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On the Invertibility of Parabolic Pseudodifferential Operators in General Exponential Weighted Spaces

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Abstract

We consider the invertibility of parabolic pseudodifferential operators in exponential weighted Sobolev spaces. We suppose that the symbol a of the operator $Op(a)$ is analytically extended with respect to the impulse variable in an unbounded tube domain $\mathbb{R}^n + iD$ and satisfies conditions of uniform parabolicity . We prove that under these conditions the pseudodifferential operator $Op(a)$ is invertible in admissible weighted Sobolev spaces with weights connected with the domain D.

As an application we obtain exponential estimates of solutions (including estimates of the fundamental solution) for parabolic differential operators.

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Keywords: pseudodifferential operators, parabolicity , exponential weighted estimates.

1 Introduction

We consider the invertibility of parabolic pseudodifferential operators $Op(a)$ in exponential weighted Sobolev spaces. We suppose that the symbol a of $Op(a)$ is analytically extended with respect to the impulse variables to an unbounded tube domain $\mathbb{R}^{n+1} + i\mathcal{D}$ and satisfies conditions of uniform parabolicity. We prove that under these conditions the pseudodifferential operator $Op(a)$ is invertible in admissible weighted Sobolev spaces with weights connected with the domain $\mathcal{D}$.

As an application we obtain exponential estimates of solutions (including estimates for the fundamental solutions) for parabolic differential operators.

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Various aspects of the Cauchy problem for differential and pseudodifferential operators have been considered by many authors. See for instance the classical I. Petrovskiĭ paper [20], the well known paper of M.Agranovich and M.Vishik [1] and the references cited there. A good survey of the state-of-art before 1990 see in [11]. Parabolic pseudodifferential Boutet de Monvel problems in the spaces without weights are considered in [10]. In the papers [12], [13] parabolic pseudodifferential boundary value problems have been considered in domains with singular boundaries.

We note also the works of S. Gindikin and L. Volevich devoted to the well-posedness classes for the Cauchy problems for exponentially correct differential operators of the constant strength (see [4], [6], [7], [8] and references sited there). Our weighted classes are closed to the well-posedness classes of these works.

The exponential estimates of solutions of elliptic pseudodifferential equations have been studied in [17], [22], [23], [25]. The methods of these papers are based on formulas of the composition of pseudodifferential operators with exponential weights. These results were extended in the papers [18], [19] to parabolic differential and pseudodifferential operators acting in the exponential weighted spaces with weights of the form

$$w(x) = \exp(\mu x_0 + v(x')) \tag{1.1}$$

where $x_0 \in \mathbb{R}_+$ is the time variable and $x' \in \mathbb{R}^n$ is the spatial variable. In the distinction from [18], [19] we consider here the general weights of the form

$$w(x) = \exp v(x_0, x') \tag{1.2}$$

connected with domain $\mathcal{D}$.

The paper is organized as follows. In Section 2.1 we following [2], [3], [15] summarize in a convenient for us form necessary facts of the calculus of pseudodifferential operators acting in admissible Sobolev spaces.

Next, in Section 2.2 we formulate some results from [18], [19] concerning the invertibility of parabolic pseudodifferential operators in Sobolev spaces with the simplest weights e^{hx_0}, $h \leq 0$.

In Section 3, which is the main in the paper, we study parabolic pseudodifferential operators in exponential weighted spaces. We introduce a class $\mathcal{W}_b(\mathcal{D}, q)$ of the weights of the form (1.2), give examples of such weights, and prove the theorem on the composition of pseudodifferential operators with weights in $\mathcal{W}_b(\mathcal{D}, q)$. Applying this theorem we reduce the study of pseudodifferential operators in Sobolev spaces with general weights of the form (1.2) to the investigation of pseudodifferential operators in Sobolev spaces with the simplest weights e^{hx_0}, and following [18], [19] obtain results on the invertibility of parabolic pseudodifferential operators in general weighted Sobolev spaces on $\mathbb{R}^{n+1}_+ = \mathbb{R}_+ \times \mathbb{R}^n$.

In the Section 4 we illustrate the results of Section 3 by the uniformly parabolic differential operators of the form

$$p(x, D) = \partial_{x_0} + \sum_{0<|\alpha|\leq 2m} a_\alpha(x) D^\alpha_{x'} + b(x) \tag{1.3}$$

acting in weighted Sobolev spaces with general weights of form (1.2).

2 Auxiliary result

2.1 Pseudodifferential operators on $\mathbb{R}^{n+1}$

We use the following notations:

- $x = (x_0, x')$ are the variables of $\mathbb{R}^{n+1}$ where $x_0 \in \mathbb{R}$ is the time variable and $x' = (x_1, ..., x_n) \in \mathbb{R}^{n+1}$ are the spatial variables, $\mathbb{R}_+ = \{x_0 \in \mathbb{R} : x_0 > 0\}$, $\mathbb{R}_+^{n+1} = \mathbb{R}_+ \times \mathbb{R}^n$, $\langle x \rangle = (1 + |x|^2)^{1/2}$.
- $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ where $\mathbb{N}$ is the set of natural numbers;

$$\begin{aligned} \partial_j &= \frac{\partial}{\partial x_j}, \nabla_{x'} = \left(\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}\right), \nabla = \left(\frac{\partial}{\partial x_0}, \nabla_{x'}\right), \\ D_j &= -i\frac{\partial}{\partial \xi_j}, j = 0, 1, ...n, \; D = (D_0, D_1 ..., D_n). \end{aligned}$$

- Let $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_n) = (\alpha_0, \alpha') \in \mathbb{N}_0^{n+1}$ be a multi-index, then $|\alpha| = \sum_{j=0}^{n} \alpha_j$ its length. We set

$$\begin{aligned} \xi^\alpha &= \prod_{j=0}^{n} \xi_j^{\alpha_j}, \; \partial_x^\alpha = \prod_{j=0}^{n} \partial_{x_j}^{\alpha_j}, \; D_x^\alpha = \prod_{j=0}^{n} D_j^{\alpha_j}; \\ p_{(\beta)}^{(\alpha)}(x,\xi) &= \partial_\xi^\alpha D_x^\beta p(x,\xi); \end{aligned}$$

- Sometime we write a function a as $a(x,\xi)$ and this expression have to explaine from which variables the a depends, but not a value of a at the point (x,ξ). We think that it does not lead to a misunderstanding.
- We denote by $E(\mathbb{R}^{n+1})$ the class of function $q \in C^\infty\left(\mathbb{R}^{n+1}\right)$ satisfying the following conditions : (a) $q(x) \geq 1$ for all $x \in \mathbb{R}^{n+1}$; (b) There exists $L > 0$ such that for every multi-index β and every $x, y \in \mathbb{R}^{n+1}$

$$\left|\partial^\beta q(x+y)\right| \leq C_\beta q(x) \langle y \rangle^L, \tag{2.1}$$

with some constants $C_\beta > 0$. Important example of $q \in E(\mathbb{R}^{n+1})$ is

$$q(x) = 1 + \frac{\langle x' \rangle^l}{\langle x_0 \rangle^m}, \; l \geq 0, m \geq 0. \tag{2.2}$$

Applying the elementary inequality

$$\langle x+y \rangle^m \leq 2^{\frac{|m|}{2}} \langle x \rangle^m \langle y \rangle^{|m|}, \; m \in \mathbb{R} \tag{2.3}$$

one can prove that this q satisfies conditions (a) and (b).

- Further, we set
$$\lambda_{q,b}(x,\xi) = |\xi_0| + \left|\xi'\right|^b + q(x) \tag{2.4}$$
where $b \in \mathbb{N}$, $q \in E\left(\mathbb{R}^{n+1}\right)$. Applying (2.1) and (2.3) one can show that there exists $C > 0$ and $L > 0$ such that
$$\left|\partial_x^\beta \partial_\xi^\alpha \lambda_{q,b}^m(x+y,\xi+\omega)\right| \le C\lambda_{q,b}^m(x,\xi)(1+|y|+|\omega|)^L \tag{2.5}$$
for every α,β, and $m \in \mathbb{R}$.

Definition 2.1. Let $a \in C^\infty\left(\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}\right)$, $m \in \mathbb{R}$. We say that a belongs to the class $S^m(\lambda_{q,b})$ if for all $l_1, l_2, \in \mathbb{N}_0$
$$|a|_{l_1,l_2} = \sum_{|\alpha|\le l_1, |\beta|\le l_2} \sup_{(x,\xi)\in\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}} \frac{\left|a_{(\beta)}^{(\alpha)}(x,\xi)\right|}{\lambda_{q,b}^{m-\left(\alpha_0+\frac{|\alpha'|}{b}\right)}(x,\xi)} < \infty. \tag{2.6}$$
The constants $|a|_{l_1,l_2}$ *define the Frechet topology on* $S^m(\lambda_{q,b})$.

