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An implication of Gödel's incompleteness theorem II: Not referring to the validity of oneself's assertion

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Abstract

In [10] we reviewed Gödel's incompleteness theorem and gave a new proof along with an application which leads to a contradiction when applying the Gödel's discussion to the set theory ZFC itself. We stated a possible solution to avoid contradiction by removing the self-reference by appealing to the axiomatic formulation of a theory with referring to its validity in no explicit ways. We will in this paper give a more specific possible solution that one can avoid the Gödel type self-contradiction by preventing oneself from telling anything definite about the validity of oneself's assertion.

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1 Introduction

Gödel's first incompleteness theorem says "If a formal theory S including number theory is consistent, there is a proposition G both of whose affirmation G and negation $\neg G$ are not provable in S." The second theorem says "If a formal theory S including number theory is consistent, the consistency of S is not provable by the method which is formalizable in the theory S itself." In [10] we gave a new proof of the theorems by giving a direct translation of Gödel predicate into a numeralwise expression inside the formal number theory S without assuming the recursive construction of the Gödel predicate. Utilizing this recursive-free construction of Gödel predicate, we proceeded to argue what happens when adding the undecidable propositions to S as additional axioms by transfinite induction with assuming that the theory S is the usual set theory ZFC as well as we can use ZFC in the argument on the meta level, and deduced a contradiction, which looks like telling as if mathematics itself is inconsistent. The answer we gave there is to restrict the self-reference with treating

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theories as ones which are formulated as axiomatic formal systems and leave the judgement of the theory to the inner development of the formal systems with the reference about the truthness of the theories untouched. We will give in this part II a more specific solution to the problem that one can avoid the contradiction of Gödel type argument by prohibiting the explicit statement about the validity of oneself's assertions. We will freely make repetitions of some of the descriptions in part I [10] whenever necessary and appropriate.

1.1 Outline of the proof of incompleteness theorem

The outline of the proof of the incompleteness theorem is as follows. The work which should be done is to construct a proposition G whose affirmation and negation is not provable in the number theory S. Such a proposition is generally a self-referential proposition whose meaning is

$$G = \text{“}G \text{ is not provable.”}$$

Suppose that such a G is constructed in S. If we assume that G is provable, by the meaning of G, G would not be provable, a contradiction. On the other hand, if we assume that the negation $\neg G$ is provable, G is provable by the meaning of negation, contradicting $\neg G$. In either case S is inconsistent, contradicting our assumption that S is consistent. Thus we have to conclude that both of G and $\neg G$ are not provable.

As we see in this argument, Gödel's proof of the incompleteness is by reductio ad absurdum. Gödel proves that both of G and its negation $\neg G$ imply the contradiction with the assumption of the consistency of the number theory S. Then he concludes by reductio ad absurdum that both of G and $\neg G$ are not provable. We will call this type of contradiction of G and $\neg G$ with themselves a Gödel type self-contradiction.

To prove the incompleteness theorem, it is therefore necessary to write down the axioms of logic and mathematics and the rules of inferences, and need to show that it is not possible to prove the proposition G and the negation $\neg G$ by using those axioms and rules. To grasp the usage of axioms and rules of inferences, it is necessary to determine the primitive symbols and to give the rules to construct propositions by using the symbols. Then it needs to explicitly list the rules of inferences with using those symbols. To do so, we introduce the primitive symbols which are necessary to write down the number theory. Primitive symbols consist of primitive logical symbols, primitive predicate symbols, primitive function symbols, primitive object symbols, variable symbols, parentheses, and comma, as follows:

1. primitive logical symbols:

$$\begin{array}{l} \Rightarrow \text{ (imply)}, \ \wedge \text{ (and)}, \ \vee \text{ (or)}, \ \neg \text{ (not)}, \\ \forall \text{ (for all)}, \ \exists \text{ (there exists)} \end{array}$$

2. primitive predicate symbols:

$$= \text{ (equals)}$$

3. primitive function symbols:

$$+ \text{ (plus)}, \ \cdot \text{ (times)}, \ ' \text{ (successor (prime))}$$

4. primitive object symbols:
$$0 \text{ (zero)}$$

5. variable symbols:
$$a,\ b,\ c,\ \ldots,\ x,\ y,\ z,\ \ldots$$

6. parentheses:
$$(\ ),\ \ \{\ \},\ \ [\ ],\ \ \ldots$$

7. comma:
$$,$$

When x is a variable, the logical expression $\forall x$ is called a universal quantifier and $\exists x$ is called an existential quantifier.

From those symbols, we first define terms which will denote the objects in number theory as follows. This type of definition is called a recursive or inductive definition.

1. 0 is a term.

2. A variable is a term.

3. If s is a term, $(s)'$ is also a term.

4. If s, t are terms, $(s)+(t)$ is a term.

5. If s, t are terms, $(s)\cdot(t)$ is a term.

6. The only expressions defined by 1-5 are the terms of the number theory.

In particular, the terms in whose construction there does not appear any variable are called numerals or numeral terms.

We next define formula, or well-formed formula (wff) as follows:

1. If s and t are terms, then $(s) = (t)$ is a formula or wff. The formula of this form is called an atomic formula.

2. If A, B are formulae, then
$$(A) \Rightarrow (B)$$
is also a formula.

3. If A, B are formulae,
$$(A) \wedge (B)$$
is also a formula.

4. If A, B are formulae,
$$(A) \vee (B)$$
is also a formula.

5. If A is a formula,

$$\neg(A)$$

   is also a formula.

6. If x is a variable and A is a formula, then $\forall x(A)$ is a formula.

7. If x is a variable and A is a formula, $\exists x(A)$ is also a formula.

8. The only expressions defined by 1-7 are the formulae of the number theory.

1.2 Axioms and rules of inference

As stated in the previous subsection we adopt some of the formulae as the axioms of the number theory, and define theorems or provable formulae as the formulae obtained by applying the rules of inferences to the axioms.

We assume the following three rules. In the following we assume that the formula C does not contain the variable x.

I_1: Modus ponens. (Syllogism): If the formula A is true and $A \Rightarrow B$ is true, then the formula B is also true.

$$\frac{A, \quad (A) \Rightarrow (B)}{B}$$

I_2: Generalization: For any variable x, from F follows $\forall x(F)$.

$$\frac{(C) \Rightarrow (F)}{(C) \Rightarrow (\forall x(F))}$$

I_3: Specialization: For any variable x, from F follows $\exists x(F)$.

$$\frac{(F) \Rightarrow (C)}{(\exists x(F)) \Rightarrow (C)}$$

The axioms of number theory are as follows. In the followings, we omit the unnecessary and obvious parentheses.

The first group consists of the axioms of propositional calculus.

A1. Axioms of propositional calculus. (A, B, C are arbitrary formulae.)

1. $A \Rightarrow (B \Rightarrow A)$

2. $(A \Rightarrow B) \Rightarrow ((A \Rightarrow (B \Rightarrow C)) \Rightarrow (A \Rightarrow C))$

3. $A \Rightarrow ((A \Rightarrow B) \Rightarrow B)$

   (a rule of inference)

4. $A \Rightarrow (B \Rightarrow A \wedge B)$

5. $A \wedge B \Rightarrow A$

6. $A \wedge B \Rightarrow B$

7. $A \Rightarrow A \vee B$

8. $B \Rightarrow A \vee B$

9. $(A \Rightarrow C) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \vee B \Rightarrow C))$

10. $(A \Rightarrow B) \Rightarrow ((A \Rightarrow \neg B) \Rightarrow \neg A)$

11. $\neg\neg A \Rightarrow A$

The second group consists of the axioms of predicate calculus.

We introduce the following terminologies. If an occurrence of a variable x is in the scope of influence of a quantifier $\forall x$ or $\exists x$, the occurrence is called a bound variable. Otherwise, it is called a free variable.

We call a term t free for x in a formula $A(x)$ which has x as a free variable, if no free occurrence of x in $A(x)$ is in the scope of a quantifier $\forall y$ or $\exists y$ for any variable y of t.

A2. Axioms of predicate calculus. (A is an arbitrary formula, B is a formula which does not contain the variable x free, $F(x)$ is a formula which contains a free variable x, and the term t is free for x in the formula $F(x)$.)

1. $(B \Rightarrow A) \Rightarrow (B \Rightarrow (\forall x A))$
   (a rule of inference)

2. $\forall x F(x) \Rightarrow F(t)$

3. $F(t) \Rightarrow \exists x F(x)$

4. $(A \Rightarrow B) \Rightarrow ((\exists x A) \Rightarrow B)$
   (a rule of inference)

The same rules of inferences appear in the list of axioms to make the same rules of inferences effective inside the formal system of number theory.

The third and fourth groups consist of the axioms of number theory.

A3. Axioms of number theory. (a, b, c are arbitrary variables.)

1. $a' = b' \Rightarrow a = b$

2. $\neg(a' = 0)$

3. $a = b \Rightarrow (a = c \Rightarrow b = c)$

4. $a = b \Rightarrow a' = b'$

5. $a + 0 = a$

6. $a + b' = (a + b)'$

7. $a \cdot 0 = 0$

8. $a \cdot b' = a \cdot b + a$

A4. Axiom of mathematical induction. (F is an arbitrary formula.)

$$(F(0) \wedge \forall x(F(x) \Rightarrow F(x'))) \Rightarrow \forall x F(x)$$

1.3 Proof, theorems, and deducibility

We make the following definition to define the theorems and the proofs of the formal number theory.

Definition 1.1. A formula C is called an immediate consequence of a formula A or two formulae A, B if C is below the line and the other(s) are above the line, in the rules I_1, I_2 or I_3.

We then define proof, provability and theorem as follows.

