Communications in **M**athematical **A**nalysis
Special Volume in Honor of Prof. Stephen Smale
Volume 10, Number 2, pp. 97–117 (2011)
ISSN 1938-9787
www.math-res-pub.org/cma

Multivalued Integral Manifolds in Banach Spaces

Michal Fečkan *
Department of Mathematical Analysis and Numerical Mathematics
Comenius University in Bratislava
Bratislava, 842 48, Mlynská dolina, Slovakia
and
Mathematical Institute
Slovak Academy of Sciences
Bratislava, 814 73, Štefánikova 49, Slovakia

Sándor Kelemen †
Mathematical Institute
Slovak Academy of Sciences
Bratislava, 814 73, Štefánikova 49, Slovakia

(Communicated by Toka Diagana)

Abstract

We consider a differential inclusion

$$\dot{x} \in A(t)x + f(t,x) + g(t,x,X_1)$$

in an arbitrary Banach space X with a general exponential dichotomy, where X_1 is the closed unit ball of X. The right-hand side is strongly measurable in the time variable and Lipschitz continuous in the others. We prove the existence and uniqueness of quasibounded solutions corresponding to suitable selectors. The stable and unstable sets of these quasibounded solutions are characterised as graphs of certain multifunctions. Exponential dichotomy criteria are also presented.

AMS Subject Classification: 34E10, 34G20, 37D10

Keywords: integral manifolds, differential inclusions, exponential dichotomy, quasibounded functions, principle of uniform contraction

1 Introduction

It is well-known that invariant manifolds like stable and unstable ones play an important role in understanding saddle dynamics for smooth nonlinear dynamical systems (DS)

*E-mail address: Michal.Feckan@fmph.uniba.sk
†E-mail address: kuglof@gmail.com

[1, 13, 14, 20–22]. To the best of our knowledge, instead, there are only few papers dealing with saddle dynamics for non-smooth or even multivalued DS. The simplest example of multivalued DS is the *inflated dynamics*, which was introduced in [15] and it was used in a fairly large number of papers since then, for details, see [12]. This paper is a continuation of [5], and we refer the reader for more results and a discussion on multivalued hyperbolical dynamics to that paper. Like in [5], our multivalued DS takes a special form of a *parametrized*, i.e., *controlled* form with Lipschitzian nonlinearities/multifunctions. In view of a parameterization result by Ornelas [16], this is not a loss of generality in finite dimensional cases and with convex valued Lipschitzian multifunctions. However, in the general case such a parameterization does not exist, see the Appendix of [5] for a short discussion of the parameterization problem for multifunctions.

Hence we consider parametrized Lipschitzian and Carathéodorian semilinear differential inclusions in Banach spaces with exponentially dichotomous linear parts. Under additional assumptions, we prove the existence and uniqueness of quasibounded sets of those differential inclusions. Then stable and unstable sets of these quasibounded sets are shown to be graphs of suitable multifunctions. We also introduce and study more general weighted quasibounded sets and discuss their hierarchy like in [1]. The paper is concluded with presenting some criteria on the existence of exponential dichotomy.

2 Preliminaries

2.1 Measure theory

Throughout the whole paper (unless otherwise is stated) we suppose that X is a real Banach space. We say that an interval $I \subset \mathbb{R}$ of arbitrary type is *positive* if its (Lebesgue) measure is positive (the case ∞ is also involved). For this subsection assume that $I \subset \mathbb{R}$ is a nonempty interval.

The function $f : I \to X$ is *strongly measurable* (according to [1]) if the range $f(I)$ is separable and f is measurable (f is measurable if the preimage $f^{-1}(B)$ is a Borel set for arbitrary Borel set $B \subset X$). The fundamental fact about *s.m.* functions (means strongly measurable) is that: if $f : I \to X$ is s.m. then there is a sequence $\{f_n\}_{n=1}^{\infty}$ of *simple* functions which converges pointwise to f and satisfy the estimate $\|f_n(t)\| \le \|f(t)\|$ for *all* $t \in I$. The function is *simple* if it has only finitely many values and is (strongly) measurable. It is possible to reverse this fact in some sense: the function which is a pointwise limit of s.m. functions is also a s.m. function. These facts together we call as *the approximation property* of s.m. functions, for the proof see Appendix E of [4].

A function $f : I \times X \to X$ has a *Carathéodory property* if at one hand $f(t, \cdot) : X \to X$ is continuous for all fixed $t \in I$ and on the other hand $f(\cdot, x) : I \to X$ is s.m. for all fixed $x \in X$. We denote the set of these function by $\mathcal{CAR}(I, X)$. The following consequence of the approximation property will be useful for us (the proof is the same as the proof of Lemma 2.2 in [1] where continuous μ is investigated).

Lemma 2.1. *Suppose that* $f \in \mathcal{CAR}(I, X)$ *and* $\mu : I \to X$ *is a s.m. function, then the "partially composed" mapping* $g : I \to X$ *defined as* $g(t) := f(t, \mu(t))$ *is also a s.m. function.*

From measure theory we need also the theory of Bochner integrals, which can be found in many books on measures and integrals (cf. [4]). The brief definition by the help of Lebesgue integrals is a following one: a s.m. function $f : I \to X$ is (Bochner) *integrable* if the norm function $\|f\| : I \to \mathbb{R}$ defined as $\|f\|(t) := \|f(t)\|$ is Lebesgue integrable. The function f is called *locally integrable* if it is integrable on every compact subinterval of I. The definition of the integral for integrable simple function is straightforward. For an arbitrary integrable function we use the approximation property to get simple integrable functions f_n such that $f = \lim_{n\to\infty} f_n$, $\|f_n(t)\| \le \|f(t)\|$ and then the well known Lebesgue's Dominated Convergence Theorem for real-valued functions implies the well-definitness of $\int_I f \mathrm{d}t := \lim_{n\to\infty} \int_I f_n \mathrm{d}t$.

2.2 The Uniform Contraction Principle

One of the most often used tools in the theory of differential equations is the Uniform Contraction Principle. We do not formulate it in the most general version, but only the case what we use. The proof is easy and can be found in almost every textbook on functional analysis (cf. [9]).

Theorem 2.2. *Assume that X is a Banach space, $\mathcal{P}$ is a nonempty metric space with a metric d and $f : X \times \mathcal{P} \to X$ is a uniform contraction (there exist an $\alpha \in [0;1)$ such that $\|f(x,p) - f(\tilde{x},p)\| \le \alpha\|x - \tilde{x}\|$ holds for all $x,\tilde{x} \in X, p \in \mathcal{P}$). Then for all $p \in \mathcal{P}$ there exists a unique fixed point $x_{\text{fix}} = x_{\text{fix}}(p)$ of the function $f(\cdot,p) : X \to X$. In addition $x_{\text{fix}} : \mathcal{P} \to X$ is continuous if $f(x,\cdot) : \mathcal{P} \to X$ is continuous for all $x \in X$ and the Lipschitz property $\|f(x,p) - f(x,\tilde{p})\| \le Ld(p,\tilde{p})$ for all $x \in X, p,\tilde{p} \in \mathcal{P}$ implies*

$$\|x_{\text{fix}}(p) - x_{\text{fix}}(\tilde{p})\| \le \frac{L}{1-\alpha} d(p,\tilde{p}).$$

2.3 Solution concept for ordinary differential equation

We adopt the following quite general definition (from [1]) for the solutions of ordinary differential equations. Suppose for this subsection that I is a positive interval and $\mathcal{P}$ is a topological space.

Definition 2.3. Assume that J is a positive subinterval of I and $f : I \times X \times \mathcal{P} \to X$ is such that $f(\cdot,\cdot,p) \in \mathcal{CAR}(I,X)$ for all $p \in \mathcal{P}$. A continuous function $\lambda : J \to X$ is a *solution* of the ordinary differential equation $\dot{x} = f(t,x,p)$ at the parameter value $p \in \mathcal{P}$ if the function $f(\cdot,\lambda(\cdot),p) : J \to X$ is locally integrable and

$$\lambda(s) - \lambda(t) = \int_t^s f(\tau,\lambda(\tau),p)\mathrm{d}\tau$$

holds for all $t,s \in J$. In addition λ satisfies the *initial condition* $x(t_0) = x_0$ for some $t_0 \in I, x_0 \in X$ if $t_0 \in J$ and $\lambda(t_0) = x_0$.

Let us recall a following fundamental theorem (Theorem 2.4 from [1]) about existence, uniqueness and continuous dependence of the above defined solution type.

Theorem 2.4 (Theorem 2.4 in [1]). *Suppose that $f : I \times X \times \mathcal{P} \to X$ is such that $f(\cdot,\cdot,p) \in$ $\mathrm{Car}(I,X)$ for all $p \in \mathcal{P}$. Assume also with locally integrable functions $l, l_0 : I \to \mathbb{R}_0^+$ the following conditions*

$$\begin{aligned} \|f(t,x,p) - f(t,\tilde{x},p)\| &\leq l(t)\|x-\tilde{x}\|, \\ \|f(t,0,p)\| &\leq l_0(t), \end{aligned}$$

for almost all $t \in I$, for all $x, \tilde{x} \in X$ and $p \in \mathcal{P}$. Finally, suppose that $f(t,x,\cdot) : \mathcal{P} \to X$ is continuous for all $(t,x) \in I \times X$. Then the initial value problem

$$\dot{x} = f(t,x,p), \quad x(t_0) = x_0$$

has a unique solution $\lambda(\cdot;t_0,x_0,p) : I \to X$ for all $(t_0,x_0,p) \in I \times X \times \mathcal{P}$. In addition the so-defined mapping $\lambda : I \times I \times X \times \mathcal{P} \to X$ is continuous.

3 Differential inclusions

Now we begin to deal with differential inclusions (or *inflated differential equations*). For this we have to introduce some notations. For the arbitrary Banach space $\mathcal{V}$ denote by $\mathcal{V}_1$ the closed unit ball $\{v \in \mathcal{V} : \|v\|_{\mathcal{V}} \leq 1\}$. A new space of function - the selector space - will be

$$\mathbb{S} := \{h : \mathbb{R} \to X \,:\, h \text{ is strongly measureable and } \|h\|_\infty < \infty\}$$

where $\|h\|_\infty = \sup_{t\in\mathbb{R}} \|h(t)\|$. It is easy to prove, that $\mathbb{S}$ is a Banach space with the norm $\|\cdot\|_\infty$. As usual $L(X)$ is the Banach space of bounded linear operators from X into itself with an operator norm $\|T\|_o := \sup_{x\in X_1} \|Tx\|$ for $T \in L(X)$.

