive n-tuple. Assume that* $\psi, \varphi : [0, \infty) \to \mathbb{R}$ *are such that* ψ *is a continuous and strictly increasing function and* φ *is a convex function with respect to* ψ *i.e.,* $\varphi \circ \psi^{-1}$ *is convex. Let* z_1 *and* z_2 *be such that*

$$\sum_{i=1}^{n} w_i \psi[(i-1)\, z_1] = \sum_{i=1}^{n} w_i \psi(a_i), \tag{2.24}$$

and

$$\sum_{i=1}^{n} w_i \psi[(n-i)\, z_2] = \sum_{i=1}^{n} w_i \psi(a_i). \tag{2.25}$$

If $\boldsymbol{a}$ *is a positive increasing concave n-tuple, then*

$$\sum_{i=1}^{n} w_i \varphi(a_i) \leq \sum_{i=1}^{n} w_i \varphi[(i-1)\, z_1]. \tag{2.26}$$

If $\boldsymbol{a}$ *is an increasing convex n-tuple and* $a_1 = 0$*, then the reverse inequality in* (2.26) *holds.*

If $\mathbf{a}$ *is a positive decreasing concave n-tuple, then*

$$\sum_{i=1}^{n} w_i \varphi(a_i) \leq \sum_{i=1}^{n} w_i \varphi[(n-i)\, z_2]. \tag{2.27}$$

If $\mathbf{a}$ *is a decreasing convex n-tuple and* $a_n = 0$*, then the reverse inequality in* (2.27) *holds.*

Proof. Take $b_1 = \varepsilon < a_1/a_2$, $b_i = i-1$ $(2 \leq i \leq n)$. So, a_i/b_i $(1 \leq i \leq n)$ is a decreasing n-tuple. Therefore, (2.24) can be written as

$$w_1 \psi(\varepsilon z_1) + \sum_{i=2}^{n} w_i \psi[(i-1)\, z_1] = \sum_{i=1}^{n} w_i \psi(a_i).$$

Using Theorem 2.8 (2.20), we have

$$\sum_{i=1}^{n} w_i \varphi(a_i) \leq w_1 \varphi(\varepsilon z_1) + \sum_{i=2}^{n} w_i \varphi[(i-1)\, z_1],$$

when $\epsilon \to 0$, then

$$\begin{aligned} \sum_{i=1}^{n} w_i \varphi(a_i) &\leq w_1 \varphi(0) + \sum_{i=2}^{n} w_i \varphi[(i-1)\, z_1] \\ &= \sum_{i=1}^{n} w_i \varphi[(i-1)\, z_1]. \end{aligned}$$

Since $\mathbf{a}$ is an increasing convex n-tuple and $a_1 = 0$, then $a_i/(i-1)$ $(2 \leq i \leq n)$ is an increasing n-tuple. Therefore, (2.24) can be written as

$$\sum_{i=2}^{n} w_i \psi[(i-1)\, z_1] = \sum_{i=2}^{n} w_i \psi(a_i).$$

Using Theorem 2.8 (2.21), we have

$$\sum_{i=2}^{n} w_i \varphi[(i-1)\, z_1] \leq \sum_{i=2}^{n} w_i \varphi(a_i).$$

$$w_1 \varphi(0 z_1) + \sum_{i=2}^{n} w_i \varphi[(i-1)\, z_1] \leq w_1 \varphi(0) + \sum_{i=1}^{n} w_i \varphi(a_i).$$

$$\sum_{i=1}^{n} w_i \varphi[(i-1)\, z_1] \leq \sum_{i=1}^{n} w_i \varphi(a_i).$$

The second case be proved by similar procedure as in the first case. □

The following corollary is an application of Theorem 2.8.

Corollary 2.10. *Let* $\boldsymbol{w}$, $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *be positive n-tuples. Also let* $\psi(x) = x^q$, $\varphi(x) = x^p$ *be such that* $0 < q \leq p$.

Let $\boldsymbol{a}/\boldsymbol{b}$ *be a decreasing n-tuple. If* $\boldsymbol{a}$ *is an increasing n-tuple, then*

$$\left(\frac{\sum_{i=1}^{n} w_i a_i^p}{\sum_{i=1}^{n} w_i b_i^p} \right)^{\frac{1}{p}} \leq \left(\frac{\sum_{i=1}^{n} w_i a_i^q}{\sum_{i=1}^{n} w_i b_i^q} \right)^{\frac{1}{q}}. \tag{2.28}$$

If $\boldsymbol{b}$ *is a decreasing n-tuple, then the reverse inequality holds in* (2.28).

Let $\boldsymbol{a}/\boldsymbol{b}$ *be an increasing n-tuple. If* $\boldsymbol{b}$ *is an increasing n-tuple, then*

$$\left(\frac{\sum_{i=1}^{n} w_i a_i^q}{\sum_{i=1}^{n} w_i b_i^q} \right)^{\frac{1}{q}} \leq \left(\frac{\sum_{i=1}^{n} w_i a_i^p}{\sum_{i=1}^{n} w_i b_i^p} \right)^{\frac{1}{p}}. \tag{2.29}$$

If $\boldsymbol{a}$ *is a decreasing n-tuple, then the reverse inequality holds in* (2.29).

The following result from [16] is an application of Corollary 2.9.

Corollary 2.11. *Let* $\boldsymbol{w}$ *be a positive n-tuple. Also let* $\psi(x) = x^q$, $\varphi(x) = x^p$ *be such that* $0 < q \leq p$.