Definition 1.2. A finite sequence of formulae, each consecutive pair of which is divided by a comma, is called a formal proof, if each formula F of the sequence is an axiom of number theory or is an immediate consequence of the formula(e) which appear(s) before F. A formal proof is said to be the proof of the formula E which appears at the end of the proof, and the formula E is said to be provable in number theory or is called a theorem of number theory.

If a formula E is deducible in the system in which some formulae are added, E is called deducible from the added assumption formulae.

Definition 1.3. Given a finite number of formulae $D_1, \cdots, D_\ell$ ($\ell \geq 0$), a finite sequence of formulae is called a formal deduction from the assumption formulae $D_1, \cdots, D_\ell$, if each formula F of the sequence is an axiom or one of the formulae $D_1, \cdots, D_\ell$, or an immediate consequence of the formula(e) which appear(s) before F. A deduction is said to be a deduction of its last formula E, and the formula E is said to be deducible from the assumption formulae or is called the conclusion of the deduction. We write this as follows.

$$D_1, \cdots, D_\ell \vdash E.$$

In the case when $\ell = 0$, this is written

$$\vdash E.$$

This is equivalent with that E is a theorem of number theory.

2 Gödel numbering

As we have seen in section 1, the terms, formulae, and theorems are defined by applying a finite number of rules repeatedly to some number of symbols in mechanical way. This procedure of construction is called a recursive or inductive construction as it constructs things by applying the rules of the same form repeatedly.

On the other hand, the procedure which can be described in the formal number theory S is the operation of finite natural numbers, and the mathematical inductions is assumed as an axiom in S. Therefore, the recursive procedure of construction of terms, formulae, theorems will be able to be mapped to the operation of natural numbers inside the formal number theory S. Namely it will be possible to assign a fixed natural number to each symbol, and from it one can form a rule to assign a unique natural number to each of terms, formulae, proof sequences, etc. If such a rule is made, it will be possible to map the fact that a given sequence of formulae is a proof to a proposition about natural numbers. That a formula A is provable means that there is a proof whose last formula is A. Thus it will be possible to express the fact that a given formula A is provable as a proposition in the number theory S. As well, it will be possible to express the fact that a given formula A is refutable, i.e. that the negation $\neg A$ of A is provable as a proposition in S. A rule that assigns a natural number to each primitive symbol and from this assigns a natural number to a general expression constructed from the primitive symbols in a recursive way is called Gödel numbering. We denote the natural number which is assigned by this rule to an expression E by $g(E)$, and call it the Gödel number of the expression E. An expression E which has Gödel number n is expressed as E_n. When E is a formula A, it is written as A_n. Thus $n = g(E_n)$, $n = g(A_n)$, etc. This map g from the totality of expressions to the set $\mathbb{N} = \{0, 1, 2, \dots\}$ of natural numbers is defined as one to one mapping, but is not onto mapping. Namely g is defined as an injection but is not a surjection in general. Hence for some natural number m, there can be the case that there is no expression E such that $g(E) = m$.

When proving Gödel's incompleteness theorem, we denote the formula which is obtained by substituting the natural number or numeral $\lceil n \rceil$ in the formal system:

$$\lceil n \rceil = 0^{\overbrace{\prime\prime\cdots\prime}^{n \text{ factors}}} \tag{2.1}$$

that corresponds to the natural number n on the meta level to the variable x of a formula $F(x)$ by

$$F(\lceil n \rceil).$$

This operation of substitution itself is the one on the meta level. The formula $F(\lceil n \rceil)$ that is obtained by this substitution is defined by

$$F(\lceil n \rceil) \overset{def}{=} \forall x(x = n \Rightarrow F). \tag{2.2}$$

We note that in the definition of the formula $F(\lceil n \rceil)$ which must be a formula in the formal system S, there appears the natural number n on the meta level. This fact corresponds to the fact that there appears the natural number n on the meta level to specify the number of primes in the definition of a numeral $\lceil n \rceil$ of the formal system S in equation (2.1) above (the number n above the $\prime\prime\dots\prime$ on the upper right side of 0 in the definition (2.1) of $\lceil n \rceil$). "Substitution" whatever it looks natural is inevitably a subjective and artificial deed performed by some subject on the meta level. Namely the construction of the numeral $\lceil n \rceil$ corresponding to n and the substitution of it to x are possible only when some subject recognizes a number n on the meta level.

2.1 Gödel numbering

We now give a concrete Gödel numbering g. There are infinitely many ways of giving mappings g, and we can take whatever g if it satisfies the properties stated above. We adopt here the method given in [10] of assigning binary numbers to expressions. Namely we first assign natural numbers to primitive symbols as follows.

$$\begin{array}{cccccccccc} ' & 0 & (&) & \{ & \} & [&] & + & \cdot \\ 2^0 & 2^1 & 2^2 & 2^3 & 2^4 & 2^5 & 2^6 & 2^7 & 2^8 & 2^9 \end{array}$$

$$\begin{array}{cccccccc} = & \Rightarrow & \wedge & \vee & \neg & \forall & \exists & , \\ 2^{10} & 2^{11} & 2^{12} & 2^{13} & 2^{14} & 2^{15} & 2^{16} & 2^{17} \end{array}$$

To the expressions constructed from those primitive symbols, we assign Gödel numbers inductively as follows. First we assign 0 to the empty expression. I.e. when the Gödel number is $x = 0$, the expression E_x corresponding to $x = 0$ is empty, and we regard that there is no expression which corresponds to 0. Next for two natural numbers n, m, letting $\ell(m)$ denote the number of figures of the binary expression of m, we define the product operation $\star$ by

$$n \star m = 2^{\ell(m)} \cdot n + m.$$

Here we define $\ell(m) = 0$ for $m = 0$. Now for two expressions A_1, A_2 with Gödel numbers $g(A_1)$, $g(A_2)$, we define the Gödel number $g(A_1A_2)$ for the connected expression A_1A_2 of A_1 and A_2 in this order by

$$g(A_1A_2) = g(A_1) \star g(A_2).$$

This mapping g is obviously one to one.

For example, the Gödel number of $(0)'$ is calculated as follows. First as the Gödel number of (is 2^2 and that of 0 is 2^1, we have $n = 2^2 = (100)_2$, $m = 2^1 = (10)_2$ and $\ell(m) = 2$. Thus the Gödel number of (0 is

$$n \star m = 2^2 \cdot 2^2 + 2^1 = 2^4 + 2^1 = (10010)_2.$$

Namely in binary number the Gödel number of (is 100, and that of 0 is 10. Connecting these consecutively we obtain the Gödel number 10010 of (0. Similarly the Gödel number of (0) is 100101000, and that of $(0)'$ is 1001010001.

In the definition of Gödel number above, there is no definition of the Gödel number of the variables $a, b, c, \ldots, x, y, z, \ldots$. This is because we can express variables by connecting the primitive symbols without overlapping with other expressions like terms or formulae as follows.

$$\begin{array}{ccc} a & \text{is} & (0'), \\ b & \text{is} & (0''), \\ c & \text{is} & (0'''), \\ & \cdots & \end{array} \tag{2.3}$$

In the following we follow this convention.

The following lemma will be crucial in the proof of Theorem 2.4. The proof is found in [10].

Lemma 2.1. *Let $a \geq 0$ be a natural number and let w be the natural number such that $w' = 2^a$, where w' is a successor of the natural number w (i.e. $w' = w + 1$). Then we have*

$$g(a) = 2^1 \star w. \tag{2.4}$$

2.2 Incompleteness theorem

First we define the following two predicates.

Definition 2.2. 1) The predicate $\mathbf{G}(a,b)$ means the following.

"A formula A_a with Gödel number a has just one free variable x, and an expression E_b with Gödel number b is a proof of the formula $A_a = A_a(\ulcorner a \urcorner)$ obtained from $A_a = A_a(x)$ by substituting $\ulcorner a \urcorner$ into x."

2) The predicate $\mathbf{H}(a,b)$ means the following.

"A formula A_a with Gödel number a has just one free variable x, and an expression E_b with Gödel number b is a proof of the formula $\neg A_a = \neg A_a(\ulcorner a \urcorner)$ obtained from $\neg A_a = \neg A_a(x)$ by substituting $\ulcorner a \urcorner$ into x."

In this paper we call the predicate $\mathbf{G}(a,b)$ Gödel predicate. We introduce the following notion.

Definition 2.3. Let $\mathbf{R}(x_1,\dots,x_n)$ be a predicate (or relation) about $n(\geq 0)$ objects. This predicate is said to be numeralwise expressible in the formal system S if there is a formula $r(u_1,\dots,u_n)$ in S with exactly n free variables $u_1,\dots,u_n$ such that for an arbitrarily given n-tuple of natural numbers $x_1,\dots,x_n$, the followings hold.

i) If $\mathbf{R}(x_1,\dots,x_n)$ is true, then $\vdash r(\ulcorner x_1 \urcorner,\dots,\ulcorner x_n \urcorner)$.

ii) If $\mathbf{R}(x_1,\dots,x_n)$ is false, then $\vdash \neg r(\ulcorner x_1 \urcorner,\dots,\ulcorner x_n \urcorner)$.

In this case, $\mathbf{R}(x_1,\dots,x_n)$ is said to be numeralwise expressed by the formula $r(u_1,\dots,u_n)$.

It will be shown that the following holds.

Theorem 2.4. *The predicates $\mathbf{G}(a,b)$ and $\mathbf{H}(a,b)$ in Definition 2.2 are both numeralwise expressed in S by some formulae $g(a,b)$ and $h(a,b)$ respectively.*

We now define Rosser formula.