Our goal is to study the following ordinary differential inclusion

$$\dot{x} \in F(t,x,X_1) \tag{3.1}$$

where $F : I \times X \times X_1 \to X$ is an arbitrary function, I is a positive interval and

$$F(t,x,X_1) = \{F(t,x,u) \,:\, u \in X_1\}.$$

Definition 3.1. Assume that $J \subset I$ for positive intervals J, I. We say that the continuous function $\lambda : J \to X$ is a *solution* of the differential inclusion (3.1) corresponding to the *selector* $h \in \mathbb{S}_1$ if λ is a solution (in the sense of Definition 2.3) of the ordinary differential equation $\dot{x} = F(t,x,h(t))$. In addition λ satisfies the *initial condition* $x(t_0) = x_0$ for $t_0 \in I, x_0 \in X$ if we have $t_0 \in J$ and $\lambda(t_0) = x_0$.

Now we are able to state a theorem which can be derived from Theorem 2.4. However we give a short proof (without technical details) which follows the lines of the proof of Theorem 2.4.

Theorem 3.2. *Assume that I is a positive interval and the right-hand side function F : $I \times X \times X_1 \to X$ satisfies the following three requirements:*

(i) $F(\cdot,x,u) : I \to X$ *is s.m. for all fixed* $(x,u) \in X \times X_1$,

(ii) $F(t,\cdot,\cdot) : X \times X_1 \to X$ *is continuous for all* $t \in I$,

(iii) *there are locally integrable functions* $l_0, l_1, l_2 : I \to [0;\infty)$ *such that*

$$\begin{aligned} \|F(t,x,u) - F(t,\tilde{x},\tilde{u})\| &\leq l_1(t)\|x-\tilde{x}\| + l_2(t)\|u-\tilde{u}\|, \\ \|F(t,0,0)\| &\leq l_0(t), \end{aligned}$$

for a.e. $t \in I$ *and for all* $x, \tilde{x} \in X, u, \tilde{u} \in X_1$.

Under these conditions for every triple $(t_0, x_0, h) \in I \times X \times \mathbb{S}_1$ *there exists a unique solution* $\lambda(\cdot) = \lambda(\cdot; t_0, x_0, h) : I \to X$ *of the initial value problem*

$$\dot{x} = F(t,x,h(t)), \qquad x(t_0) = x_0.$$

In other words, for every $(t_0, x_0) \in I \times X$ *the inclusion initial value problem*

$$\dot{x} \in F(t,x,X_1), \qquad x(t_0) = x_0$$

has a unique solution $\lambda(\cdot) = \lambda(\cdot; t_0, x_0, h)$ *corresponding to the arbitrarily chosen selector* $h \in \mathbb{S}_1$.

In addition the mapping $\lambda : I \times I \times X \times \mathbb{S}_1 \to X$ *is continuous.*

Proof. Without loss of generality we may restrict our attention to the case $I = [a;b]$ for arbitrary but fixed $a < b, a, b \in \mathbb{R}$ (simple reason is that every positive interval I can be written in the form $I = \bigcup_{j\in\mathbb{N}} [a_j, b_j]$ where $a_{j+1} \leq a_j < b_j \leq b_{j+1}$ for $j \in \mathbb{N}$ and $a_j, b_j \in \mathbb{R}$).

Denote by $C(I,X)$ the Banach space of continuous functions $x : I \to X$ with a norm $\|\cdot\|_\infty$. Define $\mathcal{T}$ for $x \in C(I,X)$ and $t_0, t \in I, x_0 \in X, h \in \mathbb{S}_1$ as follows

$$\mathcal{T}(x; t_0, x_0, h)(t) := x_0 + \int_{t_0}^{t} F(s, x(s), h(s))\mathrm{d}s.$$

It can be shown (as in the proof of Theorem 4.2 of the paper [1]) that this operator viewed as

$$\mathcal{T} : C(I,X) \times I \times X \times \mathbb{S}_1 \to C(I,X)$$

is well-defined and continuous. Moreover for $n \in \mathbb{N}$ sufficiently large the iterated mapping $\mathcal{T}^n$ (defined as $\mathcal{T}^k(x; t_0, x_0, h) := \mathcal{T}(\mathcal{T}^{k-1}(x; t_0, x_0, h); t_0, x_0, h)$ for $k \geq 2$) is a uniform contraction at the first variable. Therefore Theorem 2.2 can be used to get a unique fixed point of $\mathcal{T}^n$ which depends continuously on the "parameters" t_0, x_0, h. As a straightforward corollary we obtain the statement of our theorem. □

4 Inclusions with exponential dichotomy

Suppose that $A : \mathbb{R} \to L(X)$ is a locally integrable function. In the paper [1] there is a detailed proof of the existence of the evolution operator $\Phi : \mathbb{R} \times \mathbb{R} \to L(X)$ to the linear differential equation $\dot{x} = A(t)x$. This is defined as $\Phi(t,s) := B(t)$ where $B : \mathbb{R} \to L(X)$ is the unique solution of the operator differential equation $\dot{B} = A(t)B$ with initial condition $B(s) = id_X$. In [1] there are also proved some important properties of this operator, we collect them to the following lemma.

Lemma 4.1. *For all $t,s \in \mathbb{R}$ we have $\Phi(t,s) \in GL(X)$ which is the group of invertible operators in $L(X)$. Moreover $\Phi : \mathbb{R}\times\mathbb{R} \to L(X)$ is continuous, $\|\Phi(t,s)\|_o \le e^{|\int_s^t \|A(\tau)\| d\tau|}$ for all $s,t \in \mathbb{R}$ and the following* cocycle property *is valid*

$$\Phi(t,r)\circ\Phi(r,s) = \Phi(t,s), \quad t,r,s \in \mathbb{R}.$$

In addition for any locally integrable function $f : \mathbb{R} \to X$ there exists a unique solution of the inhomogeneous linear differential equation $\dot{x} = A(t)x + f(t)$ with initial condition $x(t_0) = x_0 \in X, t_0 \in \mathbb{R}$. This is given by the variation of constants *formula*

$$x(t) = \Phi(t,t_0)x_0 + \int_{t_0}^{t} \Phi(t,s)f(s)\mathrm{d}s.$$

Sometimes it is convenient to use also $\tilde{\Phi} : \mathbb{R} \to X$ which is the solution of the operator differential equation $\dot{x} = A(t)x$ with initial condition $x(0) = id_X$. The relation between Φ and $\tilde{\Phi}$ is

$$\tilde{\Phi}(t) = \Phi(t,0), \qquad \Phi(t,s) = \tilde{\Phi}(t)\circ(\tilde{\Phi}(s))^{-1}.$$

Definition 4.2. We say that A posses an exponential dichotomy on $\mathbb{R}$ if there are constants

$$K \ge 1, \qquad \alpha,\beta \in \mathbb{R}, \alpha < \beta$$

and a projection $P \in L(X)$ (projection means $P^2 = P$) such that

$$\|\tilde{\Phi}(t)\circ P^+ \circ (\tilde{\Phi}(s))^{-1}\|_o \le K\mathrm{e}^{\alpha(t-s)}, t \ge s, \tag{4.1}$$
$$\|\tilde{\Phi}(t)\circ P^- \circ (\tilde{\Phi}(s))^{-1}\|_o \le K\mathrm{e}^{\beta(t-s)}, t \le s, \tag{4.2}$$

where $P^+ := P$ and $P^- := id_X - P^+$. We denote by $\mathcal{ED}_{\alpha,\beta}(X)$ the set of all locally integrable $A : \mathbb{R} \to L(X)$ which posses an exponential dichotomy on $\mathbb{R}$ with $\alpha < \beta$ (we also use shorter notation $\mathcal{ED}_{\alpha,\beta}$ when it is clear which Banach space is investigated; note that $A \in \mathcal{ED}_{\alpha,\beta}$ implies also the existence of the constant K and projection P although for simplicity the notation does not includes these parameters). Furthermore we introduce for $t \in \mathbb{R}$ notations

$$P^{\pm}(t) := \tilde{\Phi}(t)\circ P^{\pm}\circ(\tilde{\Phi}(t))^{-1}, \quad \mathbb{P}_t^{\pm} := P^{\pm}(t)(X).$$

Note that $P^{\pm}(t)$ are projections and $P^+(t) + P^-(t) = I$. It is easy to establish the following quasi-commutation

$$P^{\pm}(t)\circ\Phi(t,s) = \Phi(t,s)\circ P^{\pm}(s), \quad s,t \in \mathbb{R}.$$

These notions are well-known, for more details see the book [6].

4.1 Bounded solutions revisited

For this subsection assume that

$$\alpha < 0 < \beta \text{ and } A \in \mathcal{ED}_{\alpha,\beta}. \tag{4.3}$$

This can be interpreted as the generalization of the hyperbolic matrix. The set of bounded solutions and integral manifolds corresponding to them of the differential inclusion

$$\dot{x} \in A(t)x + f(t,x) + g(t,x,X_1) \tag{4.4}$$

was analysed in [5] (cf. section 4; mainly Theorems 3 and 4). We briefly recall these results. For the sake of completness we also mention the main tools of the proofs, again without technical details.

Lemma 4.3. *Assume that $f:\mathbb{R}\to X$ is a s.m. function and M is a constant such that $\|f(t)\|\le M$ for a.e. $t\in\mathbb{R}$. Then there is a unique bounded solution $y:\mathbb{R}\to X$ of the inhomogeneous differential equation $\dot{x}=A(t)x+f(t)$ and it is given by the formula*

$$y(t)=\int_{-\infty}^{t}\Phi(t,s)P^{+}(s)f(s)\mathrm{d}s-\int_{t}^{\infty}\Phi(t,s)P^{-}(s)f(s)\mathrm{d}s. \tag{4.5}$$

In addition $\|y\|_\infty\le MK(\frac{1}{\beta}-\frac{1}{\alpha})$.

Proof. At first we show the uniqueness part of the statement. If $\mu_{1,2}$ are two bounded solution, then $u:=\mu_1-\mu_2$ is a bounded solution of $\dot{u}=A(t)u$. Therefore

$$\|P^{+}(t)u(t)\|=\|\tilde{\Phi}(t)P^{+}(\tilde{\Phi}(s))^{-1}u(s)\|\le K\,\mathrm{e}^{\alpha(t-s)}\|u\|_\infty$$

is valid for all $t,s\in\mathbb{R}, s\le t$. If s tends to $-\infty$ we obtain $P^{+}(t)u(t)=0$ for all $t\in\mathbb{R}$. Similarly we get $P^{-}(t)u(t)=0$. These facts together yields $\mu_1=\mu_2$.

