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Arcwise Connectedness of Efficient Sets

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Abstract

Let E be a topological vector space and C a pointed closed convex cone. For a set Q in E we prove arcwise connectedness of the efficient point set $Max(Q|C)$ between any two points of a closed set $M \subset Max(Q|C)$ with a compact closed convex hull and having certain additional property. An application to a class of non-convex in general sets is given. The method generalizes the one from [6] concerning compact convex sets and allows also for such sets to obtain a more general result.

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1 Introduction

In this paper E denotes a real Hausdorff topological vector space (abbreviated TVS), C is a pointed closed convex cone in E and $Q \subset E$ is a given set. The efficient point set of Q with respect to C is defined by $Max(Q|C) = \{x \in Q \mid (x+C) \cap Q = \{x\}\}$. The purpose of this paper is to investigate the arcwise connectedness of $Max(Q|C)$ between fixed two points $x_0, x_1 \in Max(Q|C)$. In Makarov, Rachkovski, Song [6] this problem is considered for convex compact sets with closed efficient point set $Max(Q|C)$. The practical aspect of the problem is to derive some nice property (arcwise connectedness) of the solutions of multicriterial decision problems. The criteria in such problems arising in practice satisfy often some generalized concavity property, say quasi-concavity seems to be a natural condition if the criteria are interpreted as utility functions. The solutions then are the points of the efficient set for sets being usually not convex, but such for which the following condition is more appropriate.

Property 1.1. For arbitrary two points $x_0, x_1 \in Q$ there exists an element $a = a(x_0, x_1) \in E$ such that $(x+C) \cap Q \neq \emptyset$ for arbitrary $x \in [x_0, a] \cup [a, x_1]$.

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Here we use the notation $[y_0, y_1] = \{y = (1-t)y_0 + ty_1 \mid 0 \le t \le 1\}$ for the segment with end points $y_0, y_1 \in E$. Each convex set satisfies Property 1.1 with $a(x_0, x_1) = \frac{1}{2}(x_0 + x_1)$. It seems that Property 1.1 is satisfied in the case $\operatorname{int} C \ne \emptyset$ by the so called simply-shaded sets considered in the paper of Benoist, Popovici [1]. They use simply-shaded sets to prove contractibility of the efficient point set in the case of three-criterial decision problem with criteria being semistrictly quasi-concave on a convex compact subset of a TVS. The cited paper refines an earlier result of Daniilidis, Hadjisavvas, Schaible [2]. Let us underline that recently the topological and in particular the connectedness properties of the efficient point set have been intensively studied. The few cited papers give further references.

Here we generalize the approach of Makarov, Rachkovski, Song [6] in a way making it possible for applications also for classes of non-convex sets. In Section 2 we formulate the main result and the precise properties assumed for the cone and the considered sets. We give also some auxiliary matter. In Section 3 we prove the main result and obtain a similar result for a particular case under simplified assumptions. In Section 4 we apply the method to a class of sets in R^n. In Section 5 we compare for convex sets the results with those of [6].

2 Preliminaries

The following proposition states the existence of efficient points in compact sets.

Proposition 2.1. *Let $Q \subset E$ be a compact set in a TVS. Then each nonempty section $(x + C) \cap Q$, $x \in E$, contains points from $Max(Q|C)$.*

Proof The conclusion is a particular case of Theorem 3.3 in Luc [5, page 46] in view of Lemma 3.5 also there [5, page 47]. This result follows also straightforward from the Zorn Lemma. ∎

Proposition 2.1 assures that for a compact set in a TVS Property 1.1 is true for all $x_0, x_1 \in Q$ if it is assumed only for all $x_0, x_1 \in Max(Q|C)$. For this reason making more precise the assumptions for Q we put conditions similar to those of Property 1.1 only to points x_0, x_1 from $Max(Q|C)$ or a subset of $Max(Q|C)$. In this paper we restrict the considerations to sets $Q \subset E$ obeying the following property.

Property 2.2. The set M is a subset of $Max(Q|C)$ such that for any unordered pair $x_0, x_1 \in M$ an element $a(x_0, x_1) \in E$ and a nonempty set $\Gamma(x_0, x_1) \subset M$ can be chosen satisfying:

1^0. For arbitrary $x \in [x_0, a(x_0, x_1)] \cup [a(x_0, x_1), x_1]$ it holds $(x + C) \cap M \ne \emptyset$.

2^0. $\Gamma(x_0, x_1) \subset (a(x_0, x_1) + C) \cap M$ and for $\hat{x} \in \Gamma(x_0, x_1)$ it holds

$$a(x_0, \hat{x}) \in \frac{1}{2}(x_0 + a(x_0, x_1)) + C$$

(and by symmetry $a(\hat{x}, x_1) \in \frac{1}{2}(a(x_0, x_1) + x_1) + C$).

Saying above that x_0, x_1 is unordered pair we mean that an interchanging of x_0 and x_1 does not affect $a(x_0, x_1)$ (otherwise one can think of a as of a symmetric function of its arguments).

For the closed convex cone C we impose the following property.

Property 2.3. For arbitrary set $K \subset C$ the condition $0 \in \operatorname{cl}\operatorname{conv} K$ implies that $0 \in \operatorname{cl} K$.

Recall that the convex cone C is said to be pointed if zero is an extreme point for C, that is there does not exist a segment $[x_0, x_1]$ with $x_0, x_1 \in C \setminus \{0\}$ such that $0 \in [x_0, x_1]$.

Proposition 2.4. *If the cone C in a Hausdorff TVS has Property 2.3 then it is pointed.*

Proof Assume that a segment $[x_0, x_1]$ exists with $x_0, x_1 \in C \setminus \{0\}$ such that $0 \in [x_0, x_1]$. Then for the set $K = \{x_0, x_1\}$ we get $0 \in \operatorname{cl}\operatorname{conv} K$ while $0 \notin \operatorname{cl} K$. ∎

Proposition 2.4 shows that Property 2.3 is a strengthening of the property that C is pointed. The following proposition shows that for finite dimensional spaces both concepts are equivalent.

Proposition 2.5. *If E is finite dimensional, then each pointed closed convex cone C has Property 2.3.*

Proof We prove the following more general statement: *If C is a pointed closed convex cone with a bounded base in a Banach space then $K \subset C$ and $0 \notin w\text{-}\operatorname{cl} K$ implies $0 \notin \operatorname{cl}\operatorname{conv} K$.* Here $w\text{-}\operatorname{cl} K$ is the weak closure of K. For the definition and properties of the cone bases see e.g. [3].

Obviously, this statement implies our proposition, since in finite dimensional spaces the strong and weak topologies coincide. We base the proof on Proposition 2 and Theorem 1 in [7] which says that *for a closed convex pointed cone C in a Banach space, $0 \notin \operatorname{cl}\operatorname{conv}(C \setminus B(0, \epsilon))$ for any $\epsilon > 0$ is equivalent to that C has a bounded base*. Suppose that $0 \notin w\text{-}\operatorname{cl} K$. Then there exists a weak neighborhood U of 0 such that $U \cap w\text{-}\operatorname{cl} K = \emptyset$. Since every weak neighborhood U of 0 must contain a ball $B(0, \epsilon)$, it follows that $K \subset C \setminus B(0, \epsilon)$ and that $\operatorname{cl}\operatorname{conv} K \subset \operatorname{cl}\operatorname{conv}(C \setminus B(0, \epsilon))$. Since C has a bounded base, we have $0 \notin \operatorname{cl}\operatorname{conv}(C \setminus B(0, \epsilon))$ This implies that $0 \notin \operatorname{cl}\operatorname{conv} K$. ∎

We investigate the arcwise connectedness of $Max(Q|C)$ between two points $x_0, x_1 \in Max(Q|C)$. Recall that $Max(Q|C)$ is arcwise connected between x_0 and x_1 if there exists a continuous function $f : [0, 1] \to Max(Q|C)$ such that $f(0) = x_0$ and $f(1) = x_1$. Our main result is the following.

Theorem 2.6. *Suppose that Q is a set in the real Hausdorff TVS E and C is a closed convex cone in E possessing Property 2.3. Assume that a closed set $M \subset Max(Q|C)$ obeying Property 2.2 exists and such that $\operatorname{cl}\operatorname{conv} M$ is compact. Then $Max(Q|C)$ is arcwise connected between any points $x_0, x_1 \in M$, $x_0 \neq x_1$. Moreover, there exists an arc between x_0 and x_1 contained entirely in M.*

Assuming that Property 2.2 holds and x_0, x_1 are fixed points in M we construct a sequence of functions $f_n : [0, 1] \to E$, whose description is given below.

Put $T_n = \{m/2^n \mid m = 0, 1, \dots 2^n\}$ for $n = 0, 1, \dots$. We introduce inductively the points $c(t) \in M$, $t \in T$, where $T = \bigcup_n T_n$ and the functions $f_n : [0, 1] \to E$ as follows.

For $n = 0$ we define $c(0) = x_0$, $c(1) = x_1$ and

$$f_0(t) = \tfrac{1/2 - t}{1/2}\, c(0) + \tfrac{t - 0}{1/2}\, a(c(0), c(1)), \quad 0 \le t \le 1/2,$$
$$f_0(t) = \tfrac{1 - t}{1/2}\, a(c(0), c(1)) + \tfrac{t - 1/2}{1/2}\, c(1), \quad 1/2 \le t \le 1.$$

Obviously $f_0(0)=c(0)=x_0\in M$, $f_0(1)=c(1)=x_1\in M$ and $f(1/2)=a(c(0),c(1))=a(x_0,x_1)$.