We associate with $a \in S^m(\lambda_{q,b})$ the pseudodifferential operator (ψdo)
$$Op(a)u(x) = \int_{\mathbb{R}^{n+1}} d'\xi \int_{\mathbb{R}^{n+1}} a(x,\xi)u(y)e^{i(x-y)\cdot\xi}dy,\ u \in S(\mathbb{R}^{n+1}), \tag{2.7}$$
where $d'\xi = (2\pi)^{-n}d\xi$. We denote the class of such $\psi do's$ by $OPS^m(\lambda_{q,b})$.

Note that the general classes of pseudodifferential operators have been studied in [2], [3] [14], [15]. The class $OPS^m(\lambda_{q,b})$ is contained among $\psi do's$ considered in the cited works. We will give some definitions and results following these papers in a convenient for us form.

Proposition 2.2. *Let* $A_1 = Op(a_1) \in OPS^{m_1}(\lambda_{q,b})$, $A_2 = Op(a_2) \in OPS^{m_2}(\lambda_{q,b})$. *Then:*

a) operator $A = A_1A_2 \in OPS^{m_1+m_2}(\lambda_{q,b})$ *and its symbol* a *is given as*
$$a(x,\xi) = \int\int_{\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}} a_1(x,\xi+\eta)a_2(x+y,\xi)e^{iy\cdot\eta}dyd'\eta$$

b) for any natural N
$$a(x,\xi) = \sum_{|\alpha|<N} \frac{1}{\alpha!} a_1^{(\alpha)}(x,\xi)a_{2(\alpha)}(x,\xi) + r_N(x,\xi), \tag{2.8}$$
where $r_N(x,\xi) \in S^{m_1+m_2-N/b}(\lambda_{q,b})$.

Definition 2.3. We denote by $H^s(\lambda_{q,b},\mathbb{R}^{n+1})$, $s \in \mathbb{N}$ the closure of $C_0^\infty(\mathbb{R}^{n+1})$ in the norm
$$\|u\|_{H^s(\lambda_{q,b},\mathbb{R}^{n+1})} = \left(\sum_{|\alpha|\le s} \left\|q^{s-|\alpha|}\partial_{x_0}^{\alpha_0}\partial_{x_1}^{b\alpha_1}\dots\partial_{x_n}^{b\alpha_n}u\right\|_{L_2(\mathbb{R}^{n+1})}^2\right)^{\frac{1}{2}}.$$

For real $s \geq 0$ the space $H^s(\lambda_{q,b}, \mathbb{R}^{n+1})$ is defined by means of the complex interpolation (see [3]) and for the negative s by the duality with respect to the standard inner product in $L_2\left(\mathbb{R}^{n+1}\right)$, i.e. $H^s(\lambda_{q,b}, \mathbb{R}^{n+1}) = \left(H^{-s}(\lambda_{q,b}, \mathbb{R}^{n+1}))\right)^*$.

Let $\mathcal{S}\left(\mathbb{R}^{n+1}\right)$ be the space of C^∞-functions *decreasing at infinity* with all their derivatives rapidly than $|x|^{-N}$ for every $N \in \mathbb{N}$, and let $\mathcal{S}'\left(\mathbb{R}^{n+1}\right)$ be the dual space of the *tempered distributions.*

Proposition 2.4. *The following statements hold:*

a) $H^0(\lambda_{q,b}, \mathbb{R}^{n+1}) \equiv L_2\left(\mathbb{R}^{n+1}\right)$;

b) the embedding $\mathcal{S}\left(\mathbb{R}^{n+1}\right) \subset H^s(\lambda_{q,b}, \mathbb{R}^{n+1}) \subset \mathcal{S}'\left(\mathbb{R}^{n+1}\right)$ *are continuous and the left embedding is dense;*

c) if $s_1 \geq s_2$ *then* $H^{s_1}(\lambda_{q,b}, \mathbb{R}^{n+1}) \subset H^{s_2}(\lambda_{q,b}, \mathbb{R}^{n+1})$.

d) there exists an operator $\Lambda \in OPS^s(\lambda_{q,b})$ *such that*

$$\Lambda : H^s(\lambda_{q,b}, \mathbb{R}^{n+1}) \to L^2(\mathbb{R}^n)$$

is a topological isomorphism and $\Lambda^{-1} \in OPS^{-s}(\lambda_{q,b})$.

Proposition 2.5. *Operator* $Op(a) \in OPS^m(\lambda_{q,b})$ *is bounded from* $H^s(\lambda_{q,b}, \mathbb{R}^{n+1})$ *into* $H^{s-m}(\lambda_{q,b}, \mathbb{R}^{n+1})$ *and there exist constants* $C > 0$ *and* $l_1, l_2 \in \mathbb{N}$ *depending only on s and m such that*

$$\|Op(a)\|_{H^s(\lambda_{q,b}, \mathbb{R}^{n+1}) \to H^{s-m}(\lambda_{q,b}, \mathbb{R}^{n+1})} \leq C\, |a|_{l_1, l_2}. \tag{2.9}$$

2.2 Invertibility of parabolic pseudodifferential operators on the half-space $\mathbb{R}^{n+1}_+$

Let:

- $r_- = \left\{\eta \in \mathbb{R}^{n+1} : \eta = (\eta_0, 0, \ldots, 0),\ \eta_0 < 0\right\}$ be the ray in $\mathbb{R}^{n+1}$,
- $\Pi_- = \{\zeta_0 = \xi_0 + i\eta_0 \in \mathbb{C} : \xi_0 \in \mathbb{R},\ \eta_0 < 0\}$ be the lower complex half-plane,
- $$\lambda_{q,b,\eta_0}(x, \xi) = |\xi_0| + |\eta_0| + \left|\xi'\right|^b + q(x),\ \ \eta_0 \leq 0.$$

Definition 2.6. Let $a \in S^m(\lambda_{q,b})$, $m \in \mathbb{R}$. We say that $a \in S^m(\lambda_{q,b}, \Pi_-)$ if the symbol $a(x, \xi_0, \xi')$ has an analytic extension with respect to the variable ξ_0 in Π_-, and for all $l_1, l_2 \in \mathbb{N}_0$

$$[a]_{l_1, l_2} = \sup_{(x, \xi_0 + i\eta_0, \xi') \in\ \mathbb{R}^{n+1} \times \Pi_- \times \mathbb{R}^n} \sum_{|\alpha| \leq l_1, |\beta| \leq l_2} \frac{\left|\partial_x^\beta \partial_\xi^\alpha a(x, \xi_0 + i\eta_0, \xi')\right|}{\lambda_{q,b,\eta_0}(x, \xi)^{m - \left(\alpha_0 + \frac{|\alpha'|}{b}\right)}} < \infty.$$

We denote the class of $\psi do's$ with symbols in $S^m(\lambda_{q,b}, \Pi_-)$ by $OPS^m(\lambda_{q,b}, \Pi_-)$, and by $S^m(\lambda_{q,b}, \mathbb{R}^{n+1}_+, \Pi_-)$ the class of symbols which are the restrictions of symbols in $S^m(\lambda_{q,b}, \Pi_-)$ on $\mathbb{R}^{n+1}_+$, and by $OPS^m(\lambda_{q,b}, \mathbb{R}^{n+1}_+, \Pi_-)$ the associated class of $\psi do's$.

Proposition 2.7. *Let* $A_1 = Op(a_1) \in OPS^{m_1}(\lambda_{q,b}, \mathbb{R}^{n+1}_+, \Pi_-)$, $A_2 = Op(a_2) \in OPS^{m_2}(\lambda_{q,b}, \mathbb{R}^{n+1}_+, \Pi_-)$. *Then the operator* $A = A_1 A_2 \in OPS^{m_1+m_2}(\lambda_{q,b}, \mathbb{R}^{n+1}_+, \Pi_-)$, *and for any natural N the symbol a of A has the following representation*

$$a(x,\xi) = \sum_{|\alpha|<N} \frac{1}{\alpha!} a_1^{(\alpha)}(x,\xi) a_{2(\alpha)}(x,\xi) + r_N(x,\xi)$$

where $r_N(x,\xi) \in S^{m_1+m_2-N/b}(\lambda_{q,b}, \mathbb{R}^{n+1}_+, \Pi_-)$.

