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The Multiplicative Anomaly for Determinants Revisited; Locality

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Abstract

Observing that the logarithm of a product of two elliptic operators differs from the sum of the logarithms by a finite sum of operator brackets, we infer that regularised traces of this difference are local as finite sums of noncommutative residues. From an explicit local formula for such regularised traces, we derive an explicit local formula for the multiplicative anomaly of ζ-determinants which sheds light on its locality and yields back previously known results.

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1 Introduction

The determinant on the linear group $\mathrm{Gl}(\mathbb{C}^n)$ reads

$$\det A = e^{\mathrm{tr}\left(\log A\right)}$$

where tr is the matrix trace and log is the multivalued inverse map of the exponential mapping $\exp : \mathrm{gl}(\mathbb{C}^n) \to \mathrm{Gl}(\mathbb{C}^n)$ on the Lie algebra of $n\times n$ matrices with complex coefficients. When the logarithm is defined by a Cauchy integral along a contour around a given spectral

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cut of the matrix, the determinant is independent of the choice of spectral cut [14]. It is moreover multiplicative as a result of the Campbell-Hausdorff formula and the cyclicity of the trace, namely:

$$\det(AB) = e^{\operatorname{tr}\left(\log AB\right)} = e^{\operatorname{tr}\left(\log A + \log B\right)} = \det A \det B.$$

In contrast, the ζ-determinant

$$\det_\zeta(A) = e^{-\zeta_A'(0)}$$

is not multiplicative. Here A is an admissible elliptic classical pseudodifferential operator (with appropriate spectral cut) acting on sections of a vector bundle E over a closed n-dimensional manifold M and $\zeta_A(s)$ is the zeta function associated with A, which corresponds to the unique meromorphic extension of the map $s \mapsto \operatorname{Tr}(A^{-s})$ given by the L^2-trace of A^{-s} defined on the domain of holomorphicity $\operatorname{Re}(as) > n$ where a is the order of A. It presents a multiplicative anomaly

$$\mathcal{M}_\zeta(A,B) = \frac{\det_\zeta(AB)}{\det_\zeta(A)\det_\zeta(B)}$$

studied independently by Okikiolu in [22] and by Kontsevich and Vishik in [15].

The multiplicative anomaly of ζ-determinants was expressed in terms of noncommutative residues of classical pseudodifferential operators in the following situations:

- by Wodzicki [34] (see also [16] for a review) for positive definite commuting elliptic differential operators,
- by Friedlander [8] for positive definite elliptic pseudodifferential operators,
- by Okikiolu [22] for operators with scalar leading symbols,
- by Kontsevich and Vishik [15] for operators sufficiently close to self-adjoint positive pseudodifferential operators.
- The multiplicative anomaly was further studied by Ducourtioux [4] in the context of weighted determinants also discussed in this paper.

The noncommutative residue (see formula (2.7)) introduced independently by Guillemin [10] and Wodzicki [34], which defines a trace on the algebra $C\ell(M,E)$ of classical pseudodifferential operators acting on smooth sections of the vector bundle E, is local in so far as it is the integral over M of a local residue $\operatorname{res}_x(A)$ which only depends on a finite number of homogeneous components of the symbol of the operator A. Consequently, the multiplicative anomaly is local.

Locality of the multiplicative anomaly for ζ-determinants relates to the locality of regularised traces[1] of the difference

$$L(A,B) := \log(AB) - \log A - \log B,$$

[1]Regularised traces are linear extensions to the algebra $C\ell(M,E)$ of the ordinary L^2-trace on smoothing operators, which are non tracial since the L^2-trace does not extend to a trace on the algebra $C\ell(M,E)$.

on which we focus in this paper, investigating their local feature which follows from the vanishing of the residue of $L(A,B)$.
To see these links, one first observes that regularised traces of $L(A,B)$ correspond to the multiplicative anomaly of another type of regularised determinants, namely weighted determinants (see [4])

$$\det{}^Q(A) = e^{\mathrm{tr}^Q(\log A)},$$

defined via a regularised trace tr^Q (see Definition 4.6) which uses the regulator Q, called a weight [2]. They differ from ζ-determinants by a local expression involving the Wodzicki residue, as can explicitly be seen from the relation (see [4] Proposition III.1.7):

$$\frac{\det_\zeta(A)}{\det^Q(A)} = e^{-\frac{1}{2a}\mathrm{res}\left[\left(\log A - \frac{a}{q}\log Q\right)^2\right]},$$

where a is the order of A, q the order of Q. Consequently, the multiplicative anomaly for ζ-determinants differs from the multiplicative anomaly for weighted determinants by a local expression so that $\log \mathcal{M}_\zeta(A,B) - \mathrm{tr}^Q(L(A,B))$ is local.