If $\boldsymbol{a}$ *is a positive increasing concave n-tuple, then*

$$\left(\frac{\sum_{i=1}^{n} w_i a_i^p}{\sum_{i=1}^{n} w_i (i-1)^p} \right)^{\frac{1}{p}} \leq \left(\frac{\sum_{i=1}^{n} w_i a_i^q}{\sum_{i=1}^{n} w_i (i-1)^q} \right)^{\frac{1}{q}}. \tag{2.30}$$

If $\boldsymbol{a}$ *is an increasing convex n-tuple and* $a_1 = 0$*, then the reverse inequality holds in* (2.30).

If $\boldsymbol{a}$ *is a positive decreasing concave n-tuple, then*

$$\left(\frac{\sum_{i=1}^{n} w_i a_i^p}{\sum_{i=1}^{n} w_i (n-i)^p} \right)^{\frac{1}{p}} \leq \left(\frac{\sum_{i=1}^{n} w_i a_i^q}{\sum_{i=1}^{n} w_i (n-i)^q} \right)^{\frac{1}{q}}. \tag{2.31}$$

If $\boldsymbol{a}$ *is a decreasing convex n-tuple and* $a_n = 0$*, then the reverse inequality holds in* (2.31).

3 Exponential Convexity, Lyapunov's and Dresher's type of inequalities

Throughout the paper we will frequently use the following family of convex functions with respect to $\psi(x) = x^q$, $q > 0$, on $(0, \infty)$:

$$\varphi_s(x) := \begin{cases} \frac{q^2}{s(s-q)} x^s, & s \neq 0, q; \\ -q \log x, & s = 0; \\ q x^q \log x, & s = q. \end{cases} \tag{3.1}$$

The following theorem gives positive semi-definite matrix, exponentially convex and log-convex functions for the difference deduced from generalized Berwald's inequality given in Theorem 2.8 and also Lyapunov's inequality for this difference.

Theorem 3.1. *Let **w**, **a** and **b** be positive n-tuples. Suppose **a**/**b** is a decreasing n-tuple, **a** is an increasing n-tuple and z_1 is defined as in* (2.19) *for $\psi(x) = x^q$, $q > 0$, and also let*

$$\Omega_s := \sum_{i=1}^{n} \varphi_s(z_1 b_i) - \sum_{i=1}^{n} \varphi_s(a_i).$$

Then the following statements are valid:

(a) For every $n \in \mathbb{N}$ and $s_1, ..., s_n \in \mathbb{R}$, the matrix $\left[\Omega_{\frac{s_i+s_j}{2}}\right]_{i,j=1}^{n}$ is positive semi-definite.

(b) The function $s \to \Omega_s$ is exponentially convex.

(c) The function $s \to \Omega_s$ is a log*-convex.*

Proof. (a) Consider the function

$$\phi(x) = \sum_{i,j}^{k} u_i u_j \varphi_{p_{ij}}(x),$$

for $k = 1, ..., n$, $x > 0$, $u_i \in \mathbb{R}$, $p_{ij} \in \mathbb{R}$, where $p_{ij} = \frac{p_i + p_j}{2}$ and $\varphi_{p_{ij}}$ is defined in (3.1).
Here, we shall show that $\phi(x)$ is convex with respect to $\psi(x) = x^q$, $q > 0$.
Set

$$F(x) = \phi(x^{\frac{1}{q}}) = \sum_{i,j}^{k} u_i u_j \varphi_{p_{ij}}\left(x^{\frac{1}{q}}\right).$$

We have

$$\begin{aligned} F''(x) &= \sum_{i,j}^{k} u_i u_j x^{\frac{p_{ij}}{q} - 2} \\ &= \left(\sum_{i}^{k} u_i x^{\frac{p_i}{2q} - 1}\right)^2 > 0, \ x > 0. \end{aligned}$$

Therefore, $\phi(x)$ is convex with respect to $\psi(x) = x^q$ $(q > 0)$ for $x > 0$. Using Theorem 2.8,

$$\sum_{i=1}^{n} w_i \phi(z_1 b_i) \geq \sum_{i=1}^{n} w_i \phi(a_i),$$

where z_1 is defined as in (2.19) for $\psi(x) = x^q$, $(q > 0)$. We have

$$\sum_{i=1}^{n}\left(\sum_{i,j}^{k} u_i u_j \varphi_{p_{ij}}(z_1 b_i)\right) - \sum_{i=1}^{n}\left(\sum_{i,j}^{k} u_i u_j \varphi_{p_{ij}}(a_i)\right) \geq 0,$$

or equivalently

$$\sum_{i,j}^{k} u_i u_j \Big[\sum_{i=1}^{n} \varphi_{p_{ij}}(z_1 b_i) - \sum_{i=1}^{n} \varphi_{p_{ij}}(a_i)\Big] \geq 0,$$

implies

$$\sum_{i,j}^{k} u_i u_j \Omega_{p_{ij}}(x) \geq 0.$$

From last inequality, it follows that the matrix $\left[\Omega_{\frac{p_i+p_j}{2}}\right]_{i,j=1}^{k}$ is a positive semi-definite matrix.
(b) Note that Ω_s is continuous for $s \in \mathbb{R}$ since

$$\lim_{s\to 0} \Omega_s = \Omega_0 \text{ and } \lim_{s\to 1} \Omega_s = \Omega_1.$$

Then by using Proposition 1.9, we get exponential convexity of the function $s \to \Omega_s$.
(c) It is a simple consequence of Corollary 1.11. □

The following theorem gives the Dresher's type inequality for difference deduced from generalized Berwald's inequality given in Theorem 2.8.

