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Existence and Uniqueness of Solutions of a Boundary Value Problem of Fractional Order

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Abstract

In this paper, we investigate some new existence and uniqueness results for nonlinear fractional differential equations with four-point nonlocal integral boundary conditions by applying fixed point theorems.

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1 Introduction

Fractional differential equations can be extensively applied to various disciplines such as physics, mechanics, chemistry and engineering, see [16, 18, 19, 20]. Recently, boundary value problems for fractional differential equations have been addressed by several researchers. Some recent work on boundary value problems of fractional order can be found in [1, 2, 3, 4, 8, 9, 10, 14] and the references therein.

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Very recently, in [7] Ahmad and Ntouyas studied the boundary value problem for nonlinear fractional differential equations of order $q \in (1,2]$ with four-point integral boundary condition

$$\begin{cases} {}^cD^q x(t) = f(t,x(t)), \quad 0 < t < 1, \ 1 < q \le 2 \\ x(0) = \alpha \displaystyle\int_0^\xi x(s)\,ds, \quad x(1) = \beta \displaystyle\int_0^\eta x(s)\,ds, \quad 0 < \xi, \eta < 1 \end{cases} \tag{1.1}$$

where ${}^cD^q$ denote the Caputo fractional derivative of order q, $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ and $\alpha, \beta \in \mathbb{R}$. It is an important consideration as the integral boundary conditions are quite important in the mathematical modeling of applied problems. For a detailed description of the integral boundary conditions, we refer the reader to the papers [5], [6], [11] and references therein.

Existence and uniqueness results for the boundary value problem (1.1) was proved in [7], by using Banach's and Krasnoselskii's fixed point theorems or Leray-Schauder degree theory.

In this paper we supplement and extend the results proved in [7], by proving some new existence and uniqueness results for the boundary value problem (1.1), by using different fixed point theorems. Thus, in Theorem 3.1 we prove an existence and uniqueness result by using a fixed point theorem of Boyd and Wong [12] for nonlinear contractions, while in Theorem 4.1 we prove the existence of a unique positive solution by using a fixed point theorem on partially order sets due to Harjani and Sadarangani [15]. Finally some extensions to a boundary value problem with first-order dependence derivative are given in Theorem 5.2 by using Leray-Schauder nonlinear alternative.

2 Preliminaries

For the reader's convenience, let us recall some basic definitions and preliminary results of fractional calculus and fixed point theory.

Definition 2.1. For a continuous function $g : [0,\infty) \to \mathbb{R}$, the Caputo derivative of fractional order q is defined as

$${}^cD^q g(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} g^{(n)}(s)ds, \ \ n-1 < q < n, n = [q]+1,$$

where $[q]$ denotes the integer part of the real number q.

Definition 2.2. The Riemann-Liouville fractional integral of order q is defined as

$$I^q g(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{g(s)}{(t-s)^{1-q}} ds, \ \ q > 0,$$

provided the integral exists.

Lemma 2.3. *([16]) For $q > 0$, the general solution of the fractional differential equation ${}^cD^q x(t) = 0$ is given by*

$$x(t) = c_0 + c_1 t + c_2 t^2 + \ldots + c_{n-1} t^{n-1},$$

where $c_i \in \mathbb{R}$, $i = 0,1,2,\ldots,n-1$ ($n = [q]+1$).

In view of Lemma 2.3, it follows that

$$I^q\,{}^cD^q x(t) = x(t) + c_0 + c_1 t + c_2 t^2 + \ldots + c_{n-1} t^{n-1}, \tag{2.1}$$

for some $c_i \in \mathbb{R},\ i = 0,1,2,\ldots,n-1$ $(n = [q]+1)$.

The following lemma was proved in [7].