On the other hand y defined by formula (4.5) really gives the unique bounded solution (the omitted technicalities can be done in the following way: the right definitness of y follows from considerations as in Lemma 2.1; to prove the continuity of y we have to use Lebesgue's theorem about dominated convergence; an application of Fubini's theorem about double integrals gives after some computation that y is a solution; the boundedness of this solution is straightforward from the definition of exponential dichotomy on $\mathbb{R}$). □

Now we state the nonlinear analogues of this lemma belonging to inclusion (4.4). Let we introduce the notation

$$\kappa_{\alpha,\beta}:=\frac{1}{\beta}-\frac{1}{\alpha}.$$

Theorem 4.4 (Section 4, Theorem 3 in [5])**.** *Suppose that for a functions*

$$f:\mathbb{R}\times X\to X,\quad g:\mathbb{R}\times X\times X_1\to X$$

the following is valid:

(i) Smoothness: *$f(\cdot,x),g(\cdot,x,u):\mathbb{R}\to X$ are s.m. for all $x\in X,u\in X_1$ and $f(t,\cdot):X\to X,g(t,\cdot,\cdot):X\times X_1\to X$ are continuous for all $t\in\mathbb{R}$,*

(ii) Boundedness: *there are constants $M_1,M_2\ge 0$ such that*

$$\|f(t,0)\|\le M_1 \text{ and } \|g(t,0,0)\|\le M_2$$

for a.e. $t\in\mathbb{R}$,

(iii) Lipschitz condition: *there are constants L_1,L_2,L_3 such that*

$$\|f(t,x)-f(t,\tilde{x})\|\le L_1\|x-\tilde{x}\| \text{ and } \|g(t,x,u)-g(t,\tilde{x},\tilde{u})\|\le L_2\|x-\tilde{x}\|+L_3\|u-\tilde{u}\|$$

is valid for a.e. $t\in\mathbb{R}$ and for all $x,\tilde{x}\in X,u,\tilde{u}\in X_1$.

Finally suppose that

(iv) $K(L_1+L_2)\kappa_{\alpha,\beta}<1$.

Then for every $h\in\mathbb{S}_1$ *there exists a unique bounded solution* $\Gamma(\cdot,h):\mathbb{R}\to X$ *of the problem* (4.4) *corresponding to the selector h. In addition the mapping* $\Gamma:\mathbb{R}\times\mathbb{S}_1\to X$ *is continuous with an estimate*

$$\|\Gamma(\cdot,h)-\Gamma(\cdot,\tilde{h})\|_\infty\le C(K,L_1,L_2,L_3,\alpha,\beta)\|h-\tilde{h}\|_\infty$$

where

$$C(K,L_1,L_2,L_3,\alpha,\beta):=\frac{KL_3\kappa_{\alpha,\beta}}{1-K(L_1+L_2)\kappa_{\alpha,\beta}}. \tag{4.6}$$

Proof. For $x\in C(\mathbb{R},X),h\in\mathbb{S}_1$ and $t\in\mathbb{R}$ we set

$$\begin{aligned}\mathcal{T}(x,h)(t) \quad := \quad & \int_{-\infty}^t\Phi(t,s)P^+(s)[f(s,x(s))+g(s,x(s),h(s))]\mathrm{d}s\\ & -\int_t^\infty\Phi(t,s)P^-(s)[f(s,x(s))+g(s,x(s),h(s))]\mathrm{d}s.\end{aligned} \tag{4.7}$$

The form of this operator is motivated from the previous lemma. Application of Theorem 2.2 gives our assertation, where $\Gamma(\cdot,h)$ is the unique fixed point of $\mathcal{T}(\cdot,h)$. □

For the later use we introduce another sets of selectors

$$\mathbb{S}_\epsilon:=\{h\in\mathbb{S}\,:\,\|h\|_\infty\le\epsilon\},$$

$$\mathbb{S}^+_{\tau,\epsilon}:=\{h:\mathbb{R}^+_\tau\to X\text{ is s.m. and }\|h\|^+_\tau:=\sup_{t\ge\tau}\|h(t)\|\le\epsilon\},$$

$$\mathbb{S}^-_{\tau,\epsilon}:=\{h:\mathbb{R}^-_\tau\to X\text{ is s.m. and }\|h\|^-_\tau:=\sup_{t\le\tau}\|h(t)\|\le\epsilon\},$$

where $\tau\in\mathbb{R},\epsilon\in[0,1]$ and $\mathbb{R}^+_\tau:=[\tau,\infty),\mathbb{R}^-_\tau:=(-\infty,\tau]$. Note that $\mathbb{S}_\epsilon,\mathbb{S}^+_{\tau,\epsilon},\mathbb{S}^-_{\tau,\epsilon}$ are complete metric spaces with corresponding metrics induced by $\|\cdot\|_\infty,\|\cdot\|^+_\tau,\|\cdot\|^-_\tau$.

In the situation of Theorem 4.4 we put down for $\epsilon\in[0,1]$ and $\tau\in\mathbb{R}$ an important set of initial positions of the bounded solutions

$$\mathbb{IP}_{\tau,\epsilon}:=\{\Gamma(\tau,h)\,:\,h\in\mathbb{S}_\epsilon\}.$$

Note that an application of Theorem 3.2 gives the existence, uniqueness and continuous dependence of the solution $\lambda(\cdot,t_0,x_0,h)$ of the problem

$$\dot{x}=A(t)x+f(t,x)+g(t,x,h(t)),\quad x(t_0)=x_0$$

for every triple $(t_0,x_0,h)\in\mathbb{R}\times X\times\mathbb{S}_1$.

Now we define the stable set of $\mathbb{IP}_{\tau,\epsilon}$ ("stable positions") as

$$\mathbb{SP}_{\tau,\epsilon}:=\{\xi\in X\,:\,\exists h\in\mathbb{S}_\epsilon\text{ such that }\lim_{t\to\infty}\|\lambda(t,\tau,\xi,h)-\Gamma(t,h)\|=0\}$$

and similarly an unstable set ("unstable positions") as

$$\mathbb{UP}_{\tau,\epsilon}:=\{\xi\in X\,:\,\exists h\in\mathbb{S}_\epsilon\text{ such that }\lim_{t\to-\infty}\|\lambda(t,\tau,\xi,h)-\Gamma(t,h)\|=0\}.$$

Generally for functions $k : \mathbb{R}^+_\tau \to X, l : \mathbb{R}^-_\tau \to X$ we introduce extensions $k^+, l^- : \mathbb{R} \to X$ as

$$k^+(t) := \begin{cases} k(t) & \text{if } t \geq \tau, \\ 0 & \text{if } t < \tau, \end{cases} \qquad l^-(t) := \begin{cases} 0 & \text{if } t > \tau, \\ l(t) & \text{if } t \leq \tau. \end{cases}$$

Introduce

$$\overline{\mathbb{SP}}_{\tau,\epsilon} := \{\xi \in X : \exists h \in \mathbb{S}^+_{\tau,\epsilon} \text{ such that } \|\lambda(\cdot,\tau,\xi,h^+)\|^+_\tau < \infty\}$$

and

$$\overline{\mathbb{UP}}_{\tau,\epsilon} := \{\xi \in X : \exists h \in \mathbb{S}^-_{\tau,\epsilon} \text{ such that } \|\lambda(\cdot,\tau,\xi,h^-)\|^-_\tau < \infty\}.$$

Theorem 4.5 (Section 4, Theorem 4 in [5])**.** *Suppose that all the assumptions of Theorem 4.4 are satisfied and choose $\tau \in \mathbb{R}, \epsilon \in [0,1]$. Then there are Lipschitz continuous functions*

$$w^s : \mathbb{P}^+_\tau \times \mathbb{S}^+_{\tau,\epsilon} \to \mathbb{P}^-_\tau, \quad w^u : \mathbb{P}^-_\tau \times \mathbb{S}^-_{\tau,\epsilon} \to \mathbb{P}^+_\tau$$

such that

$$\begin{array}{rcccl} \mathbb{SP}_{\tau,\epsilon} & = & \overline{\mathbb{SP}}_{\tau,\epsilon} & = & \{\xi^+ + w^s(\xi^+,h) : \xi^+ \in \mathbb{P}^+_\tau, h \in \mathbb{S}^+_{\tau,\epsilon}\}, \\ \mathbb{UP}_{\tau,\epsilon} & = & \overline{\mathbb{UP}}_{\tau,\epsilon} & = & \{\xi^- + w^u(\xi^-,h) : \xi^- \in \mathbb{P}^-_\tau, h \in \mathbb{S}^-_{\tau,\epsilon}\}. \end{array} \tag{4.8}$$

Exact Lipschitz constants are expressed in the formulaes

$$\|w^s(\xi^+_1,h_1) - w^s(\xi^+_2,h_2)\| \leq \frac{K}{1-K(L_1+L_2)\kappa_{\alpha,\beta}}\|\xi^+_1 - \xi^+_2\| + C(K,L_1,L_2,L_3,\alpha,\beta)\|h_1 - h_2\|^+_\tau,$$

$$\|w^u(\xi^-_1,h_1) - w^u(\xi^-_2,h_2)\| \leq \frac{K}{1-K(L_1+L_2)\kappa_{\alpha,\beta}}\|\xi^-_1 - \xi^-_2\| + C(K,L_1,L_2,L_3,\alpha,\beta)\|h_1 - h_2\|^-_\tau.$$

Proof. We deal with the stable case - the unstable one is analogical. At first we show the characterisation (4.8) for $\overline{\mathbb{SP}}_{\tau,\epsilon}$. Arguments as in Lemma 4.3 yields that the "right" operator is

$$\mathcal{T}(x,\xi^+,h)(t) := \Phi(t,\tau)\xi^+ + \int_\tau^t \Phi(t,s)P^+(s)\Lambda_{x,h}(s)\mathrm{d}s - \int_t^\infty \Phi(t,s)P^-(s)\Lambda_{x,h}(s)\mathrm{d}s$$

viewed as $\mathcal{T} : C^+_\tau \times \mathbb{P}^+_\tau \times \mathbb{S}^+_{\tau,\epsilon} \to C^+_\tau$, where $\Lambda_{x,h}(s) := f(s,x(s)) + g(s,x(s),h(s))$ and C^+_τ is the Banach space of bounded and continuous functions from $\mathbb{R}^+_\tau$ to X supplied with a norm $\|\cdot\|^+_\tau$ (of course this is not obvious at all, for more details see the series of Lemmas 3.2 – 3.7 of the paper [1]). The operator $\mathcal{T}$ is "right" in the sense that $\mu \in C^+_\tau$ is a *bounded* solution of the problem $\dot{x} = A(t)x + f(t,x) + g(t,x,h(t))$ if and only if $\mathcal{T}(\mu, P^+(\tau)\mu(\tau), h) = \mu$.

Now a standard application of Theorem 2.2 shows the existence of the unique fixed point $x_{\xi^+,h}$ of $\mathcal{T}(\cdot,\xi^+,h)$ and also the corresponding Lipschitz bounds. The statement of (4.8) for $\overline{\mathbb{SP}}_{\tau,\epsilon}$ then follows if we set $w^s(\xi^+,h) := x_{\xi^+,h}(\tau)$.