Theorem 3.2. *Let Ω_s be defined as in* Theorem 3.1 *and $t, s, u, v \in \mathbb{R}$ such that $s \leq u, t \leq v, s \neq t, u \neq v$. Then*

$$\left(\frac{\Omega_t}{\Omega_s}\right)^{\frac{1}{t-s}} \leq \left(\frac{\Omega_v}{\Omega_u}\right)^{\frac{1}{v-u}}. \tag{3.2}$$

Proof. For a convex function φ, it holds (see [14, p.2])

$$\frac{\varphi(x_2) - \varphi(x_1)}{x_2 - x_1} \leq \frac{\varphi(y_2) - \varphi(y_1)}{y_2 - y_1}, \tag{3.3}$$

where $x_1 \leq y_1$, $x_2 \leq y_2$, $x_1 \neq x_2$, $y_1 \neq y_2$. Since by Theorem 3.1, Ω_s is log-convex, we can set in (3.3): $\varphi(x) = \log \Omega_x$, $x_1 = s$, $x_2 = t$, $y_1 = u$, $y_2 = v$. We get

$$\frac{\log \Omega_t - \log \Omega_s}{t-s} \leq \frac{\log \Omega_v - \log \Omega_u}{v-u},$$

from which (3.2) trivially follows. □

Remark 3.3. Similarly as in Theorem 3.1 and Theorem 3.2, we can get positive semi-definiteness, exponential convexity, log-convexity, Lyapunov's inequalities and Dresher's inequalities for the cases when $\boldsymbol{a/b}$ is a decreasing and $\boldsymbol{b}$ is a decreasing, $\boldsymbol{a/b}$ is an increasing and $\boldsymbol{a}$ is a decreasing, and $\boldsymbol{a/b}$ is an increasing and $\boldsymbol{b}$ is an increasing by using Theorem 2.8.

The following theorem gives positive semi-definite matrix, exponentially convex and log-convex functions for the difference deduced from majorization type results given in Theorem 2.7 and also Lyapunov's inequality for this difference.

Theorem 3.4. *Let* $\mathbf{w}$, $\boldsymbol{a}$ *and* $\mathbf{b}$ *be positive n-tuples. Suppose* $\mathbf{b}$ *is a decreasing and*

$$\Gamma_s := \sum_{i=1}^{n} \varphi_s(b_i) - \sum_{i=1}^{n} \varphi_s(a_i),$$

such that conditions (2.15) *and* (2.16) *are satisfied. Then the following statements are valid:*

(a) For every $n \in \mathbb{N}$ *and* $s_1, ..., s_n \in \mathbb{R}$, *the matrix* $\left[\Gamma_{\frac{s_i+s_j}{2}}\right]_{i,j=1}^{n}$ *is a positive semi-definite.*

(b) The function $s \to \Gamma_s$ *is exponentially convex.*

(c) The function $s \to \Gamma_s$ *is a* log*-convex.*

Proof. As in the proof of Theorem 3.1, we use Theorem 2.7 instead of Theorem 2.8. □

The following theorem gives the Dresher's type inequality for difference deduced from majorization type results given in Theorem 2.7.

Theorem 3.5. *Let* Γ_s *be defined as in* Theorem 3.4 *and* $t, s, u, v \in \mathbb{R}$ *such that* $s \leq u, t \leq v, s \neq t, u \neq v$. *Then*

$$\left(\frac{\Gamma_t}{\Gamma_s}\right)^{\frac{1}{t-s}} \leq \left(\frac{\Gamma_v}{\Gamma_u}\right)^{\frac{1}{v-u}}. \tag{3.4}$$

Proof. Similar to the proof of Theorem 3.2. □

Remark 3.6. Similarly as in Theorem 3.4 and Theorem 3.5, we can get positive semi-definite matrix, exponential convexity, log-convexity, Lyapunov's inequality and Dresher's inequality in the case when **a** is increasing using Theorem 2.7.

Remark 3.7. We can get positive semi-definiteness of matrix, exponential convexity, log-convexity and Lyapunov's inequalities for differences deduced from generalized Favard's inequality (see Theorem 2.3) and majorization type results (see Theorem 2.1) by substituting $q = 1$ in Theorem 3.1 and Theorem 3.4 respectively. We can also get Dresher's inequalities for differences deduced from generalized Favard's inequality and majorization type results by substituting $q = 1$ in Theorem 3.2 and Theorem 3.5 respectively.

