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An Iterative Method for Mixed Point Problems of Nonexpansive and Monotone Mappings and Generalized Equilibrium Problems

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Abstract

In this paper, we introduce an iterative scheme for finding the common element of the set of fixed points of a countable family of nonexpansive mappings, the set of solutions of variational inequality for μ-Lipschitzian, relaxed (λ,γ)-cocoercive mapping and the set of solutions of a generalized equilibrium problem. We show that the iterative sequence converges strongly to a common element of the three sets. Our results generalize many recent results, for example, the results of B. Ali [2].

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1 Introduction

Let K be a nonempty closed convex subset of a real Hilbert space H. A mapping $A : K \to H$ is called *monotone* if

$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall x, y \in K. \tag{1.1}$$

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A mapping $A : K \to H$ is called *inverse-strongly monotone* (see, for example, [5], [10]) if there exists a positive real number λ such that

$$\langle Ax - Ay, x - y\rangle \geq \lambda \|Ax - Ay\|^2, \quad \forall x, y \in K.$$

For such a case, A is called λ-inverse-strongly monotone. A λ-inverse-strongly monotone is sometime called *λ-cocoercive*.

A mapping A is said to be *relaxed λ-cocoercive* if there exists $\lambda > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq -\lambda \|Ax - Ay\|^2, \quad \forall x, y \in K.$$

A mapping A is said to be *relaxed (λ, γ)-cocoercive* if there exist $\lambda, \gamma > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq -\lambda \|Ax - Ay\|^2 + \gamma \|x - y\|^2, \quad \forall x, y \in K.$$

A mapping $A : H \to H$ is said to be μ-Lipschitzian if there exists $\mu \geq 0$ such that

$$\|Ax - Ay\| \leq \mu \|x - y\|, \quad x, y \in H.$$

Let $A : K \to H$ be a nonlinear mapping. The variational inequality problem is to find an $x^* \in K$ such that (See, for example, [4]-[6])

$$\langle Ax^*, y - x^*\rangle \geq 0, \quad \forall y \in K. \tag{1.2}$$

We shall denote the set of solutions of the variational inequality problem (1.2) by $VI(K, A)$. Finding common element of set of fixed point of nonexpansive mappings and the set of solution of variational inequality problem has been studied extensively in the literature. See, for example [13, 10, 15, 24, 27] and the references contained therein.

A monotone mapping A is said to be *maximal* if the graph $G(A)$ is not properly contained in the graph of any other monotone map, where

$$G(A) := \{(x, y) \in H \times H : y \in Ax\}$$

for a multi-valued mapping A. It is also known that A is maximal if and only if for

$$(x, f) \in H \times H, \quad \langle x - y, f - g\rangle \geq 0$$

for every $(y, g) \in G(A)$ implies $f \in Ax$.

Let A be a monotone mapping defined from K into H and $N_K q$ be a normal cone to K at $q \in K$, i.e., $N_K q = \{p \in H : \langle q - u, p\rangle \geq 0, \ \forall u \in K\}$. Define a mapping M by

$$Mq = \begin{cases} Aq + N_K q, & q \in K \\ \emptyset, & q \notin K. \end{cases}$$

Then, M is maximal monotone and $x^* \in M^{-1}(0)$ if and only if $x^* \in VI(K, A)$, see, for example, [20].

A mapping $T : K \to K$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad x, y \in K. \tag{1.3}$$

For each $i \in \mathbb{N}$, we define $F(T_i) := \{x^* \in K : T_i x^* = x^*\}$ and denote by $\bigcap_{i=1}^{\infty} F(T_i)$, the set of common fixed points of countable family of nonexpansive mappings. The problem of finding common fixed points of countable family of nonexpansive mappings has been studied extensively in the literature. See, for example [1, 12, 18] and the references contained therein.

The computation of fixed points is important in the study of many problems including inverse problems. For instance, it is not hard to show that the split feasibility problem and the convex feasibility problem can both be formulated as a problem of finding fixed points of certain operators. In particular, construction of fixed points of nonexpansive mappings is applied in image recovery and signal processing and transition operators for initial value problems of differential inclusions (see, for example, [7], [19], [28]).

Iterative approximation of a common element of the set of fixed points of nonexpansive mappings and set of solutions of variational inequality has been studied extensively by many authors (see e.g., [10], [11], [15-17], [21], [24], [27], and the references contained therein). Several physical problems such as the theories of lubrications, filterations and flows, moving boundary problems, see, for example, [11], [16], can be reduced to variational inequality problems. Consequently, these problems have solutions as the solutions of these resultant variational inequality problems.

Let F be a bifunction of $K \times K$ into $\mathbb{R}$, the set of reals and $\psi : K \to H$ be a nonlinear mapping. The generalized equilibrium problem is to find $x \in K$ such that

$$F(x,y) + \langle \psi x, y - x \rangle \geq 0, \tag{1.4}$$

for all $y \in K$. The set of solutions of this generalized equilibrium problem is denoted by EP. Thus

$$EP := \{x^* \in K : F(x^*, y) + \langle \psi x^*, y - x^* \rangle \geq 0, \ \ \forall y \in K\}.$$

In the case of $\psi \equiv 0$, EP is denoted by $EP(F)$ and in the case of $F \equiv 0$, EP is denoted by $VI(K,A)$. The problem (1.4) includes as special cases, optimization problems, variational inequalities, minimax problems, Nash equilibrium problems in noncooperative games, etc (see, for example, [3], [14]). Numerous problems in physics, optimization and economics can be reduced to find a solution of the equilibrium problem. Some methods have been proposed to solve the equilibrium problem. See, for example [9, 23, 25].

Recently, Mainge [12] studied the Halpern-type scheme for approximation of a common fixed point of a countable family of nonexpansive mappings in a Hilbert space. He proved the following theorem.

Theorem 1.1. *(Mainge, [12]) Let K be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_i\}_{i=1}^{\infty}$ be a countable family of self-mappings of K, $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\sigma_{i,n}\}_{n=1}^{\infty}$ be sequences in $(0,1)$ satisfying the following conditions:*

(1) $\sum_{i=1}^{\infty} \alpha_n = \infty, \quad \sum_{i=1}^{\infty} \sigma_{i,n} = (1 - \alpha_n),$

(2)

$$\begin{cases} \frac{1}{\sigma_{i,n}}\left|1-\frac{\alpha_{n-1}}{\alpha_n}\right| \to 0, \text{ or } \sum_{n=1}^{\infty}\frac{1}{\sigma_{i,n}}|\alpha_{n-1}-\alpha_n| < \infty, \\ \frac{1}{\alpha_n}\left|\frac{1}{\sigma_{i,n}}-\frac{1}{\sigma_{i,n-1}}\right| \to 0, \text{ or } \sum_{n=1}^{\infty}\left|\frac{1}{\sigma_{i,n}}-\frac{1}{\sigma_{i,n-1}}\right| < \infty, \\ \frac{1}{\sigma_{i,n}\alpha_n}\sum_{k=0}^{\infty}\left|\sigma_{k,n}-\sigma_{k,n-1}\right| \to 0, \text{ or } \sum_{n=1}^{\infty}\frac{1}{\sigma_{i,n}}\sum_{k=0}^{\infty}\left|\sigma_{k,n}-\sigma_{k,n-1}\right| < \infty. \end{cases}$$

(3) $\forall i \in N_I := \{i \in \mathbb{N} : T_i \neq I\}$

(4) $\lim_{n\to\infty}\frac{\alpha_n}{\sigma_{i,n}} = 0.$

Let $C : K \to K$ be a contraction map. Then, the sequence $\{x_n\}_{n=1}^{\infty}$ defined iteratively by $x_1 \in K$,

$$x_{n+1} = \alpha_n C x_n + \sum_{i=1}^{\infty}\sigma_{i,n}T_i x_n \tag{1.5}$$

converges strongly to the unique fixed point of P_{FoC}, where P_F is the metric projection from H onto F.