The relation $\mathbb{SP}_{\tau,\epsilon} \subset \overline{\mathbb{SP}}_{\tau,\epsilon}$ is trivial. For $\overline{\mathbb{SP}}_{\tau,\epsilon} \subset \mathbb{SP}_{\tau,\epsilon}$ we use again Theorem 2.2. Let $\xi \in \overline{\mathbb{SP}}_{\tau,\epsilon}$. Therefore there is a $h \in \mathbb{S}^+_{\tau,\epsilon}$ such that $\|\lambda(\cdot,\tau,\xi,h^+)\|^+_\tau < \infty$. Introduce

$$u : \mathbb{R}^+_\tau \to X, \qquad u(t) = \lambda(t,\tau,\xi,h^+) - \Gamma(t,h^+).$$

Then u is a solution of $\dot{x} = A(t)x + F(t,x)$ on $\mathbb{R}^+_\tau$ where

$$F(t,x) := f(t,x+\Gamma(t,h^+)) - f(t,\Gamma(t,h^+)) + g(t,x+\Gamma(t,h^+),h(t)) - g(t,\Gamma(t,h^+),h(t)).$$

In addition u is bounded on $\mathbb{R}^+_\tau$ therefore (cf. Lemmas 3.2 – 3.7 in [1])

$$u(t) = \Phi(t,\tau)P^+(\tau)u(\tau) + \int_\tau^t \Phi(t,s)P^+(s)F(s,u(s))\mathrm{d}s - \int_t^\infty \Phi(t,s)P^-(s)F(s,u(s))\mathrm{d}s.$$

Set for $\gamma \in (0,-\alpha)$ a Banach space

$$X_\gamma := \{x \in C(\mathbb{R}^+_\tau, X) \,:\, \|x\|^+_{\tau,-\gamma} := \sup_{t\geq\tau} \|x(t)\| e^{\gamma t} < \infty\}$$

with a norm $\|\cdot\|^+_{\tau,-\gamma}$. Define

$$\hat{\mathcal{T}}(x,\xi^+)(t) := \Phi(t,\tau)\xi^+ + \int_\tau^t \Phi(t,s)P^+(s)F(s,x(s))\mathrm{d}s - \int_t^\infty \Phi(t,s)P^-(s)F(s,x(s))\mathrm{d}s$$

for $(x,\xi^+,t) \in X_\gamma \times \mathbb{P}^+_\tau \times \mathbb{R}^+_\tau$.

Apply Theorem 2.2 for $\hat{\mathcal{T}} : X_\gamma \times \mathbb{P}^+(\tau) \to X_\gamma$ (it is possible for $\gamma > 0$ sufficiently small – we need exactly the condition $K(L_1 + L_2)\kappa_{\alpha+\gamma,\beta+\gamma} < 1$ which can be satisfied due to the assumption (iv) of Theorem 4.4). Therefore $\|u\|^+_{\tau,-\gamma} < \infty$ which yields $\overline{\mathbb{SP}}_{\tau,\epsilon} \subset \mathbb{SP}_{\tau,\epsilon}$. □

4.2 Quasibounded solution

The main assumption for this subsection is

$$A \in \mathcal{ED}_{\alpha,\beta} \text{ with } \alpha,\beta \in \mathbb{R}, \alpha < \beta, \tag{4.9}$$

note the difference from previous subsection in (4.3) where $\alpha < 0 < \beta$ was crucial.

We adopt the exact definition of *quasiboundedness* from [1].

Definition 4.6. Assume that I is unbounded to the left (or to the right) – I is unbounded to the left if I is one of the interval types $(-\infty,a), (-\infty,a], \mathbb{R}$ and similarly I is unbounded to the right if I is one of the following interval types $(a,\infty), [a,\infty), \mathbb{R}$. Let $g : I \to X, \gamma \in \mathbb{R}$ be an arbitrary function. We say that g is γ^--quasibounded (or γ^+-quasibounded) if $\|g\|^-_{\tau,\gamma} < \infty$ (or $\|g\|^+_{\tau,\gamma} < \infty$) for some $\tau \in I$, where

$$\|g\|^-_{\tau,\gamma} := \sup_{t\leq\tau} \|g(t)\| \mathrm{e}^{-\gamma t}, \quad \|g\|^+_{\tau,\gamma} := \sup_{t\geq\tau} \|g(t)\| \mathrm{e}^{-\gamma t}.$$

In the special case $I = \mathbb{R}$ we say that g is γ-quasibounded if $\|g\|_\gamma < \infty$ where

$$\|g\|_\gamma := \sup_{t\in\mathbb{R}} \|g(t)\| \mathrm{e}^{-\gamma t}.$$

The main tool of this subsection will be the transformation discussed in the next lemma.

Lemma 4.7. *Assume that I is a positive interval and $f \in \mathcal{CAR}(I,X)$. Let $\mu : I \to X$ be a solution (in the sense of Definition 2.3) of $\dot{x} = f(t,x)$ and $\rho : I \to \mathbb{R} \setminus \{0\}$ is a C^1 scalar function. Then $\tilde{\mu} : I \to X$ defined as $\tilde{\mu}(t) := \rho(t)\mu(t)$ is a solution of $\dot{x} = g(t,x)$ for $g(t,x) = \frac{\dot{\rho}(t)}{\rho(t)}x + \rho(t)f(t,\frac{1}{\rho(t)}x)$.*

Proof. μ is a solution, therefore $f(\cdot,\mu(\cdot)) : I \to X$ is locally integrable and $\mu(t)-\mu(s) = \int_s^t f(\tau,\mu(\tau))\mathrm{d}\tau$ is valid for $t,s \in I$. Fix an arbitrary functional $\phi \in X^*$ and elements $t,s \in I$ such that $s < t$. With a notation $u(r) := \phi(\mu(r))$ we have $u(r_2)-u(r_1) = \int_{r_1}^{r_2} \phi(f(\tau,\mu(\tau)))\mathrm{d}\tau$ for $r_1,r_2 \in [s,t]$. Then $u : [s,t] \to \mathbb{R}$ is absolutely continuous on $[s,t]$. So $\dot{u}(r)$ exists for a.e. $r \in [s,t]$ and $\dot{u}(r) = \phi(f(r,\mu(r)))$ for these $r \in [s,t]$. This means that $v(r) := \rho(r)u(r)$ is also absolutely continuous on $[s,t]$ with derivative

$$\dot{v}(r) = \dot{u}(r)\rho(r) + u(r)\dot{\rho}(r) = \phi(f(r,\mu(r)))\rho(r) + \phi(\mu(r))\dot{\rho}(r)$$

for a.e. $r \in [s,t]$. Therefore for all $r_1,r_2 \in [s,t]$ we have

$$v(r_2)-v(r_1) = \int_{r_1}^{r_2} \Big(\phi(f(\tau,\mu(\tau)))\rho(\tau) + \phi(\mu(\tau))\dot{\rho}(\tau)\Big)\mathrm{d}\tau.$$

After elementary computations we obtain for all $\phi \in X^*$ and $s,t \in I$ the following equality

$$\phi(\rho(t)\mu(t)-\rho(s)\mu(s)) = \phi\left[\int_s^t \Big(\rho(\tau)f(\tau,\mu(\tau)) + \dot{\rho}(\tau)\mu(\tau)\Big)\mathrm{d}\tau\right]$$

or with a notation $\tilde{\mu}(t) = \rho(t)\mu(t)$

$$\phi(\tilde{\mu}(t)-\tilde{\mu}(s)) = \phi\left[\int_s^t \Big(\rho(\tau)f(\tau,\frac{1}{\rho(\tau)}\tilde{\mu}(\tau)) + \frac{\dot{\rho}(\tau)}{\rho(\tau)}\tilde{\mu}(\tau)\Big)\mathrm{d}\tau\right].$$

A consequence of the Hahn-Banach Theorem says that functionals separates points, so we have

$$\tilde{\mu}(t)-\tilde{\mu}(s) = \int_s^t \Big(\rho(\tau)f(\tau,\frac{1}{\rho(\tau)}\tilde{\mu}(\tau)) + \frac{\dot{\rho}(\tau)}{\rho(\tau)}\tilde{\mu}(\tau)\Big)\mathrm{d}\tau$$

which means exactly that $\tilde{\mu} : I \to X$ is a solution of $\dot{x} = g(t,x)$. □

Therefore the problem of finding a $\gamma-$quasibounded solution $\mu : \mathbb{R} \to X$ of $\dot{x} = A(t)x + F(t,x)$ with $\gamma \in (\alpha,\beta)$ can be transformed with a transformation $y(t) = x(t)\mathrm{e}^{-\gamma t}$ (apply the above lemma with $\rho(t) := \mathrm{e}^{-\gamma t}$) to the problem of finding bounded solutions $y : \mathbb{R} \to X$ of

$$\dot{y} = (A(t)-\gamma I)y + \mathrm{e}^{-\gamma t}F(t,\mathrm{e}^{\gamma t}y).$$

Noting the bijective correspondence between these solution sets and the fact

$$(A(t)-\gamma I) \in \mathcal{ED}_{\gamma-\alpha,\beta-\gamma}$$

with $\gamma-\alpha < 0 < \beta-\gamma$, we can generalize Theorem 4.4 as follows.

Theorem 4.8. *Assume that we have functions $f : \mathbb{R}\times X \to X, g : \mathbb{R}\times X\times X_1 \to X$ and a constant $\gamma \in (\alpha,\beta)$ such that*

(i) Smoothness: *$f(\cdot,x), g(\cdot,x,u) : \mathbb{R} \to X$ are s.m. for all $x \in X, u \in X_1$ and $f(t,\cdot) : X \to X, g(t,\cdot,\cdot) : X\times X_1 \to X$ are continuous for all $t \in \mathbb{R}$,*

(ii) Quasiboundedness a.e.: *there are constants M_1, M_2 such that*

$$\|f(t,0)\| \leq M_1 \mathrm{e}^{\gamma t}, \quad \|g(t,0,0)\| \leq M_2 \mathrm{e}^{\gamma t}$$

for a.e. $t \in \mathbb{R}$,

(iii) Lipschitz condition: *there are constants L_1, L_2, L_3 such that*

$$\|f(t,x) - f(t,\tilde{x})\| \leq L_1 \|x - \tilde{x}\|$$

and

$$\|g(t,x,u) - g(t,\tilde{x},\tilde{u})\| \leq L_2 \|x - \tilde{x}\| + L_3 \mathrm{e}^{\gamma t} \|u - \tilde{u}\| \tag{4.10}$$

are valid for a.e. $t \in \mathbb{R}$ and for all $x, \tilde{x} \in X, u, \tilde{u} \in X_1$,

(iv) $K(L_1 + L_2)\kappa_{\alpha-\gamma,\beta-\gamma} < 1$.

Then for every $h \in \mathbb{S}_1$ there exists a unique $\gamma-$quasibounded solution $\Gamma_\gamma(\cdot,h) : \mathbb{R} \to X$ of the problem (4.4) *corresponding to the selector h. In addition the mapping $\Gamma_\gamma : \mathbb{R} \times \mathbb{S}_1 \to X$ is continuous and the following is hold*

$$\|\Gamma_\gamma(\cdot,h) - \Gamma_\gamma(\cdot,\tilde{h})\|_\gamma \leq C(K, L_1, L_2, L_3, \alpha - \gamma, \beta - \gamma)\|h - \tilde{h}\|_\infty$$

where the function C is defined by (4.6).