Remark 3.8. As in Theorem 3.1, we proved exponential convexity and log-convexity for positive n-tuples **a** and **b** by using φ_s but there are several our corollaries in which one of the n-tuple is non-negative. So, we can not prove exponential convexity and log-convexity for these cases by using φ_s. Then we define the following family of convex functions with respect to $\psi(x) = x^q$, $q > 0$, on $[0,\infty)$ with using the convention $0\log 0 = 0$:

$$\overline{\varphi}_s(x) := \begin{cases} \frac{q^2}{s(s-q)}\, x^s, & s \neq q; \\ q\, x^q \log x, & s = q. \end{cases}$$

We give the following result for the convenience of a reader: Let $\boldsymbol{w}$ be a positive n-tuple and z_1 be defined as in (2.19) for $\psi(x) = x^q$,, $q > 0$. If $\mathbf{a}$ is a positive increasing concave n-tuple and

$$\Upsilon_s := \sum_{i=1}^{n} \overline{\varphi}_s(z_1 b_i) - \sum_{i=1}^{n} \overline{\varphi}_s(a_i).$$

Then the following statements are valid:

(a) For every $n \in \mathbb{N}$ and $s_1, ..., s_n \in \mathbb{R}$, the matrix $\left[\Upsilon_{\frac{s_i+s_j}{2}}\right]_{i,j=1}^{n}$ is a positive semi-definite.

(b) The function $s \to \Upsilon_s$ is exponentially convex.

(c) The function $s \to \Upsilon_s$ is a log-convex.

We can obtain Lyapunov's and Dresher's type inequalities for the difference deduced from Corollary 2.9. We can also introduce corresponding Cauchy's means and prove monotonicity of their means. Similarly, we can get positive semi-definite matrix, exponential convexity, log-convexity and obtain Lyapunov's and Dresher's type inequalities for differences deduced from Corollary 2.10 and Corollary 2.11. $\overline{\varphi}_s$ has a stronger condition than φ_s. So, we can prove positive semi-definiteness of matrix, exponential convexity, log-convexity, Lyapunov's inequality and Dresher's inequality for the difference deduced from all our results by using $\overline{\varphi}_s$.

4 Mean Value Theorems

Let us note that (3.2) and (3.4) have the form of some known inequalities between means (eg. Stolarsky means, Gini means). Here we will prove that expressions on both sides of (3.2) and (3.4) are also means. The proofs in the remaining cases are analogous.

Lemma 4.1. *Let $\psi, \varphi \in C^2(I)$, I interval in $\mathbb{R}$, be such that $\psi'(y) > 0$ for every $y \in I$ and*

$$m \leq \frac{\psi'(y)\varphi''(y) - \varphi'(y)\psi''(y)}{(\psi'(y))^3} \leq M. \tag{4.1}$$

Then the functions ϕ_1 and ϕ_2 defined by

$$\phi_1(x) = \frac{1}{2} M \psi^2(x) - \varphi(x),$$

and

$$\phi_2(x) = \varphi(x) - \frac{1}{2} m \psi^2(x),$$

are convex functions with respect to ψ.

Proof. Set

$$G(x) = \phi_1\left[\psi^{-1}(x)\right] = \frac{1}{2} M x^2 - \varphi\left[\psi^{-1}(x)\right].$$

We have

$$G''(x) = M - \frac{\psi'\left[\psi^{-1}(x)\right]\varphi''\left[\psi^{-1}(x)\right] - \varphi'\left[\psi^{-1}(x)\right]\psi''\left[\psi^{-1}(x)\right]}{\left(\psi'\left[\psi^{-1}(x)\right]\right)^3},$$

which shows that ϕ_1 is a convex function with respect to ψ.
Similarly, we can prove the same result for ϕ_2. □

Theorem 4.2. *Let* $\boldsymbol{w}$, $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *be positive n-tuples,* $\psi \in C^2([0,\infty))$ *and* $\varphi \in C^2([0,z_1])$. *Let* $\boldsymbol{a}/\boldsymbol{b}$ *be a decreasing n-tuple and* $\boldsymbol{a}$ *be an increasing n-tuple. Also let* $\psi'(y) > 0$ *for* $y \in [0,z_1]$ *and* z_1 *be defined as in* Theorem 2.8, *then there exists* $\xi \in [0,z_1]$ *such that*

$$\begin{aligned} &\sum_{i=1}^{n} w_i\varphi(z_1 b_i) - \sum_{i=1}^{n} w_i\varphi(a_i) \\ &= \frac{\psi'(\xi)\varphi''(\xi) - \varphi'(\xi)\psi''(\xi)}{2(\psi'(\xi))^3}\Big[\sum_{i=1}^{n} w_i\psi^2(z_1 b_i) \\ &- \sum_{i=1}^{n} w_i\psi^2(a_i)\Big]. \end{aligned} \tag{4.2}$$

Proof. Set $m = \min_{y\in[0,z_1]} \Psi(y)$ and $M = \max_{y\in[0,z_1]} \Psi(y)$, where

$$\Psi(y) = \frac{\psi'(y)\varphi''(y) - \varphi'(y)\psi''(y)}{(\psi'(y))^3}.$$

Applying (2.20) for ϕ_1 and ϕ_2 defined in Lemma 4.1, we get

$$\begin{aligned} &\frac{M}{2}\left[\sum_{i=1}^{n} w_i\psi^2(z_1 b_i) - \sum_{i=1}^{n} w_i\psi^2(a_i)\right] \\ &\geq \sum_{i=1}^{n} w_i\varphi(z_1 b_i) - \sum_{i=1}^{n} w_i\varphi(a_i) \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} &\sum_{i=1}^{n} w_i\varphi(z_1 b_i) - \sum_{i=1}^{n} w_i\varphi(a_i) \\ &\geq \frac{m}{2}\left[\sum_{i=1}^{n} w_i\psi^2(z_1 b_i) - \sum_{i=1}^{n} w_i\psi^2(a_i)\right]. \end{aligned} \tag{4.4}$$