Take some $n=1,2,\dots$ and assume that $c(t)$, $t\in T_{n-1}$, and $f_{n-1}:[0,1]\to E$ have been defined. Assume that they obey the properties that $f_{n-1}(t)=c(t)\in M$ for $t\in T_{n-1}$ and $f_{n-1}(t)=a(c(t-1/2^n),c(t+1/2^n))$ for $t\in T_n\setminus T_{n-1}$. Observe here that for $t\in T_n\setminus T_{n-1}$ it holds $t=m/2^n$, $m=1,3,\dots,2^n-1$, and both $t-1/2^n=(m-1)/2^n$ and $t+1/2^n=(m+1)/2^n$ belong to T_{n-1}. Define first $c(t)\in M$ for $t\in T_n\setminus T_{n-1}$ putting $c(t)$ to be arbitrary point from $\Gamma(c(t-1/2^n),c(t+1/2^n))$. Now define f_n to be piecewise linear on the consecutive intervals $[(m-1)/2^n,(2m-1)/2^{n+1}]$ and $[(2m-1)/2^{n+1},m/2^n]$, $m=1,2,\dots,2^n$, putting

$$\begin{aligned} f_n(t) &= \tfrac{(2m-1)/2^{n+1}-t}{1/2^{n+1}}\, c\left(\tfrac{m-1}{2^n}\right)+\tfrac{t-(m-1)/2^n}{1/2^{n+1}}\, a\left(c\left(\tfrac{m-1}{2^n}\right),c\left(\tfrac{m}{2^n}\right)\right), && \tfrac{m-1}{2^n}\le t\le \tfrac{2m-1}{2^{n+1}}\,,\\ f_n(t) &= \tfrac{m/2^n-t}{1/2^{n+1}}\, a\left(c\left(\tfrac{m-1}{2^n}\right),c\left(\tfrac{m}{2^n}\right)\right)+\tfrac{t-(2m-1)/2^{n+1}}{1/2^{n+1}}\, c\left(\tfrac{m}{2^n}\right), && \tfrac{2m-1}{2^{n+1}}\le t\le \tfrac{m}{2^n}\,. \end{aligned}$$

Obviously $f_n(\frac{m-1}{2^n})=c(\frac{m-1}{2^n})\in M$, $f_n(\frac{m}{2^n})=c(\frac{m}{2^n})\in M$ and $f_n(\frac{2m-1}{2^{n+1}})=a\left(c\left(\frac{m-1}{2^n}\right),c\left(\frac{m}{2^n}\right)\right)$.

Lemma 2.7. *The defined sequence of functions $f_n:[0,1]\to E$ has the properties:*
1^0. *For $t\in T_n$ all the values $f_k(t)$, $k\ge n$, coincide, are equal to $c(t)$ and belong to M.*
2^0. *The sequence $\{f_n(t)+C\}_n$ is a monotone decreasing sequence of closed sets in E.*

Proof Conclusion 1^0 is obvious by the definition of $c(t)$ and $f_n(t)$. The closedness is also obvious. We prove now the monotonicity, which means that $f_{n-1}(t)+C\supset f_n(t)+C$. In order to prove this inclusion it suffices to show that $f_n(t)\in f_{n-1}(t)+C$ for $\frac{m-1}{2^{n-1}}\le t\le \frac{m}{2^{n-1}}$, $m=1,2,\dots 2^{n-1}$. We confine to the case $\frac{m-1}{2^{n-1}}\le t\le\frac{2m-1}{2^n}$ (the case $\frac{2m-1}{2^n}\le t\le\frac{m}{2^{n-1}}$ is similar). Then $f_n\left(\frac{m-1}{2^{n-1}}\right)=f_{n-1}\left(\frac{m-1}{2^{n-1}}\right)=c\left(\frac{m-1}{2^{n-1}}\right)$ and $c\left(\frac{2m-1}{2^n}\right)=f_n\left(\frac{2m-1}{2^n}\right)\in f_{n-1}\left(\frac{2m-1}{2^n}\right)+C=a\left(c\left(\frac{m-1}{2^{n-1}}\right),c\left(\frac{m}{2^{n-1}}\right)\right)+C$. These equalities and inclusions follow from the definition of f_{n-1} and f_n. Now Property 2.2 implies that

$$\begin{aligned} f_n\left(\tfrac{4m-3}{2^{n+1}}\right) &= a\left(c\left(\tfrac{m-1}{2^{n-1}}\right),c\left(\tfrac{m}{2^{n-1}}\right)\right)\in \tfrac12\left(c\left(\tfrac{m-1}{2^{n-1}}\right)+a\left(c\left(\tfrac{m-1}{2^{n-1}}\right),c\left(\tfrac{m}{2^{n-1}}\right)\right)\right)+C\\ &= \tfrac12 f_{n-1}\left(\tfrac{m-1}{2^{n-1}}\right)+\tfrac12 f_{n-1}\left(\tfrac{2m-1}{2^n}\right)+C=f_{n-1}\left(\tfrac{4m-3}{2^{n+1}}\right)+C. \end{aligned}$$

Therefore the inclusion $f_n(t)\in f_{n-1}+C$ is established for t equal to $\frac{m-1}{2^{n-1}}=\frac{4m-4}{2^{n+1}}$, $\frac{4m-3}{2^{n+1}}$ and $\frac{2m-1}{2^n}=\frac{4m-2}{2^{n+1}}$. The piecewise linearity of both f_{n-1} and f_n on the intervals $[\frac{m-1}{2^{n-1}},\frac{4m-3}{2^{n+1}}]$ and $[\frac{4m-3}{2^{n+1}},\frac{2m-1}{2^n}]$ implies this inclusion for the whole interval $[\frac{m-1}{2^{n-1}},\frac{2m-1}{2^n}]$. ■

Introduce a more convenient representation for the functions f_n. For $t=\frac{m}{2^{n+1}}\in T_{n+1}$, $n=0,1,\dots$, we write $b_n(\frac{m}{2^{n+1}})=c(\frac{m}{2^{n+1}})$ in case of an even $m=0,2,\dots,2^{n+1}$ and $b_n(\frac{m}{2^{n+1}})=a(c(\frac{m-1}{2^{n+1}}),c(\frac{m+1}{2^{n+1}}))$ in case of an odd $m=1,3,\dots,2^{n+1}-1$. For given $t\in[0,1]$ denote also by $t_n^-=\max([0,t]\cap T_n)$ and $t_n^+=\min([t,1]\cap T_n)$ (we use this notation also in the sequel). Obviously $t_n^-\le t\le t_n^+$ and both sequences $\{t_n^-\}_n$ and $\{t_n^+\}_n$ converge to t. Now the function f_n can be represented as follows:

$$\begin{aligned} f_n(t) &= b_n(t) \quad \text{if}\quad t_{n+1}^-=t_{n+1}^+=t,\\ f_n(t) &= \tfrac{t-t_{n+1}^-}{t_{n+1}^+-t_{n+1}^-}\, b_n(t_{n+1}^-)+\tfrac{t_{n+1}^+-t}{t_{n+1}^+-t_{n+1}^-}\, b_n(t_{n+1}^+) \quad\text{if}\quad t_{n+1}^-\ne t_{n+1}^+. \end{aligned}$$

The sequence f_n can be useful in studying the arcwise connectedness of $Max(Q|C)$ between the points x_0 and x_1 using the following idea. If the sequence f_n converges to a continuous function f whose values are in $Max(Q|C)$ then $Max(Q|C)$ is arcwise connected between x_0 and x_1. We use this approach in Section 3 for the proof of Theorem 3.3, while for the proof of Theorem 2.6 we find more convenient the sequence of functions $g_n : [0, 1] \to E$ described below. The construction of the functions f_n and g_n is illustrated on Figure 1.

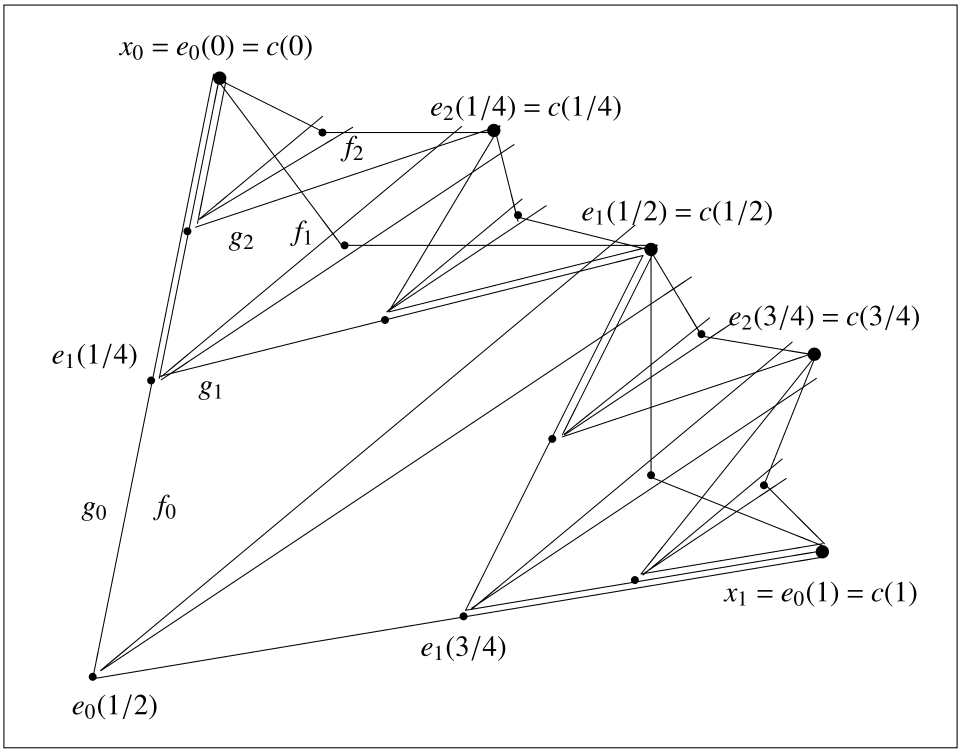

Figure 1. Construction of the sequences f_n and g_n.

Define the points $e_n(t)$, $t \in T_{n+1}$, $n = 0, 1, \dots,$ as follows: If $n = 0$ put $e_0(t) = b_0(t)$ for $t \in T_1 = \{0, 1/2, 1\}$. If $n \geq 1$ put $e_n(t) = b_n(t)$ for $n \in T_n$ (and hence $e_n(t) = c(t)$) and $e_n(t) = \frac{1}{2}\left(e_{n-1}(t - 1/2^{n+1}) + e_{n-1}(t + 1/2^{n+1})\right)$ for $t \in T_{n+1} \setminus T_n$.

Define now the functions $g_n : [0, 1] \to E$ putting

$$\begin{aligned} g_n(t) = &\quad e_n(t) \quad \text{if} \quad t^-_{n+1} = t^+_{n+1} = t, \\ g_n(t) = &\quad \tfrac{t - t^-_{n+1}}{t^+_{n+1} - t^-_{n+1}}\, e_n(t^-_{n+1}) + \tfrac{t^+_{n+1} - t}{t^+_{n+1} - t^-_{n+1}}\, e_n(t^+_{n+1}) \quad \text{if} \quad t^-_{n+1} \neq t^+_{n+1}. \end{aligned}$$

The next lemma resembles Lemma 2.7 and shows that the functions g_n obey properties similar to those of f_n.