Motivated by the results of Mainge [12] (Theorem 1.1 above), our aim in this paper is to introduce a new viscosity iterative method for approximation of a common fixed point for a countable family of nonexpansive mappings which is also a solution to generalized equilibrium problem and a variational inequality problem for a μ-Lipschitzian, relaxed (λ,γ)-cocoercive mapping in a real Hilbert space H. In our results, some conditions in Theorem 1.1 are dispensed with (see Remark 3.3). Furthermore, our results extend many important recent results.

2 Preliminaries

Let H be a real Hilbert space with inner product $\langle .,. \rangle$ and norm $\|.\|$ and let K be a nonempty closed convex subset of H. The weak convergence of $\{x_n\}_{n=1}^{\infty}$ to x is denoted by $x_n \rightharpoonup x$ as $n \to \infty$, while the strong convergence of $\{x_n\}_{n=1}^{\infty}$ to x is written $x_n \to x$ as $n \to \infty$.

For any point $u \in H$, there exists a unique point $P_K u \in K$ such that

$$\|u - P_K u\| \le \|u - y\|, \quad \forall y \in K. \tag{2.1}$$

P_K is called the *metric projection* of H onto K. We know that P_K is a nonexpansive mapping of H onto K. It is also known that P_K satisfies

$$\langle x - y, P_K x - P_K y\rangle \ge \|P_K x - P_K y\|^2, \tag{2.2}$$

for all $x, y \in H$. Furthermore, $P_K x$ is characterized by the properties $P_K x \in K$ and

$$\langle x - P_K x, P_K x - y\rangle \ge 0, \tag{2.3}$$

for all $y \in K$ and

$$\|x - P_K x\|^2 \leq \|x - y\|^2 - \|y - P_K x\|^2, \quad \forall x \in H, \quad y \in K. \tag{2.4}$$

In the context of the variational inequality problem,

$$x^* \in VI(K,A) \Leftrightarrow x^* = P_K(x^* - \lambda A x^*), \quad \forall \lambda > 0.$$

In what follows, we shall make use of the following lemmas.

Lemma 2.1. *(Suzuki, [22]) Let* $\{x_n\}_{n=1}^{\infty}$ *and* $\{y_n\}_{n=1}^{\infty}$ *be bounded sequences in a Banach space E and let* $\{\beta_n\}_{n=1}^{\infty}$ *be a sequence in [0,1] with* $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. *Suppose* $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ *for all integers* $n \geq 0$ *and* $\limsup_{n\to\infty}\Big(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|\Big) \leq 0$. *Then,* $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Lemma 2.2. *(Xu, [26]) Let* $\{a_n\}_{n=1}^{\infty}$ *be a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\beta_n)a_n + \beta_n \gamma_n, \; n \in \mathbb{N}$$

where $\{\beta_n\}_n \subset (0,1)$, $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=0}^{\infty} \beta_n = \infty$ *and* $\limsup_n \gamma_n \leq 0$. *Then,* $\lim_{n\to\infty} a_n = 0$.

For solving the equilibrium problem for a bifunction $F : K \times K \to \mathbb{R}$, let us assume that F satisfies the following conditions:
(A1) $F(x,x) = 0$ for all $x \in K$;
(A2) F is monotone, i.e., $F(x,y) + F(y,x) \leq 0$ for all $x, y, \in K$;
(A3) for each $x, y \in K$, $\lim_{t\to 0} F(tz + (1-t)x, y) \leq F(x,y)$;
(A4) for each $x \in K$, $y \mapsto F(x,y)$ is convex and lower semicontinuous.

Lemma 2.3. *(Blum and Oettli, [3]) Let K be a nonempty closed convex subset of H and let F be a bifunction of* $K \times K$ *into* $\mathbb{R}$ *satisfying (A1)-(A4). Let* $r > 0$ *and* $x \in H$. *Then, there exists* $z \in K$ *such that*

$$F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0 \quad \text{for all} \;\; y \in K.$$

Lemma 2.4. *(Combettes and Hirstoaga, [8]) Assume that* $F : K \times K \to \mathbb{R}$ *satisfies (A1)-(A4). For* $r > 0$ *and* $x \in H$, *define a mapping* $T_r : H \to K$ *as follows:*

$$T_r(x) = \{z \in K : F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in K\}$$

for all $z \in H$. *Then, the following hold:*

(1) T_r *is single-valued;*

(2) T_r *is firmly nonexpansive, i.e., for any* $x, y \in H$,

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle;$$

(3) $F(T_r) = EP(F)$;

(4) $EP(F)$ *is closed and convex.*

3 Main Results

We now prove the following theorems.

Theorem 3.1. *Let K be a closed convex subset of a real Hilbert space H. Let F be a bi-function from $K \times K$ satisfying $(A1)-(A4)$, A be a μ-Lipschitzian, relaxed (λ,γ)-cocoercive mapping of K into H and ψ be an α-inverse, strongly monotone mapping of K into H. Let $\{T_i\}_{i=1}^{\infty}$ be a countable family of nonexpansive mappings of K into H and let*

$$\Omega := \bigcap_{i=1}^{\infty} F(T_i) \bigcap VI(K,A) \bigcap EP \neq \emptyset.$$

Let $f : H \to H$ be a contraction map with constant $\beta \in (0,1)$. For a fixed $\delta \in (0,1)$, let $\{x_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ be generated by $x_1 \in H$,

$$\begin{cases} F(u_n,y) + \langle \psi x_n, y-u_n \rangle + \frac{1}{r_n}\langle y-u_n, u_n-x_n \rangle \geq 0 \quad \forall y \in K \\ \rho_n = P_K(u_n - s_n A u_n) \\ x_{n+1} = \alpha_n f(x_n) + (1-\delta)(1-\alpha_n)x_n + \delta \sum\limits_{i=1}^{\infty} \sigma_{i,n} T_i \rho_n; \end{cases} \tag{3.1}$$

for all $n \geq 1$, where $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$, $\{\sigma_{i,n}\}_{n=1}^{\infty}$ are sequences in $[0,1]$, for all $i \geq 1$, $\{s_n\}_{n=1}^{\infty}$, $\{r_n\}_{n=1}^{\infty} \subset (0,\infty)$ satisfying:

(1) $\sum\limits_{i=1}^{\infty} \sigma_{i,n} = (1-\alpha_n)$

(2) $\lim\limits_{n\to\infty} \alpha_n = 0, \quad \sum\limits_{n=1}^{\infty} \alpha_n = \infty$

(3) $0 < c \leq r_n \leq d < 2\alpha, \quad \lim\limits_{n\to\infty} |r_{n+1} - r_n| = 0$

(4) $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{\infty} \left|\sigma_{i,n+1} - \sigma_{i,n}\right| = 0$

(5) $0 < a \leq s_n \leq b < \frac{2(\gamma-\lambda\mu^2)}{\mu^2}, \quad \lim\limits_{n\to\infty} |s_{n+1} - s_n| = 0,$

then $\{x_n\}_{n=1}^{\infty}$ converges strongly to z, where $z = P_\Omega f(z)$.

Proof. We break the proof into steps.