Proposition 2.8. *Let* $A = Op(a) \in S^m(\lambda_{q,b}, \mathbb{R}^{n+1}_+, \Pi_-)$, *and* $h < 0$. *We set* $a_h(x,\xi) = a(x, \xi_0 + ih, \xi')$. *Then*

$$A_h = e^{hx_0} A e^{-hx_0} = Op(a_h) \in OPS^m(\lambda_{q,b}, \mathbb{R}^{n+1}_+, \Pi_-).$$

We denote by $H_0^s(\lambda_{q,b}, \mathbb{R}^{n+1}_+)$ the closure of $C_0^\infty(\mathbb{R}^{n+1}_+)$ in the space $H^s(\lambda_{q,b}, \mathbb{R}^{n+1})$, and by $H_0^s(\lambda_{q,b}, \mathbb{R}^{n+1}_+, e^{hx_0})$ $(h \le 0)$ the space with norm

$$\|u\|_{H_0^s(\lambda_{q,b}, \mathbb{R}^{n+1}_+, e^{hx_0})} = \left\| e^{hx_0} u \right\|_{H_0^s(\lambda_{q,b}, \mathbb{R}^{n+1}_+)}.$$

Repeating the argument in [18], [19] and taking into account Propositions 2.5 and 2.8 we obtain the following statement.

Proposition 2.9. *Let* $a \in S^m(\lambda_{q,b}, \mathbb{R}^{n+1}_+, \Pi_-)$. *Then the operator*

$$Op(a) : H_0^s(\lambda_{q,b}, \mathbb{R}^{n+1}_+, e^{hx_0}) \to H_0^{s-m}(\lambda_{q,b}, \mathbb{R}^{n+1}_+, e^{hx_0}) \tag{2.10}$$

is bounded for all $h \le 0$ *and*

$$\|Op(a)\|_{H_0^s(\lambda_{q,b}, \mathbb{R}^{n+1}_+, e^{hx_0}) \to H_0^{s-m}(\lambda_{q,b}, \mathbb{R}^{n+1}_+, e^{hx_0})} \le C\, |a_h|_{l_1, l_2}.$$

where $C > 0$, *and* $l_1, l_2 \in \mathbb{N}_0$ *are independent of a.*

Definition 2.10. We say that $Op(a) \in OPS^m(\lambda_{q,b}, \mathbb{R}^{n+1}_+, \Pi_-)$ is a uniformly parabolic pseudodifferential operator if

$$\lim_{\eta_0 \to -\infty} \inf_{(x,\xi) \in \mathbb{R}^{n+1}_+ \times \mathbb{R}^{n+1}} \frac{|a(x, \xi_0 + i\eta_0, \xi')|}{\lambda_{q,b,\eta_0}(x,\xi)^m} > 0. \tag{2.11}$$

The following result gives the sufficient conditions for the invertibility of uniformly parabolic pseudodifferential operators in the spaces $H_0^s(\lambda_{q,b}, \mathbb{R}^{n+1}_+, e^{hx_0})$ for $h < 0$ with $|h|$ is large enough.

Theorem 2.11. *Let* $Op(a) \in OPS^m(\lambda_{q,b}, \mathbb{R}^{n+1}_+, \Pi_-)$ *be a uniformly parabolic ψdo. Then for any* $s \in \mathbb{R}$ *there exists* $h_0 = h_0(s) < 0$ *such that for all* $h < h_0$

$$Op(a) : H_0^s(\lambda_{q,b}, \mathbb{R}^{n+1}_+, e^{hx_0}) \to H_0^{s-m}(\lambda_{q,b}, \mathbb{R}^{n+1}_+, e^{hx_0}) \tag{2.12}$$

is invertible.

Theorem 2.11 is proved following [18], [19], applying Propositions 2.7-2.9.

3 Parabolic pseudodifferntial operators in exponential weighted spaces

3.1 Weight functions

Definition 3.1. Let $\mathcal{D}$ be a convex unbounded domain in $\mathbb{R}^n$, $q \in E\left(\mathbb{R}^{n+1}\right)$. We say that the weight function $w(x) = e^{v(x)}$, $x \in \overline{\mathbb{R}^{n+1}_+}$ belongs to the class $\mathcal{W}_b(\mathcal{D}, q)$ if the following conditions holds:

(i) $\nabla v(x) \in \mathcal{D}$ for every $x \in \overline{\mathbb{R}^{n+1}_+}$;

(ii) there are constants $\gamma_1, \gamma_2 > 0$ and $\widetilde{\gamma} \geq 0$ such that

$$-\gamma_1 q(x) \leq \partial_{x_0} v(x) \leq -\gamma_2 q(x) + \widetilde{\gamma}; \tag{3.1}$$

(iii) $v \in C^\infty\left(\overline{\mathbb{R}^{n+1}_+}\right)$, and for every multi-index α there exist constants $C_\alpha > 0$ and $\widetilde{C_\alpha} > 0$ such that for every $x \in \overline{\mathbb{R}^{n+1}_+}$

$$\begin{aligned} \left|\partial^\alpha \left(\partial_{x_0} v(x)\right)\right| &\leq C_\alpha q(x)^b, \\ \left|\partial^\alpha \left(\nabla_{x'} v(x)\right)\right| &\leq \widetilde{C_\alpha} q(x). \end{aligned} \tag{3.2}$$

3.1.1 Examples of weight functions

In this section we construct weight functions in the class $W_b(D, q)$ applying the theory of convex functions.

Let $\chi(\eta')$, $\eta' \in \mathbb{R}^n$ be a differentiable strictly convex function(see [27],pp.253,259). We suppose also that χ is co-finite, that is

$$\lim_{\eta \to \infty} \frac{\chi(\eta')}{|\eta'|} = +\infty.$$

We associate with the function χ the convex domain ([4], p.39)

$$\mathcal{D}_\chi = \left\{(\eta_0, \eta') \in \mathbb{R}^{n+1} : \eta_0 < -\chi(\eta')\right\}, \tag{3.3}$$

and the function

$$\chi^*(x') = \sup_{\eta' \in \mathbb{R}^n} \{x' \cdot \eta' - \chi(\eta')\},\ x \in \mathbb{R}^n \tag{3.4}$$

which is called the *conjugate* ([27], p.104) or the *Young dual* ([4], p.11) function for χ. The function χ^* (see [27], Theorems 26.5, 26.6) has the following properties:

- the function $\chi^*$is differentiable, strictly convex, and co-finite;
- the *gradient mapping* $\nabla\chi : \mathbb{R}^n \to \mathbb{R}^n$ is invertible, and $(\nabla\chi)^{-1} = \nabla\chi^*$.
- Moreover,

$$\chi^*(x') = x' \cdot \nabla\chi^*(x') - \chi(\nabla\chi^*(x')). \tag{3.5}$$

Let χ be a strictly convex and co-finite function. We set

$$v(x) = (x_0+\delta)\chi^*\left(\frac{x'}{x_0+\delta}\right),\ x = (x_0, x') \in \overline{\mathbb{R}^{n+1}_+}, \delta > 0, \tag{3.6}$$

and

$$v_h(x) = v(x) + hx_0,\ h < 0.$$

Then,

$$\frac{\partial v(x)}{\partial x_0} = \left(\chi^*(y') - y'\cdot\left(\nabla_{y'}\chi^*\right)(y')\right)|_{y'=\frac{x'}{x_0+\delta}} \tag{3.7}$$

and

$$\nabla_{x'} v(x) = \left(\nabla_{y'}\chi^*\right)(y')\,|_{y'=\frac{x'}{x_0+\delta}}\ . \tag{3.8}$$

Equalities (3.5), (3.7) and (3.8) yield that for every $h < 0$

$$\begin{aligned} \frac{\partial v_h(x)}{\partial x_0} &= \frac{\partial v(x)}{\partial x_0} + h < \frac{\partial v(x)}{\partial x_0} = \\ -\chi(\nabla_{x'} v(x)) &= -\chi(\nabla_{x'} v_h(x)),\ x \in \mathbb{R}^{n+1}_+. \end{aligned}$$

Hence $\nabla v_h(x) \in \mathcal{D}_\chi$ for every $x \in \mathbb{R}^{n+1}_+$ and $h < 0$. Moreover if conditions (3.1), (3.2) hold then $w(x) = e^{v(x)} \in \mathcal{W}_b(\mathcal{D}_\chi, q)$.

Example 3.2. Let $\chi(\eta') = \frac{1}{2}A\eta'\cdot\eta'$ where A is a positively defined symmetric matrix. Hence ([27], page 108)

$$\chi^*(x') = \frac{1}{2}A^{-1}x'\cdot x',$$

and

$$\mathcal{D}_\chi = \left\{(\eta_0, \eta') \in \mathbb{R}^{n+1} : \eta_0 < -\frac{1}{2}A\eta'\cdot\eta'\right\}.$$

The associated weight is

$$v(x_0, x') = \frac{1}{2(x_0+\delta)}(A^{-1}x'\cdot x'), \delta > 0.$$

Then $\nabla v_h(x) \in \mathcal{D}_\chi$ for every $x = (x_0, x') \in \mathbb{R}^{n+1}_+$ and $h < 0$. Note if $q(x) = 1 + \frac{\langle x'\rangle}{\langle x_0\rangle}$ then $w_h(x) = e^{v_h(x)} \in \mathcal{W}_2(\mathcal{D}_\chi, q)$.