On the other hand, one infers the locality of regularised traces $\mathrm{tr}^Q(L(A,B))$ of $L(A,B)$ from the vanishing of the noncommutative residue of $L(A,B)$ (see (6.1)), a property shown in [29] which implies the multiplicativity of the residue determinant. Indeed, since all traces on the algebra of classical pseudodifferential operators on a closed connected manifold of dimension larger than one are proportional to the noncommutative residue [34], it follows that $L(A,B)$ is a finite sum of commutators of classical pseudodifferential operators. Combining this with the expression of regularised traces of brackets in terms on the noncommutative residue (see (4.9)), yields the locality of regularised traces $\mathrm{tr}^Q(L(A,B))$ as finite sums of noncommutative residues.

Explicitly, in Theorem 6.2 we show that for two admissible elliptic operators A,B with positive orders a and b, such that the product AB is also admissible, there is an operator $W(\tau)(A,B) := \frac{d}{dt}_{|t=0} L(A^t, A^\tau B)$ depending continuously on τ such that (see (6.3))

$$\mathrm{tr}^Q(L(A,B)) = \int_0^1 \mathrm{res}\left(W(\tau)(A,B)\left(\frac{\log(A^\tau B)}{a\tau+b} - \frac{\log Q}{q}\right)\right) d\tau. \tag{1.1}$$

The multiplicative anomaly for weighted determinants derived in Proposition 7.3 follows in a straightforward manner. From (1.1), in Theorem 7.6 we then derive an explicit local formula for the multiplicative anomaly for operators A and B with positive orders a and b (see equation (7.4)):

$$\begin{aligned} \log \mathcal{M}_\zeta(A,B) &= \int_0^1 \mathrm{res}\left(W(\tau)(A,B)\left(\frac{\log(A^\tau B)}{a\tau+b} - \frac{\log B}{b}\right)\right) d\tau \\ &+ \mathrm{res}\left(\frac{L(A,B)\log B}{b} - \frac{\log^2 AB}{2(a+b)} - \frac{\log^2 A}{2a} - \frac{\log^2 B}{2b}\right) \end{aligned} \tag{1.2}$$

and similarly with the roles of A and B interchanged. When the operators A and B commute, $L(A,B)$ vanishes and formula (1.2) yields back Wodzicki's formula:

$$\log \mathcal{M}_\zeta(A,B) = -\mathrm{res}\left(\frac{\log^2 AB}{2(a+b)} + \frac{\log^2 A}{2a} + \frac{\log^2 B}{2b}\right).$$

[2]A weight is any admissible elliptic operator in $C\ell(M,E)$ with positive order.

The r.h.s in the first line of (1.2) comes from a regularised trace $\mathrm{tr}^B(L(A,B))$ described in (1.1) with weight $Q = B$. The r.h.s in the second line of (1.2), which corresponds to $\log \mathcal{M}_\zeta(A,B) - \mathrm{tr}^Q(L(A,B))$, arises from a combination of two types of local terms; (i) local residues $\mathrm{res}_x(\log^2 AB)$, $\mathrm{res}_x(\log^2 A)$ and $\mathrm{res}_x(\log^2 B)$ arising in formula (7.3) for the zeta determinants established in [24]; (ii) the local residue $\mathrm{res}_x(L(A,B)\log B)$ arising in a "defect formula" for regularised traces (4.6) also established in [24], applied here to the regularised trace $\mathrm{tr}^B(L(A,B))$ with regulator B. Since the operator

$$\frac{L(A,B)\log B}{b} - \frac{\log^2 AB}{2(a+b)} - \frac{\log^2 A}{2a} - \frac{\log^2 B}{2b}$$

turns out to be classical (see Lemma 7.5), combining these local residues yields a well-defined noncommutative residue.

Our approach to the multiplicative anomaly of ζ-determinants is inspired by Okikiolu's in [22]. Before actually computing the multiplicative anomaly, she first showed in [21] that for operators A and B with scalar leading symbols,

$$L(A,B) \simeq \sum_{k=2}^{\infty} C^{(k)}(\log A, \log B),$$

i.e. that $L(A,B) - \sum_{k=2}^{n+1} C^{(k)}(\log A, \log B)$ is of order $< -n$, thus generalising the usual Campbell-Hausdorff formula to classical pseudodifferential operators with scalar leading symbols. Here $C^{(k)}(\log A, \log B)$ are Lie monomials given by iterated brackets[3].