(i) We first show that $I - s_n A$ and $I - r_n \psi$ are nonexpansive. For all $x,y \in K$ and $s_n \in (0, \frac{2(\gamma-\lambda\mu^2)}{\mu^2}]$, we obtain

$$\begin{aligned} \|(I-s_nA)x - (I-s_nA)y\|^2 &= \|x-y-s_n(Ax-Ay)\|^2 \\ &= \|x-y\|^2 - 2s_n\langle x-y, Ax-Ay \rangle + s_n^2\|Ax-Ay\|^2 \\ &\leq \|x-y\|^2 - 2s_n[-\lambda\|Ax-Ay\|^2 + \gamma\|x-y\|^2] + s_n^2\|Ax-Ay\|^2 \\ &\leq \|x-y\|^2 + 2s_n\mu^2\lambda\|x-y\|^2 - 2s_n\gamma\|x-y\|^2 + \mu^2 s_n^2\|x-y\|^2 \\ &= (1 + 2s_n\mu^2\lambda - 2s_n\gamma + \mu^2 s_n^2)\|x-y\|^2 \leq \|x-y\|^2. \end{aligned}$$

This shows that $I - s_nA$ is nonexpansive for each $n \geq 1$. Also, for all $x, y \in K$ and $0 < c \leq r_n \leq d < 2\alpha$, we obtain

$$\begin{aligned}
\|(I - r_n\psi)x - (I - r_n\psi)y\|^2 &= \|x - y - r_n(\psi x - \psi y)\|^2 \\
&= \|x - y\|^2 - 2r_n\langle x - y, \psi x - \psi y\rangle + r_n^2\|\psi x - \psi y\|^2 \\
&\leq \|x - y\|^2 - 2r_n\alpha\|\psi x - \psi y\|^2 + r_n^2\|\psi x - \psi y\|^2 \\
&\leq \|x - y\|^2 + r_n(r_n - 2\alpha)\|\psi x - \psi y\|^2 \\
&\leq \|x - y\|^2.
\end{aligned}$$

Hence, $I - r_n\psi$ is nonexpansive.

(ii) Next, we show that $\{x_n\}_{n=1}^{\infty}$ is bounded. Now, let $x^* \in \Omega$ and if $\{T_{r_n}\}_{n=1}^{\infty}$ is a sequence of mappings defined as in Lemma 2.4, then we have $x^* = P_K(x^* - s_nAx^*) = T_{r_n}(x^* - r_n\psi x^*)$. Since $\rho_n = P_K(u_n - s_nAu_n)$, for each $n \geq 1$ and ψ is an α-inverse, strongly monotone mapping, we have

$$\begin{aligned}
\|\rho_n - x^*\|^2 &= \|P_K(u_n - s_nAu_n) - P_K(x^* - s_nAx^*)\|^2 \\
&\leq \|(u_n - s_nAu_n) - (x^* - s_nAx^*)\|^2 \\
&\leq \|u_n - x^*\|^2 = \|T_{r_n}(x_n - r_n\psi x_n) - T_{r_n}(x^* - r_n\psi x^*)\|^2 \\
&\leq \|(x_n - r_n\psi x_n) - (x^* - r_n\psi x^*)\|^2 \\
&\leq \|(I - r_n\psi)x_n - (I - r_n\psi)x^*\|^2 \\
&\leq \|x_n - x^*\|^2.
\end{aligned}$$

Furthermore, we obtain

$$\begin{aligned}
&\|x_{n+1} - x^*\| = \|\alpha_n f(x_n) + (1-\delta)(1-\alpha_n)x_n + \delta\sum_{i=1}^{\infty}\sigma_{i,n}T_i\rho_n - x^*\| \\
\leq\ & \alpha_n\|f(x_n) - x^*\| + (1-\delta)(1-\alpha_n)\|x_n - x^*\| + \delta\sum_{i=1}^{\infty}\sigma_{i,n}\|\rho_n - x^*\| \\
\leq\ & \alpha_n(\|f(x_n) - f(x^*)\| + \|f(x^*) - x^*\|) + (1-\delta)(1-\alpha_n)\|x_n - x^*\| + \delta\sum_{i=1}^{\infty}\sigma_{i,n}\|x_n - x^*\| \\
\leq\ & [1 - \alpha_n(1-\beta)]\|x_n - x^*\| + \alpha_n(1-\beta)\frac{1}{(1-\beta)}\|f(x^*) - x^*\| \\
\leq\ & \max\{\|x_n - x^*\|, \frac{1}{(1-\beta)}\|f(x^*) - x^*\|\} \\
&\vdots \\
\leq\ & \max\{\|x_1 - x^*\|, \frac{1}{(1-\beta)}\|f(x^*) - x^*\|\}. \qquad (3.2)
\end{aligned}$$

Therefore, $\{x_n\}_{n=1}^{\infty}$ is bounded. Furthermore, $\{u_n\}_{n=1}^{\infty}$, $\{\rho_n\}_{n=1}^{\infty}$, $\{T_i\rho_n\}_{n=1}^{\infty}$ and $\{Au_n\}_{n=1}^{\infty}$ are bounded.

(iii) Next, we show that $\|x_{n+1} - x_n\| \to 0$, $n \to \infty$.

From $u_n = T_{r_n}(x_n - r_n\psi x_n)$ and $u_{n+1} = T_{r_{n+1}}(x_{n+1} - r_{n+1}\psi x_{n+1})$, we obtain

$$F(u_n, y) + \langle\psi x_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in K \qquad (3.3)$$

and

$$F(u_{n+1},y)+\langle\psi x_{n+1},y-u_{n+1}\rangle+\frac{1}{r_{n+1}}\langle y-u_{n+1},u_{n+1}-x_{n+1}\rangle\geq 0. \tag{3.4}$$

Substituting $y=u_{n+1}$ in (3.3) and $y=u_n$ in (3.4), we have

$$F(u_n,u_{n+1})+\langle\psi x_n,u_{n+1}-u_n\rangle+\frac{1}{r_n}\langle u_{n+1}-u_n,u_n-x_n\rangle\geq 0$$

and

$$F(u_{n+1},u_n)+\langle\psi x_{n+1},u_n-u_{n+1}\rangle+\frac{1}{r_{n+1}}\langle u_n-u_{n+1},u_{n+1}-x_{n+1}\rangle\geq 0.$$

So, from (A2), we have

$$\langle\psi x_{n+1}-\psi x_n,u_n-u_{n+1}\rangle+\langle u_{n+1}-u_n,\frac{u_n-x_n}{r_n}-\frac{u_{n+1}-x_{n+1}}{r_{n+1}}\rangle\geq 0$$

and hence,

$$\begin{aligned}
0 &\leq \langle u_n-u_{n+1},r_n(\psi x_{n+1}-\psi x_n)+\frac{r_n}{r_{n+1}}(u_{n+1}-x_{n+1})-(u_n-x_n)\rangle\\
&= \langle u_{n+1}-u_n,u_n-u_{n+1}+(1-\frac{r_n}{r_{n+1}})u_{n+1}+(x_{n+1}-r_n\psi x_{n+1})\\
&\quad -(x_n-r_n\psi x_n)-x_{n+1}+\frac{r_n}{r_{n+1}}x_{n+1}\rangle\\
&= \langle u_{n+1}-u_n,u_n-u_{n+1}+(1-\frac{r_n}{r_{n+1}})(u_{n+1}-x_{n+1})+(x_{n+1}-r_n\psi x_{n+1})\\
&\quad -(x_n-r_n\psi x_n)\rangle.
\end{aligned}$$

It then follows that

$$\|u_{n+1}-u_n\|^2\leq\|u_{n+1}-u_n\|\Big\{\Big|1-\frac{r_n}{r_{n+1}}\Big|\|u_{n+1}-x_{n+1}\|+\|x_{n+1}-x_n\|\Big\}$$

and so we have

$$\|u_{n+1}-u_n\|\leq\Big|1-\frac{r_n}{r_{n+1}}\Big|\|u_{n+1}-x_{n+1}\|+\|x_{n+1}-x_n\|.$$