Definition 2.5. Let q be the Gödel number of the following formula.

$$\forall b\,(g(a,b) \Rightarrow \exists c\,(c \leq b \wedge h(a,c)))\,.$$

Namely

$$A_q(a) = \forall b\,(g(a,b) \Rightarrow \exists c\,(c \leq b \wedge h(a,c)))\,.$$

Then

$$A_q(\ulcorner q \urcorner) = \forall b\,(g(\ulcorner q \urcorner,b) \Rightarrow \exists c\,(c \leq b \wedge h(\ulcorner q \urcorner,c)))\,.$$

Here

$$g(\ulcorner q \urcorner,b) = \forall a\,(a = q \Rightarrow g(a,b))\,,$$
$$h(\ulcorner q \urcorner,c) = \forall a\,(a = q \Rightarrow h(a,c))\,.$$

$A_q(\ulcorner q \urcorner)$ is called Rosser formula.

Utilizing Theorem 2.4 we can prove the following.

Theorem 2.6. *(Gödel's incompleteness theorem of Rosser type [18]) Let S be consistent. Then neither $A_q(\ulcorner q \urcorner)$ nor the negation $\neg A_q(\ulcorner q \urcorner)$ is provable in S.*

For the proof see e.g. Theorem 3.6 in [10].

It thus suffices to prove Theorem 2.4 in order to prove Theorem 2.6.

3 Numeralwise expression of proof

In the present section we will show that the predicates $\mathbf{G}(a,b)$, $\mathbf{H}(a,b)$ defined in section 2 are numeralwise expressible in number theory S. To show this Gödel [4] proved the following theorem, and used the fact that the predicates $\mathbf{G}(a,b)$, $\mathbf{H}(a,b)$ are recursive.

Theorem 3.1. *For any recursive relation $\mathbf{R}(x_1,\dots,x_n)$ there exists a number-theoretic formula $r(u_1,\dots,u_n)$ with n free variables $u_1,\dots,u_n$ such that for any n-tuple of natural numbers $x_1,\dots,x_n$ the following i) and ii) hold.*

i) If $\mathbf{R}(x_1,\dots,x_n)$ is true, then $\vdash r(\ulcorner x_1 \urcorner,\dots,\ulcorner x_n \urcorner)$holds.

ii) If $\mathbf{R}(x_1,\dots,x_n)$ is false, then $\vdash \neg r(\ulcorner x_1 \urcorner,\dots,\ulcorner x_n \urcorner)$ holds.

In this paper we do not prove this theorem. Instead we will prove directly that the predicates $\mathbf{G}(a,b)$, $\mathbf{H}(a,b)$ are numeralwise expressed by some formulae $g(a,b)$, $h(a,b)$, respectively. The procedure we will describe below is the same stated in section 5 of [10] so that it might sound redundant. However having seen that even an expert [20] overlooks ([11]) the point that the translation is done independently of the recursiveness, we dare repeat the procedure in this paper. That we can show that the predicates $\mathbf{G}(a,b)$, $\mathbf{H}(a,b)$ are numeralwise expressible by some formulae $g(a,b)$, $h(a,b)$ in S without using Theorem 3.1 implies that Theorem 2.4 holds independently of the recursive construction of the predicates $\mathbf{G}(a,b)$, $\mathbf{H}(a,b)$. This will show that the infinite and transfinite extensions of the number theory $S^{(0)}$ in section 4 are possible irrespective of the recursiveness of the predicates $\mathbf{G}(a,b)$, $\mathbf{H}(a,b)$, which will justify the argument in section 4. (See the footnotes 1, 2.)

3.1 Numeralwise expression of being terms and formulae

First of all we will show that it is possible to express the procedure of constructing Gödel number in a recursive way. For this purpose it suffices to prove that, for any given natural numbers x,y,z, it is possible to express the fact that z is equal to the product $x \star y$ by a proposition in the number theory S in a recursive way. Likewise the following definitions **1** – **28** are all recursive.

1. $\mathrm{Div}(x,y)$: x is a factor of y.

$$(\exists z \le y)\,(x \cdot z = y)$$

2. $2^{\times}(x)$: x is a power of 2.

$$(\forall z \le x)((\mathrm{Div}(z,x) \wedge (z \neq 1)) \Rightarrow \mathrm{Div}(2,z))$$

3. $y = 2^{\ell(x)}$: y is the least power of 2 which is greater than x.

$$(2^{\times}(y) \wedge (y > x) \wedge (y > 1)) \wedge (\forall z < y) \neg (2^{\times}(z) \wedge (z > x) \wedge (z > 1))$$

4. $z = x \star y$: z is the numeral resulting from the $\star$-product of x and y.

$$(\exists w \leq z)(z = (w \cdot x) + y \wedge w = 2^{\ell(y)})$$

We next decompose numerals expressed in binary numbers, and express the procedure to extract a subsequence from a sequence of primitive symbols in number-theoretic way.

5. $\mathrm{Begin}(x,y)$: x is the numeral which expresses a left-most part of the sequence of symbols which has Gödel number y.

$$x = y \vee (x \neq 0 \wedge (\exists z \leq y)(x \star z = y))$$

6. $\mathrm{End}(x,y)$: x is the numeral which expresses a right-most part of the sequence of symbols which has Gödel number y.

$$x = y \vee (x \neq 0 \wedge (\exists z \leq y)(z \star x = y))$$

7. $\mathrm{Part}(x,y)$: x is the numeral which expresses a part of the sequence of symbols which has Gödel number y.

$$x = y \vee (x \neq 0 \wedge (\exists z \leq y)(\mathrm{End}(z,y) \wedge \mathrm{Begin}(x,z)))$$

Using these, we can construct a predicate which classifies the nature of terms.

8. $\mathrm{Succ}(x)$: E_x is a sequence of $'$.

$$(x \neq 0) \wedge (\forall y \leq x)(\mathrm{Part}(y,x) \Rightarrow \mathrm{Part}(1,y))$$

9. $\mathrm{Var}(x)$: E_x is a variable.

$$(\exists y \leq x)\Big(\mathrm{Succ}(y) \wedge x = 2^2 \star 2^1 \star y \star 2^3\Big)$$

Her we recall that we follow the convention stated in subsection 2.1 such that the variables $a, b, c, \ldots$ are supposed to be expressed as $(0')$, $(0'')$, $(0''')$, $\ldots$.

10. $\mathrm{Num}(x)$: E_x is a numeral.

$$(x = 2^1) \vee (\exists y \leq x)\Big(\mathrm{Succ}(y) \wedge x = 2^1 \star y\Big)$$

Gödel number of a sequence of (formal) expressions E_{x_1}, E_{x_2}, $\ldots$, E_{x_n} is written as follows.

$$x_1 \star 2^{17} \star x_2 \star 2^{17} \star \ldots \star 2^{17} \star x_n$$

The facts that an expression is a sequence of formal expressions and that an expression is included in a sequence of expressions are expressed by the following propositions.

11. $\mathrm{Seq}(x)$: E_x is a sequence of formal expressions.

$$\mathrm{Part}(2^{17}, x)$$

12. $x \in y$: E_y is a sequence of expressions, and E_x is an element of it.

$$\begin{aligned}&\mathrm{Seq}(y) \wedge \neg\mathrm{Part}(2^{17}, x) \wedge \\ &\qquad \Big(\mathrm{Begin}(x \star 2^{17}, y) \vee \mathrm{End}(2^{17} \star x, y) \vee \mathrm{Part}(2^{17} \star x \star 2^{17}, y)\Big)\end{aligned}$$

13. $x \prec_z y$: For two sequences E_x, E_y of expressions which are elements of a sequence E_z of expressions, E_x appears before E_y.

$$(x \in z) \wedge (y \in z) \wedge (\exists w \leq z)\mathrm{Part}(x \star w \star y, z)$$

Using those, the fact that an expression which has Gödel number x is a formula is expressed in the formal system S.

14. $\mathrm{Term}(x)$: E_x is a term.

$$\begin{aligned}&\exists y\Big((x \in y) \wedge (\forall z \in y)\{\mathrm{Var}(z) \vee \mathrm{Num}(z) \vee \\ &(\exists v \prec_y z)(\exists w \prec_y z)[(2^2 \star v \star 2^3 \star 2^8 \star 2^2 \star w \star 2^3 = z) \vee \\ &(2^2 \star v \star 2^3 \star 2^9 \star 2^2 \star w \star 2^3 = z) \vee (2^2 \star v \star 2^3 \star 2^0 = z)]\}\Big)\end{aligned}$$

15. $\mathrm{Atom}(x)$: E_x is an atomic formula.

$$(\exists y \leq x)(\exists z \leq x)\Big(\mathrm{Term}(y) \wedge \mathrm{Term}(z) \wedge \big((x = y \star 2^{10} \star z) \vee (x = \mathrm{leq}(y, z))\big)\Big)$$

Here the function leq is defined as follows recursively.

1. $\mathrm{neq}(x, y)$ is the following Gödel number of the expression $E_x \neq E_y$.

$$2^{14} \star 2^2 \star x \star 2^{10} \star y \star 2^3$$

2. $\mathrm{leq}(x, y)$ is the following Gödel number of the expression $E_x \leq E_y$.

$$\begin{aligned}&2^{14} \star 2^2 \star 2^{15} \star 2^2 \star 2^1 \star 2^0 \star 2^3 \star 2^2 \star \\ &\mathrm{neq}(x \star 2^8 \star 2^2 \star 2^1 \star 2^0 \star 2^3, y) \star 2^3 \star 2^3\end{aligned}$$

16. $\mathrm{Gen}(x, y)$: For a variable E_u, E_y is equal to $\forall E_u(E_x)$.

$$(\exists u \leq y)\big(\mathrm{Var}(u) \wedge y = 2^{15} \star u \star 2^2 \star x \star 2^3\big)$$

17. Form(x) : E_x is a (well-formed) formula.

$$\exists y\Big((x \in y) \wedge (\forall z \in y)\{\mathrm{Atom}(z) \vee$$
$$(\exists v \prec_y z)(\exists w \prec_y z)[(z = v \star 2^{11} \star w) \vee \left(z = 2^{14} \star 2^2 \star v \star 2^3\right) \vee \mathrm{Gen}(w,z)]\}\Big)$$

Here we regard the logical symbols of propositional calculus consisting of only $\neg$ and $\Rightarrow$ with noting that logical symbols $\wedge$ and $\vee$ are expressed by using $\neg$ and $\Rightarrow$ as follows:

$$A \wedge B \text{ is } \neg(A \Rightarrow \neg B),$$
$$A \vee B \text{ is } \neg A \Rightarrow B.$$

As well we regard that the existential quantifier is expressed as follows with using the universal quantifier:

$$\exists x F(x) \text{ is } \neg\forall x \neg F(x).$$

3.2 Numeralwise expression of being axioms of propositional calculus

We next show that the fact that an expression with Gödel number x is an axiom of number theory is expressed by a formula in S. First we consider the axioms of propositional calculus.

