Communications in **M**athematical **A**nalysis
Volume 12, Number 1, pp. 117–133 (2012) www.math-res-pub.org/cma
ISSN 1938-9787

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(Communicated by Michal Fečkan)

Abstract

In this paper we investigate the global convergence result, boundedness, and periodicity of solutions of the recursive sequence

$$x_{n+1} = ax_n + \frac{bx_{n-l} + cx_{n-k}}{dx_{n-l} + ex_{n-k}}, \quad n = 0, 1, ...,$$

where the parameters a, b, c, d and e are positive real numbers and the initial conditions $x_{-k}, x_{-k+1}, ..., x_{-l}, x_{-l+1}, ..., x_{-1}$ and x_0 are positive real numbers.

AMS Subject Classification: 39A10

Keywords: stability, periodic solutions, boundedness, difference equations.

1 Introduction

Difference equations appear as natural descriptions of observed evolution phenomena because most measurements of time evolving variables are discrete and as such these equations are in their own right important mathematical models. More importantly, difference

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equations also appear in the study of discretization methods for differential equations. Several results in the theory of difference equations have been obtained as more or less natural discrete analogues of corresponding results of differential equations.

The study of rational difference equations of order greater than one is quite challenging and rewarding because some prototypes for the development of the basic theory of the global behavior of nonlinear difference equations of order greater than one come from the results for rational difference equations. However, there have not been any effective general methods to deal with the global behavior of rational difference equations of order greater than one so far. Therefore, the study of rational difference equations of order greater than one is worth further consideration.

Recently there has been a lot of interest in studying the global attractivity, boundedness character, periodicity and the solution form of nonlinear difference equations. For some results in this area, for example: Agarwal et al. [2] studied the global stability, periodicity character and gave the solution form of some special cases of the recursive sequence

$$x_{n+1} = a + \frac{dx_{n-l}x_{n-k}}{b - cx_{n-s}}.$$

Aloqeili [3] obtained the form of the solution of the difference equation

$$x_{n+1} = \frac{x_{n-1}}{a - x_n x_{n-1}}.$$

Elabbasy et al. [6] investigated the global stability character, boundedness and the periodicity of solutions of the difference equation

$$x_{n+1} = \frac{\alpha x_n + \beta x_{n-1} + \gamma x_{n-2}}{Ax_n + Bx_{n-1} + Cx_{n-2}}.$$

In [7] Elabbasy et al. studied the dynamics such that the global stability, periodicity character and gave the solution of special case of the following recursive sequence

$$x_{n+1} = ax_n - \frac{bx_n}{cx_n - dx_{n-1}}.$$

Elabbasy et al. [8] investigated the behavior of the difference equation especially global stability, boundedness, periodicity character and gave the solution of some special cases of the difference equation

$$x_{n+1} = \frac{\alpha x_{n-k}}{\beta + \gamma \prod_{i=0}^{k} x_{n-i}}.$$

El-Metwally et al. [15] dealt with the following difference equation

$$y_{n+1} = \frac{y_{n-(2k+1)} + p}{y_{n-(2k+1)} + qy_{n-2l}}.$$

Saleh et al. [30] investigated the difference equation

$$y_{n+1} = A + \frac{y_n}{y_{n-k}}.$$

Simsek et al. [32] obtained the solution of the difference equation

$$x_{n+1} = \frac{x_{n-3}}{1 + x_{n-1}}.$$

Yalçınkaya et al. [36], [39] considered the dynamics of the difference equations

$$x_{n+1} = \frac{a x_{n-k}}{b + c x_n^p}, \quad x_{n+1} = \alpha + \frac{x_{n-m}}{x_n^k}.$$

Zayed et al. [41]-[42] studied the behavior of the following rational recursive sequences

$$a x_n - \frac{b x_n}{c x_n - d x_{n-k}}, \quad x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{A + B x_n + C x_{n-1}}.$$

Other related results on rational difference equations can be found in refs. [1-40].