By combining (4.3) and (4.4), (4.2) follows from continuity of Ψ. □

Theorem 4.3. *Let $\boldsymbol{w}$, $\boldsymbol{a}$ and $\boldsymbol{b}$ be positive n-tuples, $\psi \in C^2([0,\infty))$ and $\varphi_1, \varphi_2 \in C^2([0,z_1])$. Let $\boldsymbol{a}/\boldsymbol{b}$ be a decreasing n-tuple and $\boldsymbol{a}$ be an increasing n-tuple. Also let $\psi'(y) > 0$ for $y \in [0,z_1]$ and $\boldsymbol{a} \neq z_1\boldsymbol{b}$, where z_1 is defined as in* Theorem 2.8*, then there exists $\xi \in [0,z_1]$ such that*

$$\frac{\psi'(\xi)\varphi_1''(\xi) - \varphi_1'(\xi)\psi''(\xi)}{\psi'(\xi)\varphi_2''(\xi) - \varphi_2'(\xi)\psi''(\xi)} = \frac{\sum_{i=1}^n w_i\varphi_1(z_1 b_i) - \sum_{i=1}^n w_i\varphi_1(a_i)}{\sum_{i=1}^n w_i\varphi_2(z_1 b_i) - \sum_{i=1}^n w_i\varphi_2(a_i)} \tag{4.5}$$

provided that $\psi'(y)\varphi_2''(y) - \varphi_2'(y)\psi''(y) \neq 0$ for every $y \in [0,z_1]$.

Proof. Define the functional $\Theta : C^2([0,z_1]) \to \mathbb{R}$ with:

$$\Theta(\varphi) = \sum_{i=1}^n w_i\varphi(z_1 b_i) - \sum_{i=1}^n w_i\varphi(a_i)$$

and set $\varphi_0 = \Theta(\varphi_2)\varphi_1 - \Theta(\varphi_1)\varphi_2$. Obviously $\Theta(\varphi_0) = 0$. Using Theorem 4.2, there exists $\xi \in [0,z_1]$ such that

$$\Theta(\varphi_0) = \frac{\psi'(\xi)\varphi_0''(\xi) - \varphi_0'(\xi)\psi''(\xi)}{2(\psi'(\xi))^3}\left[\sum_{i=1}^n w_i\psi^2(z_1 b_i) - \sum_{i=1}^n w_i\psi^2(a_i)\right]. \tag{4.6}$$

We give a proof that the expression in square brackets in (4.6) is non-zero due to $\boldsymbol{a} \neq z_1\boldsymbol{b}$. Suppose that the expression in square brackets in (4.6) is equal to zero, i.e.,

$$0 = \sum_{i=1}^n w_i\psi^2(z_1 b_i) - \sum_{i=1}^n w_i\psi^2(a_i). \tag{4.7}$$

In Theorem 2.8, we have that

$$\frac{a_i}{b_i} \geq z_1, \quad i = 1,\dots,m \quad \text{and} \quad \frac{a_i}{b_i} \leq z_1, \quad i = m+1,\dots,n, \tag{4.8}$$

and also

$$\sum_{i=1}^k w_i\psi(z_1 b_i) \leq \sum_{i=1}^k w_i\psi(a_i), \quad k = 1,\dots,n. \tag{4.9}$$

By (4.7), (4.8) and (4.9), we have

$$\begin{aligned} 0 &= \sum_{i=1}^n w_i\psi^2(z_1 b_i) - \sum_{i=1}^n w_i\psi^2(a_i) \\ &\geq \sum_{i=1}^n w_i\,(2\psi(z_1 b_i))\,[\psi(z_1 b_i) - \psi(a_i)] \geq 0. \end{aligned}$$

This implies

$$\sum_{i=1}^n w_i\psi^2(z_1 b_i) - \sum_{i=1}^n w_i\psi^2(a_i) = \sum_{i=1}^n w_i\,(2\psi(z_1 b_i))\,[\psi(z_1 b_i) - \psi(a_i)]$$

or equivalently

$$\sum_{i=1}^n w_i\,(\psi(z_1 b_i) - \psi(a_i))^2 = 0.$$

Which obviously implies that $\boldsymbol{a} \neq z_1\boldsymbol{b}$.
Since $\boldsymbol{a} \neq z_1\boldsymbol{b}$, the expression in square brackets in (4.6) is non-zero which implies that $\psi'(\xi)\varphi_0''(\xi) - \varphi_0'(\xi)\psi''(\xi) = 0$, and this gives (4.5). Notice that Theorem 4.2 for $\varphi = \varphi_2$ implies that the denominator of the right-hand side of (4.5) is non-zero. □

Corollary 4.4. *Let* $\boldsymbol{w}$, $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *be positive n-tuples. Also let* $\boldsymbol{a}/\boldsymbol{b}$ *be a decreasing n-tuple,* $\boldsymbol{a}$ *be an increasing n-tuple and* z_1 *be defined as in* (2.19) *for* $\psi(x) = x^q$ $(q > 0)$ *or explicitly* z_1 *is defined in* (2.19)*, then for distinct* $s, t, q \in \mathbb{R} \setminus \{0\}$*, there exists* $\xi \in (0, z_1]$ *such that*

$$\xi^{t-s} = \frac{s(s-q)}{t(t-q)} \frac{\sum_{i=1}^n w_i (z_1 b_i)^t - \sum_{i=1}^n w_i a_i^t}{\sum_{i=1}^n w_i (z_1 b_i)^s - \sum_{i=1}^n w_i a_i^s}. \tag{4.10}$$