Lemma 2.8. *The functions $g_n : [0, 1] \to E$ have the properties:*

1^0. *It holds $f_n(t) \in g_n(t) + C$ for all $t \in [0, 1]$ and nonnegative integers n.*

2^0. *For $t \in T_n$ all the values $g_k(t)$, $k \geq n$, coincide, are equal to $c(t)$ and belong to M.*

3^0. *The sequence $\{g_n(t) + C\}_n$ is a monotone decreasing sequence of closed sets in E.*

Proof Case 1^0 is a result of the assumptions in Property 2.2 and the other conclusions follow straightforward similarly to Lemma 2.7. ∎

Let $t \in [0, 1]$ We define for n nonnegative integer the sets $C_n(t) = \{c(t) \mid t_n^- \le t \le t_n^+\}$, $C(t) = \bigcap_n \operatorname{cl} C_n(t)$. The sets $\{C_n(t)\}_n$ form obviously a decreasing sequence of nonempty sets and they all are contained in M. Under the assumption that M is compact (which is the case if as in Theorem 2.6 M is closed and $\operatorname{cl}\operatorname{conv} M$ is compact) the sets $\operatorname{cl} C_n(t)$ are compact and decreasing in n, therefore their intersection $C(t)$ is not empty. Define also $E_n(t) = \{e_k(s) \mid s \in [t_n^-, t_n^+] \cap T_{k+1},\, k = n, n+1, \dots\}$ and $E_0 = E_0(t)$, $t \in [0, 1]$, where obviously $E_0(t)$ does not depend on t, since $t_0^- = 0$, $t_0^+ = 1$.

For a given n denote $\hat{E}_n = E_0 \setminus \bigcup_{t\in[0,1]} E_n(t)$ (actually in this definition we may use $t \in T_{n+1} \setminus T_n$ instead of $t \in [0, 1]$ in the union). The set $\hat{E}_n$ is obviously finite.

Lemma 2.9. *Let $n \ge 1$ be fixed. If $\operatorname{cl}\operatorname{conv} M$ is compact then for each neighbourhood U of zero in E there exists a number m_0 independent on t such that $E_{n+m}(t) \subset \operatorname{conv} C_n(t) + U$ for all $t \in [0, 1]$ and all $m \ge m_0$.*

Proof We prove the lemma in three steps.

Step 1 *Let $n \ge 1$ be fixed and m be nonnegative integer. Then the elements $e_{n+m}(s)$ from $E_n(t)$ can be expressed as convex combinations $e_{n+m}(s) = (1-\beta)c + \beta\hat{e}$, where $c \in \operatorname{conv} C_n(t)$, $\hat{e} \in \hat{E}_n$ and $0 \le \beta \le 1/2^{m+1}$.*

Obviously, it suffices to consider only $s \in T_{n+m+1} \setminus T_{n+m}$ (if $s \in T_{n+m}$ then $e_{n+m}(s) \in C_n(t)$). The proof is done by induction. We skip it giving only the first two steps which outline the idea. The main difficulty in the inductive step is just choosing a convenient notation.

For $m = 0$ if $t_n^- = t_n^+ = t$ then $e_n(t) = c_n(t) \in C_n(t)$. It holds also $t_{n+m}^- = t_{n+m}^+ = t$ and $e_{n+m}(t) = c_n(t) \in C_n(t)$ for all m and our conclusion is true with $\beta = 0$. If $t_n^- \ne t_n^+$ then $t_n^+ - t_n^- = 1/2^n$ and $s_n^0 = \frac{1}{2}(t_n^- + t_n^+)$ is the unique number in $[t_n^-, t_n^+]$ from $T_{n+1} \setminus T_n$. By definition we have $e_n(s_n^0) = \frac{1}{2}\Big(e_{n-1}(s_n^0 - 1/2^n) + e_{n-1}(s_n^0 + 1/2^n)\Big) = \frac{1}{2}c_1 + \frac{1}{2}\hat{e}$ for some $c_1 \in C_n(t)$ and $\hat{e} \in \hat{E}_n$.

For $m = 1$ there are two numbers in $[t_n^-, t_n^+]$ from $T_{n+2} \setminus T_{n+1}$, namely $s_{n+1}^0 = \frac{1}{2}(t_n^- + s_n^0)$ and $s_{n+1}^1 = \frac{1}{2}(s_n^0 + t_n^+)$. Take one of them, say s_{n+1}^0. Then $e_{n+1}(s_{n+1}^0) = \frac{1}{2}c_2 + \frac{1}{2}e_n(s_n^0) = \frac{3}{4}\Big(\frac{2}{3}c_2 + \frac{1}{3}c_1\Big) + \frac{1}{4}\hat{e} = \frac{3}{4}c + \frac{1}{4}\hat{e}$, where $c \in \operatorname{conv} C_n(t)$ and $\hat{e} \in \hat{E}_n(t)$.

Step 2 *Let $n \ge 1$ be fixed and m be nonnegative integer and $t \in [0, 1]$ be arbitrary. Then $E_{n+m}(t) \subset \operatorname{conv} C_n(t) + [0, 1/2^{m+1}] \cdot \Big(\hat{E}_n - \operatorname{conv} M\Big)$.*

This is an immediate consequence of Step 1, since an element $e \in E_{n+m}(t)$ is of the form $e = e_{n+k}(s)$ with $k \ge m$ and therefore $e = (1-\beta)c + \beta\hat{e} = c + \beta(\hat{e} - c)$ with $c \in \operatorname{conv} C_n(t) \subset \operatorname{conv} M$, $\hat{e} \in \hat{E}_n$ and $0 \le \beta \le 1/2^{k+1} \le 1/2^{m+1}$.

Step 3 *It holds $E_{n+m}(t) \subset \operatorname{conv} C_n(t) + U$ for some m and all $t \in [0, 1]$.*

Actually this step proves the lemma, since the sets $\{E_{n+m}(t)\}_m$ form a decreasing sequence of sets. The set $\hat{E}_n - \operatorname{cl}\operatorname{conv} M$ is compact, since $\hat{E}_n$ is finite and $\operatorname{cl}\operatorname{conv} M$ is compact. Therefore there exists m, such that $\beta\Big(\hat{E}_n - \operatorname{cl}\operatorname{conv} M\Big) \subset U$ for all $\beta \in [0, 1/2^{m+1}]$. For such an m accordingly to the previous step our conclusion holds. ■

Now for $t \in [0, 1]$ we define the sets $\mathcal{G}_n(t) = \bigcup\{g_k(s) \mid t_n^- \le s \le t_n^+,\, k \ge n\} \cup \operatorname{cl}\operatorname{conv} C(t)$. We put also $\mathcal{G} = \mathcal{G}_0(t)$ for in this case $\mathcal{G}_0(t)$ does not depend on $t \in [0, 1]$.

Lemma 2.10. *Let $n \geq 1$ be fixed. If* $\operatorname{cl}\operatorname{conv} M$ *is compact then for each neighbourhood U of zero in E there exists a number m_0 independent on t such that $\mathcal{G}_{n+m}(t) \subset \operatorname{conv} C_n(t) + U$ for all $t \in [0, 1]$ and all $m \geq m_0$. Further the sets $\mathcal{G}_n(t)$ are compact.*

Proof Take a neighbourhood V of zero in E satisfying $V + V \subset U$ and $\alpha V \subset V$ for $|\alpha| \leq 1$ (existence of such neighbourhoods is well known, see e.g. Theorem 5.1 in Kelley, Namioka [4, page 34]). According to Lemma 2.9 there exists a number m_0 independent on t such that $E_{n+m_0}(t) \subset \operatorname{conv} C_n(t) + V$. Take $s \in [t_n^-, t_n^+]$ and consider $g_k(s)$, $k \geq n + m_0$. If $s_k^- = s_k^+$ then $g_k(s) \in C_k(s) \subset C_n(t) \subset \operatorname{conv} C_n(t) + U$. If $s_k^- \neq s_k^+$ then by definition for $\beta = (s - s_k^-)/(s_k^+ - s_k^-) \in [0, 1]$ it holds $g_k(s) \in (1-\beta)e_k(s_k^-) + \beta e(s_k^+)$. From the choice of k we have $g_k(s) \in (1-\beta)(\operatorname{conv} C_n(t) + V) + \beta(\operatorname{conv} C_n(t) + V) = ((1-\beta)\operatorname{conv} C_n(t) + \beta \operatorname{conv} C_n(t)) + ((1-\beta)V + \beta V) \subset \operatorname{conv} C_n(t) + (V + V) \subset \operatorname{conv} C_n(t) + U$.

To prove the compactness of $\mathcal{G}_n(t)$ take an arbitrary open covering of $\mathcal{G}_n(t)$. Since $\operatorname{cl}\operatorname{conv} C(t)$ is compact, a finite number of the sets in the covering covers $\operatorname{cl}\operatorname{conv} C(t)$ and hence their union contains a set of the type $\operatorname{cl}\operatorname{conv} C(t) + U$, where U is a neighbourhood of zero in E. Let $V + V \subset U$ for some neighbourhood of zero and take n_0 such that $\operatorname{conv} C_{n_0}(s) \subset \operatorname{cl}\operatorname{conv} C_{n_0}(s) \subset \operatorname{cl}\operatorname{conv} C(t) + V$. The possibility of such a choice holds, since for each k the sets $\bigcup\{\operatorname{cl}\operatorname{conv} C_k(s) \mid s \in [t_n^-, t_n^+]\} \subset \operatorname{cl}\operatorname{conv} M$ are compact (actually this union is finite) and their intersection is $\operatorname{cl}\operatorname{conv} C(t)$. Now according to the proved part of this lemma we can find m_0 such that $\mathcal{G}_{n_0+m_0}(s) \subset C_{n_0}(t) + V \subset \operatorname{cl}\operatorname{conv} C(t) + V + V \subset \operatorname{cl}\operatorname{conv} C(t) + U$ for all $s \in [t_n^-, t_n^+]$. Therefore the set $\mathcal{G}_n(t) \setminus (\operatorname{cl}\operatorname{conv} C(t) + U)$ can have a nonempty intersection only with the functions $g_k(s)$, $t_n^- \leq s \leq t_n^+$, with $k \leq n_0 + m_0$. The union of the images of finitely many continuous functions defined on a compact interval is a compact set, that is it can be covered by a finitely many of the sets of the given covering. Thus, we have shown that $\mathcal{G}_n(t)$ admits a finite subcovering, therefore it is compact. ∎

Further we need a property of the least points of sets derived in Proposition 2.11 below. Recall that if G is a subset of E then the point $x \in G$ is said to be the least element of G with respect to C if $G \subset x + C$.