Our goal in this paper is to investigate the global stability character and the periodicity of solutions of the recursive sequence

$$x_{n+1} = a x_n + \frac{b x_{n-l} + c x_{n-k}}{d x_{n-l} + e x_{n-k}}, \tag{1.1}$$

where the parameters a, b, c, d and e are positive real numbers and the initial conditions $x_{-k}, x_{-k+1}, ..., x_{-l}, x_{-l+1}, ..., x_{-1}$ and x_0 are positive real numbers.

2 Some Basic Properties and Definitions

Here, we recall some basic definitions and some theorems that we need in the sequel.

Let I be some interval of real numbers and let

$$F : I^{k+1} \to I,$$

be a continuously differentiable function. Then for every set of initial conditions $x_{-k}, x_{-k+1}, ..., x_0 \in I$, the difference equation

$$x_{n+1} = F(x_n, x_{n-1}, ..., x_{n-k}), \quad n = 0, 1, ..., \tag{2.1}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

Definition 2.1. (Equilibrium Point) A point $\overline{x} \in I$ is called an equilibrium point of Eq.(2.1) if

$$\overline{x} = F(\overline{x}, \overline{x}, ..., \overline{x}).$$

That is, $x_n = \overline{x}$ for $n \geq 0$, is a solution of Eq.(2.1), or equivalently, $\overline{x}$ is a fixed point of F.

Definition 2.2. (Periodicity) A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if $x_{n+p} = x_n$ for all $n \geq -k$.

Definition 2.3. (Stability) (i) The equilibrium point $\overline{x}$ of Eq.(2.1) is locally stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + ... + |x_0 - \overline{x}| < \delta,$$

we have

$$|x_n - \overline{x}| < \epsilon \quad \text{for all} \quad n \geq -k.$$

(ii) The equilibrium point $\overline{x}$ of Eq.(2.1) is locally asymptotically stable if $\overline{x}$ is locally stable solution of Eq.(2.1) and there exists $\gamma > 0$, such that for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + ... + |x_0 - \overline{x}| < \gamma,$$

we have

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(iii) The equilibrium point $\overline{x}$ of Eq.(2.1) is global attractor if for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$, we have

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(iv) The equilibrium point $\overline{x}$ of Eq.(2.1) is globally asymptotically stable if $\overline{x}$ is locally stable, and $\overline{x}$ is also a global attractor of Eq.(2.1).

(v) The equilibrium point $\overline{x}$ of Eq.(2.1) is unstable if $\overline{x}$ is not locally stable.

The linearized equation of Eq.(2.1) about the equilibrium $\overline{x}$ is the linear difference equation

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial F(\overline{x}, \overline{x}, ..., \overline{x})}{\partial x_{n-i}} y_{n-i}. \tag{2.2}$$

Theorem A [26] *Assume that* $p_i \in R$, $i = 1, 2, ...$ *and* $k \in \{0, 1, 2, ...\}$. *Then*

$$\sum_{i=1}^{k} |p_i| < 1, \tag{2.3}$$

is a sufficient condition for the asymptotic stability of the difference equation

$$y_{n+k} + p_1 y_{n+k-1} + ... + p_k y_n = 0, \quad n = 0, 1, ...$$

Consider the following equation

$$x_{n+1} = g(x_n, x_{n-1}, x_{n-2}). \tag{2.4}$$

The following two theorems will be useful for the proof of our results in this paper.

Theorem B [27] Let $[\alpha, \beta]$ be an interval of real numbers and assume that

$$g : [\alpha, \beta]^3 \to [\alpha, \beta],$$

is a continuous function satisfying the following properties :

(a) $g(x, y, z)$ is non-decreasing in x and y in $[\alpha, \beta]$ for each $z \in [\alpha, \beta]$, and is non-increasing in $z \in [\alpha, \beta]$ for each x and y in $[\alpha, \beta]$;

(b) If $(m,M) \in [\alpha,\beta]\times[\alpha,\beta]$ is a solution of the system

$$M = g(M,M,m) \quad and \quad m = g(m,m,M),$$

then

$$m = M.$$

Then Eq.(2.4) has a unique equilibrium $\overline{x} \in [\alpha,\beta]$ and every solution of Eq.(2.4) converges to $\overline{x}$.