18. Pro(x): E_x is an axiom of propositional calculus.

$$\mathrm{Prop}_1(x) \vee \mathrm{Prop}_2(x) \vee \mathrm{Prop}_3(x) \vee \mathrm{Prop}_4(x) \vee \mathrm{Prop}_5(x) \vee \mathrm{Prop}_6(x) \vee$$
$$\mathrm{Prop}_7(x) \vee \mathrm{Prop}_8(x) \vee \mathrm{Prop}_9(x) \vee \mathrm{Prop}_{10}(x) \vee \mathrm{Prop}_{11}(x)$$

Here $\mathrm{Prop}_1(x)$, $\mathrm{Prop}_2(x)$, $\mathrm{Prop}_3(x)$, $\mathrm{Prop}_4(x)$, $\mathrm{Prop}_5(x)$, $\mathrm{Prop}_6(x)$, $\mathrm{Prop}_7(x)$, $\mathrm{Prop}_8(x)$, $\mathrm{Prop}_9(x)$, $\mathrm{Prop}_{10}(x)$, $\mathrm{Prop}_{11}(x)$ are defined as follows.

1. $\mathrm{Prop}_1(x)$: E_x is axiom 1 of propositional calculus.

$$(\exists a < x)(\exists b < x)(\mathrm{Form}(a) \wedge \mathrm{Form}(b) \wedge x = a \star 2^{11} \star 2^2 \star b \star 2^{11} \star a \star 2^3)$$

2. $\mathrm{Prop}_2(x)$: E_x is axiom 2 of propositional calculus.

$$\begin{aligned}&(\exists a < x)(\exists b < x)(\exists c < x)(\mathrm{Form}(a) \wedge \mathrm{Form}(b) \wedge \\ &\qquad \mathrm{Form}(c) \wedge x = 2^2 \star a \star 2^{11} \star b \star 2^3 \star 2^{11} \star 2^2 \star 2^2 \star a \star 2^{11} \\ &\qquad \star 2^2 \star b \star 2^{11} \star c \star 2^3 \star 2^3 \star 2^{11} \star 2^2 \star a \star 2^{11} \star c \star 2^3 \star 2^3)\end{aligned}$$

3. $\mathrm{Prop}_3(x)$: E_x is axiom 3 of propositional calculus.

$$\begin{aligned}&(\exists a < x)(\exists b < x)(\mathrm{Form}(a) \wedge \mathrm{Form}(b) \wedge \\ &\qquad x = a \star 2^{11} \star 2^2 \star 2^2 \star a \star 2^{11} \star b \star 2^3 \star 2^{11} \star b \star 2^3)\end{aligned}$$

4. $\mathrm{Prop}_4(x)$: E_x is axiom 4 of propositional calculus.

$$(\exists a < x)(\exists b < x)(\mathrm{Form}(a) \wedge \mathrm{Form}(b) \wedge \\ x = a \star 2^{11} \star 2^2 \star b \star 2^{11} \star a \star 2^{12} \star b \star 2^3)$$

5. $\mathrm{Prop}_5(x)$: E_x is axiom 5 of propositional calculus.

$$(\exists a < x)(\exists b < x)(\exists c < x)(\mathrm{Form}(a) \wedge \mathrm{Form}(b) \wedge x = a \star 2^{12} \star b \star 2^{11} \star a)$$

6. $\mathrm{Prop}_6(x)$: E_x is axiom 6 of propositional calculus.

$$(\exists a < x)(\exists b < x)(\exists c < x)(\mathrm{Form}(a) \wedge \mathrm{Form}(b) \wedge x = a \star 2^{12} \star b \star 2^{11} \star b)$$

7. $\mathrm{Prop}_7(x)$: E_x is axiom 7 of propositional calculus.

$$(\exists a < x)(\exists b < x)(\mathrm{Form}(a) \wedge \mathrm{Form}(b) \wedge x = a \star 2^{11} \star a \star 2^{13} \star b)$$

8. $\mathrm{Prop}_8(x)$: E_x is axiom 8 of propositional calculus.

$$(\exists a < x)(\exists b < x)(\mathrm{Form}(a) \wedge \mathrm{Form}(b) \wedge x = b \star 2^{11} \star a \star 2^{13} \star b)$$

9. $\mathrm{Prop}_9(x)$: E_x is axiom 9 of propositional calculus.

$$(\exists a < x)(\exists b < x)(\exists c < x)(\mathrm{Form}(a) \wedge \mathrm{Form}(b) \wedge \mathrm{Form}(c) \wedge \\ x = 2^2 \star a \star 2^{11} \star c \star 2^3 \star 2^{11} \star \ 2^2 \star 2^2 \star b \star 2^{11} \star c \star 2^3 \star 2^{11} \star 2^2 \\ \star\ a \star 2^{13} \star b \star 2^{11} \star c \star 2^3 \star 2^3)$$

10. $\mathrm{Prop}_{10}(x)$: E_x is axiom 10 of propositional calculus.

$$(\exists a < x)(\exists b < x)(\mathrm{Form}(a) \wedge \mathrm{Form}(b) \wedge \\ x = 2^2 \star a \star 2^{11} \star b \star 2^3 \star 2^{11} \star 2^2 \star 2^2 \star a \star 2^{11} \star 2^{14} \star \\ b \star 2^3 \star 2^{11} \star 2^{14} \star a \star 2^3)$$

11. $\mathrm{Prop}_{11}(x)$: E_x is axiom 11 of propositional calculus.

$$(\exists a < x)(\mathrm{Form}(a) \wedge x = 2^{14} \star 2^{14} \star a \star 2^{11} \star a)$$

3.3 Numeralwise expression of being axioms of predicate calculus

We now express in S that an expression E_x is an axiom of predicate calculus. As is easily seen, axiom 1 and axiom 4 are equivalent, and axiom 2 and axiom 3 are equivalent. Thus we have only to give expressions to axioms 1 and 2. In axiom 2, we need to consider the replacement of all occurrences of a free variable in a formula by a term.

19. $\mathrm{Free}(x, y)$: Every variable in a term E_x is not bounded in an expression E_y.

$$\mathrm{Term}(x) \wedge (\forall z < x)\Big([\mathrm{Var}(z) \wedge \mathrm{Part}(z, x)] \Rightarrow \big[\neg \mathrm{Part}(2^{15} \star z, y)\big]\Big)$$

20. $\mathrm{Pred}_1(x)$: E_x is axiom 1 of predicate calculus.

$$(\exists a < x)(\exists b < x)(\exists c < x)(\mathrm{Form}(a) \wedge \mathrm{Form}(b) \wedge \mathrm{Var}(c) \wedge (\neg \mathrm{Part}(c,b)) \wedge x = 2^2 \star b \star 2^{11} \star a \star 2^3 \star 2^{11} \star 2^2 \star b \star 2^{11} \star 2^2 \star 2^{15} \star c \star a \star 2^3 \star 2^3)$$

21. $\mathrm{Seq}(x,y,u)$: An expression u includes a pair $\{E_x, E_y\}$ as a consecutive pair of E_x and E_y in this order.

$$\neg\mathrm{Seq}(x) \wedge \neg\mathrm{Seq}(y) \wedge (x \neq 0) \wedge (y \neq 0) \wedge \mathrm{Part}(x \star 2^{17} \star y, u)$$

22. $x = \mathrm{alt}_y(u,t)$: A formula E_x is obtained from a formula E_y by substituting a free term E_t at every occurrence of a free variable E_u.

$$\begin{aligned}&\mathrm{Form}(x) \wedge \mathrm{Form}(y) \wedge \mathrm{Var}(u) \wedge \mathrm{Free}(u,y) \wedge \mathrm{Term}(t) \wedge \mathrm{Free}(t,y) \wedge \mathrm{Part}(u,y) \wedge \\ &\neg\mathrm{Part}(u,x) \wedge \exists w \Big\{ \mathrm{Seq}(y,x,w) \wedge (\forall a < w)(\forall b < w)\Big(\mathrm{Seq}(a,b,w) \\ &\qquad \Rightarrow \{(\neg\mathrm{Part}(u,a) \wedge a = b) \vee (\exists c_1 < a)(\exists c_2 < b)(\exists d_1 < a)(\exists d_2 < b) \\ &\qquad [\mathrm{Seq}(c_1,c_2,w) \wedge \mathrm{Seq}(d_1,d_2,w) \wedge a = c_1 \star u \star d_1 \wedge b = c_2 \star t \star d_2]\}\Big)\Big\}\end{aligned}$$

23. $\mathrm{Pred}_2(x)$: E_x is axiom 2 of predicate calculus.

$$\begin{aligned}&(\exists a < x)(\exists b < x)(\exists c < x)(\exists t < x)(\mathrm{Form}(a) \wedge \\ &\qquad \mathrm{Var}(b) \wedge \mathrm{Term}(t) \wedge c = \mathrm{alt}_a(b,t) \wedge x = 2^{15} \star b \star a \star 2^{11} \star c)\end{aligned}$$

3.4 Numeralwise expression of being axioms of number theory

Finally we express that E_x is an axiom of number theory.