Lemma 2.4. *Let $g : [0,1] \to \mathbb{R}$ be a given continuous function. Then a unique solution of the boundary value problem*

$$\begin{cases} {}^cD^q x(t) = g(t), \quad 0 < t < 1, \quad 1 < q \le 2, \\ x(0) = \alpha \displaystyle\int_0^\xi x(s)ds, \quad x(1) = \beta \int_0^\eta x(s)ds, \quad 0 < \xi,\ \eta < 1, \end{cases} \tag{2.2}$$

is given by

$$\begin{aligned} x(t) \;=\; & \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} g(s)ds \\ & + \frac{\alpha}{\gamma\Gamma(q)} \left(\frac{2-\beta\eta^2}{2} + (\beta\eta - 1)t \right) \int_0^\xi \left(\int_0^s (s-m)^{q-1} g(m)dm \right) ds \\ & + \frac{\beta}{\gamma\Gamma(q)} \left(\frac{\alpha\xi^2}{2} + (1-\xi\alpha)t \right) \int_0^\eta \left(\int_0^s (s-m)^{q-1} g(m)dm \right) ds \\ & - \frac{1}{\gamma\Gamma(q)} \left(\frac{\alpha\xi^2}{2} + (1-\xi\alpha)t \right) \int_0^1 (1-s)^{q-1} g(s)ds, \end{aligned} \tag{2.3}$$

where

$$\gamma = \frac{1}{2}[(\alpha\xi - 1)(\beta\eta^2 - 2) - \alpha\xi^2(\beta\eta - 1)] \neq 0.$$

For more details on fractional calculus we refer to [16, 18, 20].

Now we present some results from fixed point theory. Firstly we recall Boyd and Wong's lemma.

Definition 2.5. Let E be a Banach space and let $F : E \to E$ be a mapping. F is said to be a nonlinear contraction if there exists a continuous nondecrasing function $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\Psi(0) = 0$ and $\Psi(\xi) < \xi$ for all $\xi > 0$ with the property:

$$\|Fx - Fy\| \le \Psi(\|x-y\|), \quad \forall x, y \in E.$$

Lemma 2.6. *(Boyd and Wong)[12]. Let E be a Banach space and let $F : E \to E$ be a nonlinear contraction. Then F has a unique fixed point in E.*

Next we state a known result from ordered sets.

Lemma 2.7. *[15]. Let $(E, \le)$ be a partially ordered set and suppose that there exists a metric d in E such that (E,d) is a complete metric space. Assume that E satisfies the* *following condition: if (x_n) is a nondecreasing sequence in E such that $x_n \to x$ then $x_n \leq x$, for all n. Let $F: E \to E$ be a nondecreasing mapping such that*

$$d(Fx, Fy) \leq d(x,y) - \Psi(d(x,y)), \quad for\, x \geq y$$

where $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and nondecreasing function such that Ψ is positive in $(0,\infty)$, $\Psi(0) = 0$ and $\lim_{\xi\to\infty} \Psi(\xi) = \infty$. If there exists $x_0 \in E$ with $x_0 \leq F(x_0)$, then F has a fixed point.

If we consider the following condition

($\star$) for $x, y \in E$ there exists $\xi \in E$ which is comparable to x and y

then we have the following lemma ([15]).

Theorem 2.8. *Adding condition ($\star$) to the hypotheses of Lemma 2.7, one obtain uniqueness of the fixed point of F.*

3 Existence of sign changing solution

Theorem 3.1. *Assume that*

$$|f(t,x) - f(t,y)| \leq h(t)\frac{|x-y|}{H^* + |x-y|}, \quad t \in (0,1), \;\; x,y \in \mathbb{R},$$