Proposition 2.11. *Let $G_0 \supset G_1 \supset \ldots$ be a decreasing sequence of compact subsets in the real Hausdorff TVS Let $x_n \in G_n$ be the least element of G_n with respect to the pointed closed cone C. Then the intersection $G = \bigcap_n G_n$ is not empty and the sequence x_n is convergent to the unique least element x of G.*

Proof This proposition constitutes Lemma 2.3 in Makarov, Rachkovski, Song [6]. For the sake of self-containment we adduce its short proof. The nonemptiness of G is a consequence of the compactness (hence closedness for E is Hausdorff) of the sets in the decreasing sequence. Further the sequence $\{x_n\}$ has an adherent point $x \in G$. Suppose that there exists $z \in G$ such that $z \notin x + C$. Since $x + C$ is closed, there exists a neighbourhood U of zero in E such that $(z - U) \cap (x + C) = \emptyset$. Hence $z \notin x + U + C$. Take nonnegative integer n such that $x_n \in x + U$. Then $x_n + C \subset x + U + C$ and therefore $z \notin x_n + C$. On the other hand we have $z \in G \subset G_n \subset x_n + C$. This contradiction asserts that $G \subset x + C$. Hence x is the least element of G. Since C is pointed, the set G has x as the unique least element. ∎

With the help of the introduced above sequence of functions g_n we define the sequence of set-valued functions $G_n : [0, 1] \to \mathcal{G}$. The set $\mathcal{G} = \mathcal{G}_0(t)$ is compact according to Lemma

2.10. We put $G_n(t) = (g_n(t)+C)\cap\mathcal{G}$. We define further the set-valued function $G : [0, 1] \to \mathcal{G}$ putting $G(t) = \bigcap_n G_n(t)$. According to Lemma 2.8 the sequence $\{G_n(t)\}$ is a decreasing sequence of compact subsets of $\mathcal{G}$. Since $g_n(t)$ is the least point of $G_n(t)$, from Proposition 2.11 the sequence $g_n(t)$ converges to the unique least point of $G(t)$ which we denote by $g(t)$. From the definition of G we get $G(t) = (g(t)+C)\cap\mathcal{G}$. The function $g : [0, 1] \to \mathcal{G}$ satisfies $g(t) = \lim_n g_n(t)$.

In order to check arcwise connectedness of $M \subset Max(Q|C)$ we will show that $g : [0, 1] \to \mathcal{G}$ is continuous and has values in M. Let us first see some properties of G_n. Recall that the graph of the set-valued function $\Phi : [0, 1] \to E$ is the set $\operatorname{graph}\Phi = \{(t,\phi) \mid t \in [0,1],\ \phi \in \Phi(t)\}$. The set-valued function $\Phi : [0, 1] \to E$ is said to be closed if its graph is closed in the product space $[0, 1]\times E$.

Lemma 2.12. *The set-valued functions $G_n : [0, 1] \to E$ and $G : [0, 1] \to E$ are closed.*

Proof The graph of G_n is the finite union $\operatorname{graph} G_n = \bigcup\{\operatorname{graph} G_{n,m} \mid m = 1,\dots 2^{n+1}\}$ where $G_{n,m}$ is the restriction of G_n on the interval $[(m-1)/2^{n+1}, m/2^{n+1}]$. Therefore the closedness of G_n follows by the closedness of $G_{n,m}$ defined by $G_{n,m}(t) = (g_{n,m}(t)+C)\cap\mathcal{G}$ with

$$g_{n,m}(t) = g_n(t) = \frac{t-(m-1)/2^{n+1}}{1/2^{n+1}}\, e\Big(\frac{m-1}{2^{n+1}}\Big) + \frac{m/2^{n+1}-t}{1/2^{n+1}}\, e\Big(\frac{m}{2^{n+1}}\Big), \quad \frac{m-1}{2^{n+1}} \le t \le \frac{m}{2^{n+1}}\,.$$

Let $(t_k,y_k) \to (t_0,y_0)$ and $(t_k,y_k) \in \operatorname{graph} G_{n,m}$. This means that $(m-1)/2^{n+1} \le t_k \le m/2^{n+1}$, $y_k - g_{n,m}(t_k) \in C$, $y_k \in \mathcal{G}$. Since C and $\mathcal{G}$ are closed and $g_{n,m}$ continuous, we get the same conditions with (t_k,y_k) replaced by (t_0,y_0). Therefore $G_{n,m}$ is closed. The closedness of $\operatorname{graph} G$ follows from the representation $\operatorname{graph} G = \bigcap_n \operatorname{graph} G_n$. ∎

Lemma 2.13. *It holds $C(t) \subset G(t) \subset \bigcap_k \mathcal{G}_k(t) \subset \operatorname{cl}\operatorname{conv} C(t) \subset g(t)+C$.*

Proof First we prove $G(t) \subset \bigcap_k \mathcal{G}_k(t) \subset \operatorname{cl}\operatorname{conv} C(t)$. Fix a nonnegative integer n. Let U be a neighbourhood of zero in E and V be a neighbourhood of zero such that $V+V \subset U$. Choose now according to Lemma 2.10 the nonnegative integer m_0 such that $\mathcal{G}_{n+m}(s) \subset \operatorname{conv} C_n(t)+V$ for all $s \in [t_n^-, t_n^+]$ and all $m \ge m_0$. Similarly as in the proof of Lemma 2.10 we show that eventually diminishing m_0 we will have $\operatorname{conv} C_n(t) \subset \operatorname{cl}\operatorname{conv} C(t)+V$. Therefore for $k \ge n+m_0$ we have $\mathcal{G}_k(t) \subset \operatorname{conv} C_n(t)+V \subset \operatorname{cl}\operatorname{conv} C(t)+V+V \subset \operatorname{cl}\operatorname{conv} C(t)+U$. Since $G_k(t) \subset \mathcal{G}_k(t)$ we get $G(t) \subset \bigcap_k \mathcal{G}_k(t) \subset \operatorname{cl}\operatorname{conv} C(t)+U$. Since this is true for arbitrary neighbourhood U of zero and E is Hausdorff and $\operatorname{cl}\operatorname{conv} C(t)$ is closed we get $G(t) \subset \operatorname{cl}\operatorname{conv} C(t)$.

Now we prove $C(t) \subset G(t) \subset g(t)+C$, whence immediately it follows $\operatorname{cl}\operatorname{conv} C(t) \subset \operatorname{cl}\operatorname{conv}(g(t)+C) = g(t)+C$. Let $c \in C(t) = \bigcap_n \operatorname{cl} C_n(t)$. For each neighbourhood U of zero in E and each $k \in N$, where N is the set of nonnegative integers, choose an element $c(k,U) \in C_k(t)\cap(c+U)$, which is possible from $c \in \operatorname{cl} C_k(t)$. Now $c(k,U) \in C_k(t)$ means that $c(k,U) = c(s(k,U))$ for some $s(k,U) \in [t_k^-, t_k^+]\cap T$. Let $s(k,U) \in T_r$ with $r \ge k$. Then $c(s(k,U)) = g_r(s(k,U)) \in (g_k(s(k,U))+C)\cap\mathcal{G} = G_k(s(k,U))$. We consider $s(k,U)$ as a net over the directed set $N\times\mathcal{U}$, where $\mathcal{U}$ is a base of neighbourhoods of zero in E (we define $(k_1,U_1) \prec (k_2,U_2)$ if $k_1 < k_2$ and $U_1 \supset U_2$). Let n be fixed. Then for $k > n$ and $U \in \mathcal{U}$ we have $c(s(k,U)) \in G_k(s(k,U)) \subset G_n(s(k,U))$. Thus $s(k,U) \to t$, $c(s(k,U)) \to c$ and $c(s(k,U)) \in G_n(s(k,U))$, $k \ge n$. Since G_n is closed we have $c \in G_n(t)$ for each n. Therefore $c \in \bigcap_n G_n(t) = G(t)$. ∎

3 Main Result

In the previous section we formulated our main result as Theorem 2.6. Here we give the proof.

Proof of Theorem 2.6 Fix the points $x_0, x_1 \in M$, $x_0 \neq x_1$, and construct the functions $g_n, g : [0, 1] \to \mathcal{G}$ and the set-valued functions $G_n, G : [0, 1] \to \mathcal{G}$ as in the previous section with $c(0) = x_0$, $c(1) = x_1$ and $c(t) \in M$ for $t \in T$. Recall the properties $G_n(t) = (g_n(t)+C) \cap \mathcal{G}$, $g(t) = \lim_n g_n(t) \in G(t)$, $G(t) = \bigcap_n G_n(t) = (g(t)+C) \cap \mathcal{G} \subset \mathrm{cl}\,\mathrm{conv}\,C(t)$. We use here also the other notations introduced in the previous section.

We do the proof in several steps.

Step 1 *There exists a sequence of nonnegative integers $n_i \to \infty$ and a sequence of numbers $s_i \to t$, $s_i \in [t^-_{n_i}, t^-_{n_i}]$ and $s_i \in T_{n_i+1}$, such that $g(t) = \lim_i e_{n_i}(s_i)$.*