Without loss of generality, we assume that there exists $d_1\in\mathbb{R}$ such that $r_n>d_1>0,\ \forall n\geq 1$. Then

$$\begin{aligned}
\|u_{n+1}-u_n\| &\leq \|x_{n+1}-x_n\|+\frac{1}{r_{n+1}}|r_{n+1}-r_n|\|u_{n+1}-x_{n+1}\|\\
&\leq \|x_{n+1}-x_n\|+\frac{1}{d_1}|r_{n+1}-r_n|M_1,
\end{aligned} \tag{3.5}$$

where $M_1:=\sup_{n\geq 1}\|u_n-x_n\|$. Define $\beta_n:=(1-\delta)\alpha_n+\delta$. Suppose $x_{n+1}=(1-\beta_n)x_n+\beta_n y_n$. Then

$$\begin{aligned}
y_n &= \frac{x_{n+1}-x_n+\beta_n x_n}{\beta_n}\\
&= \frac{\alpha_n f(x_n)+\delta\sum_{i=1}^{\infty}\sigma_{i,n}T_iP_K(u_n-s_nAu_n)}{\beta_n}=\frac{\alpha_n f(x_n)+\delta\sum_{i=1}^{\infty}\sigma_{i,n}T_i\rho_n}{\beta_n}
\end{aligned}$$

Hence, we obtain,

$$
\begin{aligned}
&\|y_{n+1}-y_n\|-\|x_{n+1}-x_n\| \leq \frac{\alpha_{n+1}}{\beta_{n+1}}\|f(x_{n+1})\|+\frac{\alpha_n}{\beta_n}\Big\|f(x_n)\|\\
&+\frac{\delta(1-\alpha_{n+1})}{\beta_{n+1}}\|u_{n+1}-u_n\|-\|x_{n+1}-x_n\|+\frac{\delta}{\beta_{n+1}}\sum_{i=1}^{\infty}\Big|\sigma_{i,n+1}-\sigma_{i,n}\Big|\|T_iP_K(I-s_{n+1}A)u_n\|\\
&+\frac{\delta}{\beta_n\beta_{n+1}}|\beta_{n+1}-\beta_n|\sum_{i=1}^{\infty}\sigma_{i,n+1}\|T_iP_K(I-s_{n+1}A)u_n\|+\frac{\delta(1-\alpha_n)\|Au_n\|}{\beta_n}|s_{n+1}-s_n|. \qquad (3.6)
\end{aligned}
$$

Using (3.5) in (3.6), we obtain that

$$
\begin{aligned}
&\|y_{n+1}-y_n\|-\|x_{n+1}-x_n\| \leq \frac{\alpha_{n+1}}{\beta_{n+1}}\|f(x_{n+1})\|+\frac{\alpha_n}{\beta_n}\Big\|f(x_n)\|+\Big|\frac{\delta(1-\alpha_{n+1})}{\beta_{n+1}}-1\Big|\|x_{n+1}-x_n\|\\
&+\frac{\delta(1-\alpha_{n+1})}{\beta_{n+1}}\frac{1}{d_1}|r_{n+1}-r_n|M_1+\frac{\delta}{\beta_{n+1}}\sum_{i=1}^{\infty}\Big|\sigma_{i,n+1}-\sigma_{i,n}\Big|M+\frac{\delta}{\beta_n\beta_{n+1}}|\beta_{n+1}-\beta_n|\sum_{i=1}^{\infty}\sigma_{i,n+1}M\\
&+\frac{\delta(1-\alpha_n)M}{\beta_n}|s_{n+1}-s_n|,
\end{aligned}
$$

where $M := \sup_{n\geq 1}\{\|Au_n\|, \|T_iP_K(I-s_nA)u_n\|\}$. This implies that $\limsup(\|y_{n+1}-y_n\|-\|x_{n+1}-x_n\|) \leq 0$. Hence, by Lemma 2.1, we have $\lim\|y_n-x_n\| = 0$. Consequently, we have

$$
\lim_{n\to\infty}\|x_{n+1}-x_n\| = \lim_{n\to\infty}\beta_n\|y_n-x_n\| = 0. \qquad (3.7)
$$

Using (3.7), we have

$$
\lim_{n\to\infty}\|u_{n+1}-u_n\| = 0. \qquad (3.8)
$$

Now,

$$
\begin{aligned}
\|\rho_{n+1}-\rho_n\| &= \|P_K(u_{n+1}-s_{n+1}Au_{n+1})-P_K(u_n-s_nAu_n)\|\\
&\leq \|(u_{n+1}-s_{n+1}Au_{n+1})-(u_n-s_nAu_n)\|\\
&= \|(u_{n+1}-s_{n+1}Au_{n+1})-(u_n-s_{n+1}Au_n)+(s_n-s_{n+1})Au_n\|\\
&\leq \|u_{n+1}-u_n\|+|s_n-s_{n+1}|\|Au_n\|.
\end{aligned}
$$

Again, from (3.8), we have

$$
\lim_{n\to\infty}\|\rho_{n+1}-\rho_n\| = 0. \qquad (3.9)
$$

(iv) Next, we show that $\lim_{n\to\infty}\|T_i\rho_n-u_n\| = 0, \;\; i = 1,2,....$

Now,

$$
\begin{aligned}
\|x_{n+1}-x^*\|^2 &= \|\alpha_n(f(x_n)-x^*)+(1-\delta)(1-\alpha_n)(x_n-x^*)+\delta\sum_{i=1}^{\infty}\sigma_{i,n}(T_i\rho_n-x^*)\|^2\\
&\le \alpha_n\|f(x_n)-x^*\|^2+(1-\delta)(1-\alpha_n)\|x_n-x^*\|^2+\delta\sum_{i=1}^{\infty}\sigma_{i,n}\|\rho_n-x^*\|^2\\
&\le \alpha_n\|f(x_n)-x^*\|^2+(1-\delta)(1-\alpha_n)\|x_n-x^*\|^2+\delta\sum_{i=1}^{\infty}\sigma_{i,n}\|T_{r_n}(x_n-r_n\psi x_n)-T_{r_n}(x^*-r_n\psi x^*)\|^2\\
&\le \alpha_n\|f(x_n)-x^*\|^2+(1-\delta)(1-\alpha_n)\|x_n-x^*\|^2+\delta\sum_{i=1}^{\infty}\sigma_{i,n}\|(x_n-r_n\psi x_n)-(x^*-r_n\psi x^*)\|^2\\
&\le \alpha_n\|f(x_n)-x^*\|^2+(1-\delta)(1-\alpha_n)\|x_n-x^*\|^2+\delta\sum_{i=1}^{\infty}\sigma_{i,n}\Big[\|x_n-x^*\|^2+r_n(r_n-2\alpha)\|\psi x_n-\psi x^*\|^2\Big]\\
&= \alpha_n\|f(x_n)-x^*\|^2+\|x_n-x^*\|^2+\delta\sum_{i=1}^{\infty}\sigma_{i,n}r_n(r_n-2\alpha)\|\psi x_n-\psi x^*\|^2.
\end{aligned}
$$

Hence,

$$
\delta\sum_{i=1}^{\infty}\sigma_{i,n}r_n(2\alpha-r_n)\|\psi x_n-\psi x^*\|^2 \quad\le\quad \alpha_n\|f(x_n)-x^*\|^2+\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2
$$