where $h : (0,1) \to \mathbb{R}^+$ and $H^* < \infty$ with

$$\begin{aligned}
H^* &= \frac{1}{\Gamma(q)}\int_0^1 (1-s)^{q-1}h(s)ds \\
&\quad + \left|\frac{\alpha}{\gamma\Gamma(q)}\left(\frac{2-\beta\eta^2}{2} + (\beta\eta - 1)\right)\right| \int_0^\xi \left(\int_0^s (s-m)^{q-1}h(m)dm\right)ds \\
&\quad + \left|\frac{\beta}{\gamma\Gamma(q)}\left(\frac{\alpha\xi^2}{2} + (1-\alpha\xi)\right)\right| \int_0^\eta \left(\int_0^s (s-m)^{q-1}h(m)dm\right)ds \\
&\quad + \left|\frac{1}{\gamma\Gamma(q)}\left(\frac{\alpha\xi^2}{2} + (1-\alpha\xi)\right)\right| \int_0^1 (1-s)^{q-1}h(s)ds.
\end{aligned}$$

Then the boundary value problem (1.1) has a unique solution.

Proof. In view of Lemma 2.4 we define the operator $F : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ by

$$\begin{aligned}
Fx(t) &= \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}f(s,x(s))ds \\
&\quad + \frac{\alpha}{\gamma\Gamma(q)}\left(\frac{2-\beta\eta^2}{2} + (\beta\eta - 1)t\right)\int_0^\xi \left(\int_0^s (s-m)^{q-1}f(m,x(m))dm\right)ds \\
&\quad + \frac{\beta}{\gamma\Gamma(q)}\left(\frac{\alpha\xi^2}{2} + (1-\xi\alpha)t\right)\int_0^\eta \left(\int_0^s (s-m)^{q-1}f(m,x(m))dm\right)ds
\end{aligned}$$

$$-\frac{1}{\gamma\Gamma(q)}\left(\frac{\alpha\xi^2}{2}+(1-\xi\alpha)t\right)\int_0^1(1-s)^{q-1}f(s,x(s))ds,$$

where

$$\gamma=\frac{1}{2}[(\alpha\xi-1)(\beta\eta^2-2)-\alpha\xi^2(\beta\eta-1)]\neq 0.$$

Let $E_1=C([0,1],\mathbb{R})$ with norm $\|u\|=\sup_{t\in[0,1]}|u(t)|$. Let the continuous nondecrasing function $\Psi:\mathbb{R}^+\to\mathbb{R}^+$ satisfying $\Psi(0)=0$ and $\Psi(\xi)<\xi$ for all $\xi>0$ defined by

$$\Psi(\xi)=\frac{H^*\xi}{H^*+\xi},\quad \forall\xi\geq 0.$$

Let $x,y\in E$. Then

$$|f(s,x(s))-f(s,y(s))|\leq\frac{h(s)}{H^*}\Psi(\|x-y\|)$$

so that

$$\begin{aligned}
&|Fx(t)-Fy(t)|\\
\leq\ &\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}h(s)\frac{|x(s)-y(s)|}{H^*+|x(s)-y(s)|}ds\\
&+\left|\frac{\alpha}{\gamma\Gamma(q)}\left(\frac{2-\beta\eta^2}{2}+(\beta\eta-1)\right)\right|\int_0^\xi\left(\int_0^s(s-m)^{q-1}h(m)\frac{|x(m)-y(m)|}{H^*+|x(m)-y(m)|}dm\right)ds\\
&+\left|\frac{\beta}{\gamma\Gamma(q)}\left(\frac{\alpha\xi^2}{2}+(1-\alpha\xi)\right)\right|\int_0^\eta\left(\int_0^s(s-m)^{q-1}h(m)\frac{|x(m)-y(m)|}{H^*+|x(m)-y(m)|}dm\right)ds\\
&+\left|\frac{1}{\gamma\Gamma(q)}\left(\frac{\alpha\xi^2}{2}+(1-\alpha\xi)\right)\right|\int_0^1(1-s)^{q-1}h(s)\frac{|x(s)-y(s)|}{H^*+|x(s)-y(s)|}ds\\
\leq\ &\Psi(\|x-y\|).
\end{aligned}$$

Then $\|Fx-Fy\|\leq\Psi(\|x-y\|)$ and F is a nonlinear contraction and it has a unique fixed point in E, by Lemma 2.6. □