Let $\left(\overline{\mathbb{R}}_+\right)^n = \overline{\mathbb{R}}_+ \times \ldots \times \overline{\mathbb{R}}_+$, and a function $\chi \in C^1(\mathbb{R}^n)$ be of the form

$$\chi(\eta') = g(|\eta_1|, \ldots, |\eta_n|), \tag{3.9}$$

where

(a) $g \in C^1\left((\overline{\mathbb{R}}_+)^n\right) \cap C^\infty((\mathbb{R}_+)^n)$ be strictly convex and co-finite;

(b) g satisfies the condition

$$\det\left(\frac{\partial^2 g(t')}{\partial t_i \partial t_j}\right)^n_{i,j=1} \neq 0, \tag{3.10}$$

for every $t' \in (\mathbb{R}_+)^n$;

Then $\chi(\eta') \in C^1(\mathbb{R}^n) \cap C^\infty((\mathbb{R}_+)^n)$ is a strictly convex, co-finite function on $\mathbb{R}^n$, and $\frac{\partial \chi(\eta')}{\partial \eta_j} > 0, j = 1,...,n$ for all $\eta' \in (\mathbb{R}_+)^n$. One can see that the mapping

$$\nabla\chi : (\mathbb{R}_+)^n \to (\mathbb{R}_+)^n$$

is well defined.

The function $\chi^*(x)$ conjugate to function $\chi(\eta')$ is (see [27], p.111) a strictly convex and co-finite function of the form

$$\chi^*(x') = g^+(|x_1|,...,|x_n|),$$

where

$$g^+(z') = \sup_{t' \in \left(\overline{\mathbb{R}_+}\right)^n} \{z' \cdot t' - g(t')\},\ z' \in \left(\overline{\mathbb{R}_+}\right)^n$$

is *the monotone conjugate function of* $g(t')$. Moreover

$$(\nabla\chi)^{-1} = \nabla\chi^* : (\mathbb{R}_+)^n \to (\mathbb{R}_+)^n. \tag{3.11}$$

Condition 3.10 provides that $\nabla\chi : (\mathbb{R}_+)^n \to (\mathbb{R}_+)^n$, and $\nabla\chi^* : (\mathbb{R}_+)^n \to (\mathbb{R}_+)^n$ are $C^\infty-$diffeomorphisms.

Let $\langle y\rangle_\nu = (\nu^2 + y^2)^{1/2}, \nu > 0,\ y \in \mathbb{R}$, and χ be of the form (3.9), and satisfy condition (3.10). We introduce a function $v : \mathbb{R}^{n+1}_+ \to \mathbb{R}$ as

$$\begin{aligned} v(x) &= (x_0+\delta)\chi^*\left(\frac{\langle x_1\rangle_\nu}{x_0+\delta},...,\frac{\langle x_n\rangle_\nu}{x_0+\delta}\right) \\ &= (x_0+\delta)g^+\left(\frac{\langle x_1\rangle_\nu}{x_0+\delta},...,\frac{\langle x_n\rangle_\nu}{x_0+\delta}\right), \\ x &= (x_0,x') \in \mathbb{R}^{n+1}_+, \delta > 0. \end{aligned}$$

Note that $v \in C^\infty\left(\overline{\mathbb{R}^{n+1}_+}\right)$.

Let the domain $\mathcal{D}_\chi$ be defined by (3.4) and $v_h(x) = v(x) + hx_0,\ h < 0$. Now we will prove that $\nabla v_h(x) \in \mathcal{D}_\chi$. Indeed, applying (3.3), and (3.5) we obtain

$$\begin{gathered} \frac{\partial v_h(x)}{\partial x_0} < \frac{\partial v(x_0,x')}{\partial x_0} = \\ g^+\left(\frac{\langle x_1\rangle_\nu}{x_0+\delta},...,\frac{\langle x_n\rangle_\nu}{x_0+\delta}\right) - \sum_{j=1}^n y_j \frac{\partial g^+(y)}{\partial y_j} \Big|_{y=\left(\frac{\langle x_1\rangle_\nu}{x_0+\delta},...,\frac{\langle x_n\rangle_\nu}{x_0+\delta}\right)} = \\ \left(g^+(y) - y\cdot\nabla g^+(y)\right) \Big|_{y=\left(\frac{\langle x_1\rangle_\nu}{x_0+\delta},...,\frac{\langle x_n\rangle_\nu}{x_0+\delta}\right)} = \\ -g\left(\nabla_y g^+(y)\right) \Big|_{y=\left(\frac{\langle x_1\rangle_\nu}{x_0+\delta},...,\frac{\langle x_n\rangle_\nu}{x_0+\delta}\right)}. \end{gathered} \tag{3.12}$$

Further,

$$\frac{\partial v(x_0,x')}{\partial x_j} = \frac{x_j}{\left\langle x_j\right\rangle_\nu} \cdot \frac{\partial g^+(y)}{\partial y_j} \Big|_{y=\left(\frac{\langle x_1\rangle_\nu}{x_0+\delta},...,\frac{\langle x_n\rangle_\nu}{x_0+\delta}\right)}.$$

Hence

$$\left|\frac{\partial v(x_0,x')}{\partial x_j}\right| \le \left|\frac{\partial g^+(y)}{\partial y_j} \Big|_{y=\left(\frac{\langle x_1\rangle_\nu}{x_0+\delta},...,\frac{\langle x_n\rangle_\nu}{x_0+\delta}\right)}\right|. \tag{3.13}$$

The monotonic property of g and (3.13) imply that

$$\begin{gathered} -g\left(\nabla_y g^+(y)\right)|_{y=\left(\frac{\langle x_1\rangle_\nu}{x_0+\delta},\dots,\frac{\langle x_n\rangle_\nu}{x_0+\delta}\right)} \leq \\ -g(\nabla_{x'} v(x_0,x')) = -\chi(\nabla_{x'} v(x_0,x')) = -\chi(\nabla_{x'} v_h(x_0,x')) \end{gathered}. \tag{3.14}$$

Applying formulas (3.12) and (3.14) we obtain

$$\frac{\partial v_h(x)}{\partial x_0} < -\chi(\nabla_{x'} v_h(x)),\ x \in \mathbb{R}^{n+1}_+.$$

Therefore $\nabla v_h(x) \in \mathcal{D}_\chi$ for every $x \in \mathbb{R}^{n+1}_+$, $h < 0$.

Example 3.3. Let

$$p(\xi) = i\xi_0 + \sum_{|\alpha|=2m} a_{\alpha'}\xi'^{\alpha'} \equiv i\xi_0 + Q_{2m}(\xi')$$

be a $2m$-parabolic polynomial ([9], p.12),$m \geq 1$, that is

$$\inf_{\xi' \in \mathbb{R}^n / \{0\}} \frac{\Re(Q_{2m}(\xi'))}{|\xi'|^{2m}} = \nu > 0.$$

Following [4], pp. 39-40 and [5] we introduce the function

$$\chi_{p_0}(\eta') = \sup_{\xi' \in \mathbb{R}^n} \{-\Re(Q_{2m}(\xi' + i\eta'))\}.$$

The function $\chi_{p_0}(\eta')$ is a convex, continuous, homogeneous of the degree $2m$ and there exist positive constants C_1 and C_2 such that

$$C_1\left|\eta'\right|^{2m} \leq \chi_{p_0}(\eta') \leq C_2\left|\eta'\right|^{2m},\ \eta' \in \mathbb{R}^n.$$

(see [5] ,Theorem 1.1). Moreover ([5], Theorem 1.16) $\chi^*_{p_0}(x')$ is a convex, co-finite, homogeneous of the order $\frac{2m}{2m-1}$ function and there exist positive constants c_1 and c_2, such that

$$c_1\left|x'\right|^{\frac{2m}{2m-1}} \leq \chi^*_{p_0}(x') \leq c_2\left|x'\right|^{\frac{2m}{2m-1}}.$$

Let χ_{p_0} be of the form (3.9) and the conditions $(a),(b)$ hold. Then the function v defined by (3.6) is of the form

$$v(x) = \frac{\chi^*_{p_0}(\langle x_1\rangle_\nu,\dots,\langle x_n\rangle_\nu)}{(x_0+\delta)^{\frac{1}{2m-1}}} \in C^\infty\left(\overline{\mathbb{R}^{n+1}_+}\right), \delta > 0, \tag{3.15}$$

and $\nabla v(x) \in \mathcal{D}_\chi$. Moreover there exist constants $\gamma_1, \gamma_2 > 0$, $\widetilde{\gamma} \geq 0$, such that

$$-\gamma_1\left[\widehat{q}_m(x)\right]^{2m} \leq \partial_0 v(x) \leq -\gamma_2\left[\widehat{q}_m(x)\right]^{2m} + \widetilde{\gamma} \tag{3.16}$$

and for every multi-index α

$$\begin{gathered} \left|\partial^\alpha(\partial_{x_0} v(x))\right| \leq C_\alpha\left[\widehat{q}_m(x)\right]^{2m}, \\ |\partial^\alpha(\nabla_{x'} v(x))| \leq \widetilde{C}_\alpha \widehat{q}_m(x) \end{gathered} \tag{3.17}$$

where $\widehat{q_m}(x) = \left(1 + \frac{\langle x' \rangle}{\langle x_0 \rangle}\right)^{\frac{1}{2m-1}}$ if $m > 1$ and $\widehat{q_1}(x) \equiv \widehat{q}(x) = 1 + \frac{\langle x' \rangle}{\langle x_0 \rangle}$. Hence the weight function $w_h(x) = e^{v_h(x)} \in \mathcal{W}_{2m}(\mathcal{D}_\chi, \widehat{q_m})$, $h < 0$.