Proof. Set $\varphi_1(x) = x^t$, $\varphi_2(x) = x^s$ and $\psi(x) = x^q$, $t \neq s \neq 0, q$ in (4.5), then we get (4.10). □

Remark 4.5. Since the function $\xi \to \xi^{t-s}$ is invertible, then from (4.10) we have

$$0 < \left(\frac{s(s-q)}{t(t-q)} \frac{\sum_{i=1}^n w_i (z_1 b_i)^t - \sum_{i=1}^n w_i a_i^t}{\sum_{i=1}^n w_i (z_1 b_i)^s - \sum_{i=1}^n w_i a_i^s} \right)^{\frac{1}{t-s}} \leq z_1. \tag{4.11}$$

In fact, similar result can also be given for (4.5). Namely, suppose that $\Lambda(y) = \left(\psi'(y)\varphi_1''(y) - \varphi_1'(y)\psi''(y)\right) / \left(\psi'(y)\varphi_2''(y) - \varphi_2'(y)\psi''(y)\right)$ has inverse function. Then from (4.5), we have

$$\xi = \Lambda^{-1}\left(\frac{\sum_{i=1}^n w_i \varphi_1 (z_1 b_i) - \sum_{i=1}^n w_i \varphi_1 (a_i)}{\sum_{i=1}^n w_i \varphi_2 (z_1 b_i) - \sum_{i=1}^n w_i \varphi_2 (a_i)} \right). \tag{4.12}$$

By inequality (4.11), we can consider

$$M_{t,s} = \left(\frac{\Omega_t}{\Omega_s} \right)^{\frac{1}{t-s}} \quad \text{for } s, t \in \mathbb{R} \setminus \{0\},\ s \neq t, \tag{4.13}$$

as means in a broader sense. Moreover we can extend these means in other cases. So by passing to the limit, we have

$$\log M_{s,s} = \frac{z_1^s \log z_1 \sum_{i=1}^n w_i b_i^s + z_1^s \sum_{i=1}^n w_i b_i^s \log b_i}{z_1^s \sum_{i=1}^n w_i b_i^s - \sum_{i=1}^n w_i a_i^s} - \frac{\sum_{i=1}^n w_i a_i^s \log a_i}{z_1^s \sum_{i=1}^n w_i b_i^s - \sum_{i=1}^n w_i a_i^s} - \frac{2s-q}{s(s-q)}, \quad s \neq 0, q.$$

$$\log M_{q,q} = \frac{z_1^q \log^2 z_1^q \frac{1}{q^2} \sum_{i=1}^n w_i b_i^q + 2 z_1^q \log z_1 \sum_{i=1}^n w_i b_i^q \log b_i + z_1^q \sum_{i=1}^n w_i b_i^q \log^2 b_i}{2\left(z_1^q \log z_1 \sum_{i=1}^n w_i b_i^q + z_1^q \sum_{i=1}^n w_i b_i^q \log b_i - \sum_{i=1}^n w_i a_i^q \log a_i\right)} - \frac{\sum_{i=1}^n w_i a_i^q \log^2 a_i}{2\left(z_1^q \log z_1 \sum_{i=1}^n w_i b_i^q + \gamma \sum_{i=1}^n w_i b_i^q \log b_i - \sum_{i=1}^n w_i a_i^q \log a_i\right)} - \frac{1}{q}.$$

$$\log M_{0,0} = \frac{\log^2 z_1^q \frac{1}{q^2} \sum_{i=1}^n w_i + 2\log z_1 \sum_{i=1}^n w_i \log b_i + \sum_{i=1}^n w_i \log^2 b_i + \sum_{i=1}^n w_i \log^2 a_i}{2\left(\log z_1 \sum_{i=1}^n w_i + \sum_{i=1}^n w_i \log b_i + \sum_{i=1}^n w_i \log a_i\right)} + \frac{1}{q}.$$

Theorem 4.6. *Let $t \leq u$, $r \leq s$, then the following inequality is valid*

$$M_{t,r} \leq M_{u,s}. \tag{4.14}$$

Proof. Since Ω_s is log-convex, therefore by (3.2) we get (4.14). □

Denote,

$$m_{\boldsymbol{a},\boldsymbol{b}} = \min\{m_{\boldsymbol{a}}, m_{\boldsymbol{b}}\} \text{ and } M_{\boldsymbol{a},\boldsymbol{b}} = \max\{M_{\boldsymbol{a}}, M_{\boldsymbol{b}}\},$$

where, $m_{\boldsymbol{a}}$ and $m_{\boldsymbol{b}}$ denote minima of $\boldsymbol{a}$ and $\boldsymbol{b}$ respectively, and $M_{\boldsymbol{a}}$ and $M_{\boldsymbol{b}}$ denote maxima of $\boldsymbol{a}$ and $\boldsymbol{b}$ respectively.