For $t \in T = \cup T_n$ this is obviously true, since then $G_i(t) = \{c(t)\}$ and $s_i = t$ for all sufficiently large i. Suppose now that $t \in [0, 1] \setminus T$. Then for each nonnegative integer n it holds $g_n(t) = (1-\beta_n)e^-_n + \beta_n e^+_n$ with $\beta_n = (t - t^-_{n+1})/(t^+_{n+1} - t^-_{n+1})$, $e^-_n = e_n(t^-_{n+1})$, $e^+_n = e_n(t^+_{n+1})$. The triple (β_n, e^-_n, e^+_n) belongs to the compact set $[0, 1] \times \mathcal{G} \times \mathcal{G}$. Therefore there exists a subsequence $n = n_i$ such that $\beta_{n_i} \to \beta \in [0, 1]$, $e^-_{n_i} \to u^- \in \mathcal{G}$, $e^+_{n_i} \to u^+ \in \mathcal{G}$. For $k \geq n$ we have $(t^-_{k+1}, e^-_k) = (t^-_{k+1}, g_k(t^-_{k+1})) \in \mathrm{graph}\, G_k \in \mathrm{graph}\, G_n$ and similarly $(t^+_{k+1}, e^+_k) = (t^+_{k+1}, g_k(t^+_{k+1})) \in \mathrm{graph}\, G_n$. Further $(t^-_{k+1}, e^-_k) \to (t, u^-)$ and $(t^+_{k+1}, e^+_k) \to (t, u^+)$. Since $\mathrm{graph}\, G_n$ is closed, therefore (t, u^-) and (t, u^+) belong to $\mathrm{graph}\, G_n$ for each nonnegative integer n. In other words $u^-, u^+ \in G_n(t)$ for all n, whence $u^-, u^+ \in \bigcap_n G_n(t) = G(t)$. Accounting that $g(t) = \lim_n g_n(t) = \lim_i g_{n_i}(t) = (1-\beta)u^- + \beta u^+$ we get $u^-, u^+ \in G(t) \subset g(t) + C = (1-\beta)u^- + \beta u^+ + C$. These inclusions imply that $g(t)$ is either u^- or u^+. Indeed, if $\beta = 0$ then $g(t) = u^-$ and if $\beta = 1$ then $g(t) = u^+$. Let now $0 < \beta < 1$. Then $u^- \in (1-\beta)u^- + \beta u^+ + C$ implies $u^- - u^+ \in C$. Similarly $u^+ \in (1-\beta)u^- + \beta u^+ + C$ implies $u^+ - u^- \in C$. Since according to Proposition 2.4 the cone is pointed, we get $u^- = u^+$, whence also $g(t) = u^- = u^+$. Since $u^- = \lim_i e^-_{n_i} = \lim_i g_{n_i}(t^-_{n_i+1})$ and $u^+ = \lim_i e^+_{n_i} = \lim_i g_{n_i}(t^+_{n_i+1})$ we have shown actually that there exists a sequence of nonnegative integers $n_i \to \infty$ and a sequence of numbers $s_i \to t$, $s_i \in [t^-_{n_i+1}, t^-_{n_i+1}]$ such that $g(t) = \lim_i e_{n_i}(s_i)$.

Step 2 *It holds $g(t) \in M$.*

This is the most difficult part of the proof. Like in the proof of Step 1 we may confine to $t \in [0, 1] \setminus T$. We divide this step into several substeps.

1^0. In Step 1 it was shown that there exists a sequence of numbers of nonnegative integers $n_i \to \infty$ and a sequence $s_i \in [t^-_{n_i}, t^+_{n_i}]$, $s_i \in T_{n_i}$, such that $g(t) = \lim_i e_{n_i}(s_i)$. Taking if needed a subsequence we may assume that $s_i \in T_{n_i+1} \setminus T_{n_i}$. Indeed, if such a subsequence does not exists, then for all sufficiently large i it holds $e_{n_i}(s_i) = c(s_i) \in M$ and consequently $g(t) \in M$ by the closedness of M.

2^0. We reformulate the proved in Step 1 as follows. If $t \in [0, 1] \setminus T$ there is a strictly increasing sequence of different nonnegative integers $\{\nu^0\}$, which we consider as a directed set (more detailed written as $\{\nu^0\} = \{\nu^0_1, \nu^0_2, \dots\}$ with $\nu^0_1 < \nu^0_2 < \dots$, $\nu^0_k \to \infty$), and a net $\{s^0_{\nu^0},\ \nu^0 \in \{\nu^0\}\}$, such that for $\nu^0 \in \{\nu^0\}$ it holds

a. $s^0_{\nu^0} \in T_{\nu^0+1} \setminus T_{\nu^0}$, $s^0_{\nu^0} \in [t^-_{\nu^0}, t^+_{\nu^0}]$,

b. $e_{\nu^0}(s^0_{\nu^0}) \to g(t)$.

3^0. For any $\nu^0 \in \{\nu^0\}$, $\nu^0 \geq 1$, we have

$$e_{\nu^0}(s^0_{\nu^0}) = \tfrac{1}{2}e_{\nu^0-1}(s^0_{\nu^0-1} - \tfrac{1}{2^{\nu^0+1}}) + \tfrac{1}{2}e_{\nu^0-1}(s^0_{\nu^0-1} + \tfrac{1}{2^{\nu^0+1}}) = \tfrac{1}{2}c(\bar{s}^0_{\nu^0}) + \tfrac{1}{2}e_{\nu^0-1}(s^1_{\nu^0}), \tag{3.1}$$

where $\bar{s}^0_{\nu^0} = s^0_{\nu^0} \pm \frac{1}{2^{\nu^0+1}} \in T_{\nu^0-1}$, $s^1_{\nu^0} = s^0_{\nu^0} \mp \frac{1}{2^{\nu^0+1}} \in T_{\nu^0} \setminus T_{\nu^0-1}$. It is true also that $\bar{s}^0_{\nu^0} \in [t^-_{\nu^0-1}, t^+_{\nu^0-1}]$, $s^1_{\nu^0} \in [t^-_{\nu^0-1}, t^+_{\nu^0-1}]$. Actually accounting $t^+_{\nu^0} - t^-_{\nu^0} = \frac{1}{2^{\nu^0}}$ the conditions $s^0_{\nu^0} \in [t^-_{\nu^0}, t^+_{\nu^0}]$, $s^0_{\nu^0} \in T_{\nu^0+1} \setminus T_{\nu^0}$ determine $s^0_{\nu^0}$ uniquely, namely $s^0_{\nu^0} = \frac{1}{2}(t^-_{\nu^0} + t^+_{\nu^0})$. We have now that $s^0_{\nu^0} \pm \frac{1}{2^{\nu^0+1}}$ is either $t^-_{\nu^0}$ or $t^+_{\nu^0}$. Both these numbers are contained in the interval $[t^-_{\nu^0-1}, t^+_{\nu^0-1}]$.

It holds $c(\bar{s}^0_{\nu^0}) \in M$ and M is compact. Therefore there exists a subsequence $\{\nu^1\} \subset \{\nu^0\}$ such that the net $\{c(\bar{s}^0_{\nu^1}), \nu^1 \in \{\nu^1\}\}$ is convergent $c(\bar{s}^0_{\nu^1}) \to c^1$ and $c^1 \in M$. From (3.1) we get that the sequence $e_{\nu^1-1}(s^1_{\nu^1})$ is also convergent. Let $e_{\nu^1-1}(s^1_{\nu^1}) \to e^1$, where $e^1 \in \mathcal{G}$. The equalities (3.1) restricted to the sequence $\{\nu^1\}$ are

$$e_{\nu^1}(s^0_{\nu^1}) = \tfrac{1}{2}c(\bar{s}^0_{\nu^1}) + \tfrac{1}{2}e_{\nu^1-1}(s^1_{\nu^1}), \quad \nu^1 \in \{\nu^1\}. \tag{3.2}$$

A passing to a limit in (3.2) gives $g(t) = \frac{1}{2}c^1 + \frac{1}{2}e^1$.

Resume now the above considerations. For the sequence $\{\nu^1\} \subset \{\nu^0\}$ written as $\{\nu^1\} = \{\nu^1_1, \nu^1_2, \ldots\}$ with $1 \leq \nu^1_1 < \nu^1_2 < \ldots$, $\nu^1_k \to \infty$, considered as a directed set, there exist nets $\bar{s}^0_{\nu^1}$ and $s^1_{\nu^1}$ of numbers satisfying for $\nu^1 \in \{\nu^1\}$ the conditions:

a. $s^1_{\nu^1} \in T_{\nu^1} \setminus T_{\nu^1-1}$, $s^1_{\nu^1} \in [t^-_{\nu^1-1}, t^+_{\nu^1-1}]$ (true also for $\nu^1 \in \{\nu^0\}$),

b. $\bar{s}^0_{\nu^1} \in T_{\nu^0-1}$, $\bar{s}^0_{\nu^1} \in [t^-_{\nu^1-1}, t^+_{\nu^1-1}]$ (true also for $\nu^1 \in \{\nu^0\}$),

c. $c(\bar{s}^0_{\nu^1}) \to c^1 \in M$, $e_{\nu^1-1}(s^1_{\nu^1}) \to e^1 \in \mathcal{G}$,

d. $\frac{1}{2}c(\bar{s}^0_{\nu^1}) + \frac{1}{2}e_{\nu^1-1}(s^1_{\nu^1}) \to g(t)$ and accounting the limit in the left hand side $\frac{1}{2}c^1 + \frac{1}{2}e^1 = g(t)$,

e. $c^1 \in G(t)$ and in particular $c^1 \in g(t) + C$.

Actually only point **e** has not been shown so far and now we prove it. For $\nu^1 \in \{\nu^1\}$ we have $\bar{s}^0_{\nu^1} \in [t^-_{\nu^1-1}, t^+_{\nu^1-1}]$, whence $\bar{s}^0_{\nu^1} \to t$. It holds also $c(\bar{s}^0_{\nu^1}) = e_{\nu^1-1}(\bar{s}^0_{\nu^1}) = g_{\nu^1-1}(\bar{s}^0_{\nu^1}) \in G_{\nu^1-1}(\bar{s}^0_{\nu^1})$. Since $\{G_{\nu^1-1}(t)\}$ is a monotone decreasing sequence, then by fixing $\nu^1_* \in \{\nu^1\}$ we get $c(\bar{s}^0_{\nu^1}) = g_{\nu^1-1}(\bar{s}^0_{\nu^1}) \in G_{\nu^1_*-1}(\bar{s}^0_{\nu^1})$ for $\nu_1 \geq \nu^1_*$ and $c(\bar{s}^0_{\nu^1}) = g_{\nu^1-1}(\bar{s}^0_{\nu^1}) \to c^1$. Since $G_{\nu^1_*-1}$ is closed, we get $c^1 \in G_{\nu^1_*-1}(t)$ for each $\nu^1_* \in \{\nu^1\}$. Therefore $c^1 \in \bigcap\{G_{\nu^1-1}(t) \mid \nu_1 \in \{\nu^1\}\} = G(t) = (g(t) + C) \cap \mathcal{G}$, in particular $c^1 \in g(t) + C$.