24. $\mathrm{Nat}(x)$: E_x is an axiom of number theory.

$$\mathrm{Nat}_1(x) \vee \mathrm{Nat}_2(x) \vee \mathrm{Nat}_3(x) \vee \mathrm{Nat}_4(x) \vee \mathrm{Nat}_5(x) \vee \mathrm{Nat}_6(x) \vee \mathrm{Nat}_7(x) \vee \mathrm{Nat}_8(x)$$

Here $\mathrm{Nat}_1(x)$, $\mathrm{Nat}_2(x)$, $\mathrm{Nat}_3(x)$, $\mathrm{Nat}_4(x)$, $\mathrm{Nat}_5(x)$, $\mathrm{Nat}_6(x)$, $\mathrm{Nat}_7(x)$, $\mathrm{Nat}_8(x)$ are defined as follows.

1. $\mathrm{Nat}_1(x)$: E_x is axiom 1 of number theory.

$$\begin{aligned}&(\exists a < x)(\exists b < x)(\mathrm{Term}(a) \wedge \mathrm{Term}(b) \wedge \\ &\qquad x = 2^2 \star a \star 2^0 \star 2^{10} \star b \star 2^0 \star 2^3 \star 2^{11} \star 2^2 \star a \star 2^{10} \star b \star 2^3)\end{aligned}$$

2. $\mathrm{Nat}_2(x)$: E_x is axiom 2 of number theory.

$$(\exists a < x)(\mathrm{Term}(a) \wedge x = 2^{14} \star 2^2 \star a \star 2^0 \star 2^{10} \star 2^1 \star 2^3)$$

3. $\mathrm{Nat}_3(x)$: E_x is axiom 3 of number theory.

$$\begin{aligned}&(\exists a < x)(\exists b < x)(\exists c < x)(\mathrm{Term}(a) \wedge \mathrm{Term}(b) \wedge \mathrm{Term}(c) \wedge \\ &\qquad x = a \star 2^{10} \star b \star 2^{11} \star 2^2 \star a \star 2^{10} \star c \star 2^{11} \star b \star 2^{10} \star c \star 2^3)\end{aligned}$$

4. $\mathrm{Nat}_4(x)$: E_x is axiom 4 of number theory.

$$(\exists a < x)(\exists b < x)(\mathrm{Term}(a) \wedge \mathrm{Term}(b) \wedge \\ x = a \star 2^{10} \star b \star 2^{11} \star a \star 2^0 \star 2^{10} \star b \star 2^0)$$

5. $\mathrm{Nat}_5(x)$: E_x is axiom 5 of number theory.

$$(\exists a < x)(\mathrm{Term}(a) \wedge x = a \star 2^8 \star 2^1 \star 2^{10} \star a)$$

6. $\mathrm{Nat}_6(x)$: E_x is axiom 6 of number theory.

$$(\exists a < x)(\exists b < x)(\mathrm{Term}(a) \wedge \mathrm{Term}(b) \wedge \\ x = a \star 2^8 \star b \star 2^0 \star 2^{10} \star 2^2 \star a \star 2^8 \star b \star 2^3 \star 2^0)$$

7. $\mathrm{Nat}_7(x)$: E_x is axiom 7 of number theory.

$$(\exists a < x)(\mathrm{Term}(a) \wedge x = a \star 2^9 \star 2^1 \star 2^{10} \star 2^1)$$

8. $\mathrm{Nat}_8(x)$: E_x is axiom 8 of number theory.

$$(\exists a < x)(\exists b < x)(\mathrm{Term}(a) \wedge \mathrm{Term}(b) \wedge \\ x = a \star 2^9 \star b \star 2^0 \star 2^{10} \star a \star 2^9 \star b \star 2^8 \star a)$$

That E_x is the axiom of mathematical induction is expressed as follows.

25. $\mathrm{sub}_a(x,y)$ is the Gödel number of the formula $\forall E_x((E_x = E_y) \Rightarrow (E_a))$ meaning the formal substitution of E_y into the variable E_x of E_a.

$$2^{15} \star x \star 2^2 \star 2^2 \star x \star 2^{10} \star y \star 2^3 \star 2^{11} \star 2^2 \star a \star 2^3 \star 2^3$$

26. $\mathrm{MI}(x)$: E_x is the axiom of mathematical induction.

$$(\exists a < x)(\exists b < x)(\exists c < x)(\mathrm{Form}(a) \wedge \mathrm{Var}(b) \wedge \mathrm{Var}(c) \wedge \\ x = 2^2 \star \mathrm{sub}_a(b, 2^1) \star 2^{12} \star 2^{15} \star c \star\ 2^2 \star \mathrm{sub}_a(b,c) \star 2^{11} \star \mathrm{sub}_a(b, c \star 2^0) \\ \star 2^3 \star 2^3 \star 2^{11} \star\ 2^{15} \star c \star \mathrm{sub}_a(b,c))$$

We have expressed all axioms of S in the formal system S.

3.5 Numeralwise expression of being a proof sequence

From the above, we can express in the formal system S the fact that a sequence of expressions is a proof sequence consisting of axioms and the results of the applications of rules of inference as follows.

27. $\mathrm{Axiom}(x)$: E_x is an axiom.

$$\mathrm{Pro}(x) \vee \mathrm{Pred}_1(x) \vee \mathrm{Pred}_2(x) \vee \mathrm{Nat}(x) \vee \mathrm{MI}(x)$$

28. Proof(x) : E_x is a proof sequence.

$$\mathrm{Seq}(x) \wedge \forall y\Big(y \in x \Rightarrow (\mathrm{Axiom}(y) \vee (\exists v \prec_x y)(\exists w \prec_x y)\{(w = v \star 2^{11} \star y) \vee$$
$$(\exists a < v)(\exists b < v)(\exists c < y)[v = b \star 2^{11} \star a \wedge y = b \star 2^{11} \star c \wedge \mathrm{Gen}(a,c) \wedge$$
$$(\forall z \leq a)(\mathrm{Var}(z) \Rightarrow \neg\mathrm{Part}(z,b))]\})\Big)$$

29. Pr(x) : E_x is provable.

$$\exists y\,(\mathrm{Proof}(y) \wedge (x \in y))$$

30. Re(x) : E_x is refutable.

$$\exists y\Big(\mathrm{Proof}(y) \wedge (2^{14} \star 2^2 \star x \star 2^3 \in y)\Big)$$

The predicates in **29** and **30** are not recursive predicates in the above. Other predicates from **1** to **28** are recursive because the latter predicates are all determined to be true or not by making checking for natural numbers in a finite set. In **29**, **30**, there is no restriction to a finite set of natural numbers. Thus in a finitary method one can not determine whether the predicates Pr(x) and Re(x) are true or not.

The predicates $\mathbf{G}(a,b)$, $\mathbf{H}(a,b)$ defined in section 2 are numeralwise expressible by formulae $g(a,b)$, $h(a,b)$ respectively if the following holds.

1) i) If $\mathbf{G}(a,b)$ is true, then $\vdash g(\lceil a \rceil, \lceil b \rceil)$ holds.
   ii) If $\mathbf{G}(a,b)$ is false, then $\vdash \neg g(\lceil a \rceil, \lceil b \rceil)$ holds.

2) i) If $\mathbf{H}(a,b)$ is true, then $\vdash h(\lceil a \rceil, \lceil b \rceil)$ holds.
   ii) If $\mathbf{H}(a,b)$ is false, then $\vdash \neg h(\lceil a \rceil, \lceil b \rceil)$ holds.

From the procedures **1** – **28** stated in the above follows Theorem 2.4 ([10]).

3.6 Gödel's incompleteness theorem

The Rosser formula $A_q(\lceil q \rceil)$ was defined in Definition 2.5. The results in the previous subsections have shown Theorem 2.4 in section 2. Therefore we have the incompleteness theorem of Rosser type by Theorem 2.6 in section 2.

On the other hand, Gödel's original result is as follows.

Definition 3.2. Let p be the Gödel number of the following formula.

$$\forall b \neg g(a,b).$$

Namely

$$A_p(a) = \forall b \neg g(a,b).$$

Then we call the following formula Gödel sentence or formula.

$$A_p(\lceil p \rceil) = \forall b \neg g(\lceil p \rceil, b). \tag{3.1}$$

where

$$g(\lceil p \rceil, b) = \forall a\,(a = p \Rightarrow g(a,b)).$$

Definition 3.3. A formal system S which includes the number theory is called ω-consistent if for any variable x and any formula $A(x)$, not all of

$$A(0),\ A(1),\ A(2),\ \dots \quad \text{and} \quad \neg\forall x A(x)$$

is provable. In particular, if S is ω-consistent, it is (simply) consistent.

Theorem 3.4. *(Gödel's incompleteness theorem (1931)) If the number theory S is consistent, then*

$$\text{not } \vdash A_p(\ulcorner p \urcorner).$$

If S is ω-consistent,

$$\text{not } \vdash \neg A_p(\ulcorner p \urcorner).$$

In particular if S is ω-consistent, then $A_p(\ulcorner p \urcorner)$ is neither provable nor refutable in S.

Proof. We assume that S is consistent. Suppose that

$$\vdash A_p(\ulcorner p \urcorner) \tag{3.2}$$

holds. Then there is a proof of $A_p(\ulcorner p \urcorner)$. Thus, if we let b be the Gödel number of the proof, $\mathbf{G}(p,b)$ is true. Therefore the numeralwise expressibility of the predicate $\mathbf{G}(a,b)$ implies

$$\vdash g(\ulcorner p \urcorner, \ulcorner b \urcorner).$$

From this and axiom 3 of predicate calculus we have

$$\vdash \exists b g(\ulcorner p \urcorner, b).$$

Namely

$$\vdash \neg\forall b \neg g(\ulcorner p \urcorner, b).$$

This means the following by the definition (3.1) of Gödel formula.

$$\vdash \neg A_p(\ulcorner p \urcorner).$$

This and (3.2) show that S is inconsistent. As we have made a premise that S is consistent, (3.2) is wrong. The former part is proved.