Theorem C [27] Let $[\alpha,\beta]$ be an interval of real numbers and assume that

$$g : [\alpha,\beta]^3 \to [\alpha,\beta],$$

is a continuous function satisfying the following properties :

(a) $g(x,y,z)$ is non-decreasing in x and z in $[\alpha,\beta]$ for each $y \in [\alpha,\beta]$, and is non-increasing in $y \in [\alpha,\beta]$ for each x and z in $[\alpha,\beta]$;

(b) If $(m,M) \in [\alpha,\beta]\times[\alpha,\beta]$ is a solution of the system

$$M = g(M,m,M) \quad and \quad m = g(m,M,m),$$

then

$$m = M.$$

Then Eq.(2.4) has a unique equilibrium $\overline{x} \in [\alpha,\beta]$ and every solution of Eq.(2.4) converges to $\overline{x}$.

The paper proceeds as follows. In Section 3 we show that the equilibrium point of Eq.(1.1) is locally asymptotically stable when $2|(be-dc)| < (d+e)(b+c)$. In Section 4 we prove that the solution is bounded when $a < 1$ and the solution of Eq.(1.1) is unbounded if $a > 1$. In Section 5 we prove that the there exists a period two solution of Eq.(1.1). In Section 6 we prove that the equilibrium point of Eq.(1.1) is global attractor. Finally, we give numerical examples of some special cases of Eq. (1.1) and draw it by using Matlab.

3 Local Stability of the Equilibrium Point of Eq.(1.1)

This section deals with study the local stability character of the equilibrium point of Eq.(1.1).

Theorem 3.1. *Assume that*

$$2|(be-dc)| < (d+e)(b+c).$$

Then the positive equilibrium point of Eq.(1.1) is locally asymptotically stable.

Proof. Eq.(1.1) has equilibrium point and is given by

$$\overline{x} = a\overline{x} + \frac{b+c}{d+e}.$$

If $a < 1$, then the only positive equilibrium point of Eq.(1.1) is given by

$$\overline{x} = \frac{b+c}{(1-a)(d+e)}.$$

Let $f:(0,\infty)^3 \longrightarrow (0,\infty)$ be a continuous function defined by

$$f(u,v,w) = au + \frac{bv+cw}{dv+ew}. \tag{3.1}$$

Therefore it follows that

$$\begin{aligned}
\frac{\partial f(u,v,w)}{\partial u} &= a, \\
\frac{\partial f(u,v,w)}{\partial v} &= \frac{(be-dc)w}{(dv+ew)^2}, \\
\frac{\partial f(u,v,w)}{\partial w} &= \frac{(dc-be)u}{(dv+ew)^2}.
\end{aligned}$$

Then we see that

$$\begin{aligned}
\frac{\partial f(\overline{x},\overline{x},\overline{x})}{\partial u} &= a = -a_2, \\
\frac{\partial f(\overline{x},\overline{x},\overline{x})}{\partial v} &= \frac{(be-dc)}{(d+e)^2\overline{x}} = \frac{(be-dc)(1-a)}{(d+e)(b+c)} = -a_1, \\
\frac{\partial f(\overline{x},\overline{x},\overline{x})}{\partial w} &= \frac{(dc-be)}{(d+e)^2\overline{x}} = \frac{(dc-be)(1-a)}{(d+e)(b+c)} = -a_0.
\end{aligned}$$

Then the linearized equation of Eq.(1.1) about $\overline{x}$ is

$$y_{n+1} + a_2 y_n + a_1 y_{n-l} + a_0 y_{n-k} = 0, \tag{3.2}$$

whose characteristic equation is

$$\lambda^{k+1} + a_2\lambda^{k-1} + a_1\lambda^{k-l} + a_0 = 0. \tag{3.3}$$

It follows by Theorem A that, Eq.(3.2) is asymptotically stable if all roots of Eq.(3.3) lie in the open disc $|\lambda| < 1$ that is if

$$|a_2| + |a_1| + |a_0| < 1.$$

$$|a| + \left|\frac{(be-dc)(1-a)}{(d+e)(b+c)}\right| + \left|\frac{(dc-be)(1-a)}{(d+e)(b+c)}\right| < 1,$$

and so

$$2\left|\frac{(be-dc)(1-a)}{(d+e)(b+c)}\right| < (1-a), \qquad a < 1,$$

or

$$2\,|be-dc| < (d+e)(b+c).$$

The proof is complete. □

4 Existence of Bounded and Unbounded Solutions of Eq.(1.1)

Here we study the boundedness nature of solutions of Eq.(1.1).