4 Existence of positive solution

Theorem 4.1. *Assume that $f:(0,1)\times\mathbb{R}^+\to\mathbb{R}^+$ satisfies the following condition:*

(H) $0\leq f(t,x)-f(t,y)\leq h(t)\dfrac{x-y}{H^*+x-y},\ t\in(0,1),\ x,y\in\mathbb{R}^+\ x\geq y.$

Moreover we assume that $\alpha\xi\leq 1$ and $\beta\eta\leq 1$. Then the boundary value problem (1.1) has a unique nonnegative solution.

Proof. Consider the space $E_2=C([0,1],\mathbb{R}^+)$ equipped with a partial order given by

$$x,y\in E_2\ :x\leq y\Longleftrightarrow x(t)\leq y(t),\ \text{for } t\in[0,1]$$

with the classic metric given by

$$d(x,y)=\max_{t\in[0,1]}|x(t)-y(t)|.$$

Note that, for $x, y \in E_2$, E_2 satisfies condition ($\star$) (see [17]).

The operator F is nondecreasing, since for $x \geq y$ we have $F(x) \geq F(y)$ because $0 \leq f(t,x) - f(t,y)$.

Furthermore, for $x \geq y$ we have:

$$
\begin{aligned}
d(Fx,Fy) &= \max_{t\in[0,1]} |Fx(t) - Fy(t)| \\
&\leq \max_{t\in[0,1]} \Big[\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} h(s) \frac{x(s)-y(s)}{H^* + x(s) - y(s)} ds \\
&\quad + \left| \frac{\alpha}{\gamma\Gamma(q)} \left(\frac{2-\beta\eta^2}{2} + (\beta\eta - 1) \right) \right| \int_0^\xi \left(\int_0^s (s-m)^{q-1} h(m) \frac{x(m)-y(m)}{H^* + x(m) - y(m)} dm \right) ds \\
&\quad + \left| \frac{\beta}{\gamma\Gamma(q)} \left(\frac{\alpha\xi^2}{2} + (1-\alpha\xi) \right) \right| \int_0^\eta \left(\int_0^s (s-m)^{q-1} h(m) \frac{x(m)-y(m)}{H^* + x(m) - y(m)} dm \right) ds \\
&\quad + \left| \frac{1}{\gamma\Gamma(q)} \left(\frac{\alpha\xi^2}{2} + (1-\alpha\xi) \right) \right| \int_0^1 (1-s)^{q-1} h(s) \frac{x(s)-y(s)}{H^* + x(s) - y(s)} ds \Big] \\
&\leq \frac{H^* \|x-y\|}{H^* + \|x-y\|} = \|x-y\| - \left[\|x-y\| - \frac{H^*\|x-y\|}{H^* + \|x-y\|} \right].
\end{aligned}
$$

We put

$$\Phi(\xi) = \xi - \frac{H^*\xi}{H^* + \xi}.$$

Then $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous, nondecrasing, positive on $(0,\infty)$, $\Phi(0) = 0$ and $\lim_{\xi\to\infty} \Phi(\xi) = \infty$.

Therefore for $x \geq y$ we have

$$d(Fx,Fy) \leq d(x,y) - \Phi(d(x,y)).$$

Next we prove that if $\alpha\xi \leq 1$ and $\beta\eta \leq 1$ then $Fx \geq 0, \forall x \in E_2$. Indeed, we remark that

$$(Fx)''(t) = \frac{(q-1)(q-2)}{\Gamma(q)} \int_0^t (t-s)^{q-3} f(s,x(s))\, ds \leq 0,$$

because $1 < q \leq 2$, which means that $(Fx)(t)$ is a concave function and it satisfy

$$
\begin{cases}
{}^cD^q(Fx)(t) = f(t,x(t)), \quad 0 < t < 1, \ 1 < q \leq 2 \\
(Fx)(0) = \alpha \displaystyle\int_0^\xi (Fx)(s)\, ds, \quad (Fx)(1) = \beta \displaystyle\int_0^\eta (Fx)(s)\, ds, \quad 0 < \xi, \eta < 1.
\end{cases}
$$

To prove that $Fx \geq 0, \forall x \in E_2$ it is sufficient, by concavity, to prove that $(Fx)(0) \geq 0$ and $(Fx)(1) \geq 0$.