Theorem 4.7. *Let $\boldsymbol{w}$, $\boldsymbol{a}$ and $\boldsymbol{b}$ be positive n-tuples, $\psi \in C^2([0,\infty))$ and $\varphi \in C^2([m_{\boldsymbol{a},\boldsymbol{b}}, M_{\boldsymbol{a},\boldsymbol{b}}])$ such that conditions* (2.15) *and* (2.16) *are satisfied. Let $\boldsymbol{b}$ be a decreasing n-tuple and $\psi'(y) > 0$ for $y \in ([m_{\boldsymbol{a},\boldsymbol{b}}, M_{\boldsymbol{a},\boldsymbol{b}}])$, then there exists $\xi \in ([m_{\boldsymbol{a},\boldsymbol{b}}, M_{\boldsymbol{a},\boldsymbol{b}}])$ such that*

$$\begin{aligned}\sum_{i=1}^n w_i \varphi(a_i) - \sum_{i=1}^n w_i \varphi(b_i) \\ = \frac{\psi'(\xi)\varphi''(\xi) - \varphi'(\xi)\psi''(\xi)}{2(\psi'(\xi))^3}\Big[\sum_{i=1}^n w_i \psi^2(a_i) \\ - \sum_{i=1}^n w_i \psi^2(b_i)\Big].\end{aligned} \tag{4.15}$$

Proof. Set $m = \min_{y\in[m_{\mathbf{a},\mathbf{b}},M_{\mathbf{a},\mathbf{b}}]} \Psi(y)$ and $M = \max_{y\in[m_{\mathbf{a},\mathbf{b}},M_{\mathbf{a},\mathbf{b}}]} \Psi(y)$, where

$$\Psi(y) = \frac{\psi'(y)\varphi''(y) - \varphi'(y)\psi''(y)}{(\psi'(y))^3}.$$

Applying (2.17) for ϕ_1 and ϕ_2 defined in Lemma 4.1, we get

$$\begin{aligned}\frac{M}{2}\left[\sum_{i=1}^n w_i \psi^2(a_i) - \sum_{i=1}^n w_i \psi^2(b_i)\right] \\ \geq \sum_{i=1}^n w_i \varphi(a_i) - \sum_{i=1}^n w_i \varphi(b_i)\end{aligned} \tag{4.16}$$

and

$$\begin{aligned}\sum_{i=1}^n w_i \varphi(a_i) - \sum_{i=1}^n w_i \varphi(b_i) \\ \geq \frac{m}{2}\left[\sum_{i=1}^n w_i \psi^2(a_i) - \sum_{i=1}^n w_i \psi^2(b_i)\right].\end{aligned} \tag{4.17}$$

By combining (4.16) and (4.17), (4.15) follows from continuity of Ψ. □

Theorem 4.8. *Let $\boldsymbol{w}$, $\boldsymbol{a}$ and $\boldsymbol{b}$ be positive n-tuples, $\psi \in C^2([0,\infty))$ and $\varphi_1, \varphi_2 \in C^2([m_{\boldsymbol{a},\boldsymbol{b}}, M_{\boldsymbol{a},\boldsymbol{b}}])$ such that conditions* (2.15) *and* (2.16) *are satisfied. Let $\boldsymbol{b}$ be a decreasing n-tuple and $\psi'(y) > 0$ for $y \in ([m_{\boldsymbol{a},\boldsymbol{b}}, M_{\boldsymbol{a},\boldsymbol{b}}])$, then there exists $\xi \in ([m_{\boldsymbol{a},\boldsymbol{b}}, M_{\boldsymbol{a},\boldsymbol{b}}])$ such that*

$$\frac{\psi'(\xi)\varphi_1''(\xi) - \varphi_1'(\xi)\psi''(\xi)}{\psi'(\xi)\varphi_2''(\xi) - \varphi_2'(\xi)\psi''(\xi)} = \frac{\sum_{i=1}^n w_i\varphi_1(a_i) - \sum_{i=1}^n w_i\varphi_1(b_i)}{\sum_{i=1}^n w_i\varphi_2(a_i) - \sum_{i=1}^n w_i\varphi_2(b_i)} \tag{4.18}$$

provided that $\psi'(y)\varphi_2''(y) - \varphi_2'(y)\psi''(y) \neq 0$ for every $y \in [m_{\boldsymbol{a},\boldsymbol{b}}, M_{\boldsymbol{a},\boldsymbol{b}}]$.

Proof. Define the functional $\Theta : C^2([m_{\boldsymbol{a},\boldsymbol{b}}, M_{\boldsymbol{a},\boldsymbol{b}}]) \to \mathbb{R}$ with:

$$\Theta(\varphi) = \sum_{i=1}^n w_i\varphi(a_i) - \sum_{i=1}^n w_i\varphi(b_i)$$

and set $\varphi_0 = \Theta(\varphi_2)\varphi_1 - \Theta(\varphi_1)\varphi_2$. Obviously $\Theta(\varphi_0) = 0$. Using Theorem 4.7, there exists $\xi \in [m_{\boldsymbol{a},\boldsymbol{b}}, M_{\boldsymbol{a},\boldsymbol{b}}]$ such that

$$\Theta(\varphi_0) = \frac{\psi'(\xi)\varphi_0''(\xi) - \varphi_0'(\xi)\psi''(\xi)}{2(\psi'(\xi))^3}\left[\sum_{i=1}^n w_i\psi^2(a_i) - \sum_{i=1}^n w_i\psi^2(b_i)\right]. \tag{4.19}$$