Theorem 4.1. *Every solution of Eq.(1.1) is bounded if* $a < 1$.

Proof. Let $\{x_n\}_{n=-k}^{\infty}$ be a solution of Eq.(1.1). It follows from Eq.(1.1) that

$$x_{n+1} = ax_n + \frac{bx_{n-l} + cx_{n-k}}{dx_{n-l} + ex_{n-k}} = ax_n + \frac{bx_{n-l}}{dx_{n-l} + ex_{n-k}} + \frac{cx_{n-k}}{dx_{n-l} + ex_{n-k}}.$$

Then

$$x_{n+1} \le ax_n + \frac{bx_{n-l}}{dx_{n-l}} + \frac{cx_{n-k}}{ex_{n-k}} = ax_n + \frac{b}{d} + \frac{c}{e} \quad \text{for all} \quad n \ge 1.$$

By using a comparison, we can write the right hand side as follows

$$y_{n+1} = ay_n + \frac{b}{d} + \frac{c}{e},$$

then

$$y_n = a^n y_0 + constant,$$

and this equation is locally asymptotically stable because $a < 1$, and converges to the equilibrium point $\overline{y} = \dfrac{be + cd}{de(1-a)}$.

Therefore,

$$\limsup_{n\to\infty} x_n \le \frac{be + cd}{de(1-a)}.$$

Thus the solution is bounded. □

Theorem 4.2. *Every solution of Eq.(1.1) is unbounded if* $a > 1$.

Proof. Let $\{x_n\}_{n=-k}^{\infty}$ be a solution of Eq.(1.1). Then from Eq.(1.1) we see that

$$x_{n+1} = ax_n + \frac{bx_{n-l} + cx_{n-k}}{dx_{n-l} + ex_{n-k}} > ax_n \quad \text{for all} \quad n \ge 1.$$

We see that the right hand side can write as follows

$$y_{n+1} = ay_n \quad \Rightarrow \quad y_n = a^n y_0,$$

and this equation is unstable because $a > 1$, and $\lim_{n\to\infty} y_n = \infty$. Then by using ratio test $\{x_n\}_{n=-k}^{\infty}$ is unbounded from above. □

5 Existence of Periodic Solutions

In this section we study the existence of periodic solutions of Eq.(1.1). The following theorem states the necessary and sufficient conditions that this equation has periodic solutions of prime period two.

Theorem 5.1. *Eq.(1.1) has positive prime period two solutions if and only if*

$$(i)\ (b-c)(d-e)(1+a)+4(bae+cd)>0,\ d>e,\ b>c \text{ and } l-\text{odd},\ k-\text{even}.$$

$$(ii)\ (c-b)(e-d)(1+a)+4(acd+be)>0,\ e>d,\ c>b \text{ and } k-\text{odd},\ l-\text{even}.$$

Proof. We prove that when l−odd, k−even and when l−even, k−odd is similar and will be omitted.

First suppose that there exists a prime period two solution

$$...,p,q,p,q,...,$$

of Eq.(1.1). We will prove that Condition (i) holds.