Since $0<c\le r_n\le d<2\alpha$, $\lim_{n\to\infty}\alpha_n=0$ and $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$, we have $\lim_{n\to\infty}\|\psi x_n-\psi x^*\|=0$. If $x^*\in\Omega$, then we have

$$
\begin{aligned}
\|u_n-x^*\|^2 &\le \|T_{r_n}(x_n-r_n\psi x_n)-T_{r_n}(x^*-r_n\psi x^*)\|^2\le\langle(x_n-r_n\psi x_n)-(x^*-r_n\psi x^*),u_n-x^*\rangle\\
&= \frac{1}{2}\Big[\|(x_n-r_n\psi x_n)-(x^*-r_n\psi x^*)\|^2+\|u_n-x^*\|^2-\|(x_n-r_n\psi x_n)-(x^*-r_n\psi x^*)-(u_n-x^*)\|^2\Big]\\
&\le \frac{1}{2}\Big[\|x_n-x^*\|^2+\|u_n-x^*\|^2-\|(x_n-r_n\psi x_n)-(x^*-r_n\psi x^*)-(u_n-x^*)\|^2\Big]\\
&= \frac{1}{2}\Big[\|x_n-x^*\|^2+\|u_n-x^*\|^2-\|u_n-x_n\|^2+2r_n\langle x_n-u_n,\psi x_n-\psi x^*\rangle-r_n^2\|\psi x_n-\psi x^*\|^2\Big]
\end{aligned}
$$

and hence,

$$
\begin{aligned}
\|u_n-x^*\|^2 &\le \|x_n-x^*\|^2-\|u_n-x_n\|^2+2r_n\langle x_n-u_n,\psi x_n-\psi x^*\rangle\\
&\quad -r_n^2\|\psi x_n-\psi x^*\|^2\\
&\le \|x_n-x^*\|^2-\|u_n-x_n\|^2+2r_n\|x_n-u_n\|\|\psi x_n-\psi x^*\|. \qquad (3.10)
\end{aligned}
$$

From (3.2), we have

$$\|x_{n+1}-x^*\|^2 = \|\alpha_n(f(x_n)-x^*)+(1-\delta)(1-\alpha_n)(x_n-x^*)+\delta\sum_{i=1}^{\infty}\sigma_{i,n}(T_i\rho_n-x^*)\|^2$$

$$\begin{aligned}
&\leq \alpha_n\|f(x_n)-x^*\|^2+(1-\delta)(1-\alpha_n)\|x_n-x^*\|^2+\delta\sum_{i=1}^{\infty}\sigma_{i,n}\|T_i\rho_n-x^*\|^2\\
&\leq \alpha_n\|f(x_n)-x^*\|^2+(1-\delta)(1-\alpha_n)\|x_n-x^*\|^2+\delta\sum_{i=1}^{\infty}\sigma_{i,n}\|\rho_n-x^*\|^2\\
&\leq \alpha_n\|f(x_n)-x^*\|^2+(1-\delta)(1-\alpha_n)\|x_n-x^*\|^2+\delta\sum_{i=1}^{\infty}\sigma_{i,n}\|u_n-x^*\|^2\\
&= \alpha_n\|f(x_n)-x^*\|^2+(1-\delta)(1-\alpha_n)\|x_n-x^*\|^2+\delta\sum_{i=1}^{\infty}\sigma_{i,n}\Big[\|x_n-x^*\|^2-\|x_n-u_n\|^2\\
&\quad +2r_n\|x_n-u_n\|\|\psi x_n-\psi x^*\|\Big]\\
&\leq \alpha_n\|f(x_n)-x^*\|^2+(1-\delta)(1-\alpha_n)\|x_n-x^*\|^2+\delta\sum_{i=1}^{\infty}\sigma_{i,n}\|x_n-x^*\|^2-\delta\sum_{i=1}^{\infty}\sigma_{i,n}\|x_n-u_n\|^2\\
&\quad +2r_n\delta\sum_{i=1}^{\infty}\sigma_{i,n}\|x_n-u_n\|\|\psi x_n-\psi x^*\|.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\delta\sum_{i=1}^{\infty}\sigma_{i,n}\|x_n-u_n\|^2 \leq \alpha_n\|f(x_n)-x^*\|^2+(1-\delta)(1-\alpha_n)\|x_n-x^*\|^2\\
&\quad +\delta\sum_{i=1}^{\infty}\sigma_{i,n}\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2+2r_n\delta\sum_{i=1}^{\infty}\sigma_{i,n}\|x_n-u_n\|\|\psi x_n-\psi x^*\|\\
&= \alpha_n\|f(x_n)-x^*\|^2+((1-\delta)(1-\alpha_n)+\delta\sum_{i=1}^{\infty}\sigma_{i,n})\|x_n-x^*\|^2\\
&\quad -\|x_{n+1}-x^*\|^2+2r_n\delta\sum_{i=1}^{\infty}\sigma_{i,n}\|x_n-u_n\|\|\psi x_n-\psi x^*\|\\
&\leq \alpha_n\|f(x_n)-x^*\|^2+(1-\alpha_n)\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2+2r_n\delta\sum_{i=1}^{\infty}\sigma_{i,n}\|x_n-u_n\|\|\psi x_n-\psi x^*\|\\
&\leq \alpha_n\|f(x_n)-x^*\|^2+\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2+2r_n\delta\sum_{i=1}^{\infty}\sigma_{i,n}\|x_n-u_n\|\|\psi x_n-\psi x^*\|\\
&\leq \alpha_n\|f(x_n)-x^*\|^2+\|x_n-x_{n+1}\|(\|x_n-x^*\|+\|x_{n+1}-x^*\|)+2r_n\delta\sum_{i=1}^{\infty}\sigma_{i,n}\|x_n-u_n\|\|\psi x_n-\psi x^*\|
\end{aligned}$$

By (3.7), we have

$$\lim_{n\to\infty}\|u_n-x_n\|=0. \tag{3.11}$$

Furthermore,

$$
\begin{aligned}
&\|x_{n+1}-x^*\|^2 \le \alpha_n\|f(x_n)-x^*\|^2+(1-\delta)(1-\alpha_n)\|x_n-x^*\|^2+\delta\sum_{i=1}^{\infty}\sigma_{i,n}\|\rho_n-x^*\|^2\\
\le\;& \alpha_n\|f(x_n)-x^*\|^2+(1-\delta)(1-\alpha_n)\|x_n-x^*\|^2+\delta\sum_{i=1}^{\infty}\sigma_{i,n}\Big(\|(u_n-s_nAu_n)-(x^*-s_nAx^*)\|^2\Big)\\
\le\;& \alpha_n\|f(x_n)-x^*\|^2+(1-\delta)(1-\alpha_n)\|x_n-x^*\|^2+\delta\sum_{i=1}^{\infty}\sigma_{i,n}\Big(\|u_n-x^*\|^2+\Big(2s_n\lambda+s_n^2-\frac{2s_n\gamma}{\mu^2}\Big)\|Au_n-Ax^*\|^2\Big)\\
\le\;& \alpha_n\|f(x_n)-x^*\|^2+\|x_n-x^*\|^2+\delta\sum_{i=1}^{\infty}\sigma_{i,n}\Big(2s_n\lambda+s_n^2-\frac{2s_n\gamma}{\mu^2}\Big)\|Au_n-Ax^*\|^2
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&-\sum_{i=1}^{\infty}\sigma_{i,n}\Big(2s_n\lambda+s_n^2-\frac{2s_n\gamma}{\mu^2}\Big)\|Au_n-Ax^*\|^2 \le \alpha_n\|f(x_n)-x^*\|^2+\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2\\
\le\;& \|x_n-x_{n+1}\|(\|x_n-x^*\|+\|x_{n+1}-x^*\|)+\alpha_n\|f(x_n)-x^*\|^2. \qquad (3.12)
\end{aligned}
$$