Proof. Apply the above lemma with $\rho(t) := \mathrm{e}^{-\gamma t}$. Then the assumptions of Theorem 4.4 are fulfilled and the inverse transformation gives our statement with $\Gamma_\gamma(\cdot,h) := \Gamma(\cdot,h)\mathrm{e}^{\gamma\cdot}$. $\square$

Remark 4.9. The special Lipschitz-type condition (4.10) can be omitted. For this aim we have to work with a new complete metric space of selectors $\mathbb{S}^\gamma_\epsilon := \{h \in \mathbb{S}_\epsilon : \|h\|_\gamma < \infty\}$ where the metric is induced by $\|\cdot\|_\gamma$. So, under the conditions of the previous theorem, except (4.10), which is modified to the usual one

$$\|g(t,x,u) - g(t,\tilde{x},\tilde{u})\| \leq L_2 \|x - \tilde{x}\| + L_3 \|u - \tilde{u}\|$$

the assertion of the theorem is still valid with $\Gamma_\gamma : \mathbb{R} \times \mathbb{S}^\gamma_1 \to X$ for selectors $h \in \mathbb{S}^\gamma_1$. We get an estimate

$$\|\Gamma_\gamma(\cdot,h) - \Gamma_\gamma(\cdot,\tilde{h})\|_\gamma \leq C(K, L_1, L_2, L_3, \alpha - \gamma, \beta - \gamma)\|h - \tilde{h}\|_\gamma.$$

The proof of this remark is the same application of the above defined transformation.

Remark 4.10. Now we generalize Theorem 4.5. We accomplish this as simply as above by the use of Lemma 4.7. Assume all the assumptions of Theorem 4.8. Introduce the following notations

$$\begin{array}{lcl}
\mathbb{IP}^\gamma_{\tau,\epsilon} & := & \{\Gamma_\gamma(\tau,h) : h \in \mathbb{S}_\epsilon\}, \\
\mathbb{SP}^\gamma_{\tau,\epsilon} & := & \{\xi \in X : \exists h \in \mathbb{S}_\epsilon \text{ such that } \lim_{t\to\infty} \|\lambda(t,\tau,\xi,h) - \Gamma_\gamma(t,h)\|\mathrm{e}^{-\gamma t} = 0\}, \\
\mathbb{UP}^\gamma_{\tau,\epsilon} & := & \{\xi \in X : \exists h \in \mathbb{S}_\epsilon \text{ such that } \lim_{t\to-\infty} \|\lambda(t,\tau,\xi,h) - \Gamma_\gamma(t,h)\|\mathrm{e}^{-\gamma t} = 0\}, \\
\overline{\mathbb{SP}}^\gamma_{\tau,\epsilon} & := & \{\xi \in X : \exists h \in \mathbb{S}^+_{\tau,\epsilon} \text{ such that } \|\lambda(\cdot,\tau,\xi,h^+)\|^+_{\tau,\gamma} < \infty\}, \\
\overline{\mathbb{UP}}^\gamma_{\tau,\epsilon} & := & \{\xi \in X : \exists h \in \mathbb{S}^-_{\tau,\epsilon} \text{ such that } \|\lambda(\cdot,\tau,\xi,h^-)\|^-_{\tau,\gamma} < \infty\}.
\end{array}$$

where $\lambda(\cdot, t_0, x_0, h)$ is the unique solution of the problem

$$\dot{x} = A(t)x + f(t,x) + g(t,x,h(t)), \quad x(t_0) = x_0.$$

Then there exists Lipschitz continuous functions

$$w^{s,\gamma} : \mathbb{P}^+_\tau \times \mathbb{S}^+_{\tau,\epsilon} \to \mathbb{P}^-_\tau, \quad w^{u,\gamma} : \mathbb{P}^-_\tau \times \mathbb{S}^-_{\tau,\epsilon} \to \mathbb{P}^+_\tau$$

such that

$$\mathbb{SP}^\gamma_{\tau,\epsilon} = \overline{\mathbb{SP}}^\gamma_{\tau,\epsilon} = \{\xi^+ + w^{s,\gamma}(\xi^+, h) : \xi^+ \in \mathbb{P}^+_\tau, h \in \mathbb{S}^+_{\tau,\epsilon}\},$$
$$\mathbb{UP}^\gamma_{\tau,\epsilon} = \overline{\mathbb{UP}}^\gamma_{\tau,\epsilon} = \{\xi^- + w^{u,\gamma}(\xi^-, h) : \xi^- \in \mathbb{P}^-_\tau, h \in \mathbb{S}^-_{\tau,\epsilon}\}.$$

Remark 4.11. The statement of the previous remark has again a variant for the situation when (4.10) in Theorem 4.8 is replaced with

$$\|g(t,x,u) - g(t,\tilde{x},\tilde{u})\| \le L_2\|x - \tilde{x}\| + L_3\|u - \tilde{u}\|.$$

In the light of the above mentioned two remarks it is straightforward how to achive this. At first set new selector spaces $\mathbb{S}^{\gamma,\pm}_{\tau,\epsilon} := \{h \in \mathbb{S}^\pm_{\tau,\epsilon} : h^\pm \in \mathbb{S}^\gamma_\epsilon\}$ (they are complete metric spaces with metrics induced by $\|\cdot\|^\pm_{\tau,\gamma}$; everywhere we use the standard double notation $\pm$). Introduce

$$\begin{array}{lcl}
\mathbb{IP}^\gamma_{\tau,\epsilon} & := & \{\Gamma_\gamma(\tau,h) : h \in \mathbb{S}^\gamma_\epsilon\}, \\
\mathbb{SP}^\gamma_{\tau,\epsilon} & := & \{\xi \in X : \exists h \in \mathbb{S}^\gamma_\epsilon \text{ such that } \lim_{t\to\infty} \|\lambda(t,\tau,\xi,h) - \Gamma_\gamma(t,h)\| \mathrm{e}^{-\gamma t} = 0\}, \\
\mathbb{UP}^\gamma_{\tau,\epsilon} & := & \{\xi \in X : \exists h \in \mathbb{S}^\gamma_\epsilon \text{ such that } \lim_{t\to-\infty} \|\lambda(t,\tau,\xi,h) - \Gamma_\gamma(t,h)\| \mathrm{e}^{-\gamma t} = 0\}, \\
\overline{\mathbb{SP}}^\gamma_{\tau,\epsilon} & := & \{\xi \in X : \exists h \in \mathbb{S}^{\gamma,+}_{\tau,\epsilon} \text{ such that } \|\lambda(\cdot,\tau,\xi,h^+)\|^+_{\tau,\gamma} < \infty\}, \\
\overline{\mathbb{UP}}^\gamma_{\tau,\epsilon} & := & \{\xi \in X : \exists h \in \mathbb{S}^{\gamma,-}_{\tau,\epsilon} \text{ such that } \|\lambda(\cdot,\tau,\xi,h^-)\|^-_{\tau,\gamma} < \infty\}.
\end{array}$$

Then there are uniquely determined Lipschitz continuous functions

$$w^{s,\gamma} : \mathbb{P}^+_\tau \times \mathbb{S}^{\gamma,+}_{\tau,\epsilon} \to \mathbb{P}^-_\tau, \quad w^{u,\gamma} : \mathbb{P}^-_\tau \times \mathbb{S}^{\gamma,-}_{\tau,\epsilon} \to \mathbb{P}^+_\tau$$

such that

$$\mathbb{SP}^\gamma_{\tau,\epsilon} = \overline{\mathbb{SP}}^\gamma_{\tau,\epsilon} = \{\xi^+ + w^{s,\gamma}(\xi^+, h) : \xi^+ \in \mathbb{P}^+_\tau, h \in \mathbb{S}^{\gamma,+}_{\tau,\epsilon}\},$$
$$\mathbb{UP}^\gamma_{\tau,\epsilon} = \overline{\mathbb{UP}}^\gamma_{\tau,\epsilon} = \{\xi^- + w^{u,\gamma}(\xi^-, h) : \xi^- \in \mathbb{P}^-_\tau, h \in \mathbb{S}^{\gamma,-}_{\tau,\epsilon}\}.$$

Partial answer to the question of independence of Γ_γ from γ is stated in the following corollary.

Corollary 4.12. *Let we have $\alpha < \alpha_1 < \beta_1 < \beta$ and functions $f : \mathbb{R}\times X \to X$, $g : \mathbb{R}\times X\times X_1 \to X$ such that*

(i) Smoothness: *$f(\cdot,x), g(\cdot,x,u) : \mathbb{R} \to X$ are s.m. for all $x \in X, u \in X_1$ and $f(t,\cdot) : X \to X, g(t,\cdot,\cdot) : X \times X_1 \to X$ are continuous for all $t \in \mathbb{R}$,*

(ii) Quasiboundedness a.e.: *there are constants M_1, M_2 such that*

$$\|f(t,0)\| \le M_1\eta(t), \quad \|g(t,0,0)\| \le M_2\eta(t)$$

for a.e. $t \in \mathbb{R}$, where $\eta(t) := \min\{\mathrm{e}^{\alpha_1 t}, \mathrm{e}^{\beta_1 t}\}$, *(iii)* Lipschitz condition: *there are constants L_1, L_2, L_3 such that*

$$\|f(t,x) - f(t,\tilde{x})\| \le L_1\|x - \tilde{x}\|$$

and

$$\|g(t,x,u) - g(t,\tilde{x},\tilde{u})\| \le L_2\|x - \tilde{x}\| + L_3\eta(t)\|u - \tilde{u}\|$$

are valid for a.e. $t \in \mathbb{R}$ and for all $x, \tilde{x} \in X, u, \tilde{u} \in X_1$,

(iv) for $\theta := \max\left\{\kappa_{\alpha-\alpha_1,\beta-\alpha_1}, \kappa_{\alpha-\beta_1,\beta-\beta_1}\right\}$ we have

$$K(L_1 + L_2)\theta < 1.$$

Then Γ_γ from previous theorem is well-defined for $\gamma \in [\alpha_1,\beta_1]$ and independent from γ (that is $\Gamma_{\gamma_1} = \Gamma_{\gamma_2}$ for all $\gamma_1, \gamma_2 \in [\alpha_1,\beta_1]$).

Proof. We set $|||x||| := \|x\|_{\alpha_1} + \|x\|_{\beta_1}$ for $x \in C(\mathbb{R}, X)$. The space

$$Y := \{x \in C(\mathbb{R}, X) : |||x||| < \infty\}$$

is a Banach space with a norm $|||\cdot|||$. Define $\mathcal{T} : Y \times \mathbb{S}_1 \to Y$ formally as in (4.7). Then it is well-defined, continuous and also a uniform contraction. Theorem 2.2 yields the unique solution x^* in the space Y. We have immediately $\Gamma_\gamma = x^*$ for all $\gamma \in [\alpha_1,\beta_1]$. □

Note that from this corollary without any effort we may obtain γ-independent variants (in the above mentioned interpretation) of remarks 4.9 – 4.11.