We see from Eq.(1.1) when l−odd, k−even that

$$p=aq+\frac{bp+cq}{dp+eq},$$

and

$$q=ap+\frac{bq+cp}{dq+ep}.$$

Then

$$dp^2+epq=adpq+aeq^2+bp+cq, \tag{5.1}$$

and

$$dq^2+epq=adpq+aep^2+bq+cp. \tag{5.2}$$

Subtracting (5.1) from (5.2) gives

$$d(p^2-q^2)=-ae(p^2-q^2)+(b-c)(p-q).$$

Since $p\neq q$, it follows that

$$p+q=\frac{(b-c)}{(d+ae)}. \tag{5.3}$$

Again, adding (5.1) and (5.2) yields

$$d(p^2+q^2)+2epq=2adpq+ae(p^2+q^2)+(b+c)(p+q),$$

$$(d-ae)(p^2+q^2)+2(e-ad)pq=(b+c)(p+q). \tag{5.4}$$

It follows by (5.3), (5.4) and the relation

$$p^2+q^2=(p+q)^2-2pq \quad \text{for all } p,q\in R,$$

that

$$2(e-d)(1+a)pq=\frac{2(bae+cd)(b-c)}{(d+ae)^2}.$$

Thus

$$pq=\frac{(bae+cd)(b-c)}{(d+ae)^2(e-d)(1+a)}. \tag{5.5}$$

Now it is clear from Eq.(5.3) and Eq.(5.5) that p and q are the two distinct roots of the quadratic equation

$$\begin{aligned} t^2-\left(\frac{(b-c)}{(d+ae)}\right)t+\left(\frac{(bae+cd)(b-c)}{(d+ae)^2(e-d)(1+a)}\right) &= 0, \\ (d+ae)t^2-(b-c)t+\left(\frac{(bae+cd)(b-c)}{(d+ae)(e-d)(1+a)}\right) &= 0, \end{aligned} \tag{5.6}$$

and so

$$[b-c]^2-\frac{4(bae+cd)(b-c)}{(e-d)(1+a)}>0,$$

or

$$[b-c]^2+\frac{4(bae+cd)(b-c)}{(d-e)(1+a)}>0.$$

$$(b-c)(d-e)(1+a)+4(bae+cd)>0.$$

Therefore inequalities (i) holds.

Second suppose that inequalities (i) is true. We will show that Eq.(1.1) has a prime period two solution.

Assume that

$$p=\frac{b-c+\zeta}{2(d+ae)},$$

and

$$q=\frac{b-c-\zeta}{2(d+ae)},$$

where $\zeta=\sqrt{[b-c]^2-\dfrac{4(bae+cd)(b-c)}{(e-d)(1+a)}}$.

We see from inequalities (i) that

$$(b-c)(d-e)(1+a)+4(bae+cd)>0,\ b>c,\ \ d>e,$$

which equivalents to

$$(b-c)^2>\frac{4(bae+cd)(b-c)}{(e-d)(1+a)}.$$

Therefore p and q are distinct real numbers.

Set

$$x_{-2}=q,\ x_{-1}=p \text{ and } x_0=q.$$

We wish to show that

$$x_1=x_{-1}=p \quad\text{and}\quad x_2=x_0=q.$$

It follows from Eq.(1.1) that

$$x_1=aq+\frac{bp+cq}{dp+eq}=a\left(\frac{b-c-\zeta}{2(d+ae)}\right)+\frac{b\left(\frac{b-c+\zeta}{2(d+ae)}\right)+c\left(\frac{b-c-\zeta}{2(d+ae)}\right)}{d\left(\frac{b-c+\zeta}{2(d+ae)}\right)+e\left(\frac{b-c-\zeta}{2(d+ae)}\right)}.$$

Dividing the denominator and numerator by $2(d+ae)$ gives

$$\begin{aligned} x_1 &= \frac{ab-ac-a\zeta}{2(d+ae)}+\frac{b(b-c+\zeta)+c(b-c-\zeta)}{d(b-c+\zeta)+e(b-c-\zeta)} \\ &= \frac{ab-ac-a\zeta}{2(d+ae)}+\frac{(b-c)[(b+c)+\zeta]}{(d+e)(b-c)+(d-e)\zeta}. \end{aligned}$$