We assume that S is ω-consistent. In particular, S is consistent. Thus by the result above, $A_p(\ulcorner p \urcorner)$ is not provable in S. Thus every natural number $0,1,2,\dots$ is not a Gödel number of a proof of $A_p(\ulcorner p \urcorner)$. Namely, $\mathbf{G}(p,0)$, $\mathbf{G}(p,1)$, $\mathbf{G}(p,2)$, $\dots$ are all wrong. Therefore by the numeralwise expressibility of the predicate $\mathbf{G}(a,b)$, all of

$$\vdash \neg g(\ulcorner p \urcorner, \ulcorner 0 \urcorner),\ \vdash \neg g(\ulcorner p \urcorner, \ulcorner 1 \urcorner),\ \vdash \neg g(\ulcorner p \urcorner, \ulcorner 2 \urcorner),\ \dots$$

hold. As we assume that S is ω-consistent, from this follows

$$\text{not } \vdash \neg\forall b \neg g(\ulcorner p \urcorner, b).$$

By the definition (3.1) of Gödel formula, this means

$$\text{not } \vdash \neg A_p(\ulcorner p \urcorner).$$

This proves the latter part of the theorem. $\square$

3.7 The second incompleteness theorem

The former part of Gödel's Theorem 3.4 is summarized as follows.

$$S \text{ is consistent } \Rightarrow A_p(\lceil p \rceil) \text{ is not provable.} \tag{3.3}$$

By (3.1), the fact that "$A_p(\lceil p \rceil)$ is not provable" on the right hand side is written by translating it by Gödel numbering as follows:

$$A_p(\lceil p \rceil) = \forall b \neg g(\lceil p \rceil, b). \tag{3.4}$$

Therefore if we translate and map the metamathematics by Gödel numbering into S and write the fact that "S is consistent" by a formal formula in S:

$$\mathrm{Consis}(S),$$

then together with the formula (3.4) we have from the first part of Theorem 3.4 that

$$\vdash \mathrm{Consis}(S) \Rightarrow A_p(\lceil p \rceil). \tag{3.5}$$

Now let us assume on the meta level

$$\vdash \mathrm{Consis}(S).$$

Then together with the formula (3.5), we have

$$\vdash A_p(\lceil p \rceil).$$

This contradicts the first part of Theorem 3.4. Therefore we have the following theorem.

Theorem 3.5. *(Gödel's second incompleteness theorem (1931)) If the number theory S is consistent, then*

$$\text{not } \vdash \mathrm{Consis}(S)$$

holds. Namely if S is consistent, the consistency of S is not proved by a method formalizable in S.

The proof above is an outline. A complete proof is given in Hilbert-Bernays (1939). We note that this theorem is proved without using Rosser's stronger result: Theorem 2.6.

4 Mathematics is inconsistent?

In this section we will consider the problem whether the set theory ZFC which is thought to be a basis of modern mathematics is consistent.

4.1 Incompleteness theorem of Rosser type, revisited

We now consider a formal set theory S equivalent to ZFC, and assume that we can use the same set theory ZFC also on the meta level. We can develop a number theory in this formal system S. We denote this subsystem of number theory by $S^{(0)}$. Then the Gödel predicate $\mathbf{G}^{(0)}(a,b)$ and the related predicate $\mathbf{H}^{(0)}(a,b)$ are defined as follows.

Definition 4.1. 1) The predicate $\mathbf{G}^{(0)}(a,b)$ means the following.

"A formula A_a with Gödel number a has just one free variable x, and an expression E_b with Gödel number b is a proof of the formula $A_a = A_a(\ulcorner a \urcorner)$ obtained from $A_a = A_a(x)$ by substituting $\ulcorner a \urcorner$ into x."

2) The predicate $\mathbf{H}^{(0)}(a,b)$ means the following.

"A formula A_a with Gödel number a has just one free variable x, and an expression E_b with Gödel number b is a proof of the formula $\neg A_a = \neg A_a(\ulcorner a \urcorner)$ obtained from $\neg A_a = \neg A_a(x)$ by substituting $\ulcorner a \urcorner$ into x."

For these predicates we have shown the following in the former sections.

Theorem 4.2. *By the Gödel numbering we have stated before, the predicates $\mathbf{G}^{(0)}(a,b)$, $\mathbf{H}^{(0)}(a,b)$ in Definition 4.1 are numeralwise expressed by corresponding formulae $g^{(0)}(a,b)$, $h^{(0)}(a,b)$ in $S^{(0)}$, therefore in the formal set theory S. Namely the following holds. Let the formulae $g^{(0)}(a,b)$ and $h^{(0)}(a,b)$ be defined as follows.*

1) $g^{(0)}(a,b)$: E_a has a free variable E_x, and E_b is a proof of the formula E_a when $E_x = a$.

$$\begin{aligned}&\exists x(\mathrm{Var}(x) \wedge \mathrm{Part}(x,a) \wedge \mathrm{Proof}(b) \wedge \\ &\exists w[w' = 2^a \wedge (\mathrm{sub}_a(x, 2^1 \star w) \in b)])\end{aligned}$$

2) $h^{(0)}(a,b)$: E_a has a free variable E_x, and E_b is a proof of $\neg E_a$ when $E_x = a$.

$$\begin{aligned}&\exists x(\mathrm{Var}(x) \wedge \mathrm{Part}(x,a) \wedge \mathrm{Proof}(b) \wedge \\ &\exists w[w' = 2^a \wedge (2^{14} \star 2^2 \star \mathrm{sub}_a(x, 2^1 \star w) \star 2^3 \in b)])\end{aligned}$$

Then the following holds.

(1) i) If $\mathbf{G}^{(0)}(a,b)$ is true, then $\vdash g^{(0)}(\ulcorner a \urcorner, \ulcorner b \urcorner)$ holds.
ii) If $\mathbf{G}^{(0)}(a,b)$ is false, then $\vdash \neg g^{(0)}(\ulcorner a \urcorner, \ulcorner b \urcorner)$ holds.

(2) i) If $\mathbf{H}^{(0)}(a,b)$ is true, then $\vdash h^{(0)}(\ulcorner a \urcorner, \ulcorner b \urcorner)$ holds.
ii) If $\mathbf{H}^{(0)}(a,b)$ is false, then $\vdash \neg h^{(0)}(\ulcorner a \urcorner, \ulcorner b \urcorner)$ holds.

Definition 4.3. Let $q^{(0)}$ be the Gödel number of the formula

$$\forall b[\neg g^{(0)}(a,b) \vee \exists c(c \le b \ \wedge \ h^{(0)}(a,c))].$$

Namely

$$A_{q^{(0)}}(a) = \forall b[\neg g^{(0)}(a,b) \vee \exists c(c \le b \ \wedge \ h^{(0)}(a,c))].$$

We then define Rosser formula in $S^{(0)}$ as follows.

$$A_{q^{(0)}}(\ulcorner q^{(0)} \urcorner) = \forall b[\neg g^{(0)}(\ulcorner q^{(0)} \urcorner, b) \vee \exists c(c \le b \ \wedge \ h^{(0)}(\ulcorner q^{(0)} \urcorner, c))].$$

Then the incompleteness Theorem 2.6 of Rosser type for $S^{(0)}$ is as follows.

Lemma 4.4. *If $S^{(0)}$ is consistent, both of $A_{q^{(0)}}(\ulcorner q^{(0)}\urcorner)$ and $\neg A_{q^{(0)}}(\ulcorner q^{(0)}\urcorner)$ are unprovable in $S^{(0)}$.*

4.2 Extension of $S^{(0)}$

By Lemma 4.4, if we let either of $A_{q^{(0)}}(\ulcorner q^{(0)}\urcorner)$ or $\neg A_{q^{(0)}}(\ulcorner q^{(0)}\urcorner)$ be $A_{(0)}$, and add $A_{(0)}$ to the axioms of $S^{(0)}$ as a new axiom to obtain a new system $S^{(1)}$, we have

$$S^{(1)} \text{ is consistent.} \tag{4.1}$$

We extend the Gödel numbering for $S^{(0)}$ in Theorem 4.2 to the system $S^{(1)}$, and extend definitions 4.1 and 4.3 to the system $S^{(1)}$ as follows.

1) The predicate $\mathbf{G}^{(1)}(a,b)$ means the following.

   "A formula A_a with Gödel number a has just one free variable x, and an expression E_b with Gödel number b is a proof of the formula $A_a = A_a(\ulcorner a\urcorner)$ obtained from $A_a = A_a(x)$ by substituting $\ulcorner a\urcorner$ into x."

2) The predicate $\mathbf{H}^{(1)}(a,b)$ means the following.

   "A formula A_a with Gödel number a has just one free variable x, and an expression E_b with Gödel number b is a proof of the formula $\neg A_a = \neg A_a(\ulcorner a\urcorner)$ obtained from $\neg A_a = \neg A_a(x)$ by substituting $\ulcorner a\urcorner$ into x."

In the same way as before we can show that the predicates $\mathbf{G}^{(1)}(a,b)$ and $\mathbf{H}^{(1)}(a,c)$ are numeralwise expressed by the corresponding formulae $g^{(1)}(a,b)$ and $h^{(1)}(a,c)$ in S respectively.