4^0. From (3.2) we see that for $\nu^1 \in \{\nu^1\}$, $\nu^1 \geq 2$, it holds

$$\begin{aligned} e_{\nu^1}(s^0_{\nu^1}) &= \tfrac{1}{2}c(\bar{s}^0_{\nu^1}) + \tfrac{1}{4}e_{\nu^1-2}(s^1_{\nu^1} - \tfrac{1}{2^{\nu^1}}) + \tfrac{1}{4}e_{\nu^1-2}(s^1_{\nu^1} + \tfrac{1}{2^{\nu^1}}) \\ &= \tfrac{1}{2}c(\bar{s}^0_{\nu^1}) + \tfrac{1}{4}c(\bar{s}^1_{\nu^1}) + \tfrac{1}{4}e_{\nu^1-2}(s^2_{\nu^1}), \end{aligned} \tag{3.3}$$

where $\bar{s}^1_{\nu^1} = s^1_{\nu^1} \pm \frac{1}{2^{\nu^1}} \in T_{\nu^1-2}$, $s^2_{\nu^1} = s^1_{\nu^1} \mp \frac{1}{2^{\nu^1}} \in T_{\nu^1-1} \setminus T_{\nu^1-2}$. It is true also $\bar{s}^1_{\nu^1} \in [t^-_{\nu^1-2}, t^+_{\nu^1-2}]$, $s^2_{\nu^1} \in [t^-_{\nu^1-2}, t^+_{\nu^1-2}]$. Actually accounting $t^+_{\nu^1-1} - t^-_{\nu^1-1} = \frac{1}{2^{\nu_1-1}}$ the conditions $s^1_{\nu^1} \in [t^-_{\nu^1-1}, t^+_{\nu^1-1}]$, $s^1_{\nu^1} \in T_{\nu^1} \setminus T_{\nu^1-1}$ determine $s^1_{\nu^1}$ uniquely, namely $s^1_{\nu^1} = \frac{1}{2}(t^-_{\nu^1-1} + t^+_{\nu^1-1})$. We have now that $s^1_{\nu^1} \pm \frac{1}{2^{\nu^1}}$ is either $t^-_{\nu^1-1}$ or $t^+_{\nu^1-1}$. Both these numbers are contained in the interval $[t^-_{\nu^1-2}, t^+_{\nu^1-2}]$.

It holds $c(\bar{s}^1_{\nu^1}) \in M$ and M is compact. Therefore there exists a subsequence $\{\nu^2\} \subset \{\nu^1\}$ considered as a directed set, such that the net $\{c(\bar{s}^1_{\nu^2}), \nu^2 \in \{\nu^2\}\}$ is convergent $c(\bar{s}^1_{\nu^2}) \to c^2$ and $c^2 \in M$. From (3.3) we get that the sequence $e_{\nu^2-2}(s^2_{\nu^2})$ is also convergent. Let $e_{\nu^2-2}(s^2_{\nu^2}) \to e^2$, where $e^2 \in \mathcal{G}$. The equalities (3.3) restricted to the sequence $\{\nu^2\}$ are

$$e_{\nu^2}(s^0_{\nu^2}) = \tfrac{1}{2}c(\bar{s}^0_{\nu^2}) + \tfrac{1}{4}c(\bar{s}^1_{\nu^2}) + \tfrac{1}{4}e_{\nu^2-2}(s^2_{\nu^2}), \quad \nu^2 \in \{\nu^2\}. \tag{3.4}$$

A passing to a limit in (3.4) gives $g(t) = \frac{1}{2}c^1 + \frac{1}{4}c^2 + \frac{1}{4}e^2$.

Resume now the above considerations. For the sequence $\{\nu^2\} \subset \{\nu^1\}$ written as $\{\nu^2\} = \{\nu^2_1, \nu^2_2, \dots\}$ with $2 \le \nu^2_1 < \nu^2_2 < \dots$, $\nu^2_k \to \infty$, considered as a directed set, there exist nets $\bar{s}^1_{\nu^2}$ and $s^2_{\nu^2}$ of numbers satisfying for $\nu^2 \in \{\nu^2\}$ the conditions:

a. $s^2_{\nu^2} \in T_{\nu^2-1} \setminus T_{\nu^2-2}$, $s^2_{\nu^2} \in [t^-_{\nu^2-2}, t^+_{\nu^2-2}]$ (true also for $\nu^2 \in \{\nu^1\}$),

b. $\bar{s}^1_{\nu^2} \in T_{\nu^2-2}$, $\bar{s}^1_{\nu^2} \in [t^-_{\nu^2-2}, t^+_{\nu^2-2}]$ (true also for $\nu^2 \in \{\nu^1\}$),

c. $c(\bar{s}^1_{\nu^2}) \to c^2 \in M$, $e_{\nu^2-2}(s^2_{\nu^2}) \to e^2 \in \mathcal{G}$,

d. $\frac{1}{2}c(\bar{s}^0_{\nu^2}) + \frac{1}{4}c(\bar{s}^1_{\nu^2}) + \frac{1}{4}e_{\nu^2-2}(s^2_{\nu^2}) \to g(t)$ and $\frac{1}{2}c^1 + \frac{1}{4}c^2 + \frac{1}{4}e^2 = g(t)$,

e. $c^2 \in G(t)$ and in particular $c^2 \in g(t) + C$.

Point **e** is derived quite similarly as in $\mathbf{2}^0$**.**

$\mathbf{5}^0$**.** The steps described in $\mathbf{3}^0$ and $\mathbf{4}^0$ can be continued by an obvious induction. Skipping the details we refer the final result. In the m-th step some sequence $\{\nu^m\} \subset \{\nu^{m-1}\}$ is defined. Here $\{\nu^m\} = \{\nu^m_1, \nu^m_2, \dots\}$, $m \le \nu^m_1 < \nu^m_2 < \dots$, $\nu^m_k \to \infty$, is considered as a directed set. To the already defined in the previous points nets of numbers we add the nets of numbers $\{s^m_{\nu^{m-1}}, \nu^{m-1} \in \{\nu^{m-1}\}$, such that now the following conditions are satisfied for $\nu^m \in \{\nu^m\}$.

a. $s^m_{\nu^m} \in T_{\nu^m+1-m} \setminus T_{\nu^m-m}$, $s^m_{\nu^m} \in [t^-_{\nu^m-m}, t^+_{\nu^m-m}]$ (true also for $\nu^m \in \{\nu^{m-1}\}$),

b. $\bar{s}^{m-1}_{\nu^m} \in T_{\nu^m-m}$, $\bar{s}^{m-1}_{\nu^m} \in [t^-_{\nu^m-m}, t^+_{\nu^m-m}]$ (true also for $\nu^m \in \{\nu^{m-1}\}$),

c. $c(\bar{s}^{m-1}_{\nu^m}) \to c^m \in M$, $e_{\nu^m-m}(s^m_{\nu^m}) \to e^m \in \mathcal{G}$,

d. $\frac{1}{2}c(\bar{s}^0_{\nu^m}) + \frac{1}{4}c(\bar{s}^1_{\nu^m}) + \dots + \frac{1}{2^m}c(\bar{s}^{m-1}_{\nu^m}) + \frac{1}{2^m}e_{\nu^m-m}(s^m_{\nu^m}) \to g(t)$ and passing to a limits we get

$$\frac{1}{2}c^1 + \frac{1}{4}c^2 + \cdots \frac{1}{2^m}c^m + \frac{1}{2^m}e^m = g(t), \tag{3.5}$$

e. $c^1, c^2, \dots, c^m \in G(t)$ and in particular $c^1, c^2, \dots, c^m \in g(t) + C$.

$\mathbf{6}^0$**.** Take the set $K = \{c^m - g(t) \mid m = 1, 2, \dots\}$. We prove now that $0 \in \operatorname{cl} \operatorname{conv} K$.

Rewrite (3.5) with some obvious recasting

$$\sum_{\mu=1}^{m} \frac{1}{2^\mu}\left(c^\mu - g(t)\right) + \frac{1}{2^m}\left(c^m - g(t)\right) = \frac{1}{2^m}c^m - \frac{1}{2^m}g(t) - \frac{1}{2^m}e^m \in \frac{1}{2^m}\left(M - \mathcal{G} - \mathcal{G}\right).$$

The set $M - \mathcal{G} - \mathcal{G}$ is compact and $1/2^m \to 0$ as $m \to \infty$. Therefore for arbitrary neighbourhood U of zero there exists an integer m such that $\frac{1}{2^m}(M - \mathcal{G} - \mathcal{G}) \subset U$ and consequently the convex combination of the elements of K in the left hand side of the above formula also belongs to U. Therefore $0 \in \operatorname{cl} \operatorname{conv} K$.

7^0. Now we prove that $g(t) \in M$.

Since $c^m \in g(t)+C$, $m = 1,2,\dots$, the set $K = \{c^m - g(t) \mid m = 1,2,\dots\}$ is contained in C. We have proved that $0 \in \operatorname{cl}\operatorname{conv} K$. According to Property 2.3 $0 \in \operatorname{cl} K$. From $0 = \lim_i (c^{m_i} - g(t))$ we get $g(t) = \lim_i c^{m_i}$. However $c^{m_i} \in M$ and M is closed. Therefore $g(t) \in M$.

Step 3 *Define the set-valued function $\hat{G} : [0, 1] \to \mathcal{G}$ by $\hat{G}(t) = G(t) \cap M$. Then $\hat{G}$ is closed and with a compact graph. It holds $g(t) \in \hat{G}(t)$, moreover $\hat{G}$ is single-valued and $\hat{G}(t) = \{g(t)\}$.*

The closedness of $\hat{G}$ follows from $\operatorname{graph}\hat{G} = \operatorname{graph} G \cap ([0, 1] \times M)$, where both $\operatorname{graph} G$ and $[0, 1] \times M$ are closed sets. Obviously, the graph is compact as a closed subset of the compact set $[0, 1] \times \mathcal{G}$. The inclusion $g(t) \in \hat{G}(t) = G(t) \cap M$ is true since by definition $g(t) \in G(t)$ ($g(t)$ is defined as the least element of $G(t)$) and in Step 2 we have proved $g(t) \in M$. The equality $\hat{G}(t) = \{g(t)\}$ (which in particular means single-valuedness) follows by $g(t) \in \hat{G}(t) = G(t) \cap M \subset (g(t) + C) \cap Max(Q|C) = \{g(t)\}$, the last equality is implied by $g(t) \in M \subset Max(Q|C)$.

Step 4 *The function g is continuous. The set $Max(Q|C)$ is arcwise connected between the points $x_0, x_1 \in M$. Moreover, there exists an arc connecting x_0 and x_1 which is entirely contained in M.*

Now we have $\hat{G}(t) = \{g(t)\}$ and the graph of $\hat{G}$ is closed and compact. Therefore g is continuous as a function having a closed and compact graph. Further $g : [0, 1] \to E$ is an arc connecting x_0 and x_1 and contained entirely in $M \subset Max(Q|C)$. ∎

We put as open the following problem.

Problem 3.1. Is the conclusion of Theorem 2.6 true replacing in the assumptions Property 2.3 by "C pointed"?