We give a proof that the expression in square brackets in (4.19) is non-zero due to $\boldsymbol{a} \neq \boldsymbol{b}$. Suppose that the expression in square brackets in (4.19) is equal to zero, i.e.,

$$0 = \sum_{i=1}^n w_i\psi^2(a_i) - \sum_{i=1}^n w_i\psi^2(b_i). \tag{4.20}$$

By using (4.20), (2.15) and (2.16), we have

$$\begin{aligned} 0 &= \sum_{i=1}^n w_i\psi^2(a_i) - \sum_{i=1}^n w_i\psi^2(b_i) \\ &\geq \sum_{i=1}^n w_i(2\psi(b_i))[\psi(a_i) - \psi(b_i)] \geq 0. \end{aligned}$$

This implies

$$\sum_{i=1}^n w_i\psi^2(a_i) - \sum_{i=1}^n w_i\psi^2(b_i) = \sum_{i=1}^n w_i(2\psi(b_i))[\psi(a_i) - \psi(b_i)]$$

or equivalently

$$\sum_{i=1}^n w_i(\psi(a_i) - \psi(b_i))^2 = 0.$$

Which obviously implies that $\boldsymbol{a} \neq \boldsymbol{b}$.
Since $\boldsymbol{a} \neq \boldsymbol{b}$, the expression in square brackets in (4.19) is non-zero which implies that $\psi'(\xi)\varphi_0''(\xi) - \varphi_0'(\xi)\psi''(\xi) = 0$, and this gives (4.18). Notice that Theorem 4.7 for $\varphi = \varphi_2$ implies that the denominator of the right-hand side of (4.18) is non-zero. □

Corollary 4.9. *Let* $\boldsymbol{w}$*,* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *be positive n-tuples such that conditions* (2.15) *and* (2.16) *are satisfied. Also let* $\boldsymbol{b}$ *be a decreasing n-tuple, then for distinct* $s,t,q \in \mathbb{R}\setminus\{0\}$*, there exists* $\xi \in [m_{\boldsymbol{a},\boldsymbol{b}}, M_{\boldsymbol{a},\boldsymbol{b}}]$ *such that*

$$\xi^{t-s} = \frac{s(s-q)}{t(t-q)} \frac{\sum_{i=1}^{n} w_i a_i^t - \sum_{i=1}^{n} w_i b_i^t}{\sum_{i=1}^{n} w_i a_i^s - \sum_{i=1}^{n} w_i b_i^s}. \tag{4.21}$$

Proof. Set $\varphi_1(x) = x^t$, $\varphi_2(x) = x^s$ and $\psi(x) = x^q$, $t \neq s \neq 0, q$ in (4.18), then we get (4.21). □

Remark 4.10. Since the function $\xi \to \xi^{t-s}$ is invertible, then from (4.21) we have

$$m_{\boldsymbol{a},\boldsymbol{b}} \leq \left(\frac{s(s-q)}{t(t-q)} \frac{\sum_{i=1}^{n} w_i a_i^t - \sum_{i=1}^{n} w_i b_i^t}{\sum_{i=1}^{n} w_i a_i^s - \sum_{i=1}^{n} w_i b_i^s}\right)^{\frac{1}{t-s}} \leq M_{\boldsymbol{a},\boldsymbol{b}}. \tag{4.22}$$

In fact, similar result can also be given for (4.18). Namely, suppose that $\Lambda(y) = \left(\psi'(y)\varphi_1''(y) - \varphi_1'(y)\psi''(y)\right) / \left(\psi'(y)\varphi_2''(y) - \varphi_2'(y)\psi''(y)\right)$ has inverse function. Then from (4.18), we have

$$\xi = \Lambda^{-1}\left(\frac{\sum_{i=1}^{n} w_i \varphi_1(a_i) - \sum_{i=1}^{n} w_i \varphi_1(b_i)}{\sum_{i=1}^{n} w_i \varphi_2(a_i) - \sum_{i=1}^{n} w_i \varphi_2(b_i)}\right). \tag{4.23}$$

By the inequality (4.22), we can consider

$$N_{t,s} = \left(\frac{\Gamma_t}{\Gamma_s}\right)^{\frac{1}{t-s}}, \text{ for } s,t \in \mathbb{R}\setminus\{0\},\ s \neq t, \tag{4.24}$$

as means in broader sense. Moreover we can extend these means in other cases. So by passing limit, we have

$$\log N_{s,s} = \frac{\sum_{i=1}^{n} w_i a_i^s \log a_i - \sum_{i=1}^{n} w_i b_i^s \log b_i}{\sum_{i=1}^{n} w_i a_i^s - \sum_{i=1}^{n} w_i b_i^s} - \frac{2s-q}{s(s-q)}, \quad s \neq 0, q.$$

$$\log N_{q,q} = \frac{\sum_{i=1}^{n} w_i a_i^q \log^2 a_i - \sum_{i=1}^{n} w_i b_i^q \log^2 b_i}{2\left[\sum_{i=1}^{n} w_i a_i^q \log a_i - \sum_{i=1}^{n} w_i b_i^q \log b_i\right]} - \frac{1}{q}.$$

$$\log N_{0,0} = \frac{\sum_{i=1}^{n} w_i \log^2 a_i - \sum_{i=1}^{n} w_i \log^2 b_i}{2\left[\sum_{i=1}^{n} w_i \log a_i - \sum_{i=1}^{n} w_i \log b_i\right]} + \frac{1}{q}.$$

Theorem 4.11. *Let* $t \leq u$, $r \leq s$, *then the following inequality is valid*

$$N_{t,r} \leq N_{u,s}. \tag{4.25}$$

Proof. Since Ω_s is log-convex, therefore by (3.4) we get (4.25). □

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