3) Let $q^{(1)}$ be the Gödel number of the formula

$$\forall b[\neg g^{(1)}(a,b) \vee \exists c(c \leq b \ \wedge \ h^{(1)}(a,c))].$$

   Namely

$$A_{q^{(1)}}(a) = \forall b[\neg g^{(1)}(a,b) \vee \exists c(c \leq b \ \wedge \ h^{(1)}(a,c))].$$

   Then

$$A_{q^{(1)}}(\ulcorner q^{(1)}\urcorner) = \forall b[\neg g^{(1)}(\ulcorner q^{(1)}\urcorner,b) \vee \exists c(c \leq b \ \wedge \ h^{(1)}(\ulcorner q^{(1)}\urcorner,c))].$$

Using the numeralwise expressibility of the predicates $\mathbf{G}^{(1)}(a,b)$ and $\mathbf{H}^{(1)}(a,c)$ and the consistency of $S^{(1)}$ in (4.1), we can show in the same way as in the proof of Lemma 4.4

$$\text{not } \vdash A_{q^{(1)}}(\ulcorner q^{(1)}\urcorner) \ \text{ and } \ \text{not } \vdash \neg A_{q^{(1)}}(\ulcorner q^{(1)}\urcorner) \text{ in } S^{(1)}.$$

Then we let either of $A_{q^{(1)}}(\ulcorner q^{(1)}\urcorner)$ or $\neg A_{q^{(1)}}(\ulcorner q^{(1)}\urcorner)$ be $A_{(1)}$, and can add $A_{(1)}$ as a new axiom of $S^{(1)}$ to obtain a new consistent system $S^{(2)}$.

Proceeding in the same way, we have for any natural number $n(\geq 0)$

$$S^{(n)} \text{ is consistent} \tag{4.2}$$

and

$$\text{not } \vdash A_{q^{(n)}}(\ulcorner q^{(n)}\urcorner) \ \text{ and } \ \text{not } \vdash \neg A_{q^{(n)}}(\ulcorner q^{(n)}\urcorner) \text{ in } S^{(n)}.$$

4.3 Infinite extension of $S^{(0)}$

We denote by $S^{(\omega)}$ the system obtained by adding all of the following as new axioms to the system $S^{(0)}$.

$$A_{(n)} = A_{q^{(n)}}(\ulcorner q^{(n)} \urcorner) \quad \text{or} \quad \neg A_{q^{(n)}}(\ulcorner q^{(n)} \urcorner)\ (n \geq 0)$$

Then by (4.2), the system $S^{(\omega)}$ is consistent. Let $\widehat{q}(n)$ be the Gödel number of the formula $A_{(n)}$. The formula $A_{(j)}$ is not provable in $S^{(i+1)}$ for $i < j$. Thus if $i < j$, the system $S^{(i)}$ is a proper subsystem of $S^{(j)}$, and $\widehat{q}(i) < \widehat{q}(j)$. Therefore for a given formula A_r with Gödel number r, we can decide[1] whether A_r is an axiom of the form $A_{(n)}$ by comparing the given formula A_r with a finite number of axioms $A_{(n)}$ with $\widehat{q}(n) \leq r$. From this fact we can define the following two predicates on the meta level of $S^{(\omega)}$, if we assume the same Gödel numbering for the system $S^{(\omega)}$ as the one for the system $S^{(0)}$ in Theorem 4.2.

1) The predicate $\mathbf{G}^{(\omega)}(a,b)$ means the following.

   "A formula A_a with Gödel number a has just one free variable x, and an expression E_b with Gödel number b is a proof of the formula $A_a = A_a(\ulcorner a \urcorner)$ obtained from $A_a = A_a(x)$ by substituting $\ulcorner a \urcorner$ into x."

2) The predicate $\mathbf{H}^{(\omega)}(a,b)$ means the following.

   "A formula A_a with Gödel number a has just one free variable x, and an expression E_b with Gödel number b is a proof of the formula $\neg A_a = \neg A_a(\ulcorner a \urcorner)$ obtained from $\neg A_a = \neg A_a(x)$ by substituting $\ulcorner a \urcorner$ into x."

The predicates $\mathbf{G}^{(\omega)}(a,b)$ and $\mathbf{H}^{(\omega)}(a,b)$ are numeralwise expressed[2] by corresponding formulae $g^{(\omega)}(a,b)$ and $h^{(\omega)}(a,c)$ in S.

3) Let $q^{(\omega)}$ be the Gödel number of the formula

$$\forall b[\neg g^{(\omega)}(a,b) \vee \exists c(c \leq b \ \wedge \ h^{(\omega)}(a,c))].$$

   Namely

$$A_{q^{(\omega)}}(a) = \forall b[\neg g^{(\omega)}(a,b) \vee \exists c(c \leq b \ \wedge \ h^{(\omega)}(a,c))].$$

   Then

$$A_{q^{(\omega)}}(\ulcorner q^{(\omega)} \urcorner) = \forall b[\neg g^{(\omega)}(\ulcorner q^{(\omega)} \urcorner, b) \vee \exists c(c \leq b \ \wedge \ h^{(\omega)}(\ulcorner q^{(\omega)} \urcorner, c))].$$

From these and the consistency of $S^{(\omega)}$, similarly to Lemma 4.4, we obtain

$$\text{not } \vdash A_{q^{(\omega)}}(\ulcorner q^{(\omega)} \urcorner) \ \text{ and } \ \text{not } \vdash \neg A_{q^{(\omega)}}(\ulcorner q^{(\omega)} \urcorner) \text{ in } S^{(\omega)}.$$

[1] In fact by the monotonicity of the sequence $\widehat{q}(n)$ and its recursive definition, it is possible to decide whether A_r is the axiom of the form $A_{(n)}$ in a recursive way. cf. e.g. [2], Chapter 5. However as we assume ZFC on the meta level, we can make this decision by the axioms of ZFC even if we cannot make this decision recursively.

[2] Note that in the proof of Theorem 2.4 we do not use the recursiveness of the predicates $\mathbf{G}(a,b)$, $\mathbf{H}(a,b)$. Theorem 2.4 thus yields that the set-theoretic predicates $\mathbf{G}^{(\omega)}(a,b)$, $\mathbf{H}^{(\omega)}(a,b)$ on the meta level are directly expressed by the formal formulae $g^{(\omega)}(a,b)$, $h^{(\omega)}(a,c)$ of system S which is a formalization of set theory ZFC.

4.4 Transfinite extension of $S^{(0)}$

Now we let

$$A_{(\omega)} = A_{q^{(\omega)}}(\ulcorner q^{(\omega)} \urcorner) \text{ or } \neg A_{q^{(\omega)}}(\ulcorner q^{(\omega)} \urcorner)$$

and we add this as an axiom to the system $S^{(\omega)}$ to obtain a new system $S^{(\omega+1)}$. Then in a similar way as above we obtain

If $S^{(0)}$ is consistent, then $S^{(\omega+1)}$ is consistent.

Repeating this procedure in a similar way transfinitely, we can construct[3] a consistent formal system $S^{(\alpha)}$ for any ordinal number α, which is an extension of $S^{(0)}$. Namely we have

If $S^{(0)}$ is consistent, then $S^{(\alpha)}$ is consistent.

However if we can construct a formal system $S^{(\alpha)}$ for any ordinal α, the number of the totality of axioms $A_{(\alpha)}$ added at each step will be greater than countable infinity[4]. The number of the formulae of the system S is at most countable as each formula consists of a finite number of primitive symbols. This is a contradiction. Thus the extension like this must stop at a countable ordinal β_0. Namely we have shown the following.

Theorem 4.5. *There is a countable limit ordinal β_0 such that when $\alpha = \beta_0$, the system $S^{(\alpha)}$ has no undecidable proposition, and $S^{(\beta_0)}$ is complete. In other words, any extension of $S^{(\beta_0)}$ is inconsistent.*

Proof. We have only to show that β_0 is a limit ordinal. In fact if $\beta_0 = \delta + 1$, $S^{(\beta_0)}$ is obtained by adding the axiom $A_{(\delta)}$ to $S^{(\delta)}$. In this case, by the same method mentioned above, the system $S^{(\beta_0)} = S^{(\delta+1)}$ can be extended with retaining consistency, contradicting the fact that the extension ends at β_0. □

By Theorem 4.5, the extension of our system $S^{(\alpha)}$ ends at β_0. Namely if $S^{(0)}$ is consistent, $S^{(\beta_0)}$ cannot be extended further with retaining consistency, i.e. $S^{(\beta_0)}$ is complete.

The ordinal β_0 is a countable limit ordinal by Theorem 4.5. Therefore we can take a monotone increasing sequence $\{\alpha_n\}_{n=0}^{\infty}$ of countable ordinals such that $\alpha_n < \beta_0$ $(n = 0, 1, 2, \ldots)$ and

$$\beta_0 = \bigcup_{n=0}^{\infty} \alpha_n.$$

The axioms $A_{(\gamma)}$ $(\gamma < \beta_0)$ of $S^{(\beta_0)}$ are the sum of the axioms $A_{(\gamma)}$ $(\gamma < \alpha_n)$ of $S^{(\alpha_n)}$. By the definition of $\widehat{q}(\gamma)$ for $\gamma < \alpha_n$, it is possible to decide whether a given formula A_r is an axiom of $S^{(\alpha_n)}$ by seeing whether $A_{(\gamma)} = A_r$ for a finite number of γ with $\widehat{q}(\gamma) \le r$. Therefore to see if a given formula A_r is an axiom of $S^{(\beta_0)}$, it is sufficient to see if $A_{(\gamma)} = A_r$ for a finite number of γ such that $\widehat{q}(\gamma) \le r$, $\gamma < \beta_0$. By

$$\beta_0 = \bigcup_{n=0}^{\infty} \alpha_n,$$

[3] cf. the former footnotes 1, 2.

[4] This is because we assume the axiom of choice.

we have

$$\widehat{q}(\gamma) \le r \wedge \gamma < \beta_0 \Leftrightarrow \exists n \ [\widehat{q}(\gamma) \le r \wedge \gamma < \alpha_n].$$

Thus whether a given formula A_r is an axiom of $S^{(\beta_0)}$ is decided by an induction on n. Therefore if we define the predicates $\mathbf{G}^{(\beta_0)}(a,b)$ and $\mathbf{H}^{(\beta_0)}(a,b)$ by

1) The predicate $\mathbf{G}^{(\beta_0)}(a,b)$ means the following.