Multiplying the denominator and numerator of the right side by $(d+e)(b-c)-(d-e)\zeta$ gives

$$\begin{aligned} x_1 &= \frac{ab-ac-a\zeta}{2(d+ae)}+\frac{(b-c)[(b+c)+\zeta][(d+e)(b-c)-(d-e)\zeta]}{[(d+e)(b-c)+(d-e)\zeta][(d+e)(b-c)-(d-e)\zeta]} \\ &= \frac{ab-ac-a\zeta}{2(d+ae)} \end{aligned}$$

$$+\frac{(b-c)\left\{(d+e)\left(b^2-c^2\right)+\zeta[(d+e)(b-c)-(d-e)(b+c)]-(d-e)\zeta^2\right\}}{(d+e)^2(b-c)^2-(d-e)^2\zeta^2}$$

$$= \frac{ab-ac-a\zeta}{2(d+ae)}$$

$$+\frac{(b-c)\left\{(d+e)\left(b^2-c^2\right)+2\zeta(eb-cd)-(d-e)\left([b-c]^2-\frac{4(bae+cd)(b-c)}{(e-d)(1+a)}\right)\right\}}{(d+e)^2(b-c)^2-(d-e)^2\left([b-c]^2-\dfrac{4(bae+cd)(b-c)}{(e-d)(1+a)}\right)}$$

$$= \frac{ab-ac-a\zeta}{2(d+ae)}$$

$$+\frac{(b-c)\left\{(d+e)\left(b^2-c^2\right)+2\zeta(eb-cd)-(d-e)(b-c)^2-\frac{4(bae+cd)(b-c)}{(1+a)}\right\}}{(d+e)^2(b-c)^2-(d-e)^2\left([b-c]^2-\dfrac{4(bae+cd)(b-c)}{(e-d)(1+a)}\right)}$$

$$= \frac{ab-ac-a\zeta}{2(d+ae)}+\frac{(b-c)\left\{2(b-c)\left[dc+eb-\dfrac{2(bae+cd)}{(1+a)}\right]+2\zeta(eb-cd)\right\}}{4(b-c)\left[ed(b-c)+\dfrac{(e-d)(bae+cd)}{(1+a)}\right]}.$$

Multiplying the denominator and numerator of the right side by $(1+a)$ we obtain

$$\begin{aligned} x_1 &= \frac{ab-ac-a\zeta}{2(d+ae)}+\frac{(b-c)[(dc+eb)(1+a)-2(bae+cd)]+\zeta(1+a)(eb-cd)}{2[ed(b-c)(1+a)+(e-d)(bae+cd)]} \\ &= \frac{ab-ac-a\zeta}{2(d+ae)}+\frac{(b-c)(eb-dc)(1-a)+\zeta(1+a)(eb-cd)}{2[ed(b-c)(1+a)+(e-d)(bae+cd)]} \end{aligned}$$

$$\begin{aligned} &= \frac{ab-ac-a\zeta}{2(d+ae)}+\frac{(eb-dc)\{(b-c)(1-a)+\zeta(1+a)\}}{2(eb-cd)(d+ae)} \\ &= \frac{ab-ac-a\zeta}{2(d+ae)}+\frac{(b-c)(1-a)+\zeta(1+a)}{2(d+ae)} \\ &= \frac{ab-ac-a\zeta+(b-c)(1-a)+\zeta(1+a)}{2(d+ae)}=\frac{b-c+\zeta}{2(d+ae)}=p. \end{aligned}$$

Similarly as before one can easily show that

$$x_2 = q.$$

Then it follows by induction that

$$x_{2n} = q \quad \text{and} \quad x_{2n+1} = p \quad \text{for all} \quad n \geq -1.$$

Thus Eq.(1.1) has the prime period two solution

$$...,p,q,p,q,...,$$

where p and q are the distinct roots of the quadratic equation (5.6) and the proof is complete. □

6 Global Attractivity of the Equilibrium Point of Eq.(1.1)

In this section we investigate the global asymptotic stability of Eq.(1.1).