- Assume that $(Fx)(0) = \min\{(Fx)(0), (Fx)(1)\}$. By concavity of Fx we know that

  $$(Fx)(t) \geq (Fx)(0), \quad \text{for } t \in [0,1].$$

  We have $(Fx)(0) \geq \alpha \int_0^\xi (Fx)(0)\, ds$ which implies that $(1-\alpha\xi)(Fx)(0) \geq 0$ or $(Fx)(0) \geq 0$.

- Assume that $(Fx)(1) = \min\{(Fx)(0), (Fx)(1)\}$. By concavity of Fx we know that

$$(Fx)(t) \geq (Fx)(1), \quad \text{for } t \in [0,1].$$

We have $(Fx)(1) \geq \beta \int_0^\eta (Fx)(1)\,ds$ which implies that $(1-\beta\eta)(Fx)(1) \geq 0$ or $(Fx)(1) \geq 0$.

Therefore $Fx \geq 0, \forall x \in E_2$. In particular $0 \leq F0$, where 0 denotes the zero function.

Consequently the boundary value problem (1.1) has a unique nonnegative solution, by Lemma 2.7. □

5 Some extensions

In this section we study the existence of solutions to the following boundary value problem with first-order dependence derivative

$$\begin{cases} {}^cD^q x(t) = f(t, x(t), x'(t)), \quad 0 < t < 1, \ 1 < q \leq 2 \\ x(0) = \alpha \displaystyle\int_0^\xi x(s)\,ds, \quad x(1) = \beta \displaystyle\int_0^\eta x(s)\,ds, \quad 0 < \xi, \eta < 1 \end{cases} \tag{5.1}$$

where α, β, η, ξ are as in problem (1.1) and $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous and satisfying

(H) $|f(t,x,y)| \leq h_1(t)|x| + h_2(t)|y| + h_3(t), \quad t \in [0,1], \ x, y \in \mathbb{R}$ and $h_i : [0,1] \to \mathbb{R}^+$, $i = 1,2,3$ are continuous functions.

To prove the main result of this section, we will apply the following Leray-Schauder nonlinear alternative [13].

Theorem 5.1. *Let X be a Banach space, $\Omega \subset X$ bounded and open, $0 \in \Omega$, and $F : \overline{\Omega} \to X$ be a completely continuous operator. Then, either there exists $u \in \partial\Omega$ and $\lambda > 1$ such that $Fu = \lambda u$ or F has a fixed point in $\overline{\Omega}$.*

For convenience, let us put

$$\begin{aligned}
H_i^1 &= \frac{1}{\Gamma(q)} \int_0^1 (1-s)^{q-1} h_i(s)\,ds, \ i = 1,2,3. \\
H_i^2 &= \frac{|\alpha|}{|\gamma|\Gamma(q)} \left| \frac{2-\beta\eta^2}{2} + (\beta\eta - 1) \right| \int_0^\xi \left(\int_0^s (s-m)^{q-1} h_i(m)\,dm \right) ds, i = 1,2,3. \\
H_i^3 &= \frac{|\beta|}{|\gamma|\Gamma(q)} \left| \frac{\alpha\xi^2}{2} + (1-\alpha\xi) \right| \int_0^\eta \left(\int_0^s (s-m)^{q-1} h_i(m)\,dm \right) ds, i = 1,2,3. \\
H_i^4 &= \frac{1}{|\gamma|\Gamma(q)} \left| \frac{\alpha\xi^2}{2} + (1-\alpha\xi) \right| \int_0^1 (1-s)^{q-1} h_i(s)\,ds, \ i = 1,2,3. \\
H_i^5 &= \frac{q-1}{\Gamma(q)} \int_0^1 (1-s)^{q-2} h_i(s)\,ds, \ i = 1,2,3. \\
H_i^6 &= \frac{|\alpha|}{|\gamma|\Gamma(q)} |\beta\eta - 1| \int_0^\xi \left(\int_0^s (s-m)^{q-1} h_i(m)\,dm \right) ds, i = 1,2,3.
\end{aligned}$$