By (3.7), we obtain $\lim_{n\to\infty}\|Au_n-Ax^*\|=0$. Furthermore, we have

$$
\begin{aligned}
&\|P_K(u_n-s_nAu_n)-x^*\|^2 = \|P_K(u_n-s_nAu_n)-P_K(x^*-s_nAx^*)\|^2\\
\le\;& \langle (u_n-s_nAu_n)-(x^*-s_nAx^*), P_K(u_n-s_nAu_n)-x^*\rangle\\
=\;& \frac{1}{2}\Big[\|(u_n-s_nAu_n)-(x^*-s_nAx^*)\|^2+\|P_K(u_n-s_nAu_n)-x^*\|^2\\
&-\|(u_n-s_nAu_n)-(x^*-s_nAx^*)-(P_K(u_n-s_nAu_n)-x^*)\|^2\Big]\\
\le\;& \frac{1}{2}\Big[\|u_n-x^*\|^2+\|P_K(u_n-s_nAu_n)-x^*\|^2-\|(u_n-P_K(u_n-s_nAu_n))-s_n(Au_n-Ax^*)\|^2\Big]\\
=\;& \frac{1}{2}\Big[\|u_n-x^*\|^2+\|P_K(u_n-s_nAu_n)-x^*\|^2-\|u_n-P_K(u_n-s_nAu_n)\|^2+2s_n\langle u_n-P_K(u_n-s_nAu_n), Au_n-Ax^*\rangle\\
&-{s_n}^2\|Au_n-Ax^*\|^2\Big].
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\|P_K(u_n-s_nAu_n)-x^*\|^2 \le\;& \|u_n-x^*\|^2-\|u_n-P_K(u_n-s_nAu_n)\|^2\\
&+2s_n\langle u_n-P_K(u_n-s_nAu_n), Au_n-Ax^*\rangle-{s_n}^2\|Au_n-Ax^*\|^2.
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
\|x_{n+1}-x^*\|^2 &= \|\alpha_n f(x_n)+(1-\delta)(1-\alpha_n)x_n+\delta\sum_{i=1}^{\infty}\sigma_{i,n}T_iP_K(u_n-s_nAu_n)-x^*\|^2\\
&\le \alpha_n\|f(x_n)-x^*\|^2+(1-\delta)(1-\alpha_n)\|x_n-x^*\|^2+\delta\sum_{i=1}^{\infty}\sigma_{i,n}\|P_K(u_n-s_nAu_n)-x^*\|^2\\
&\le \alpha_n\|f(x_n)-x^*\|^2+(1-\delta)(1-\alpha_n)\|x_n-x^*\|^2+\delta\sum_{i=1}^{\infty}\sigma_{i,n}\|u_n-x^*\|^2-\delta\sum_{i=1}^{\infty}\sigma_{i,n}\|u_n-P_K(u_n-s_nAu_n)\|^2\\
&+ 2s_n\delta\sum_{i=1}^{\infty}\sigma_{i,n}\langle u_n-P_K(u_n-s_nAu_n),Au_n-Ax^*\rangle-\delta\sum_{i=1}^{\infty}\sigma_{i,n}{s_n}^2\|Au_n-Ax^*\|^2\\
&\le \alpha_n\|f(x_n)-x^*\|^2+\|x_n-x^*\|^2-\delta\sum_{i=1}^{\infty}\sigma_{i,n}\|u_n-P_K(u_n-s_nAu_n)\|^2\\
&+2\delta\sum_{i=1}^{\infty}\sigma_{i,n}s_n\|u_n-P_K(u_n-s_nAu_n)\|\|Au_n-Ax^*\|
\end{aligned}
$$

This implies that

$$
\begin{aligned}
&\delta\sum_{i=1}^{\infty}\sigma_{i,n}\|u_n-P_K(u_n-s_nAu_n)\|^2 \le \alpha_n\|f(x_n)-x^*\|^2+\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2\\
&+2\delta\sum_{i=1}^{\infty}\sigma_{i,n}s_n\|u_n-P_K(u_n-s_nAu_n)\|\|Au_n-Ax^*\|\\
\le\ &\alpha_n\|f(x_n)-x^*\|^2+\|x_n-x_{n+1}\|(\|x_n-x^*\|+\|x_{n+1}-x^*\|)\\
&+2\delta\sum_{i=1}^{\infty}\sigma_{i,n}s_n\|u_n-P_K(u_n-s_nAu_n)\|\|Au_n-Ax^*\| \qquad (3.13)
\end{aligned}
$$

Using $\lim\alpha_n=0$, $\lim\|x_n-x_{n+1}\|=0$ and $\lim\|Au_n-Ax^*\|=0$ in (3.13), we have

$$\lim_{n\to\infty}\|u_n-P_K(u_n-s_nAu_n)\|=0. \qquad (3.14)$$

Now, consider the following estimates

$$
\begin{aligned}
&\|T_iP_K(I-s_nA)u_n-u_n\|^2 \le \|T_iP_K(I-s_nA)u_n-x^*+x^*-u_n\|^2\\
=\ &\|x^*-u_n\|^2+2\langle T_iP_K(I-s_nA)u_n-x^*,x^*-u_n\rangle+\|T_iP_K(I-s_nA)u_n-x^*\|^2\\
\le\ &2\|x^*-u_n\|^2+2\langle T_iP_K(I-s_nA)u_n-u_n+u_n-x^*,x^*-u_n\rangle\\
=\ &2\|x^*-u_n\|^2+2\langle T_iP_K(I-s_nA)u_n-u_n,x^*-u_n\rangle-2\|u_n-x^*\|^2\\
=\ &2\langle T_iP_K(I-s_nA)u_n-u_n,x^*-u_n\rangle. \qquad (3.15)
\end{aligned}
$$

Using (3.15), we obtain

$$
\begin{aligned}
&\langle x_{n+1}-x^*,u_n-x^*\rangle = \alpha_n\langle f(x_n)-x^*,u_n-x^*\rangle+(1-\alpha_n)(1-\delta)\langle x_n-x^*,u_n-x^*\rangle \\
&+\delta\sum_{i=1}^{\infty}\sigma_{i,n}\langle T_iP_K(I-s_nA)u_n-u_n+u_n-x^*,u_n-x^*\rangle \\
= &\ \alpha_n\langle f(x_n)-x^*,u_n-x^*\rangle+(1-\alpha_n)(1-\delta)\langle x_n-u_n+u_n-x^*,u_n-x^*\rangle \\
&+\delta\sum_{i=1}^{\infty}\sigma_{i,n}\langle T_iP_K(I-s_nA)u_n-u_n,u_n-x^*\rangle+\delta(1-\alpha_n)\|u_n-x^*\|^2 \\
= &\ \alpha_n\langle f(x_n)-x^*,u_n-x^*\rangle+(1-\alpha_n)(1-\delta)\langle x_n-u_n,u_n-x^*\rangle \\
&+\delta\sum_{i=1}^{\infty}\sigma_{i,n}\langle T_iP_K(I-s_nA)u_n-u_n,u_n-x^*\rangle+(1-\alpha_n)(1-\delta)\|u_n-x^*\|^2 \\
&+\delta(1-\alpha_n)\|u_n-x^*\|^2
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\delta\sum_{i=1}^{\infty}\sigma_{i,n}\langle T_iP_K(I-s_nA)u_n-u_n,x^*-u_n\rangle = \alpha_n\langle f(x_n)-x^*,u_n-x^*\rangle \\
&+(1-\alpha_n)(1-\delta)\langle x_n-u_n,u_n-x^*\rangle+(1-\alpha_n)\langle u_n-x^*,u_n-x^*\rangle \\
&-\langle x_{n+1}-x^*,u_n-x^*\rangle \\
= &\ \alpha_n\langle f(x_n)-x_{n+1},u_n-x^*\rangle+(1-\alpha_n)(1-\delta)\langle x_n-u_n,u_n-x^*\rangle \\
&+(1-\alpha_n)\langle u_n-x_{n+1},u_n-x^*\rangle. \qquad (3.16)
\end{aligned}
$$