Consider the parabolic symbols of the form

$$p_0(\xi) = i\xi_0 + a(\xi_1^2 + \ldots + \xi_n^2)^m.$$

In this case (see [5])

$$\chi(\eta') = a(\eta_1^2 + \ldots + \eta_n^2)^m, m \in \mathbb{N},$$

$$\chi^*(x') = c_m \left(x_1^2 + \ldots + x_n^2\right)^{\frac{m}{2m-1}} \tag{3.18}$$

where $c_m = a^{-\frac{1}{2m-1}}(2m-1)(2m)^{2m-1}$, and

$$v(x) = c_m \frac{\left(1 + x_1^2 + \ldots + x_n^2\right)^{\frac{m}{2m-1}}}{(x_0 + \delta)^{\frac{1}{2m-1}}}.$$

Thus $w_h(x) = e^{v_h(x)} \in \mathcal{W}_{2m}(\mathcal{D}_\chi, \widehat{q_m})$, $h < 0$.

3.2 Composition of pseudodifferential operators and exponential weights

Let $\mathcal{D}$ be a convex unbounded domain in $\mathbb{R}^n$. We suppose that $\mathcal{D}$ contains the ray

$$r_- = \left\{\eta \in \mathbb{R}^{n+1} : \eta = (\eta_0, 0, \ldots, 0),\ \eta_0 < 0\right\}.$$

Definition 3.4. Let $a \in S^m(\lambda_{q,b}, \mathbb{R}^{n+1}_+, \Pi_-)$, $m \in \mathbb{R}$. We say that $a \in S^m(\lambda_{q,b}, \mathbb{R}^{n+1}_+, \mathcal{D})$ if for any fixed point $x \in \mathbb{R}^{n+1}_+$ the function $a(x,\xi)$ has an analytic extension with respect to the variable ξ in the tube domain $T_{\mathcal{D}} = \mathbb{R}^n + i\mathcal{D}$, and for all $l_1, l_2 \in \mathbb{N}_0$

$$\{a\}_{l_1,l_2} = \sum_{|\alpha| \le l_1, |\beta| \le l_2} \sup_{(x,\xi+i\eta) \in \mathbb{R}^{n+1}_+ \times T_{\mathcal{D}}} \frac{\left|a^{(\alpha)}_{(\beta)}(x, \xi + i\eta)\right|}{\lambda_{q,b,\eta}(x,\xi)^{m - \left(\alpha_0 + \frac{|\alpha'|}{b}\right)}} < \infty \tag{3.19}$$

where

$$\lambda_{q,b,\eta}(x,\xi) = |\xi_0| + |\eta_0| + \left|\xi'\right|^b + \left|\eta'\right|^b + q(x).$$

We denote by $OPS^m(\lambda_{q,b}, \mathbb{R}^{n+1}_+, \mathcal{D})$ the corresponding class of $\psi do's$.

Note that $S^m(\lambda_{q,b}, \mathbb{R}^{n+1}_+, \mathcal{D}) \subset S^m(\lambda_{q,b}, \mathbb{R}^{n+1}_+, \Pi_-)$ because the ray $r_- \subset \mathcal{D}$.

Remark 3.5. Since $r_- \subset \mathcal{D}$, it follows from [27] (Theorem 8.3) that for each $\eta = (\eta_0, \eta') \in \mathcal{D}$ and $h < 0$ the point $(\eta_0 + h, \eta') \in \mathcal{D}$ also. Therefore if $a(x,\xi) \in S^m(\lambda_{q,b}, \mathbb{R}^{n+1}_+, \mathcal{D})$, then Definitions 2.6 and 3.4 imply that the symbol $a(x, \xi + i\eta) \in S^m(\lambda_{q,b}, \mathbb{R}^{n+1}_+, \Pi_-)$ for each $\eta \in \mathcal{D}$.

Theorem 3.6. *Let $q_1, q_2 \in E(\mathbb{R}^{n+1})$, $a \in S^m(\lambda_{q_1,b}, \mathbb{R}^{n+1}_+, \mathcal{D})$ and $w(x) \in \mathcal{W}_b(\mathcal{D}, q_2)$. Then the operator $A_w \equiv wOp(a)w^{-1} \in OPS^m(\lambda_{q,b}, \mathbb{R}^{n+1}_+, \Pi_-)$ where $q = q_1 + q_2$ and it symbol $a_w(x,\xi)$ can be represented of the form*

$$a_w(x,\xi) = a(x, \xi + i\nabla v(x)) + r(x,\xi), \tag{3.20}$$

where $a(x, \xi + i\nabla v(x)) \in S^m(\lambda_{q,b}, \mathbb{R}^{n+1}_+, \Pi_-)$ and $r \in S^{m-\frac{1}{b}}(\lambda_{q,b}, \mathbb{R}^{n+1}_+, \Pi_-)$.

Proof. Following to the papers [23], see (also [25], [26]) we obtain the representation

$$A_w u(x) = \int_{\mathbb{R}^{n+1}} d'\xi \int_{\mathbb{R}^{n+1}_+} e^{i(x-y)\cdot\xi} a_g(x,y,\xi)u(y)dy,\ u \in C_0^\infty(\mathbb{R}^{n+1}_+)$$

where $a_g(x,y,\xi) = a(x,\xi + ig_v(x,y))$, and

$$g_v(x,y) = \int_0^1 \nabla v((1-\theta)x + \theta y)d\theta. \tag{3.21}$$

Because the domain $\mathcal{D}$ is convex, $g_v(x,y) \in \mathcal{D}$ for every points $x, y \in \mathbb{R}^{n+1}_+$. The operator A_w can be represented as a ψdo of the form

$$A_w u(x) = \int_{\mathbb{R}^{n+1}} d'\xi \int_{\mathbb{R}^{n+1}_+} e^{i(x-y)\cdot\xi} a_w(x,\xi)u(y)dy,\ u \in C_0^\infty(\mathbb{R}^{n+1}_+)$$

where

$$a_w(x,\xi) = \int_{\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}_+} a_g(x, x+y, \xi+\omega)e^{-iy\cdot\omega}dyd'\omega \tag{3.22}$$

and the double integral is understood as oscillatory (see for instance [14], [15], [26]). The Lagrange formula imply

$$a_g(x, x+y, \xi+\omega) = a_g(x, x+y, \xi) + r(x,y,\xi,\omega), \tag{3.23}$$

where

$$r(x,y,\xi,\omega) = \sum_{j=0}^{n}\left[\int_0^1 \partial_{\xi_j} a_g(x, x+y, \xi+t\omega)dt\right]\omega_j. \tag{3.24}$$

It follows from (3.22)- (3.24) that

$$a_w(x,\xi) = \int_{\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}_+} a_g(x, x+y, \xi)e^{-iy\cdot\omega}dyd'\omega + r(x,\xi), \tag{3.25}$$

where

$$r(x,\xi) = \sum_{j=0}^{n}\int_0^1 r_{t,j}(x,\xi)dt. \tag{3.26}$$

and

$$r_{t,j}(x,\xi) = \int_{\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}_+} \partial_{\xi_j} D_{y_j} a_g(x, x+y, \xi+t\omega)e^{-iy\cdot\omega}dyd'\omega. \tag{3.27}$$

Now, applying the well known properties of the oscillatory integral (see for instance [14], [15], [26]) we obtain

$$\int_{\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}_+} a_g(x, x+y, \xi)e^{-iy\cdot\omega}dyd'\omega = a_g(x,x,\xi) = a(x, \xi + i\nabla v(x)) \tag{3.28}$$

By Definitions 3.1, 3.4 and Remark 3.5

$$a(x,\xi+i\nabla v(x)) \in S^{m}(\lambda_{q,b},\mathbb{R}^{n+1}_{+},\Pi_{-}). \tag{3.29}$$