We will show that in the special case $a(x_0, x_1) = \frac{1}{2}(x_0 + x_1)$ the answer is "yes". For this purpose we need to change the construction. Instead of working in the set $\mathcal{G}$ defined by the functions g_n we will work in a set $\mathcal{F}$ defined in a similar way by the functions f_n. Instead of the sets of points $E_n(t)$ we consider the sets $B_n(t) = \{b_k(s) \mid s \in [t_n^-, t_n^+] \cap T_{k+1},\, k \ge n\}$. We put also $\hat{B}_n = B_0(t) \setminus \bigcap_{t \in [0,1]} B_n(t)$. Now for each integers n and $m \ge 0$ it holds

$$b_{n+m}(s) \in \operatorname{conv} C_{n+m}(s) \subset \operatorname{conv} C_n(t)$$

(compare with the result of Lemma 2.9), where $s \in T_{n+m+1}$. This is true, since $b_{n+m}(s) = c_{n+m}(s)$ for $s \in T_{n+m}$ and

$$b_{n+m}(s) = \frac{1}{2} c_{n+m}(s - 1/2^{n+m+1}) + \frac{1}{2} c_{n+m}(s + 1/2^{n+m+1})$$

for $s \in T_{n+m+1} \setminus T_{n+m}$.

Now for $t \in [0, 1]$ we define the sets $\mathcal{F}_n(t) = \bigcup \{f_k(s) \mid t_n^- \le s \le t_n^+,\, k \ge n\} \cup \operatorname{cl}\operatorname{conv} C(t)$. We put also $\mathcal{F} = \mathcal{F}_0(t)$ for in this case $\mathcal{F}_0(t)$ does not depend on $t \in [0, 1]$.

Lemma 3.2. *All the sets $\mathcal{F}_n(t)$ are compact.*

Proof Take an arbitrary open covering of $\mathcal{F}_n(t)$. Since $\mathrm{cl\,conv}\,C(t)$ is compact, a finite number of the sets from the covering cover $\mathrm{cl\,conv}\,C(t)$ and their union contains a set of the type $\mathrm{cl\,conv}\,C(t)+U$, where U is a neighbourhood of zero in E. Then there is n_0 such that $C_{n_0}(s) \subset \mathrm{cl\,conv}\,C_{n_0}(s) \subset \mathrm{cl\,conv}\,C(t)+U$ (the possibility of such a choice is explained in the proof of Lemma 2.10). Since $b_{n_0}(s) \in \mathrm{conv}\left(c(s^-_{n_0}), c(s^+_{n_0})\right)$ we get $\mathcal{F}_{n_0}(s) \subset \mathrm{cl\,conv}\,C(t)+U$. Therefore in $\mathcal{F}_n(t) \setminus \bigcup\{\mathcal{F}_{n_0}(s) \mid s \in [t^-_n, t^+_n]\}$ there remain only finitely many functions $f_k(s)$, $t^-_n \le s \le t^+_n$. The union of the images of finitely many continuous functions defined on a compact interval is a compact set, that is it is contained in the union of finitely many of the sets of the given covering. Thus, we have shown that $\mathcal{F}_n(t)$ admits a finite subcovering, therefore it is compact. ∎

We define a sequence of set-valued functions $F_n : [0,1] \to \mathcal{F}$. The set $\mathcal{F} = \mathcal{F}_0(t)$ is compact according to Lemma 3.2. We put $F_n(t) = (f_n(t)+C) \cap \mathcal{F}$. We define further the set-valued function $F : [0,1] \to \mathcal{F}$ putting $F(t) = \bigcap_n F_n(t)$. The sequence $\{F_n(t)\}$ is a decreasing sequence of compact subsets of $\mathcal{F}$. Since $f_n(t)$ is the least point of $F_n(t)$, according to Proposition 2.11 the sequence $f_n(t)$ converges to the unique least point of $F(t)$ which we denote by $f(t)$. From the definition of F we get $F(t) = (f(t)+C) \cap \mathcal{F}$. The function $f : [0,1] \to \mathcal{F}$ satisfies $f(t) = \lim_n f_n(t)$.

Theorem 3.3. *Suppose that Q is a set in the real Hausdorff TVS E and C is a pointed closed convex cone in E. Assume that there exists a closed set $M \subset Max(Q|C)$ obeying Property 2.2 with $a(x_0,x_1) = \frac{1}{2}(x_0+x_1)$ such that* $\mathrm{cl\,conv}\,M$ *is compact. Then $Max(Q|C)$ is arcwise connected between any points $x_0, x_1 \in M$, $x_0 \neq x_1$. Moreover, there exists an arc between x_0 and x_1 contained entirely in M.*

Proof We do the proof in four steps following the same scheme as in the proof of Theorem 2.6.

Step 1 *For each t there exists a sequence of nonnegative integers $n_i \to \infty$ and a sequence of numbers $s_i \to t$, $s_i \in [t^-_{n_i}, t^-_{n_i}]$ and $s_i \in T_{n_i+1}$ such that $f(t) = \lim_i b_{n_i}(s_i)$.*

This step is a repeating of Step 1 from the proof of Theorem 2.6 by obvious replacements of $e_n(s)$ by $b_n(s)$, g by f etc. Turn attention that in the proof of Step 1 in Theorem 2.6 Property 2.3 has not been used. We have used only that C is pointed.

Step 2 *It holds $f(t) \in M$.*

In Step 1 it is shown that there exists a sequence of nonnegative integers $n_i \to \infty$ and a sequence $s_i \in [t^-_{n_i}, t^+_{n_i}]$ with $s_i \in T_{n_i+1}$, such that $f(t) = \lim_i b_{n_i}(s_i)$. Now either $s_i \in T_{n_i}$ and then $b_{n_i}(s_i) = c(s_i)$ or $s_i \in T_{n_i+1} \setminus T_{n_i}$ and then $b_{n_i}(s_i) = \frac{1}{2}c(s_i - 1/2^{n_i+1}) + \frac{1}{2}c(s_i + 1/2^{n_i+1})$, where $s_i \pm 1/2^{n_i+1} \in T_{n_i}$.

In each case there is a sequence $\beta_i \in [0,1]$ and sequences s'_i, $s''_i \in T_{n_i}$ such that $b_{n_i}(s_i) = (1-\beta_i)c(s'_i) + \beta_i c(s''_i)$. Using the compactness of $[0,1] \times \mathcal{F} \times \mathcal{F}$ we see that there exists a subsequence (denote it again n_i), such that $\beta_i \to \beta$, $c(s'_i) \to c'$, $c(s''_i) \to c''$ and $s'_i \to t$, $s''_i \to t$. Now using the closedness of F_n and F (obtained in quite a similar way as the closedness of G_n and G in Lemma 2.12) we get $c', c'' \in F(t) \subset f(t)+C$ and $(1-\beta)c' + \beta c'' \in F(t) \subset f(t)+C$. On the other hand $f(t) = \lim_i b_{n_i}(s_i)$ implies $(1-\beta)c' + \beta c'' = f(t)$. We assert that either $c' = f(t)$ or $c'' = f(t)$. This is obvious if $\beta = 0$ or $\beta = 1$. Let $0 < \beta < 1$. Then $c' \in (1-\beta)c' + \beta c'' + C$ implies $c' - c'' \in C$. Similarly $c'' \in (1-\beta)c' + \beta c'' + C$ implies $c'' - c' \in C$. Since C is pointed, $c' = c'' = f(t)$.

Step 3 *Define the set-valued function $\hat{F} : [0,1] \to \mathcal{F}$ by $\hat{F}(t) = F(t) \cap M$. Then $\hat{F}$ is closed and with a compact graph. It holds $f(t) \in \hat{F}(t)$, moreover $\hat{F}$ is single-valued and $\hat{F}(t) = \{f(t)\}$.*

Step 4 *The function f is continuous. The set $Max(Q|C)$ is arcwise connected between the points $x_0, x_1 \in M$. Moreover, there exists an arc connecting x_0 and x_1 which is entirely contained in M.*

The proofs of Step 3 and Step 4 are identical up to obvious replacements to those in Theorem 2.6. ∎

The following example shows that arcwise connectedness in the conclusion of Theorem 2.6 cannot be replaced by contractibility.

Example 3.4. Let $E = R^3$, $C = R^3_+$ and $Q = [a^1,a^2] \cup [a^2,a^3] \cup [a^3,a^1]$, where $a^1 = (1,0,0)$, $a^2 = (0,1,0)$, $a^3 = (0,0,1)$. Obviously $Max(Q|C) = Q$ is arcwise connected but not contractible. Let $M = [a^2,a^3] \cup [a^3,a^1]$. For $x^0, x^1 \in M$ consider the cases: Case 1, x^0 and x^1 are both in either $[a^2,a^3]$ or $[a^3,a^1]$; Case 2, x^0 and x^1 belong to different segments $[a^2,a^3]$ and $[a^3,a^1]$. Put

$$a(x^0,x^1) = \begin{cases} \frac{1}{2}\left(x^0 + x^1\right) & \text{if Case 1,} \\ x^1 + \frac{1}{2}\left(\|x^1 - a^3\| + \|a^3 - x^0\|\right)\frac{a^3 - x^1}{\|a^3 - x^1\|} & \text{if Case 2 and } \|a^3 - x^1\| \ge \|a^3 - x^0\|, \\ x^0 + \frac{1}{2}\left(\|x^1 - a^3\| + \|a^3 - x^0\|\right)\frac{a^3 - x^0}{\|a^3 - x^0\|} & \text{if Case 2 and } \|a^3 - x^1\| < \|a^3 - x^0\|. \end{cases}$$

Then Property 2.2 holds for all $x^0, x^1 \in M$ with $\Gamma(x^0,x^1) = (a(x^0,x^1) + C) \cap M$ and therefore the arcwise connectedness between any two such points can be established by means of Theorem 2.6. By obvious cyclic permutations when defining M the arcwise connectedness can be established between any two points $x^0, x^1 \in Max(Q|C)$.

4 Application

The next theorem is an application of Theorem 2.6 to establish arcwise connectedness of the efficient set for a class of in general nonconvex sets.

Theorem 4.1. *Let $E = R^n$ and $C = R^n_+$. Assume that for the set $Q \subset E$ there exists a compact set $M \subset Max(Q|C)$ such that for all $x^0, x^1 \in M$ it holds $\left(\frac{1}{2}(x^0 + x^1) + C\right) \cap M \neq \emptyset$. Then $Max(Q|C)$ is arcwise connected between any points $x^0, x^1 \in M$. Moreover, there exists an arc between x^0 and x^1 contained entirely in M.*

Proof It is natural to look for a proof based on Theorem 3.3 with $a(x^0,x^1) = \frac{1}{2}(x^0 + x^1)$. With such a choice point 2^0 in Property 2.2 if not wrong is at least not easy to be checked. So, we will apply Theorem 2.6 choosing another definition for $a(x^0,x^1)$.