   "A formula A_a with Gödel number a has just one free variable x, and an expression E_b with Gödel number b is a proof of the formula $A_a = A_a(\lceil a \rceil)$ obtained from $A_a = A_a(x)$ by substituting $\lceil a \rceil$ into x."

2) The predicate $\mathbf{H}^{(\beta_0)}(a,b)$ means the following.

   "A formula A_a with Gödel number a has just one free variable x, and an expression E_b with Gödel number b is a proof of the formula $\neg A_a = \neg A_a(\lceil a \rceil)$ obtained from $\neg A_a = \neg A_a(x)$ by substituting $\lceil a \rceil$ into x."

these predicates are numeralwise expressed by the corresponding formulae $g^{(\beta_0)}(a,b)$, $h^{(\beta_0)}(a,c)$ in S, and the Gödel number $q^{(\beta_0)}$ of the formula

$$A_{q^{(\beta_0)}}(a) = \forall b[\neg g^{(\beta_0)}(a,b) \vee \exists c(c \le b \ \wedge \ h^{(\beta_0)}(a,c))]$$

in $S^{(\beta_0)}$ is defined. Thus the system $S^{(\beta_0)}$ has an undecidable proposition

$$A_{q^{(\beta_0)}}(\lceil q^{(\beta_0)} \rceil),$$

and the incompleteness theorem holds for the system $S^{(\beta_0)}$. Hence $S^{(\beta_0)}$ is incomplete. This contradicts Theorem 4.5.

In this way we meet a contradiction if we assume that set theory holds on the meta level and discuss the set theory as an object theory. The cause that a contradiction appeared in the above argument is that we assumed the axiom of infinity on the both levels of the object world and the meta world. Namely the cause is that we assumed that the actual infinity exists in both of the object and meta worlds. If we take the standpoint that mathematical existence is only computable things and that the infinity is not an actual one but is a fictitious existence which is an auxiliary tool for the inquiry of the computability, Hilbert's thesis that consistency and completeness are the certification of the soundness of mathematics will revive.

In the next section we will see the deeper problem hidden behind these.

5 Discussion

As stated at the beginning of the paper, Gödel's first incompleteness theorem says "If a theory S including number theory is consistent, then there is a proposition G which is not provable and refutable." A theory S is called complete if for any given proposition A of S, one can decide either of A or the negation $\neg A$ is derived by logical inferences from the axioms of the theory S. Therefore the incompleteness theorem means that if a theory S is consistent, then it is incomplete. That S is consistent means that for any given proposition

B, it is not the case that both of B and the negation $\neg B$ are provable. In an inconsistent theory S, thus, there is a proposition B such that B and $\neg B$ are both provable, and hence in S every proposition C is provable. The incompleteness theorem above is rephrased as follows: "A theory S which includes number theory is either inconsistent or incomplete." Further the second incompleteness theorem says "If a theory S including number theory is consistent, then the consistency of S is not provable by the method formalizable in the theory S." The second incompleteness theorem by Gödel in 1931 is at least on its surface the one which denies the Hilbert formalism's program: "A mathematical theory is shown to be sound by proving its consistency based on the finitary standpoint." This program was proposed by D. Hilbert to cope with the intuitionism proposed by L. E. J. Brouwer as a criticism to the situation of mathematics which had met several serious difficulties in its foundation around the year 1900. The procedure formalizable in the number theory is thought to be equivalent to the procedure of formal treatment of the words based on the finitary method. Therefore if it is not possible to show the consistency by the method formalizable in the number theory, it would mean that the consistency of number theory is not provable insofar as based on finitary standpoint. This would mean that Hilbert's program is not performable. In this sense, what is essential and important is the problem of consistency and it is not essential whether or not a theory is complete. However the second incompleteness theorem is a corollary of the first incompleteness theorem, so in order to discuss the problem of consistency, it was necessary first to discuss the completeness of the number theory.

5.1 Implications of the second theorem

In this way, the Gödel's second incompleteness theorem is thought to have shown the impossibility of the Hilbert's formalism or program "if one could show the consistency of the formal axiomatic system of classical mathematics from the finitary standpoint, it shows the soundness of the formal system treating infinities." This is because the second theorem is interpreted as meaning "if a system S is consistent, it is impossible to show the consistency of S by a method having the power equivalent to that of S," and because the finitary method is thought equal to the ability of number theory S.

If the first and hence the second theorem is proved by a completely syntactic method, this interpretation would be true. However as we have seen at the beginning of section 2, already in the proof of the first incompleteness theorem, a semantic interpretation is assumed as suggested just before subsection 2.1. Namely, in the process of replacing the natural number n by a numeral $\lceil n \rceil$ of the formal system, a substitution is made with assuming an identification of the meta leveled theory and the object leveled theory. This is a self-reference stated in section 1. The proof of Gödel's theorem was made by making a complete self-reference by embedding the discussion on the meta level (which is the subject who does mathematics) into the formal system (which is his own object theory). It is at a glance a syntactic argument, however it assumes the symmetry or reflexivity between the meta level and the object level. To assume the symmetry between the meta level and the object level in the case of number theory means that the discussion is not based merely on the syntactic treatment of words, but the meaning of the object theory of natural numbers is applied to the discussion of the meta level.

How then about Cretan paradox or other self-referential paradoxes which implies the following Tarski's theorem [22], [23] asserting that, if a formal system including the usual number theory is consistent, it must be impossible to express the predicate $\mathbf{T}(a)$ for the system by a formula $t(a)$ such that $t(\lceil a \rceil) \Leftrightarrow A_a$ is provable in the system whenever $a = g(A)$ is the Gödel number of a closed formula $A = A_a$. A corollary of this theorem is the following.

> The set of true sentences of a language $\mathcal{L}$ is not referred to by a sentence inside the language $\mathcal{L}$. Namely the predicate $\mathbf{T}$ showing the truthness must not be inside the language $\mathcal{L}$.

These are about the ordinary language and are not related with numbers.

This would also be the problem in the same category as the problem of number theory however. In fact in either case of numbers or language one can do reflection and rumination only when symbolization of them has been made, and no problem as above arises without the "objectification of oneself." The problem arises only when the subject of thinking on the meta level objectifies himself and makes himself the object of thinking.

5.2 Restriction of self-reference

As is well-known, the paradoxes like the famous Russell's one which looks coming from the self-reference produced the formalism that proposed that if one writes down the mathematics in a formal system of symbols in finitary method, the contradiction would disappear. The avoidance of Russell's paradox was done in this direction by a formal axiomatic set theory. This approach also excluded the Burali-Forti's paradox of the set of all ordinals and Cantor's paradox of the set of all sets by regarding them as not-sets or proper classes.

There are many problems which arise by self-reference. For example, there is the problem of 'impredicative definition.' This refers to the situation that a set M and an object m are defined as follows. Namely on the one hand, m is an element of the set M, and on the other hand, the definition of m depends on M. Similarly the terminology is used in the case when for a property P, an object m whose definition depends on P satisfies the property P. In the latter terminology, the set M above is the set of all elements which satisfy the property P. Apparently these situations are 'cyclic.' Poincaré (1905-6) asserted that the cause of the paradoxes is the vicious circle of discussions, and Russell made the same opinion as his vicious circle principle (1906) which claims to prohibit such cyclic definitions. This principle can exclude Russell's paradox, the paradox of all sets, etc. However how about the following concrete example of analysis?

The definition of supremum $\sup M$ of a subset M of real numbers is as follows in the Dedekind's construction of real numbers by the notion 'cut.' Let $\mathbb{R}$ be the totality of real numbers and let $\mathbb{Q}$ be the totality of rational numbers. An element α of $\mathbb{R}$ is defined as a set of rational numbers which satisfies the following three properties.

1. $\alpha \neq \emptyset, \quad \alpha^c := \mathbb{Q} - \alpha = \{s | s \in \mathbb{Q} \ \wedge \ s \notin \alpha\} \neq \emptyset.$
2. $r \in \alpha, s < r, s \in \mathbb{Q} \Rightarrow s \in \alpha.$
3. α has no maximum element.

Given a set M of real numbers, the supremum $\sup M$ of M is defined as the sum set of M:

$$\bigcup M = \bigcup_{\alpha \in M} \alpha.$$

In general the set M is a set of all elements m of $\mathbb{R}$ which satisfies the given properties. In the above case, in the sense that the definition of $\sup M = \cup M \in \mathbb{R}$ starts from $\mathbb{R}$ and then defines an element $\sup M$ of $\mathbb{R}$, the definition is an impredicative definition.

One might think he seems to be able to refute the above criticism as follows. The above procedure just describes a process of choosing an element $\sup M$ from the set $\mathbb{R}$, but does not create the element $\sup M$ itself by the definition. However when writing the class of all sets by C, one can then say that the set $\{x \mid x \in C,\ x \notin x\}$ is just choosing elements $x \in C$ such that $x \notin x$. Thus if the definition of $\sup M$ is allowed, then the Russell's set must be allowed to exist.

5.3 Conclusion

Those considerations would tell that just the exclusion of cyclic arguments might exclude other necessary and useful things, even if it can exclude paradoxes. Rather these and the nature of Gödel type self-contradiction will tell that the cause is the reference to the validity of oneself's statement. Especially Gödel type self-contradiction arises only when one makes an explicit judgement that his own statement is valid, which is prohibited by Tarski's theorem stated above.

The contradictions we see around us are all caused by the assertions something the like that one is right and correct and the others are wrong and incorrect. We seem to be in the age to be able to recognize the true cause of troubles and quarrels, given the deep considerations by the predecessors like Kurt Gödel and Alfred Tarski. They would even have told us that we should not as well make judgement on others as such deeds would imply, even if implicitly, the self-justification of the advocate of any claims.

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