For the estimate of the symbol r we use the following regularization of the oscillatory double integral

$$\begin{gathered} r_{t,j}(x,\xi) = \int\limits_{\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}_{+}} \partial_{\xi_j} D_{y_j} a_g(x,x+y,\xi+t\omega)e^{-iy\omega}dyd'\omega = \\ \int\limits_{\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}_{+}} \langle y\rangle^{-2k_1}\left\langle D_\xi\right\rangle^{2k_1}\left\{\langle\xi\rangle^{-2k_2}\left\langle D_y\right\rangle^{2k_2} b_{t,j}(x,y,\xi,\omega)\right\}e^{-iy\cdot\omega}dyd'\omega, \\ j=1,\dots n,\ t\in[0.1] \end{gathered} \tag{3.30}$$

where $2k_1, 2k_2 \in 2\mathbb{N}$ are large enough, and

$$\begin{gathered} b_{t,j}(x,y,\xi,\omega) = \partial_{\xi_j} D_{y_j} a_g(x,x+y,\xi+t\omega) = \\ \partial_{\xi_j} D_{y_j} a(x,\xi+t\omega+ig_v(x,x+y)). \end{gathered}$$

In light of (2.1) there exist L such that for every α

$$\left|\partial^\alpha q_2(x+y)\right| \le C_\alpha q_2(x)\langle y\rangle^L$$

Therefore it follows from Definition 3.1 and (3.21) that

$$|g_v(x,x+y)| \le \int\limits_0^1 |\nabla v(x+\theta y)|\,d\theta \le Cq_2(x)\langle y\rangle^{L_1}$$

for some constants C and L_1.

Applying estimates (3.1) (3.2) and (3.29) we obtain

$$\begin{gathered} \left|\langle y\rangle^{-2k_1}\left\langle D_\xi\right\rangle^{2k_1}\left\{\langle\xi\rangle^{-2k_2}\left\langle D_y\right\rangle^{2k_2} b_{t,j}(x,y,\xi,\omega)\right\}\right| \le \\ C\lambda_{q,b}(x,\xi)^{m-\frac{1}{b}}\langle y\rangle^{L\left|m-\frac{1}{b}\right|-2k_1}\langle\omega\rangle^{b\left|m-\frac{1}{b}\right|-2k_2} \end{gathered}$$

Let $2k_1 > L\left|m-\frac{1}{b}\right|+n$, $2k_2 > b\left|m-\frac{1}{b}\right|+n$, then (3.26) and (3.30) imply that

$$|r(x,\xi)| \le C\lambda_{q,b}(x,\xi)^{m-\frac{1}{b}}.$$

In the same way we obtain the estimates

$$\left|\partial_x^\beta\partial_\xi^\alpha r(x,\xi)\right| \le C_{\gamma\beta}(q(x)+|\xi_0|+|\xi|)^{m-\frac{1}{b}-|\alpha_0|-\frac{|\alpha'|}{b}}.$$

Hence Remark 3.5 implies that $r \in S^{m-\frac{1}{b}}(\lambda_{q,b},\mathbb{R}^{n+1}_{+},\Pi_{-})$. Hence (3.25), (3.28) and (3.29) imply that (3.20) holds and $A_w \in OPS^{m}(\lambda_{q,b},\mathbb{R}^{n+1}_{+},\Pi_{-})$. □

Remark 3.7. Let $a \in S^{m}(\mathbb{R}^{n+1}_{+},\lambda_{q,b})$, $m \ge 0$ be a polynomial with respect to the variable ξ. Then representation (3.20) holds for every weight $w(x) = \exp v(x)$ if $\nabla v(x)$ satisfies estimates (3.1) and (3.2).

3.3 Invertibility of parabolic pseudodifferntial operators in exponential weighted spaces

Let w be a weight. We denote by $H_0^s(\lambda_{q,b},\mathbb{R}_+^{n+1},w)$ the weighted Sobolev space with the norm

$$\|u\|_{H_0^s(\lambda_{q,b},\mathbb{R}_+^{n+1},w)} = \|wu\|_{H_0^s(\lambda_{q,b},\mathbb{R}_+^{n+1})}.$$

Let

$$w_h(x) = e^{v_h(x)} \equiv e^{v(x)+hx_0},\ x = (x_0,x') \in \overline{\mathbb{R}_+^{n+1}},\ h \le 0. \tag{3.31}$$

Proposition 3.8. *Let the conditions of Theorem 3.6 be fulfilled. Then the operator*

$$Op(a) : H_0^s(\lambda_{q,b},\mathbb{R}_+^{n+1},w_h) \to H_0^{s-m}(\lambda_{q,b},\mathbb{R}_+^{n+1},w_h) \tag{3.32}$$

is bounded for every $h \le 0$.

This proposition is a corollary of Theorem 3.6 and Proposition 2.9.

Theorem 3.9. *Let the conditions of Theorem 3.6 be fulfilled, and*

$$\lim_{h\to-\infty} \inf_{(x,\xi)\in\mathbb{R}_+^{n+1}\times\mathbb{R}^{n+1}} \frac{\left|a(x,\xi_0 + i(\partial_{x_0}v(x)+h),\xi' + i\nabla_{x'}v(x))\right|}{\lambda_{q,b,h}^m(x,\xi)} > 0. \tag{3.33}$$

Then for every $s \in \mathbb{R}$ there exists $h_0 = h_0(s) < 0$ such that for all $h \le h_0$ operator (3.32) is invertible.

Proof. The invertibility of (3.32) is equivalent to the invertibility of the operator

$$Op(a_w) = wOp(a)w^{-1} : H_0^s(\lambda_{q,b},\mathbb{R}_+^{n+1},e^{hx_0}) \to H_0^{s-m}(\lambda_{q,b},\mathbb{R}_+^{n+1},e^{hx_0}). \tag{3.34}$$

It follows from (3.20) that

$$a_w(x,\xi) = a(x,\xi + i\nabla v(x)) + r(x,\xi),$$

where $r(x,\xi) \in S^{m-\frac{1}{b}}(\lambda_{q,b},\mathbb{R}_+^{n+1},\Pi_-)$. Therefore

$$\lim_{h\to-\infty} \inf_{(x,\xi)\in\mathbb{R}_+^{n+1}\times\mathbb{R}^{n+1}} \frac{|r(x,\xi_0+ih,\xi')|}{\lambda_{q,b,h}^m(x,\xi)} = 0$$

and applying condition (3.33) we obtain

$$\lim_{h\to-\infty} \inf_{(x,\xi)\in\mathbb{R}_+^{n+1}\times\mathbb{R}^{n+1}} \frac{|a_w(x,\xi_0+ih,\xi')|}{\lambda_{q,b,h}^m(x,\xi)} > 0.$$

Hence by Theorem 2.11 the operator (3.34) is invertible for all $h < -h_0$ where $|h_0| > 0$ is large enough. □

Now we apply the previous results for exponential estimates of fundamental solutions of differential operators.

We recall that a distribution $g_y(x) (\in \mathcal{S}'(\mathbb{R}_+^{n+1}))$ is called a *fundamental solution* of a differential operator $a(x,D)$ if

$$a(x,D)\,g_y(x) = \delta(x-y), x = (x_0,x'),\ y = (y_0,y') \in \mathbb{R}_+^{n+1}$$

where δ is the Dirac distribution.

Theorem 3.10. *Let a differential operator $a(x,D) \in OPS^k(\lambda_{q_1,b}, \mathbb{R}^{n+1}_+, \mathcal{D})$, $s > \frac{n+b}{2b}$, and the conditions of Theorem 3.9 be fulfilled. Then there exists $h_0 < 0$ such that the differential operator $a(x,D)$ has an unique fundamental solution g_y in the space $H_0^{-s+k}(\lambda_{q,b}, \mathbb{R}^{n+1}, w_h)$ where $w_h \in \mathcal{W}_b(\mathcal{D}, q_2)$ and $h \le h_0$.*

Proof. Easy calculations show that $\delta(\cdot - y) \in H_0^{-s}(\lambda_{q,b}, \mathbb{R}^{n+1}, w_h)$ if $s > \frac{n+b}{2b}$. Hence Theorem 3.9 yields that $g_y \in H_0^{-s+k}(\lambda_{q,b}, \mathbb{R}^{n+1}_+, w_h)$ for every $y \in \mathbb{R}^{n+1}_+$ if $s > \frac{n+b}{2b}$, and $h \le h_0$. □

4 Parabolic differential operators in general exponential weighted spaces

4.1 Convex functions corresponding to parabolic differential operators

Let $p_0(x,D)$ differential operator of the form

$$p_0(x,D) = iD_0 + Q_{2m}(x,D') \tag{4.1}$$

with symbol

$$p_0(x,\xi) = i\xi_0 + Q_{2m}(x,\xi') \tag{4.2}$$

where

$$Q_{2m}(x,\xi') = \sum_{|\alpha|=2m} a_\alpha(x)\xi'^\alpha. \tag{4.3}$$

We suppose that the coefficients $a_\alpha \in C_b^\infty(\mathbb{R}^{n+1})$ the class of functions in $C^\infty(\mathbb{R}^{n+1})$ bounded with all their partial derivatives.