Remark 4.13. From these results the so-called "hierarchy of integral manifolds" (cf. [1]) could be also established. We describe this in a simple situation (without inflation). Let $X := \mathbb{R}^3$ and $A(t) := \mathrm{diag}(a(t), 0, b(t))$ with $a, b : \mathbb{R} \to R$ continuous and

$$a(t) \le \alpha < 0 < \beta \le b(t), \quad t \in \mathbb{R}.$$

Then the evolution operator of $\dot{x} = A(t)x$ is

$$\Phi(t,s) = \mathrm{diag}(\mathrm{e}^{\int_s^t a(\tau)\mathrm{d}\tau}, 1, \mathrm{e}^{\int_s^t b(\tau)\mathrm{d}\tau})$$

which posses an exponential dichotomy on $\mathbb{R}$ in two ways

- with projection $P(x_1,x_2,x_3)^T := (x_1,0,0)^T$ and constant $K = 1$ we have $A \in \mathcal{ED}_{\alpha,0}$,
- with projection $P(x_1,x_2,x_3)^T := (x_1,x_2,0)^T$ and constant $K = 1$ we have $A \in \mathcal{ED}_{0,\beta}$.

Choose $\gamma_1, \gamma_2 \in \mathbb{R}$ such that $\alpha < \gamma_1 < 0 < \gamma_2 < \beta$. Let $f : \mathbb{R} \times X \to X$ satisfies the following conditions

- $f \in \mathcal{CAR}(\mathbb{R}, X)$,
- $\exists M \ge 0$ such that $f(t,0) \le M \min\{\mathrm{e}^{\gamma_1 t}, \mathrm{e}^{\gamma_2 t}\}$ for a.e. $t \in \mathbb{R}$,
- $\exists L \ge 0$ such that $\|f(t,x) - f(t,\tilde{x})\| \le L\|x - \tilde{x}\|$ for a.e. $t \in \mathbb{R}$ and for all $x, \tilde{x} \in X$.

Assume that

$$L\max\{\kappa_{\alpha-\gamma_1,-\gamma_1},\kappa_{-\gamma_2,\beta-\gamma_2}\}<1.$$

Then the above discussed graph characterisation is valid and we have immediately the following hierarchy

$$\begin{array}{ccccccc}\mathbb{SP}^{\gamma_1}_{0,0} & = & \overline{\mathbb{SP}}^{\gamma_1}_{0,0} & \subset & \overline{\mathbb{SP}}^{\gamma_2}_{0,0} & = & \mathbb{SP}^{\gamma_2}_{0,0},\\ \mathbb{UP}^{\gamma_1}_{0,0} & = & \overline{\mathbb{UP}}^{\gamma_1}_{0,0} & \supset & \overline{\mathbb{UP}}^{\gamma_2}_{0,0} & = & \mathbb{UP}^{\gamma_2}_{0,0}.\end{array}$$

5 Remarks on Exponential Dichotomy of ODE

Here we present simple criteria on exponential dichotomy for linear systems. We consider on $\mathbb{C}^n$ the following standard norms [23]:

$$|x|_p := \sqrt[p]{|x_1|^p+|x_2|^p+\cdots+|x_n|^p}, \quad |x|_\infty := \max\{|x_1|,|x_2|,\cdots,|x_n|\},$$

where $p\geq 1$ and $x=(x_1,x_2,\cdots,x_n)$. The corresponding norms on $L(\mathbb{C}^n)$ are denoted by $\|\cdot\|_p$ and $\|\cdot\|_\infty$. We recall the following result [23]: If $A=\left(a_{ij}\right)_{i,j=1}^n$ then

$$\|A\|_1=\max_{1\leq i\leq n}\left(\sum_{j=1}^n|a_{ji}|\right),\quad \|A\|_2=\sqrt{\max_{1\leq i\leq n}\lambda^i_{\bar{A}^TA}},\quad \|A\|_\infty=\max_{1\leq i\leq n}\left(\sum_{j=1}^n|a_{ij}|\right), \tag{5.1}$$

where $\lambda^i_{\bar{A}^TA}$, $i=1,2,\cdots,n$ are eigenvalues of $\bar{A}^TA$.

Next, by using the Hölder inequality, for $p>1$ we compute

$$|Ax|_p=\sqrt[p]{\sum_{i=1}^n\left|\sum_{j=1}^n a_{ij}x_j\right|^p}\leq\sqrt[p]{\sum_{i=1}^n\left(\sum_{j=1}^n|a_{ij}|^q\right)^{p/q}\left(\sum_{j=1}^n|x_j|^p\right)}=\sqrt[p]{\sum_{i=1}^n\left(\sum_{j=1}^n|a_{ij}|^q\right)^{p/q}}\;|x|_p$$

for $\frac{1}{p}+\frac{1}{q}=1$, which gives

$$\|A\|_p\leq\sqrt[p]{\sum_{i=1}^n\left(\sum_{j=1}^n|a_{ij}|^q\right)^{p/q}}. \tag{5.2}$$

Take $\lambda\in\mathbb{C}$. To show the invertibility of $A_\lambda:=\lambda\mathbb{I}-A$, first we suppose that

$$a_{ii}\neq\lambda,\quad \forall i=1,2,\cdots,n, \tag{5.3}$$

and then we consider the following modification of A_λ:

$$\widetilde{A}_\lambda:=\left(\frac{\lambda\delta_i^j-a_{ij}}{\lambda-a_{ii}}\right)_{i,j=1}^n. \tag{5.4}$$

Now we decompose (5.4) as follows

$$\widetilde{A}_\lambda:=\mathbb{I}+B_\lambda,\quad B_\lambda:=\left(b^\lambda_{ij}\right)_{i,j=1}^n,\quad b^\lambda_{ij}=\begin{cases}-\frac{a_{ij}}{\lambda-a_{ii}} & \text{for } i\neq j,\\ 0 & \text{for } i=j.\end{cases}$$

Note $A_\lambda=D_\lambda\widetilde{A}_\lambda$ for $D_\lambda:=\operatorname{diag}(\lambda-a_{11},\lambda-a_{22},\cdots,\lambda-a_{nn})$. Clearly A_λ is invertible if and only if $\widetilde{A}_\lambda$ is invertible, and then $A_\lambda^{-1}=\widetilde{A}_\lambda^{-1}D_\lambda^{-1}$.

Now we have the following consequences of Neumann's theorem [23].

Theorem 5.1. *Suppose* (5.3) *and set* $d := \max_{1\le i\le n}\left\{|\lambda - a_{ii}|^{-1}\right\}$. *Then the following statements hold:*

1. If

$$\eta_1 := \max_{1\le i\le n}\left\{\sum_{j=1, j\neq i}^{n} \frac{|a_{ji}|}{|\lambda - a_{jj}|}\right\} < 1\,, \tag{5.5}$$

then A_λ *is invertible and* $\|A_\lambda^{-1}\|_1 \le \frac{d}{1-\eta_1}$.

2. If

$$\eta_\infty := \max_{1\le i\le n}\left\{\frac{\sum_{j=1, j\neq i}^{n} |a_{ij}|}{|\lambda - a_{ii}|}\right\} < 1\,, \tag{5.6}$$

then A_λ *is invertible and* $\|A_\lambda^{-1}\|_\infty \le \frac{d}{1-\eta_\infty}$.

3. If

$$\tau_p := \sum_{i=1}^{n} \frac{\left(\sum_{j=1, j\neq i}^{n} |a_{ij}|^q\right)^{p/q}}{|\lambda - a_{ii}|^p} < 1\,, \tag{5.7}$$

for some $p > 1$, *where* $\frac{1}{p} + \frac{1}{q} = 1$, *then* A_λ *is invertible and* $\|A_\lambda^{-1}\|_p \le \frac{d}{1-\sqrt[p]{\tau_p}}$.

Proof. Since by Neumann's theorem

$$\|A_\lambda^{-1}\|_p \le \|D_\lambda^{-1}\|_p \|\widetilde{A}_\lambda^{-1}\|_p \le \frac{\|D_\lambda^{-1}\|_p}{1 - \|B_\lambda\|_p}$$

and $\|D_\lambda^{-1}\|_p = d$ for any $p \in [1,\infty]$, statements follow from (5.1) and (5.2), respectively. □

Remark 5.2. a) For $\lambda = 0$, condition (5.5) is the Hadamard classical assumption on invertibility of A [18], but Hadamard ones have no estimates on the norm of A^{-1}. Further results on the invertibility of matrices are presented in [11].

b) Taking the transpose A^T we get dual results of Theorem 5.1 which here we do not present explicitly.

c) Taking opposite inequalities in the above conditions (5.5), (5.6) and (5.7), we can localize the spectrum $\sigma(A)$ by obtaining Geršgoring type sets [11, 26].

d) If $\Re a_{ii} \neq 0$ for all $i = 1, 2, \cdots, n$ then using $|\lambda - a_{ii}| \ge |\Re a_{ii}|$ for all $i = 1, 2, \cdots, n$ and any $\lambda \in \mathbb{C}$ with $\Re\lambda = 0$ we see that A is hyperbolic, i.e., $\Re\sigma(A) \neq 0$, if one of the next assumptions holds

$$\max_{1\le i\le n}\left\{\sum_{j=1, j\neq i}^{n} \frac{|a_{ji}|}{|\Re a_{jj}|}\right\} < 1\,, \quad \max_{1\le i\le n}\left\{\frac{\sum_{j=1, j\neq i}^{n} |a_{ij}|}{|\Re a_{ii}|}\right\} < 1\,, \quad \sum_{i=1}^{n} \frac{\left(\sum_{j=1, j\neq i}^{n} |a_{ij}|^q\right)^{p/q}}{|\Re a_{ii}|^p} < 1\,, \tag{5.8}$$

for some $p > 1$ where $\frac{1}{p} + \frac{1}{q} = 1$. Moreover, A has the same type of hyperbolicity as $\operatorname{diag}(\Re a_{11}, \Re a_{22}, \cdots, \Re a_{nn})$. This follows from the fact that all matrices

$$\operatorname{diag}(\Re a_{11}, \Re a_{22}, \cdots, \Re a_{nn}) + \xi\left(A - \operatorname{diag}(\Re a_{11}, \Re a_{22}, \cdots, \Re a_{nn})\right), \quad \xi \in [0,1]$$

are hyperbolic.