$$
\begin{aligned}
H_i^7 &= \frac{|\beta|}{|\gamma|\Gamma(q)}|1-\alpha\xi|\int_0^\eta\left(\int_0^s (s-m)^{q-1}h_i(m)\,dm\right)ds, i=1,2,3.\\
H_i^8 &= \frac{1}{|\gamma|\Gamma(q)}|1-\alpha\xi|\int_0^1 (1-s)^{q-1}h_i(s)\,ds,\ \ i=1,2,3.
\end{aligned}
$$

Our main result in this section is:

Theorem 5.2. *Under the above hypotheses, the BVP (5.1) has at least one solution, provided*

$$\sum_{j=1}^{8}(H_1^j+H_2^j)<1 \quad \textit{and} \quad H_i^5<\infty \quad \textit{for} \quad i=1,2,3.$$

Proof. In view of Lemma 2.4 we define the operator $F_1 : C^1([0,1],\mathbb{R}) \to C^1([0,1],\mathbb{R})$ by

$$
\begin{aligned}
F_1x(t) &= \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}f(s,x(s),x'(s))ds\\
&\quad +\frac{\alpha}{\gamma\Gamma(q)}\left(\frac{2-\beta\eta^2}{2}+(\beta\eta-1)t\right)\int_0^\xi\left(\int_0^s (s-m)^{q-1}f(m,x(m),x'(m))dm\right)ds\\
&\quad +\frac{\beta}{\gamma\Gamma(q)}\left(\frac{\alpha\xi^2}{2}+(1-\xi\alpha)t\right)\int_0^\eta\left(\int_0^s (s-m)^{q-1}f(m,x(m),x'(m))dm\right)ds\\
&\quad -\frac{1}{\gamma\Gamma(q)}\left(\frac{\alpha\xi^2}{2}+(1-\xi\alpha)t\right)\int_0^1 (1-s)^{q-1}f(s,x(s),x'(s))ds,
\end{aligned}
$$

where

$$\gamma=\frac{1}{2}[(\alpha\xi-1)(\beta\eta^2-2)-\alpha\xi^2(\beta\eta-1)]\neq 0.$$

From the continuity of the nonlinear function f, it is easy to prove that the operator F_1 is completely continuous.
Let $X=C^1([0,1],\mathbb{R})$ with $\|x\|=\|x\|_0+\|x'\|_0$ and $\|x\|_0=\sup_{t\in[0,1]}|x(t)|$. We have

$$
\begin{aligned}
&|F_1x(t)|\\
\le\ & \frac{1}{\Gamma(q)}\int_0^1 (1-s)^{q-1}\left\{h_1(s)|x(s)|+h_2(s)|x'(s)|+h_3(s)\right\}ds\\
&+\frac{|\alpha|}{|\gamma|\Gamma(q)}\left|\frac{2-\beta\eta^2}{2}+(\beta\eta-1)\right|\times\\
&\qquad\times\int_0^\xi\left(\int_0^s (s-m)^{q-1}\left\{h_1(m)|x(m)|+h_2(m)|x'(m)|+h_3(m)\right\}dm\right)ds\\
&+\frac{|\beta|}{|\gamma|\Gamma(q)}\left|\frac{\alpha\xi^2}{2}+(1-\alpha\xi)\right|\int_0^\eta\left(\int_0^s \left\{h_1(m)|x(m)|+h_2(m)|x'(m)|+h_3(m)\right\}dm\right)ds\\
&+\frac{1}{|\gamma|\Gamma(q)}\left|\frac{\alpha\xi^2}{2}+(1-\alpha\xi)\right|\int_0^1 (1-s)^{q-1}\left\{h_1(s)|x(s)|+h_2(s)|x'(s)|+h_3(s)\right\}ds\\
\le\ & \left(\sum_{j=1}^4 H_1^j\right)\|x\|_0+\left(\sum_{j=1}^4 H_2^j\right)\|x'\|_0+\sum_{j=1}^4 H_3^j.
\end{aligned}
$$