We suppose that the differential operator $p_0(x,D)$ is uniformly parabolic. It implies ([9], p.74) that there exists a constant $\nu > 0$ such that

$$\inf_{x\in\mathbb{R}^n_+,\, \xi'\in\mathbb{R}^n \diagup \{0\}} \frac{\{\Re(Q_{2m}(x,\xi'))\}}{|\xi'|^{2m}} = \nu > 0. \tag{4.4}$$

For every fixed $x \in \mathbb{R}^{n+1}_+$ we introduce the function

$$\chi_{p_0}(x,\eta') = \sup_{\xi'\in\mathbb{R}^n} \{-\Re(Q_{2m}(x,\xi' + i\eta'))\}, \,. \tag{4.5}$$

It follows from Example 3.3) the function $\chi_{p_0}(x,\eta')$ is a convex continuous and homogeneous of the degree $2m$ with respect to $\eta' \in \mathbb{R}^n$ for every fixed $x \in \mathbb{R}^{n+1}_+$. Moreover there exist constants $C_j(x) \ge C_j^0 > 0$, $j = 1,2$ such that

$$C_1(x)\left|\eta'\right|^{2m} \le \chi_{p_0}(x,\eta') \le C_2(x)\left|\eta'\right|^{2m}, \eta' \in \mathbb{R}^n. \tag{4.6}$$

We set

$$\chi_{p_0}(\eta') = \sup_{x\in\mathbb{R}^{n+1}_+} \chi_{p_0}(x,\eta'). \tag{4.7}$$

Lemma 4.1. *Let the polynomial $Q_{2m}(x,\xi')$ satisfy condition (4.4). Then*

a) $\chi_{p_0}(\eta')$ is a convex continuous and homogeneous of order $2m$ function;

b) there exist positive constants d_1 and d_2 such that

$$d_1 \left|\eta'\right|^{2m} \le \chi_{p_0}(\eta') \le d_2 \left|\eta'\right|^{2m}, \ \eta' \in \mathbb{R}^n; \tag{4.8}$$

c) $\chi^*_{p_0}(\eta')$ is a convex, continuous, homogeneous of the order $\frac{2m}{2m-1}$ function, and there exist positive constants c_1 and c_2 such that

$$c_1 \left|x'\right|^{\frac{2m}{2m-1}} \le \chi^*_{p_0}(\eta') \le c_2 \left|x'\right|^{\frac{2m}{2m-1}}, \ x' \in \mathbb{R}^n \tag{4.9}$$

Proof. a) It follows from [27] (Theorem 5.5) that the function χ_{p_0} is a convex, homogeneous of the order $2m$ function, and (4.8) holds. Further, we will prove that χ_{p_0} is a finite on $\mathbb{R}^n$ function. It implies (see [27], Corollary10.1.1) that χ_{p_0} is a continuous function. Indeed, it follows from $2m-$homogeneity of $Q_{2m}(x,\xi')$, with respect to ξ' that

$$\left|D^{\varkappa}_{\xi} Q_{2m}(x,\xi')\right| < C_{\varkappa} \left|\xi'\right|^{2m-|\varkappa'|}, \ (x,\xi') \in \mathbb{R}^{n+1}_{+} \times \mathbb{R}^n \tag{4.10}$$

for all multi-indices $\varkappa$. Then applying the decomposition of $Q(x,\xi'+i\eta')$ for fixed $\eta' \in \mathbb{R}^n$ in the Taylor's series

$$Q_{2m}(x,\xi'+i\eta') = \sum_{0\le|\varkappa|\le 2m} \frac{1}{\alpha!} \frac{\partial^{\varkappa} Q_{2m}(x,\xi')}{\partial \xi'^{\varkappa}} (i\eta')^{\varkappa}$$

and condition (4.4) we obtain

$$Q_{2m}(x,\xi'+i\eta') = Q_{2m}(x,\xi')(1+G(x,\xi',\eta'))$$

where $G(x,\xi',\eta')$ is such that for every $\eta' \in \mathbb{R}^n$

$$\lim_{\xi'\to\infty} G(x,\xi',\eta') = 0$$

uniformly with respect to $x \in \mathbb{R}^{n+1}_{+}$. It yields that there exist $R = R(\eta')$ and $\delta = \delta(\eta') > 0$ such that

$$\Re(Q_{2m}(x,\xi'+i\eta')) \ge \delta$$

for all $\xi' : |\xi'| > R$ and for all $x \in x \in \mathbb{R}^{n+1}_{+}$.

$$\sup_{\{\xi' : |\xi'| \ge R\}} \{-\Re(Q_{2m}(x,\xi'+i\eta'))\} < 0$$

Since $\chi_{p_0}(x,\eta')$ is a positive function, there exists constant $K(\eta')$ such that

$$\chi_{p_0}(x,\eta') = \sup_{\{\xi' : |\xi'| \le R\}} \{-\Re(Q_{2m}(x,\xi'+i\eta')\} \le K(\eta'), \ \forall x \in \mathbb{R}^{n+1}$$

Hence applying (4.7)we obtain that $\chi_{p_0}(\eta') < \infty$, for every $\eta' \in \mathbb{R}^n$.

b) Because χ is a continuous homogeneous function of the degree $2m$ on $\mathbb{R}^n$ the restriction of $\chi\,|_{S^{n-1}}$is a continuous function. Hence there exist constants d_1, d_2 such that

$$d_1 \left|\eta'\right|^{2m} \le \chi_{p_0}(\eta') \le d_2 \left|\eta'\right|^{2m}, \ \eta' \in \mathbb{R}^n. \tag{4.11}$$

Further by the definition of $\chi_{p_0}(\eta')$ and formula (4.6) we obtain that for every fixed point $x_0 \in \mathbb{R}^{n+1}_+$

$$\chi_{p_0}(\eta') \geq \chi_{p_0}(x_0, \eta') \geq C_1(x_0) \left|\eta'\right|^{2m}$$

where $C_1(x_0) > 0$. Hence $d_1 > 0$.

Assertion c) follows from [27] (Corollary 15.3.1), (4.8), (3.18), and ([27], p.104): the inequality $f_1 \leq f_2$ implies the inequality $f_2^* \leq f_1^*$. □

Remark 4.2. a) Let

$$\mathcal{D}_{\chi_{p_0}}(x, \eta') = \left\{(\eta_0, \eta') \in \mathbb{R}^{n+1} : \eta_0 < -\chi_{p_0}(x, \eta')\right\}$$

and

$$\mathcal{D}_{\chi_{p_0}} = \left\{(\eta_0, \eta') \in \mathbb{R}^{n+1} : \eta_0 < -\chi_{p_0}(\eta')\right\}. \tag{4.12}$$

Then

$$\mathcal{D}_{\chi_{p_0}} = \bigcap_{x \in \mathbb{R}^{n+1}_+} \mathcal{D}_{\chi_{p_0}}(x, \eta'). \tag{4.13}$$

b) Note that for every point $\eta = (\eta_0, \eta') \in \overline{\mathcal{D}}_{\chi_{p_0}} \setminus \{0\}$ and for every $\varepsilon \in (0,1)$ the point $(\varepsilon\eta_0, \varepsilon\eta') \in \mathcal{D}_{\chi_{p_0}}$.

It is easy to prove the following Lemma.

Lemma 4.3. *Let operator (4.1) be uniformly* $2m-$ *parabolic. Then for any* $\varepsilon \in [0,1)$ *there exists a constant* $C = C(\varepsilon)$ *such that the follows inequality holds*

$$|p_0(x, \xi + i\varepsilon\eta)| \geq C\left(|\eta_0| + |\xi_0| + \left|\eta'\right|^{2m} + \left|\xi'\right|^{2m}\right) \tag{4.14}$$

where $(x, \xi + i\eta) \in \Omega_+ \times \mathbb{T}_{\overline{\mathcal{D}}_{\chi_{p_0}}}$ *holds.*

4.2 Invertibility of parabolic differential operators in exponential weighted spaces

We consider the differential operator of the form

$$p(x, D) = \partial_{x_0} + \sum_{0<|\alpha'|\leq 2m} a_{\alpha'}(x) D'^{\alpha'} + b(x), \tag{4.15}$$

with symbol

$$p(x, \xi) = i\xi_0 + \sum_{0<|\alpha'|\leq 2m} a_{\alpha'}(x) \xi'^{\alpha'} + b(x).$$

As above (see (4.1)-(4.3)) we set

$$p_0(x, \xi) = i\xi_0 + \sum_{|\alpha'|=2m} a_{\alpha'}(x) \xi'^{\alpha'} = i\xi_0 + Q_{2m}(x, \xi'). \tag{4.16}$$

We say that a polynomial of the form (4.15) belongs to the class $\mathcal{P}_{2m}(q_1)$, where $q_1 \in E\left(\mathbb{R}^{n+1}\right)$ if the following conditions holds:

(1) $p_0(x, D)$ is uniformly $2m-$ parabolic (see (4.4));

(2) $a_\alpha \in C_b^\infty(\mathbb{R}_+^{n+1})$;
(3) $b \in C^\infty(\mathbb{R}_+^{n+1})$,

$$\inf_{x\in\Omega_+} \frac{b(x)}{q_1(x)} = b_0 > 0, \tag{4.17}$$

and

$$\left|\partial^\beta b(x)\right| < C_\beta q_1(x) \tag{4.18}$$

for all multi-indices β.