Now we deal with infinite dimensional matrices of the form $(Bx)_i = \sum_{j=i-s}^{i+s} b_{ij}x_j$, $i \in \mathbb{Z}$ for $s \in \mathbb{N}$ and a bounded sequence $\{b_{ij}\}_{i,j\in\mathbb{Z}}^{|j-i|\le s}$, where $x = \{x_i\}_{i\in\mathbb{Z}} \in \ell_p$. Then we have

$$|Bx|_1 = \sum_{i\in\mathbb{Z}} |(Bx)_i| \le \left(\sup_{i\in\mathbb{Z}} \sum_{j=i-s}^{i+s} |b_{ji}|\right)|x|_1, \quad |Bx|_\infty = \sup_{i\in\mathbb{Z}} |(Bx)_i| \le \left(\sup_{i\in\mathbb{Z}} \sum_{j=i-s}^{i+s} |b_{ij}|\right)|x|_\infty,$$

$$|Bx|_p = \sqrt[p]{\sum_{i\in\mathbb{Z}} |(Bx)_i|^p} \le \sqrt[p]{\sup_{i\in\mathbb{Z}} \sum_{k=i-s}^{i+s} \left(\sum_{j=k-s}^{k+s} |b_{kj}|^q\right)^{p/q}}\ |x|_p$$

for $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. So if we take a matrix

$$(Ax)_i = \sum_{j=-s}^{s} a_{ij}x_j, \quad i \in \mathbb{Z} \tag{5.9}$$

for $s \in \mathbb{N}$ and a bounded sequence $\{a_{ij}\}_{i,j\in\mathbb{Z}}^{|j-i|\le s}$ such that $\inf_{i\in\mathbb{Z}} |\Re a_{ii}| = \omega > 0$, then A is hyperbolic

$$\begin{gathered} \text{in } \ell_1 \text{ if } \sup_{i\in\mathbb{Z}} \sum_{j=i-s, j\ne i}^{i+s} \frac{|a_{ji}|}{|\Re a_{jj}|} < 1, \quad \text{in } \ell_\infty \text{ if } \sup_{i\in\mathbb{Z}} \frac{\sum_{j=i-s, j\ne i}^{i+s} |a_{ij}|}{|\Re a_{ii}|} < 1, \\ \text{in } \ell_p \text{ if } \sup_{i\in\mathbb{Z}} \sum_{k=i-s}^{i+s} \frac{\left(\sum_{j=k-s, j\ne k}^{k+s} |a_{kj}|^q\right)^{p/q}}{|\Re a_{kk}|^p} < 1, \end{gathered} \tag{5.10}$$

for some $p > 1$ where $\frac{1}{p} + \frac{1}{q} = 1$. Moreover, A has the same type of hyperbolicity as $\operatorname{diag}(\Re a_{ii})_{i\in\mathbb{Z}}$. Of course, conditions (5.10) are direct generalizations of (5.8) to (5.9). More sophisticated results are presented in [24, 25] on spectra of infinite matrices.

Now we consider a first order T-periodic ODE

$$\dot{x} = A(t)x \tag{5.11}$$

with $(A(t)x)_i = \sum_{j=i-s}^{i+s} a_{ij}(t)x_j$, $i \in \mathbb{Z}$ for $s \in \mathbb{N}$ and a uniformly bounded sequence $\{a_{ij}(t)\}_{i,j\in\mathbb{Z}}^{|j-i|\le s}$ of T-periodic continuous functions. First we suppose that

$$\lim_{i\to\pm\infty} a_{ij}(t) = a_j^\pm(t)\, \forall j = i-s, \cdots, i+s \tag{5.12}$$

uniformly on $[0,T]$. Then we set

$$(A_\infty(t)x)_i = \begin{cases} \sum_{j=i-s}^{i+s} a_j^+(t)x_j & i \ge 0 \\ \sum_{j=i-s}^{i+s} a_j^-(t)x_j & i < 0 \end{cases}$$

It is easy to verify that $C(t) := A(t) - A_\infty(t)$ are compact in any ℓ_p, $p \in [1,\infty]$ for all $t \in [0,T]$. Note the fundamental matrix solution $X(t)$ of (5.11) has the form

$$X(t) = X_\infty(t) + \int_0^t X_\infty(t)(X_\infty(z))^{-1}C(z)X(z)\mathrm{d}z,$$

where X_∞ is the fundamental matrix solution of $\dot{x} = A_\infty(t)x$. Hence $X(T) - X_\infty(T)$ is compact and so $\sigma_{ess}(X(T)) = \sigma_{ess}(X_\infty(T))$. For instance, if $A_\infty(t) = 0$ then $X(T)$ is a compact perturbation of $\mathbb{I}$. We recall [7, Theorem 2.1, p. 203] that (5.11) has an exponential dichotomy on $\mathbb{R}$ if and only if $\sigma(X(T)) \cap S^1 = \emptyset$ for the unit circle S^1. This is equivalent to say that the inhomogeneous system

$$\dot{x} = A(t)x + h(t) \tag{5.13}$$

has a unique bounded solution on $\mathbb{R}$ for any bounded continuous $h \in C_b(\mathbb{R}, X)$ (here X is a complex Banach space, namely one of the ℓ_p spaces for $p \in [1;\infty]$). Now we rewrite (5.13) as a system

$$\dot{x}_i = a_{ii}(t)x_i + \sum_{j=i-s, j\neq i}^{i+s} a_{ij}(t)x_j + h_i(t), \quad i \in \mathbb{Z} \tag{5.14}$$

for $h(t) = \{h_i(t)\}_{i\in\mathbb{Z}}$. We suppose

$$\omega := \inf_{i\in\mathbb{Z}, t\in\mathbb{R}} |\Re a_{ii}(t)| > 0. \tag{5.15}$$

We want to find criteria that (5.14) has a unique bounded solution on $\mathbb{R}$. For this purpose, we rewrite it as

$$x_i(t) = \int_{a_i\infty}^t e^{A_i(t,z)} \sum_{j=i-s, j\neq i}^{i+s} a_{ij}(z)x_j(z)dz + \int_{a_i\infty}^t e^{A_i(t,z)} h_i(z)dz, \quad i \in \mathbb{Z},$$

where $a_i := \operatorname{sign} \Re a_{ii}(t)$ and $A_i(t,z) := \int_z^t a_{ii}(u)du$. Note $|\Re A_i(t,z)| \geq \omega|t-z|$.

Then for $x, h \in C_b(\mathbb{R}, \ell_\infty)$ we derive

$$\begin{aligned}
|x_i(t)| &\leq \sup_{i\in\mathbb{Z}, z\in\mathbb{R}} \sum_{j=i-s, j\neq i}^{i+s} \frac{|a_{ij}(z)|}{|\Re a_{ii}(z)|} \left(-a_i \int_{a_i\infty}^t |\Re a_{ii}(z)| e^{\Re A_i(t,z)} dz\right) |x|_\infty + |h|_\infty (-a_i) \int_{a_i\infty}^t e^{a_i\omega(t-z)} dz \\
&= \sup_{i\in\mathbb{Z}, z\in\mathbb{R}} \sum_{j=i-s, j\neq i}^{i+s} \frac{|a_{ij}(z)|}{|\Re a_{ii}(z)|} |x|_\infty + \frac{|h|_\infty}{\omega}.
\end{aligned}$$

Consequently, if

$$\sup_{i\in\mathbb{Z}, z\in\mathbb{R}} \sum_{j=i-s, j\neq i}^{i+s} \frac{|a_{ij}(z)|}{|\Re a_{ii}(z)|} < 1$$

then (5.13) has a unique solution $x \in C_b(\mathbb{R}, \ell_\infty)$ for any $h \in C_b(\mathbb{R}, \ell_\infty)$, and thus (5.11) has an exponential dichotomy on ℓ_∞.

Similarly for $x, h \in C_b(\mathbb{R}, \ell_1)$ we derive

$$\begin{aligned}
|x(t)|_1 &= \sum_{i\in\mathbb{Z}} |x_i(t)| \leq \sum_{i\in\mathbb{Z}} -a_i \int_{a_i\infty}^t e^{a_i\omega(t-z)} \sum_{j=i-s, j\neq i}^{i+s} |a_{ij}(z)||x_j(z)|dz + \sum_{i\in\mathbb{Z}} -a_i \int_{a_i\infty}^t e^{a_i\omega(t-z)} |h_i(z)|dz \\
&= \sum_{j\in\mathbb{Z}} \sum_{i=j-s, i\neq j}^{j+s} -a_i \int_{a_i\infty}^t e^{a_i\omega(t-z)} |a_{ij}(z)||x_j(z)|dz + \sum_{i\in\mathbb{Z}} -a_i \int_{a_i\infty}^t e^{a_i\omega(t-z)} |h_i(z)|dz \\
&= \sum_{j\in\mathbb{Z}} \left(\sum_{0<|i-j|\leq s}^{a_i=-1} \int_{-\infty}^t e^{-\omega(t-z)} |a_{ij}(z)||x_j(z)|dz + \sum_{0<|i-j|\leq s}^{a_i=1} \int_t^\infty e^{\omega(t-z)} |a_{ij}(z)||x_j(z)|dz \right) \\
&\quad + \sum_{i\in\mathbb{Z}}^{a_i=-1} \int_{-\infty}^t e^{-\omega(t-z)} |h_i(z)|dz + \sum_{i\in\mathbb{Z}}^{a_i=1} \int_t^\infty e^{\omega(t-z)} |h_i(z)|dz \\
&\leq \frac{\sup_{j\in\mathbb{Z}} \left(\sup_{z\in\mathbb{R}} \sum_{0<|i-j|\leq s}^{a_i=-1} |a_{ij}(z)| + \sup_{z\in\mathbb{R}} \sum_{0<|i-j|\leq s}^{a_i=1} |a_{ij}(z)| \right)}{\omega} |x|_\infty + \frac{2|h|_\infty}{\omega},
\end{aligned}$$

which implies

$$|x|_\infty \le \frac{\sup_{j\in\mathbb{Z}}\left(\sup_{z\in\mathbb{R}}\sum_{0<|i-j|\le s}^{a_i=-1}|a_{ij}(z)| + \sup_{z\in\mathbb{R}}\sum_{0<|i-j|\le s}^{a_i=1}|a_{ij}(z)|\right)}{\omega}|x|_\infty + \frac{2|h|_\infty}{\omega}.$$

Consequently, if

$$\sup_{j\in\mathbb{Z}}\left(\sup_{z\in\mathbb{R}}\sum_{0<|i-j|\le s}^{a_i=-1}|a_{ij}(z)| + \sup_{z\in\mathbb{R}}\sum_{0<|i-j|\le s}^{a_i=1}|a_{ij}(z)|\right) < \omega$$

then (5.13) has a unique solution $x \in C_b(\mathbb{R},\ell_1)$ for any $h \in C_b(\mathbb{R},\ell_1)$, and thus (5.11) has an exponential dichotomy on ℓ_1.