We agree that if $x \in E$ then the coordinate of x will be written with lower indices, that is we write e.g. $x = (x_1, \dots, x_n)$. Now $x^0 = (x^0_1, \dots, x^0_n)$, $x^1 = (x^1_1, \dots, x^1_n)$ etc. For two points $x^1, x^2 \in M$ we define $a = a(x^0,x^1)$ putting for the coordinates $a_i = \min(x^0_i, x^1_i)$, $i = 1, \dots n$. We put also $\Gamma(x^0,x^1) = \left(\frac{1}{2}(x^0 + x^1) + C\right) \cap M$. With such a choice we see that Property 2.2 is satisfied by checking separately conditions 1^0 and 2^0.

Condition 1^0. We will show that $(x+C)\cap M\neq\emptyset$ for arbitrary point $x\in[x^0,a(x^0,x^1)]$ (similar is the case $x\in[a(x^0,x^1),x^1]$. For this purpose we construct inductively a sequence of points $\hat{x}^1,\hat{x}^2,\dots$, such that $\hat{x}^1\in\left(\frac{1}{2}(x^0+x^1)+C\right)\cap M$ and if $\hat{x}^k$ is already constructed we choose $\hat{x}^{k+1}\in\left(\frac{1}{2}(x^0+x^k)+C\right)\cap M$. We prove by induction that

$$\hat{x}^k\in(1-\beta)a(x^0,x^1)+\beta x^0+C,\quad 0\le\beta\le 1-\frac{1}{2^k}.$$

This inclusion is equivalent to

$$\hat{x}^k_i\in(1-\beta)\min(x^0_i,x^1_i)+\beta x^0_i+C,\quad i=1,\dots,n,\quad 0\le\beta\le 1-\frac{1}{2^k}. \tag{4.1}$$

The inequality (4.1) gives

$$\hat{x}^k_i-x^0_i\ge 0\quad\text{if}\quad \min(x^0_i,x^1_i)=x^0_i, \tag{4.2}$$

$$\hat{x}^k_i-\left((1-\beta)x^1_i+\beta x^0_i\right)\ge 0\quad\text{if}\quad \min(x^0_i,x^1_i)=x^1_i. \tag{4.3}$$

We provide separate inductive proofs for the two cases.

Case 1. It holds $\min(x^0_i,x^1_i)=x^0_i$. Then we prove (4.2).

For $k=1$ we have

$$\hat{x}^1_i-x^0_i\ge\frac{1}{2}(x^0_i+x^1_i)-x^0_i=\frac{1}{2}(x^1_i-x^0_i)\ge 0.$$

If (4.2) is true for some k then

$$\hat{x}^{k+1}_i-x^0_i\ge\frac{1}{2}(x^0_i+\hat{x}^k_i)-x^0_i=\frac{1}{2}(\hat{x}^k_i-x^0_i)\ge 0.$$

Case 2. It holds $\min(x^0_i,x^1_i)=x^1_i$. Then we prove (4.3) for $0\le\beta\le 1-1/2^k$.

We have

$$(1-\beta)x^1_i+\beta x^0_i=x^1_i+\beta(x^0_i-x^1_i)\le x^1_i+(1-\frac{1}{2^k})(x^0_i-x^1_i).$$

Therefore (4.3) is true for each $\beta\in[0,1-1/2^k]$ if it is true for $\beta=1-1/2^k$. In this case transforms into (4.3) inequality (4.4) which we prove by induction

$$\hat{x}^k_i-\frac{1}{2^k}x^1_i-\left(1-\frac{1}{2^k}\right)x^0_i\ge 0. \tag{4.4}$$

For $k=1$ (4.4) transforms into

$$\hat{x}^1_i-\frac{1}{2}x^1_i-\frac{1}{2}x^0_i\ge 0$$

which is true from the choice of $\hat{x}^1$.

Assume now that (4.4) holds for some k. We prove it for $k+1$. Then

$$\begin{aligned}\hat{x}_i^{k+1} - \frac{1}{2^{k+1}}\,x_i^1 - \left(1-\frac{1}{2^{k+1}}\right)x_i^0 &\geq \frac{1}{2}\left(x_i^0+\hat{x}_i^k\right) - \frac{1}{2^{k+1}}\,x_i^1 - \left(1-\frac{1}{2^{k+1}}\right)x_i^0\\ &= \frac{1}{2}\left(\hat{x}_i^k - \frac{1}{2^k}\,x_i^1 - \left(1-\frac{1}{2^k}\right)x_i^0\right) \geq 0\,.\end{aligned}$$

If $x = x^0$, then $x^0 \in (x+C)\cap M$. We prove now that $(x+C)\cap M \neq \emptyset$ for $x \in [x^0, a(x^0,x^1)]$, $x \neq x^0$. Then $x = (1-\beta)a(x^0,x^1)+\beta x^0$ for some β with $0 \leq \beta \leq 1-1/2^k$. We have proved that $\hat{x}^k \in x+C$ and since $\hat{x}^k \in M$, therefore $\hat{x}^k \in (x+C)\cap M$.

Condition 2^0. We prove first that $\Gamma(x^0,x^1) \subset (a(x^0,x^1)+C)\cap M$. Since by definition $\Gamma(x^0,x^1) \subset M$ it remains to show $\frac{1}{2}(x^0+x^1) \in a(x^0,x^1)+C$ which follows by the nonnegativeness of the coordinates

$$\left(\frac{1}{2}(x^0+x^1)-a(x^0,x^1)\right)_i = \frac{1}{2}(x_i^0+x_i^1)-\min(x_i^0,x_i^1) = \frac{1}{2}\left(\max(x_i^0,x_i^1)-\min(x_i^0,x_i^1)\right) \geq 0\,.$$

Now we prove that for $\hat{x} \in \Gamma(x^0,x^1)$ it holds $a(x^0,\hat{x}) \in \frac{1}{2}\left(x^0+a(x^0,x^1)\right)+C$ (similarly $a(\hat{x},x^1) \in \frac{1}{2}\left(a(x^0,x^1)+x^1\right)+C$). From $\hat{x} \in \Gamma(x^0,x^1)$ we have $\hat{x} \in \frac{1}{2}(x^0+x^1)+C$ and therefore $\hat{x}_i \geq \frac{1}{2}(x_i^0+x_i^1)$, $i = 1,\ldots n$. To prove the above inclusion we must check that

$$\min(x_i^0,\hat{x}_i) \geq \frac{1}{2}\left(x_i^0+\min(x_i^0,x_i^1)\right), \quad i = 1,\ldots,n.$$

If the minimum on the left hand side is x_i^0 we get the equivalent inequality $x_i^0 \geq \min(x_i^0,x_i^1)$ which is obviously true. If the minimum is $\hat{x}_i$ we get the equivalent inequality

$$\hat{x}_i \geq \frac{1}{2}\left(x_i^0+\min(x_i^0,x_i^1)\right)$$

which is implied by $\hat{x}_i \geq \frac{1}{2}(x_i^0+x_i^1) \geq \frac{1}{2}\left(x_i^0+\min(x_i^0,x_i^1)\right)$.

Thus Property 2.2 is satisfied. The cone $C = R_+^n$ is a pointed cone in R^n and according to Proposition 2.5 has also Property 2.3. Since $M \subset R^n$ is compact then M is closed and $\mathrm{cl\,conv}\,M$ is compact. Therefore the hypotheses of Theorem 2.6 are satisfied, whence we get the desired arcwise connectedness. ∎

5 Convex Sets

The following theorem is a straightforward application to convex sets of Theorem 3.3.

Theorem 5.1. *Suppose that Q is a convex set in the real Hausdorff TVS E and C is a pointed closed convex cone in E. Assume that there exists a closed set $M \subset Max(Q|C)$ with* cl conv M *compact, such that $(a(x_0,x_1)+C)\cap M \neq \emptyset$ for $a(x_0,x_1) = \frac{1}{2}(x_0+x_1)$ and all $x_1,x_2 \in M$. Then $Max(Q|C)$ is arcwise connected between any points $x_0,x_1 \in M$, $x_0 \neq x_1$. Moreover, there exists an arc between x_0 and x_1 contained entirely in M.*

Proof Obviously Property 2.2 has place. Therefore the conclusion follows directly from Theorem 3.3. ∎

Corollary 5.2 (Makarov, Rachkovski, Song [6]). *Let C be a pointed closed convex cone and Q be a compact and convex set in the real Hausdorff TVS E. If the set $Max(Q|C)$ is closed, then it is arcwise connected.*

Proof We put $M = Max(Q|C)$ and $a(x_0, x_1) = \frac{1}{2}(x_0 + x_1)$ for $x_0, x_1 \in Max(Q|C)$. The inclusion $\operatorname{cl}\operatorname{conv} M \subset \operatorname{cl}\operatorname{conv} Q = Q$ implies that $\operatorname{cl}\operatorname{conv} M$ is compact. From the convexity of Q we have $a(x_0, x_1) = \frac{1}{2}(x_0 + x_1) \in Q$ and from Proposition 2.1 $(a(x_0, x_1) + C) \cap M \neq \emptyset$. Therefore the conclusion follows directly from Theorem 5.1. ∎

Example 5.3 (Makarov, Rachkovski, Song [6]). Consider the space $E = R^3$. Let $Q = \operatorname{conv}\left(\{(x,y,z) \in R^3 \mid x^2 + y^2 \leq 1, z = 0\} \cup \{b\}\right)$ with $b = (1,0,1)$ and $C = \{(x,y,z) \in R^3 \mid x = y = 0, z \geq 0\}$. Obviously, $Max(Q|C)$ is arcwise connected, but not closed.

Since $Max(Q|C)$ in Example 5.3 is not closed, its arcwise connectedness cannot be established by means of Corollary 5.2. We show now that the arcwise connectedness between any points $x', x'' \in Max(Q|C)$ can be established by means of Theorem 5.1. For this purpose we take the set $M = (\operatorname{conv}\{x', x'', b\} + C) \cap Max(Q|C)$ and put $a(x_0, x_1) = \frac{1}{2}(x_0 + x_1)$ and $\Gamma(x_0, x_1) = (a(x_0, x_1) + C) \cap M$ for $x_0, x_1 \in Max(Q|C)$. With such a choice the hypotheses of Theorem 5.1 are satisfied and therefore $Max(Q|C)$ is arcwise connected between x' and x''.

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