Remark 4.4. It follows from condition (**1**) and (4.18), that $p(x,D) \in OPS^1(\lambda_{q_1,2m}, \mathbb{R}_+^{n+1}, \mathcal{D})$ and

$$p(x,D) - p_0(x,D) \in OPS^{1-1/2m}(\lambda_{q_1,2m}, \mathbb{R}_+^{n+1}, \mathcal{D}),$$

for any convex domain $\mathcal{D}$.

Let the function χ_{p_0} be defined by (4.7) and $\chi_{p_0}^*$ be conjugate for χ_{p_0}. It follows from Lemma 4.1, c) that the above defined function v is of the form

$$v(x) = \frac{\chi_{p_0}^*(\langle x_1\rangle_\nu, ..., \langle x_n\rangle_\nu)}{(x_0+\delta)^{\frac{1}{2m-1}}}, \tag{4.19}$$

Let

$$w_{\varepsilon,h}(x) = e^{v_{\varepsilon,h}(x)},\ \varepsilon \in [0,1), h > 0.$$

where $v_{\varepsilon,h}(x) = \varepsilon v(x) + hx_0$ and $v(x)$ defined by (4.19).

Theorem 4.5. *Let* $p(x,D) \in \mathcal{P}_{2m}(q_1)$, $w_{\varepsilon,h}(x) \in \mathcal{W}_{2m}(\mathcal{D}_{\chi_{p_0}}, \widehat{q}_m)$, $\varepsilon \in (0,1)$, $h < 0$ *and* $q(x) = q_1(x) + [\widehat{q}_m(x)]^{2m}$. *Then for every* $s \in \mathbb{R}$ *there exists* $h_0 = h_0(s) < 0$ *such that the operator*

$$p(x,D) : H_0^s(\lambda_{q,2m}, \mathbb{R}_+^{n+1}, w_{\varepsilon,h}(x)) \to H_0^{s-1}(\lambda_{q,2m}, \mathbb{R}_+^{n+1}, w_{\varepsilon,h}(x)) \tag{4.20}$$

is an isomorphism for all $h \le h_0$.

Proof. The case $\varepsilon = 0$ was studied in Section 2, hence we suppose that $\varepsilon \in (0,1)$.In light of Remark 4.4, and Theorem 3.6 to prove the invertibility of operator (4.20) it is enough to prove the inequality:

$$\left|p_0(x, \xi + i\nabla(v_{\varepsilon,h}(x)))\right| \ge C\lambda_{q,2m,h},\ (x,\xi) \in \mathbb{R}_+^{n+1} \times \mathbb{R}^{n+1}, \tag{4.21}$$

with a constant $C = C(\varepsilon) > 0$, and

$$\lambda_{q,2m,h}(x,\xi) = 1 + h + |\xi_0| + \left|\xi'\right|^{2m} + q(x).$$

Since $w_{\varepsilon,h}(x) \in \mathcal{W}_{2m}(\mathcal{D}_{\chi_{p_0}}, \widehat{q})$ then for every $\varepsilon \in (0,1)$ and $h < 0$, Remark (4.2) a) implies that $\nabla v_{\varepsilon,h}(x) \in \mathcal{D}_{\chi_{p_0}}$ for every $x \in \mathbb{R}_+^{n+1}$. It follows from estimate (3.16) that $\partial_0\mu(x) < 0$, then Lemma 4.3 yields that

$$\begin{gathered} \left|p_0(x, \xi + i\nabla(v_{\varepsilon,h}(x)))\right| = \\ |p_0(x, \xi_0 + i(\varepsilon(\partial_0\mu(x)) + h), \xi' + i\varepsilon\nabla'(\mu(x)))| \ge \\ C_1\left(h + |\xi_0| + |\xi'|^{2m} + \left[\frac{\langle x'\rangle}{\langle x_0\rangle}\right]^{\frac{2m}{2m-1}}\right),\ (x,\xi) \in \mathbb{R}_+^{n+1} \times \mathbb{R}^{n+1}, \end{gathered} \tag{4.22}$$

for some constants $C_1 = C_1(\varepsilon)$. Hence (4.17) and (4.22) imply (4.21) □

As an application of Theorem 4.5 we consider the differential operator

$$p(x,D) = iD_0 + \sum_{1\le j<l} a_j(x)D_j^{2m} + a_l(x)\left(\sum_{j=l}^{n} D_j^2\right)^m + b(x) \tag{4.23}$$

where

$$\min_{j=1,\ldots,l} \inf_{x\in\mathbb{R}_+^{n+1}} a_j(x) > 0.$$

Note that in the case $l = 1$ we obtain the operator

$$p(x,D) = iD_0 + a_1(x)\left(\sum_{j=1}^{n} D_j^2\right)^m + b(x)$$

and in the case$l = n$we obtain the operator

$$p(x,D) = iD_0 + \sum_{j=1}^{n} a_j(x)D_j^{2m} + b(x).$$

It follows from [5] (p.528) that for every fixed $x \in \mathbb{R}_+^{n+1}$

$$\chi_{p_0}(x,\eta') = \sum_{1\le j<l} \widetilde{a}_j(x)\eta_j^{2m} + \widetilde{a}_l(x)\left(\sum_{j=l}^{n} \eta_j^2\right)^m,$$

where

$$\widetilde{a}_j(x) = a_j(x)\left(\sin\frac{\pi}{2(2m-1)}\right)^{1-2m}, j = 0,\ldots,l.$$

We introduce the function

$$\chi_{p_0}(\eta') = \sup_{x\in\mathbb{R}_+^{n+1}} \chi_{p_0}(x,\eta') = \sum_{1\le j<l} \widetilde{a}_j\eta_j^{2m} + \widetilde{a}_l\left(\sum_{j=l}^{n} \eta_j^2\right)^m,$$

where

$$\widetilde{a}_j = \sup_{x\in\mathbb{R}_+^{n+1}} \widetilde{a}_j(x),\ j = 0,\ldots,l.$$

Then

$$\chi^*(x') = \sum_{0<j<l} a_j^* \left|x_j\right|^{\frac{2m}{2m-1}} + a_{l+1}^*\left(\sum_{j=l}^{n} x_j^2\right)^{\frac{2m}{2m-1}}$$

where

$$a_j^* = \left(\widetilde{a}_j\right)^{\frac{1}{1-2m}}(2m-1)(2m)^{\frac{2m}{1-2m}},\ j = 0,\ldots,l,$$

and

$$v(x) = \left(\frac{1}{x_0+d}\right)^{\frac{1}{2m-1}}\left\{\sum_{0<j<l} a_j^*\left\langle x_j\right\rangle^{\frac{2m}{2m-1}} + a_{l+1}^*\left(1+\sum_{j=l}^{n} x_j^2\right)^{\frac{2m}{2m-1}}\right\},\ x \in \mathbb{R}_+^{n+1}.$$

Theorem 4.6. *Let* $\varepsilon \in [0,1)$, $h < 0$, *and*

$$\lambda_{q,2m}(x,\xi) = 1 + |\xi_0| + \left|\xi'\right|^{2m} + q(x)$$

where $q(x) = q_1(x) + [\widehat{q}_m(x)]^{2m}$. *Then:*

a) for every $s \in \mathbb{R}$ *there exists* $h_0 = h_0(s) < 0$ *such that the operator* $p(x,D)$ *defined by formula (4.23) is invertible from* $H^s(\lambda_{q,2m}, \mathbb{R}^{n+1}_+, w_{\varepsilon,h})$ *into* $H^{s-1}(\lambda_q, \mathbb{R}^{n+1}_+, w_{\varepsilon,h})$ *for all* $h < h_0$;

b) if $s > \frac{n+b}{2b}$ *then there exists* $h_0 = h_0(s) < 0$ *such that* $p(x,D)$ *has the unique fundamental solution* $g_y \in H_0^{-s+1}(\lambda_{q,2m}, \mathbb{R}^{n+1}, w_{\varepsilon,h})$ *where* $w_{\varepsilon,h}(x) \in \mathcal{W}_{2m}(\mathcal{D}_{\chi_{p_0}}, \widehat{q}_m)$ *and* $h \leq h_0$.

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