Finally for $x,h \in C_b(\mathbb{R},\ell_p)$, $p \in (1,\infty)$ we derive

$$\begin{aligned}
|x(t)|_p &= \sqrt[p]{\sum_{i\in\mathbb{Z}}\left|-a_i\int_{a_i\infty}^t \mathrm{e}^{A_i(t,s)}\sum_{j=i-s,j\ne i}^{i+s}|a_{ij}(z)|x_j(z)\mathrm{d}z - a_i\int_{a_i\infty}^t \mathrm{e}^{A_i(t,z)}h_i(z)\mathrm{d}z\right|^p} \\
&\le \sqrt[p]{\sum_{i\in\mathbb{Z}}\left|\int_{a_i\infty}^t \mathrm{e}^{a_i\omega(t-z)}\sum_{j=i-s,j\ne i}^{i+s}|a_{ij}(z)||x_j(z)|\mathrm{d}z\right|^p} + \sqrt[p]{\sum_{i\in\mathbb{Z}}\left|\int_{a_i\infty}^t \mathrm{e}^{a_i\omega(t-z)}|h_i(z)|\mathrm{d}z\right|^p} \\
&\le \sqrt[p]{\sum_{i\in\mathbb{Z}}\left|\int_{a_i\infty}^t \mathrm{e}^{a_i\omega(t-z)}\sqrt[q]{\sum_{j=i-s,j\ne i}^{i+s}|a_{ij}(z)|^q}\sqrt[p]{\sum_{j=i-s,j\ne i}^{i+s}|x_j(z)|^p\mathrm{d}z}\right|^p} \\
&\quad + \sqrt[p]{\sum_{i\in\mathbb{Z}}\left|\int_{a_i\infty}^t \mathrm{e}^{a_i\omega(t-z)}\mathrm{d}z\right|^{p/q}\left|\int_{a_i\infty}^t \mathrm{e}^{a_i\omega(t-z)}|h_i(z)|^p\mathrm{d}z\right|}
\end{aligned}$$

$$\begin{aligned}
&\le \sqrt[p]{\sum_{i\in\mathbb{Z}}\left|\int_{a_i\infty}^t \mathrm{e}^{a_i\omega(t-z)}\mathrm{d}z\right|^{p/q}\left|\int_{a_i\infty}^t \mathrm{e}^{a_i\omega(t-z)}\left(\sum_{j=i-s,j\ne i}^{i+s}|a_{ij}(z)|^q\right)^{p/q}\left(\sum_{j=i-s,j\ne i}^{i+s}|x_j(z)|^p\right)\mathrm{d}z\right|} \\
&\quad + \sqrt[q]{\frac{1}{\omega}}\sqrt[p]{\sum_{i\in\mathbb{Z}}^{a_i=-1}\int_{-\infty}^t \mathrm{e}^{-\omega(t-z)}|h_i(z)|^p\mathrm{d}z + \sum_{i\in\mathbb{Z}}^{a_i=1}\int_t^\infty \mathrm{e}^{\omega(t-z)}|h_i(z)|^p\mathrm{d}z} \\
&\le \sqrt[q]{\frac{1}{\omega}}\left(\sum_{j\in\mathbb{Z}}\int_{-\infty}^t \mathrm{e}^{-\omega(t-z)}\sum_{0<|i-j|\le s}^{a_i=-1}\left(\sum_{k=i-s,k\ne i}^{i+s}|a_{ik}(z)|^q\right)^{p/q}|x_j(z)|^p\mathrm{d}z\right. \\
&\quad \left. + \sum_{j\in\mathbb{Z}}\int_t^\infty \mathrm{e}^{\omega(t-z)}\sum_{0<|i-j|\le s}^{a_i=1}\left(\sum_{k=i-s,k\ne i}^{i+s}|a_{ik}(z)|^q\right)^{p/q}|x_j(z)|^p\mathrm{d}z\right)^{1/p} + \frac{\sqrt[p]{2}}{\omega}|h|_\infty \\
&\le \frac{1}{\omega}\left(\sup_{j\in\mathbb{Z},z\in\mathbb{R}}\sum_{0<|i-j|\le s}^{a_i=-1}\left(\sum_{k=i-s,k\ne i}^{i+s}|a_{ik}(z)|^q\right)^{p/q}\right. \\
&\quad \left. + \sup_{j\in\mathbb{Z},z\in\mathbb{R}}\sum_{0<|i-j|\le s}^{a_i=1}\left(\sum_{k=i-s,k\ne i}^{i+s}|a_{ik}(z)|^q\right)^{p/q}\right)^{1/p}|x|_\infty + \frac{\sqrt[p]{2}}{\omega}|h|_\infty.
\end{aligned}$$

Consequently, if

$$\sup_{j\in\mathbb{Z},z\in\mathbb{R}}\sum_{0<|i-j|\le s}^{a_i=-1}\left(\sum_{k=i-s,k\ne i}^{i+s}|a_{ik}(z)|^q\right)^{p/q} + \sup_{j\in\mathbb{Z},z\in\mathbb{R}}\sum_{0<|i-j|\le s}^{a_i=1}\left(\sum_{k=i-s,k\ne i}^{i+s}|a_{ik}(z)|^q\right)^{p/q} < \omega^p$$

then (5.13) has a unique solution $x \in C_b(\mathbb{R},\ell_p)$ for any $h \in C_b(\mathbb{R},\ell_p)$, and thus (5.11) has an exponential dichotomy on ℓ_p (note that in the above computations $|x|_\infty = \sup_{t\in\mathbb{R}}|x(t)|_p$ strongly depends on $p \in [1,\infty]$).

Finally we consider a second-order ODE

$$\ddot{x} = A(t)x \tag{5.16}$$

with $(A(t)x)_i = \sum_{j=i-s}^{i+s} a_{ij}(t)x_j$, $i \in \mathbb{Z}$ for $s \in \mathbb{N}$ and a uniformly bounded sequence $\{a_{ij}(t)\}_{i,j\in\mathbb{Z}}^{|j-i|\le s}$ of T-periodic continuous functions. By following [7, Theorem 5.1 p. 32, Theorem 2.4 p. 208] and a method partition of unity on $[0,T]$, we know that (5.16) is exponentially dichotomous on ℓ_2 if $\Re\sigma(A(t)) > 0$ for any $t \in \mathbb{R}$. Next if $\Re a_{ii}(t) > 0$ for all $i \in \mathbb{Z}$, $t \in \mathbb{R}$ then using $|\lambda - a_{ii}(t)| \ge \Re a_{ii}(t)$ for all $i \in \mathbb{Z}$, $t \in \mathbb{R}$ and any $\lambda \in \mathbb{C}$ with $\Re\lambda \le 0$ we see from the above arguments (see (5.10)) that $\Re\sigma(A(t)) > 0$ for any $t \in \mathbb{R}$ if the following holds

$$\omega := \inf_{i\in\mathbb{Z}, t\in\mathbb{R}} \Re a_{ii}(t) > 0 \text{ and } \sup_{i\in\mathbb{Z}} \sum_{k=i-s}^{i+s} \frac{\sum_{j=k-s, j\neq k}^{k+s} |a_{kj}(t)|^2}{\Re a_{kk}(t)^2} < 1 \text{ for all } t \in \mathbb{R}.$$

Countable systems of ODE are also studied in [3, 8, 19].

Acknowledgments

This paper is partially supported by Grants VEGA-MS 1/0507/11, VEGA-SAV 2/0124/10 and APVV-0414-07.

References

[1] B. Aulbach and T. Wanner: Integral Manifolds for Carathéodory Type Differential Equations in Banach Spaces, In: B. Aulbach, F. Colonius, *Six Lectures on Dynamical systems*, World Scientific, Singapure, 45-119.

[2] F. Batelli and C. Lazarri: Exponential dichotomies, heteroclinic orbits, and Melnikov functions, *J. Diff. Equations* **86** (1990), 342-366.

[3] A. A. Boichuk and A. A. Pokutnii: Bounded solutions of linear differential equations in a Banach space, *Nonlinear Oscillations* **9** (2006), 1-12.

[4] D. L. Cohn: *Measure Theory*, Birkhäuser, Boston, 1980.

[5] G. Colombo, M. Fečkan and B. M. Garay: Multivalued perturbation of a saddle dynamics, *Diff. Equ. Dyn. Sys.* **18** (2010), 29-56.

[6] W. A. Coppel: *Dichotomies in Stability Theory*, Springer-Verlag, Berlin, 1978.

[7] J. L. Daleckij and M. G. Krein: *Stability of Differential Equations in Banach Space*, Amer. Math. Soc., Providence RI, 1974.

[8] K. Deimling: *Ordinary Differential Equations in Banach Spaces*, Springer, Berlin, 1977.

[9] K. Deimling: *Nonlinear Functional Analysis*, Springer-Verlag, Berlin 1985.

[10] N. Dunford and J.T. Schwartz: *Linear Operators, Part 1: General Theory*, Intercience Publisher, 1958.

[11] M. I. Gil: *Operator Functions and Localization of Spectra*, Springer, Berlin, 2003.

[12] L. Grüne: *Asymptotic Behaviour of Dynamical and Control Systems under Perturbation and Discretization*, Springer, Berlin, 2002.

[13] M.C. Irwin: *Smooth Dynamical Systems*, Academic Press, New York, 1980.

[14] U. Kirchraber and K. Palmer: *Geometry in the Neighbourhood of Invariant Manifolds of Maps and Flows and Linearization*, Pitman, London, 1991.

[15] P.E. Kloeden and V.S. Kozyakin: The inflation of attractors and their discretization: the autonomous case, *Nonlinear Anal.* **40** (2000), 333-343.

[16] A. Ornelas: Parametrization of Carathéodory multifunctions, *Rend. Sem. Mat. Univ. Padova* **83** (1990), 33-44.

[17] K. J. Palmer: Exponential dichotomies and transversal homoclinic points, *J. Diff. Equations* **55** (1990), 225-256.

[18] M. Parodi: *La Localisation des Valeurs Caractéristiques des Matrices et ses Applications*, Gauthier-Villars, Paris, 1959.

[19] A. M. Samoilenko and Yu. V. Teplinskij: *Countable Systems of Differential Equations*, Brill, Utrecht (2003)

[20] M. Shub: *Global stability of dynamical systems*, Springer, Berlin, 1987.

[21] S. Smale: Stable manifold for differential equations and diffeomorphisms, *Ann. Scuola Norm. Sup. Pisa* **17** (1963), 97-116.

[22] S. Smale: Differentiable dynamical systems, *Bull. Amer. Math. Soc.* **73** (1967), 747-817.

[23] A. E. Taylor: *Linear Functional Analysis*, John Willey & Sons, Inc., New York, 1967.

[24] G. Teschl: *Jacobi Operators and Completely Integrable Nonlinear Lattices*, Amer. Math. Soc., 2000.

[25] D. Turzík and M. Dubcová: Stability of steady state and traveling waves solutions in coupled map lattices, *Int. J. Bifur. Chaos* **18** (2008), 219-225.

[26] R. S. Varga: *Geršgorin and His Circles*, Springer, Berlin, 2011.