Lemma 6.1. *For any values of the quotient* $\frac{b}{d}$ *and* $\frac{c}{e}$, *the function* $f(u,v,w)$ *defined by Eq.(3.1) has the monotonicity behavior in its two arguments.*

Proof. The proof follows by some computations and it will be omitted. □

Theorem 6.2. *The equilibrium point* $\overline{x}$ *is a global attractor of Eq.(1.1) if one of the following statements holds*

$$(1)\ be \geq dc \ \text{ and } \ c \geq b. \tag{6.1}$$

$$(2)\ be \leq dc \ \text{ and } \ c \leq b. \tag{6.2}$$

Proof. Let α and β be a real numbers and assume that $g : [\alpha,\beta]^3 \longrightarrow [\alpha,\beta]$ be a function defined by

$$g(u,v,w) = au + \frac{bv + cw}{dv + ew}.$$

Then

$$\begin{aligned} \frac{\partial g(u,v,w)}{\partial u} &= a, \\ \frac{\partial g(u,v,w)}{\partial v} &= \frac{(be - dc)w}{(dv + ew)^2}, \\ \frac{\partial g(u,v,w)}{\partial w} &= \frac{(dc - be)u}{(dv + ew)^2}. \end{aligned}$$

We consider the two cases:-

Case (1) Assume that (6.1) is true, then we can easily see that the function $g(u,v,w)$ increasing in u,v and decreasing in w.

Suppose that (m,M) is a solution of the system $M = g(M,M,m)$ *and* $m = g(m,m,M)$.Then from Eq.(1.1), we see that

$$M = aM + \frac{bM + cm}{dM + em}, \quad m = am + \frac{bm + cM}{dm + eM},$$

or

$$M(1-a)=\frac{bM+cm}{dM+em},\quad m(1-a)=\frac{bm+cM}{dm+eM},$$

then

$$d(1-a)M^2+e(1-a)Mm=bM+cm,\quad d(1-a)m^2+e(1-a)Mm=bm+cM.$$

Subtracting this two equations we obtain

$$(M-m)\{d(1-a)(M+m)+(c-b)\}=0,$$

under the conditions $c\geq b,\ a<1$, we see that

$$M=m.$$

It follows by Theorem B that $\overline{x}$ is a global attractor of Eq.(1.1) and then the proof is complete.

Case (2) Assume that (6.2) is true, let $\alpha$and β be a real numbers and assume that $g:[\alpha,\beta]^3\longrightarrow[\alpha,\beta]$ be a function defined by $g(u,v,w)=au+\dfrac{bv+cw}{dv+ew}$, then we can easily see that the function $g(u,v,w)$ increasing in u,w and decreasing in v.

Suppose that (m,M) is a solution of the system $M=g(M,m,M)$ *and* $m=g(m,M,m)$.Then from Eq.(1.1), we see that

$$M=aM+\frac{bm+cM}{dm+eM},\quad m=am+\frac{bM+cm}{dM+em},$$

or

$$M(1-a)=\frac{bm+cM}{dm+eM},\quad m(1-a)=\frac{bM+cm}{dM+em},$$

then

$$d(1-a)Mm+e(1-a)M^2=bm+cM,\quad d(1-a)mM+e(1-a)m^2=bM+cm.$$

Subtracting we obtain

$$(M-m)\{e(1-a)(M+m)+(b-c)\}=0,$$

under the conditions $b\geq c,\ a<1$, we see that

$$M=m.$$

It follows by Theorem C that $\overline{x}$ is a global attractor of Eq.(1.1) and then the proof is complete. $\square$

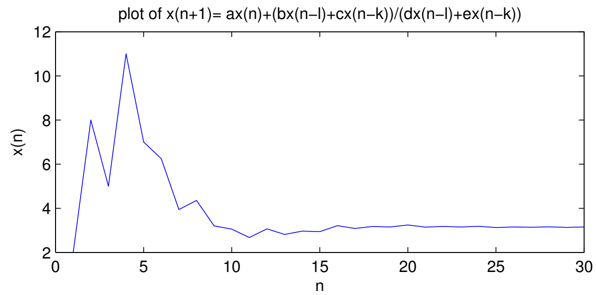

Figure 1.