Then

$$\|F_1x\|_0 \le \left[\sum_{j=1}^{4}(H_1^j + H_2^j)\right]\|x\| + \sum_{j=1}^{4} H_3^j.$$

Also by Leibnitz's formula we obtain

$$\begin{aligned}(F_1x)'(t) &= \frac{q-1}{\Gamma(q)}\int_0^t (t-s)^{q-2} f(s,x(s),x'(s))\,ds \\ &+\frac{\alpha}{\gamma\Gamma(q)}(\beta\eta-1)\int_0^\xi\left(\int_0^s (s-m)^{q-1} f(m,x(m),x'(m))\,dm\right)ds \\ &+\frac{\beta}{\gamma\Gamma(q)}(1-\alpha\xi)\int_0^\eta\left(\int_0^s (s-m)^{q-1} f(m,x(m),x'(m))\,dm\right)ds \\ &-\frac{1}{\gamma\Gamma(q)}(1-\alpha\xi)\int_0^1 (1-s)^{q-1} f(s,x(s),x'(s))\,ds,\end{aligned}$$

and then

$$\begin{aligned}&|(F_1x)'(t)| \\ \le\ & \frac{q-1}{\Gamma(q)}\int_0^1 (1-s)^{q-2}\{h_1(s)|x(s)| + h_2(s)|x'(s)| + h_3(s)\}\,ds \\ &+\frac{|\alpha|}{\gamma\Gamma(q)}|\beta\eta-1|\int_0^\xi\left(\int_0^s (s-m)^{q-1}\{h_1(m)|x(m)| + h_2(m)|x'(m)| + h_3(m)\}\,dm\right)ds \\ &+\frac{|\beta|}{|\gamma|\Gamma(q)}|1-\alpha\xi|\int_0^\eta\left(\int_0^s (s-m)^{q-1}\{h_1(m)|x(m)| + h_2(m)|x'(m)| + h_3(m)\}\,dm\right)ds \\ &+\frac{1}{|\gamma|\Gamma(q)}|1-\alpha\xi|\int_0^1 (1-s)^{q-1}\{h_1(s)|x(s)| + h_2(s)|x'(s)| + h_3(s)\}\,ds \\ \le\ & \left(\sum_{j=5}^{8} H_1^j\right)\|x\|_0 + \left(\sum_{j=5}^{8} H_2^j\right)\|x'\|_0 + \sum_{j=5}^{8} H_3^j.\end{aligned}$$

Then

$$\|(F_1x)'\|_0 \le \left[\sum_{j=5}^{8}(H_1^j + H_2^j)\right]\|x\| + \sum_{j=5}^{8} H_3^j.$$

From $\|F_1x\| = \|F_1x\|_0 + \|(F_1x)'\|_0$ we have

$$\|F_1x\| \le \left[\sum_{j=1}^{8}(H_1^j + H_2^j)\right]\|x\| + \sum_{j=1}^{8} H_3^j.$$