By (3.16) and (3.15), we obtain

$$
\begin{aligned}
&\frac{\delta}{2}\sum_{i=1}^{\infty}\sigma_{i,n}\|T_iP_K(I-s_nA)u_n-u_n\|^2 \le \alpha_n\langle f(x_n)-x_{n+1},u_n-x^*\rangle \\
&+(1-\alpha_n)(1-\delta)\langle x_n-u_n,u_n-x^*\rangle+(1-\alpha_n)\langle u_n-x_{n+1},u_n-x^*\rangle.
\end{aligned}
$$

Since $\{x_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ are bounded and by (3.11), we have that

$$\lim_{n\to\infty}\|T_iP_K(I-s_nA)u_n-u_n\|=0,\quad \forall i=1,2,... \qquad (3.17)$$

By (3.17) and (3.11), we have

$$\lim_{n\to\infty}\|T_iP_K(I-s_nA)u_n-x_n\|=0,\quad \forall i=1,2,... \qquad (3.18)$$

(v) Finally, we show that

$$\limsup_{n\to\infty}\langle f(z)-z,x_n-z\rangle \le 0,$$

where $z=P_Ff(z)$. To do this, we choose a subsequence $\{x_{n_j}\}$ of $\{x_n\}_{n=1}^{\infty}$ such that $\limsup_{n\to\infty}\langle f(z)-z,x_n-z\rangle = \lim_{j\to\infty}\langle f(z)-z,x_{n_j}-z\rangle$. As $\{u_n\}_{n=1}^{\infty}$ is bounded, there exists a subsequence $\{u_{n_j}\}_{j=1}^{\infty}$

of $\{u_n\}_{n=1}^{\infty}$ which converges weakly to w. We first show that $w \in \bigcap_{i=1}^{\infty} F(T_iP_K(I-s_nA))$. Assume the contrary that $w \neq T_iP_K(I-s_nA)w,\ \ i=1,2,\ldots$. Then by Opial condition, we obtain

$$\begin{aligned}
\liminf_{j\to\infty}\|u_{n_j}-w\| &< \liminf_{j\to\infty}\|u_{n_j}-T_iP_K(I-s_nA)w\| \\
&\leq \liminf_{j\to\infty}(\|u_{n_j}-T_iP_K(I-s_nA)u_{n_j}\| \\
&+ \|T_iP_K(I-s_nA)u_{n_j}-T_iP_K(I-s_nA)w\|) \\
&\leq \liminf_{j\to\infty}\|u_{n_j}-w\|.
\end{aligned}$$

This is a contradiction. Hence, $w \in \bigcap_{i=1}^{\infty} F(T_iP_K(I-s_nA))$.

We next show that $w \in EP$. Since $u_n = T_{r_n}(x_n - r_n\psi x_n),\ \ n \geq 1$, we have for any $y \in K$ that

$$F(u_n,y)+\langle \psi x_n, y-u_n\rangle + \frac{1}{r_n}\langle y-u_n, u_n-x_n\rangle \geq 0.$$

Furthermore, replacing n by n_j in the last inequality and using $(A2)$, we obtain

$$\langle \psi x_{n_j}, y-u_{n_j}\rangle + \frac{1}{r_{n_j}}\langle y-u_{n_j}, u_{n_j}-x_{n_j}\rangle \geq F(y,u_{n_j}). \tag{3.19}$$

Let $z_t := ty+(1-t)w$ for all $t \in (0,1]$ and $y \in K$. This implies that $z_t \in K$. Then, by (3.19), we have

$$\begin{aligned}
\langle z_t-u_{n_j},\psi z_t\rangle &\geq \langle z_t-u_{n_j},\psi z_t\rangle - \langle z_t-u_{n_j},\psi x_{n_j}\rangle - \langle z_t-u_{n_j}, \frac{u_{n_j}-x_{n_j}}{r_{n_j}}\rangle + F(z_t,u_{n_j}) \\
&= \langle z_t-u_{n_j},\psi z_t-\psi u_{n_j}\rangle + \langle z_t-u_{n_j},\psi u_{n_j}-\psi x_{n_j}\rangle - \langle z_t-u_{n_j}, \frac{u_{n_j}-x_{n_j}}{r_{n_j}}\rangle \\
&\quad + F(z_t,u_{n_j}).
\end{aligned}$$

Since $\|x_{n_j}-u_{n_j}\| \to 0,\ \ j\to\infty$, we obtain $\|\psi x_{n_j}-\psi u_{n_j}\| \to 0$. Furthermore, by the monotonicity of ψ, we obtain $\langle z_t-u_{n_j},\psi z_t-\psi u_{n_j}\rangle \geq 0$. Then, by $(A4)$ we obtain

$$\langle z_t-w,\psi z_t\rangle \geq F(z_t,w),\ \ j\to\infty. \tag{3.20}$$

Using $(A1)$, $(A4)$ and (3.20) we also obtain

$$\begin{aligned}
0 &= F(z_t,z_t) \leq tF(z_t,y)+(1-t)F(z_t,w) \\
&\leq tF(z_t,y)+(1-t)\langle z_t-w,\psi z_t\rangle \\
&= tF(z_t,y)+(1-t)t\langle y-w,\psi z_t\rangle
\end{aligned}$$

and hence

$$0 \leq F(z_t,y)+(1-t)\langle y-w,\psi z_t\rangle.$$

Letting $t\to 0$, we have, for each $y \in K$,

$$0 \leq F(w,y)+\langle y-w,\psi w\rangle. \tag{3.21}$$

This implies that $w \in EP$.

Next, we show $u \in V(K,A)$. Put

$$Mw = \begin{cases} Aw + N_K w, & w \in K \\ \emptyset, & w \notin K. \end{cases}$$

Since A is relaxed (λ,γ)-cocoercive and by condition (v), we have

$$\langle Ax - Ay, x - y\rangle \geq (-\lambda)\|Ax - Ay\|^2 + \gamma\|x - y\|^2 \geq (\gamma - \lambda\mu^2)\|x - y\|^2 \geq 0,$$

which shows that A is monotone. Thus, M is maximal monotone. Let $(w_1, w_2) \in G(M)$. Since $w_2 - Aw_1 \in N_K w_1$ and $\rho_n \in K$, we have

$$\langle w_1 - \rho_n, w_2 - Aw_1\rangle \geq 0.$$

On the other hand, from $\rho_n = P_K(I - s_n A)u_n$, we have

$$\langle w_1 - \rho_n, \rho_n - (I - s_n A)u_n\rangle \geq 0.$$

and hence

$$\langle w_1 - \rho_n, \frac{\rho_n - u_n}{s_n} + Au_n\rangle \geq 0.$$

It follows that

$$\begin{aligned}
\langle w_1 - \rho_{n_i}, w_2\rangle &\geq \langle w_1 - \rho_{n_i}, Aw_1\rangle \geq \langle w_1 - \rho_{n_i}, Aw_1\rangle \\
&\quad - \langle w_1 - \rho_{n_i}, \frac{\rho_{n_i} - u_{n_i}}{s_{n_i}} + Au_{n_i}\rangle \\
&= \langle w_1 - \rho_{n_i}, Aw_1 - \frac{\rho_{n_i} - u_{n_i}}{s_{n_i}} - Au_{n_i}\rangle \\
&= \langle w_1 - \rho_{n_i}, Aw_1 - A\rho_{n_i}\rangle + \langle w_1 - \rho_{n_i}, A\rho_{n_i} - Au_{n_i}\rangle \\
&\quad - \langle w_1 - \rho_{n_i}, \frac{\rho_{n_i} - u_{n_i}}{s_{n_i}}\rangle \\
&\geq \langle w_1 - \rho_{n_i}, A\rho_{n_i} - Au_{n_i}\rangle - \langle w_1 - \rho_{n_i}, \frac{\rho_{n_i} - u_{n_i}}{s_{n_i}}\rangle,
\end{aligned}$$

which implies that $\langle w_1 - w, w_2\rangle \geq 0$, $(i \to \infty)$. We have $w \in M^{-1}0$ and hence $w \in VI(K,A)$. Thus, $w \in \bigcap_{i=1}^{\infty} F(T_i) \bigcap EP \bigcap VI(K,A)$.