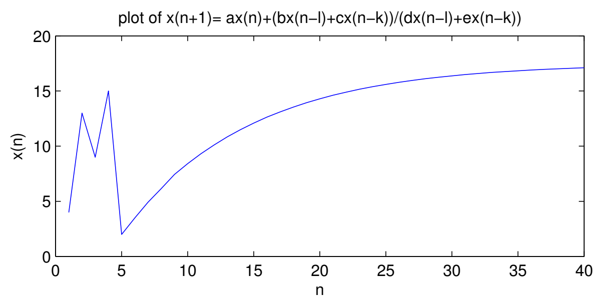

Figure 2.

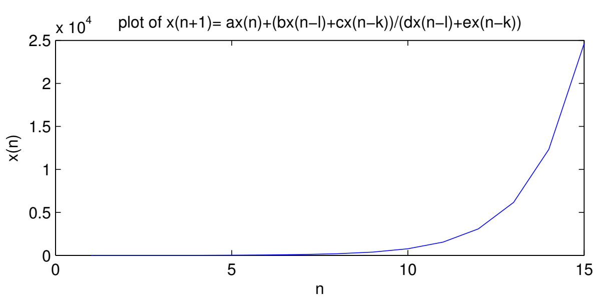

Figure 3.

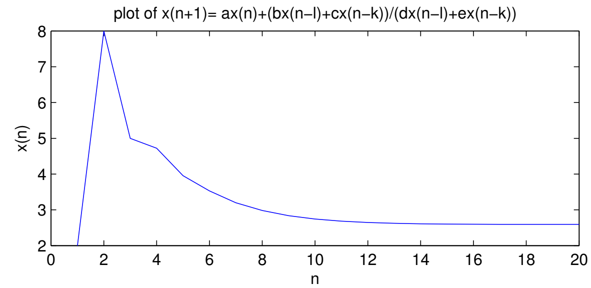

Figure 4.

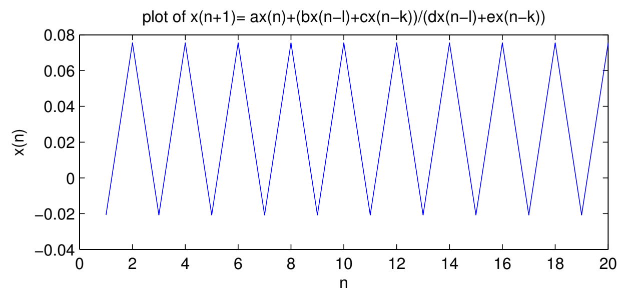

Figure 5.

7 Numerical Examples

For confirming the results of this paper, we consider numerical examples which represent different types of solutions to Eq. (1.1).

Example 1. We assume $l = 3$, $k = 4$, $x_{-4} = 2$, $x_{-3} = 8$, $x_{-2} = 5$, $x_{-1} = 11$, $x_0 = 7$, $a = 0.4$, $b = 1.5$, $c = 0.2$, $d = 0.3$, $e = 0.6$**.** See Fig. 1.

Example 2. See Fig. 2, since $l = 3$, $k = 4$, $x_{-4} = 4$, $x_{-3} = 13$, $x_{-2} = 9$, $x_{-1} = 15$, $x_0 = 2$, $a = 0.9$, $b = 5$, $c = 2$, $d = 3$, $e = 1$.

Example 3. We consider $l = 2$, $k = 3$, $x_{-3} = 2$, $x_{-2} = 8$, $x_{-1} = 5$, $x_0 = 11$, $a = 2$, $b = 5$, $c = 2$, $d = 3$, $e = 6$. See Fig. 3.

Example 4. See Fig. 4, since $l = 1$, $k = 2$, $x_{-2} = 2$, $x_{-1} = 8$, $x_0 = 5$, $a = 0.7$, $b = 5$, $c = 2$, $d = 3$, $e = 6$.

Example 5. Fig. 5. shows the solutions when $l = 1$, $k = 2$, $a = 0.8$, $b = 0.5$, $c = 0.2$, $d = 5$, $e = 0.6$, $x_{-2} = q$, $x_{-1} = p$, $x_0 = q$.

$$\left(\text{Since } p,q = \frac{b-c \pm \sqrt{[b-c]^2 - \dfrac{4(bae+cd)(b-c)}{(e-d)(1+a)}}}{2(d+ae)} \right).$$

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