Define

$$m = \begin{cases} \dfrac{\sum\limits_{j=1}^{8} H_3^j}{1-\sum\limits_{j=1}^{8}(H_1^j + H_2^j)}, & \text{if } \sum\limits_{j=1}^{8} H_3^j \ne 0, \\ 1, & \text{if } \sum\limits_{j=1}^{8} H_3^j = 0 \end{cases}$$

and

$$\Omega = \{x \in X = C^1([0,1],\mathbb{R}) : \|x\| < m\}.$$

Let $x \in \partial\Omega$ and $\lambda > 1$ with $F_1 x = \lambda x$. We have

$$\lambda m = \lambda\|x\| = \|F_1 x\| \le \left[\sum_{j=1}^{8}(H_1^j + H_2^j)\right]\|x\| + \sum_{j=1}^{8} H_3^j$$

so that

$$\begin{aligned} 1 < \lambda &\le \sum_{j=1}^{8}(H_1^j + H_2^j) + \frac{1}{m}\sum_{j=1}^{8} H_3^j \\ &= \begin{cases} \sum_{j=1}^{8}(H_1^j + H_2^j), & \text{if } \sum_{j=1}^{8} H_3^j = 0, \\ 1, & \text{if } \sum_{j=1}^{8} H_3^j \neq 0 \end{cases} \\ &\le 1, \end{aligned}$$

a contradiction. Thus F_1 has a fixed point in $\overline{\Omega}$ which is a solution in $C^1([0,1],\mathbb{R})$ of the boundary value problem (5.1). □

6 Examples

Example 6.1. Consider the following four-point integral fractional boundary value problem

$$\begin{cases} {}^cD^{3/2}x(t) = \dfrac{1}{(1-t)^{1/2}} \cdot \dfrac{|x(t)|}{H^* + |x(t)|} + 1, \quad 0 < t < 1, \\ x(0) = \displaystyle\int_0^{1/4} x(s)\,ds, \quad x(1) = \int_0^{3/4} x(s)\,ds. \end{cases} \tag{6.1}$$

Here $q = \frac{3}{2}, \xi = \frac{1}{4}, \eta = \frac{3}{4}, \alpha, \beta = 1$ and $f(t,x) = \frac{1}{(1-t)^{1/2}} \cdot \frac{|x(t)|}{H^* + |x(t)|} + 1$. Clearly $\gamma = \frac{35}{64}$ and $H^* = \frac{40}{7\sqrt{\pi}}$. With $h(t) = \frac{1}{(1-t)^{1/2}}$ we have:

$$\begin{aligned} |f(t,x) - f(t,y)| &= h(t)\frac{H^*\big||x| - |y|\big|}{(H^* + |x|)(H^* + |y|)} \\ &\le h(t)\frac{H^*|x-y|}{(H^*)^2 + H^*(|x| + |y|)} \\ &\le h(t)\frac{|x-y|}{H^* + |x-y|}. \end{aligned}$$

Thus, by Theorem 3.1, the boundary value problem (6.1) has a unique solution.

Example 6.2. Consider the following boundary value problem

$$\begin{cases} {}^cD^{3/2}x(t) = \dfrac{1}{(1-t)^{1/2}} \cdot \dfrac{x(t)}{H^* + x(t)} + 1, \quad 0 < t < 1, \\ x(0) = \displaystyle\int_0^{1/4} x(s)\,ds, \quad x(1) = \int_0^{3/4} x(s)\,ds. \end{cases} \tag{6.2}$$

By Theorem 4.1, the boundary value problem (6.2) has a unique nonnegative solution.

Example 6.3. Consider the following boundary value problem

$$\begin{cases} {}^cD^{3/2}x(t) = h(t)\left[\sqrt{x(t)} + x'(t) + 1\right], \quad 0 < t < 1, \\ x(0) = \alpha \displaystyle\int_0^{1/4} x(s)\,ds, \quad x(1) = \beta \int_0^{3/4} x(s)\,ds. \end{cases} \tag{6.3}$$

With $h(t)$ and $\alpha, \beta \in \mathbb{R}$ such that $\sum_{j=1}^{8}(H_1^j + H_2^j) < 1$, by Theorem 5.2, the boundary value problem (6.3) has at least one solution.

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