Now, from (2.3), we have

$$\begin{aligned}
\limsup_{n\to\infty}\langle f(z) - z, x_n - z\rangle &= \lim_{j\to\infty}\langle f(z) - z, x_{n_j} - z\rangle \\
&= \langle f(z) - z, w - z\rangle \leq 0. \qquad (3.22)
\end{aligned}$$

Therefore,

$$
\begin{aligned}
\|x_{n+1}-z\|^2 &= \|\alpha_n f(x_n)+(1-\delta)(1-\alpha_n)x_n+\delta\sum_{i=1}^{\infty}\sigma_{i,n}T_i\rho_n - z\|^2\\
&= \|\alpha_n(f(x_n)-z)+(1-\delta)(1-\alpha_n)(x_n-z)+\delta\sum_{i=1}^{\infty}\sigma_{i,n}(T_i\rho_n-z)\|^2\\
&\leq \|(1-\delta)(1-\alpha_n)(x_n-z)+\delta\sum_{i=1}^{\infty}\sigma_{i,n}(T_i\rho_n-z)\|^2+2\alpha_n\langle f(x_n)-z,x_{n+1}-z\rangle\\
&\leq (1-\alpha_n)^2\|x_n-z\|^2+2\alpha_n\langle f(x_n)-z,x_{n+1}-z\rangle\\
&\leq (1-\alpha_n)^2\|x_n-z\|^2+2\alpha_n\langle f(x_n)-f(z),x_{n+1}-z\rangle+2\alpha_n\langle f(z)-z,x_{n+1}-z\rangle\\
&\leq (1-\alpha_n)^2\|x_n-z\|^2+2\alpha_n\beta\|x_n-z\|\|x_{n+1}-z\|+2\alpha_n\langle f(z)-z,x_{n+1}-z\rangle\\
&\leq (1-\alpha_n)^2\|x_n-z\|^2+\alpha_n\beta(\|x_n-z\|^2+\|x_{n+1}-z\|^2)+2\alpha_n\langle f(z)-z,x_{n+1}-z\rangle.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
\|x_{n+1}-z\|^2 &\leq \frac{1-2\alpha_n+\alpha_n^2+\alpha_n\beta}{1-\alpha_n\beta}\|x_n-z\|^2+\frac{2\alpha_n}{1-\alpha_n\beta}\langle f(z)-z,x_{n+1}-z\rangle\\
&= \Big[1-\frac{2(1-\beta)\alpha_n}{1-\alpha_n\beta}\Big]\|x_n-z\|^2+\frac{\alpha_n^2}{1-\alpha_n\beta}\|x_n-z\|^2+\frac{2\alpha_n}{1-\alpha_n\beta}\langle f(z)-z,x_{n+1}-z\rangle\\
&= (1-\gamma_n)\|x_n-z\|^2+\delta_n,
\end{aligned}
$$

where $\gamma_n := \frac{2(1-\beta)\alpha_n}{1-\alpha_n\beta}$, $\delta_n := \frac{\alpha_n}{1-\alpha_n\beta}\Big[\alpha_n\|x_n-z\|^2+2\langle f(z)-z,x_{n+1}-z\rangle\Big]$. By Lemma 2.2, we get that $\{x_n\}_{n=1}^{\infty}$ converges strongly to z. This completes the proof. □

Corollary 3.2. *(Ali [2]) Let K be a closed convex subset of a real Hilbert space H. Let F be a bifunction from $K\times K$ satisfying $(A1)-(A4)$, A be a μ-Lipschitzian, relaxed (λ,γ)-cocoercive mapping of K into H and ψ be an α-inverse, strongly monotone mapping of K into H. Let $\{T_i\}_{i=1}^{\infty}$ be a countable family of nonexpansive mappings of K into H and let*

$$\bigcap_{i=1}^{\infty}F(T_i)\bigcap VI(A,C)\bigcap EP(F)\neq\emptyset.$$

Let $f: K\to K$ be a contraction map with constant $\beta\in(0,1)$. For a fixed $\delta\in(0,1)$, let $\{x_n\}$ and $\{u_n\}$ be generated by $u, x_1\in K$,

$$
\begin{cases}
F(u_n,y)+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\geq 0 \ \ \forall y\in C,\\[2ex]
x_{n+1}=\alpha_n u+(1-\delta)(1-\alpha_n)x_n+\delta\sum_{i=1}^{\infty}\sigma_{i,n}T_iP_K(u_n-s_nAu_n);
\end{cases}
$$

for all $n\geq 1$, where $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$, $\{\sigma_{i,n}\}_{n=1}^{\infty}$ are sequences in [0,1], for all $i\geq 1$, $\{s_n\}_{n=1}^{\infty}\subset [a,b]$ for some $a,b\in(0,\frac{2(\gamma-\lambda\mu^2)}{\mu^2})$ and $r_n\in(0,\infty)$ satisfying:

(1) $\sum_{i=1}^{\infty}\sigma_{i,n}=(1-\alpha_n)$

(2) $\lim_{n\to\infty}\alpha_n = 0, \quad \sum_{n=1}^{\infty}\alpha_n = \infty$

(3) $\liminf_{n\to\infty} r_n > 0, \quad \lim_{n\to\infty}|r_{n+1} - r_n| = 0$

(4) $\lim_{n\to\infty}\sum_{i=1}^{\infty}\left|\sigma_{i,n+1} - \sigma_{i,n}\right| = 0$

(5) $0 < a \le b < \frac{2(\gamma - \lambda\mu^2)}{\mu^2}, \quad \lim_{n\to\infty}|s_{n+1} - s_n| = 0,$

then $\{x_n\}_{n=1}^{\infty}$ *converges strongly to z, where* $z = P_F u$.

Proof. Put $\psi \equiv 0$ and $f(x_n) = u, \quad \forall n \ge 1$ in Theorem 3.1. Then, by Theorem 3.1 we have the desired result. □

Remark 3.3. The conditions

$$\begin{cases} \frac{1}{\sigma_{i,n}}\left|1 - \frac{\alpha_{n-1}}{\alpha_n}\right| \to 0, \text{ or } \sum\limits_{n=1}^{\infty}\frac{1}{\sigma_{i,n}}|\alpha_{n-1} - \alpha_n| < \infty, \\ \frac{1}{\alpha_n}\left|\frac{1}{\sigma_{i,n}} - \frac{1}{\sigma_{i,n-1}}\right| \to 0, \text{ or } \sum\limits_{n=1}^{\infty}\left|\frac{1}{\sigma_{i,n}} - \frac{1}{\sigma_{i,n-1}}\right| < \infty \end{cases}$$

in used Theorem 1.1 are dispensed with in all our results. Furthermore, condition

$$\frac{1}{\sigma_{i,n}\alpha_n}\sum_{k=0}^{\infty}\left|\sigma_{k,n} - \sigma_{k,n-1}\right| \to 0, \text{ or } \sum_{n=1}^{\infty}\frac{1}{\sigma_{i,n}}\sum_{k=0}^{\infty}\left|\sigma_{k,n} - \sigma_{k,n-1}\right| < \infty.$$

in Theorem 1.1 is weakened to $\lim_{n\to\infty}\sum_{i=1}^{\infty}\left|\sigma_{i,n+1} - \sigma_{i,n}\right| = 0$ in all our results.

Remark 3.4. The prototypes for the sequences $\{\alpha_n\}$ and $\{\sigma_{i,n}\}$ in this paper are the following:

$$\alpha_n := \frac{1}{n+1}; \quad \sigma_{i,n} := \frac{n}{2^i(n+1)}, \quad \forall i \in \mathbb{N}.$$

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