Under the assumption that the operators have scalar leading symbols, the iterated brackets arising in the Campbell-Hausdorff formula have decreasing order, allowing to implement ordinary traces after a certain order. In our more general situation, the leading symbols are not necessarily scalar and the iterated brackets arising in the Campbell-Hausdorff formula hence do not a priori have decreasing order which is why we use regularised traces instead of the ordinary trace and study regularised traces of $L(A,B)$. Okikiolu's proof in the case of operators with scalar leading symbols is largely based on the observation that the trace of the operator $L(A,B) - \sum_{k=2}^{n+1} C^{(k)}(\log A, \log B)$ only depends on the first n positively homogeneous components of A and B where n is the dimension of the underlying manifold M; this allows her to work with a finite dimensional space of formal symbols. Interestingly, in our more general situation regularised traces of $L(A,B)$ still only depend on the first n positively homogeneous components of A and B. Precisely, given a weight Q and two admissible operators A and B in $C\ell(M,E)$ with non negative orders, we show that (see Theorem 5.3)

$$\frac{d}{dt}\mathrm{tr}^Q(L(A(1+tS),B) = \frac{d}{dt}\mathrm{tr}^Q(L(A,B(1+tS)) = 0, \tag{1.3}$$

[3] Their precise definition is:

$$C^{(k)}(P,Q) := \frac{1}{k}\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{j}\sum_{\sum_{i=1}^{j}\alpha_i+\beta_i=k,\alpha_j,\beta_j\geq 0}\frac{(\mathrm{Ad}_P)^{\alpha_1}(\mathrm{Ad}_Q)^{\beta_1}\cdots(\mathrm{Ad}_P)^{\alpha_j}(\mathrm{Ad}_Q)^{\beta_j-1}Q}{\alpha_1!\cdots\alpha_j!\beta_1!\cdots\beta_j!},$$

with the following notational convention: $(\mathrm{Ad}_P)^{\alpha_j}(\mathrm{Ad}_Q)^{\beta_j-1}Q = (\mathrm{Ad}_P)^{\alpha_j-1}P$ if $\beta_j = 0$ in which case this vanishes if $\alpha_j > 1$.

for any operator S in $C\ell(M,E)$ of order $< -n$.
The proofs of Theorem 5.3 and Theorem 6.2 both use the fact that differentiation in t commutes with regularised traces on differentiable families of constant order, a fact we prove in Proposition 4.8.

To conclude, this approach sheds light on the locality of multiplicative anomalies for regularised determinants (weighted determinants on the one hand and ζ-determinants on the other) in so far that it relates it to the cyclicity of the noncommutative residue and hence to the multiplicativity of the residue determinant via the locality of regularised traces of $L(A,B)$, which are interesting in their own right.

2 The noncommutative residue

We recall a few basic definitions concerning classical pseudodifferential operators on closed manifolds, set some notations and define the noncommutative residue introduced by Wodzicki in [33, 34].

Let U be an open subset of $\mathbb{R}^n$. Given $a \in \mathbb{C}$, the space of symbols $S^a(U)$ consists of functions $\sigma(x,\xi)$ in $C^\infty(U\times\mathbb{R}^n)$ such that for any compact subset K of U and any two multiindices $\alpha = (\alpha_1,\cdots,\alpha_n) \in \mathbb{N}^n, \beta = (\beta_1,\cdots,\beta_n) \in \mathbb{N}^n$ there exists a constant $C_{K\alpha\beta}$ satisfying for all $(x,\xi) \in K\times\mathbb{R}^n$

$$|\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)| \leq C_{K\alpha\beta}(1+|\xi|)^{\mathrm{Re}(a)-|\beta|}, \tag{2.1}$$

where $\mathrm{Re}(a)$ is the real part of a and $|\beta| = \beta_1+\cdots+\beta_n$.

If $\mathrm{Re}(a_1) < \mathrm{Re}(a_2)$, then $S^{a_1}(U) \subset S^{a_2}(U)$.

The product $\star$ on symbols is defined as follows: if $\sigma_1 \in S^{a_1}(U)$ and $\sigma_2 \in S^{a_2}(U)$,

$$\sigma_1 \star \sigma_2(x,\xi) \sim \sum_{\alpha\in\mathbb{N}^n} \frac{(-i)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha \sigma_1(x,\xi) \partial_x^\alpha \sigma_2(x,\xi) \tag{2.2}$$

i.e. for any integer $N \geq 1$ we have

$$\sigma_1 \star \sigma_2(x,\xi) - \sum_{|\alpha|<N} \frac{(-i)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha \sigma_1(x,\xi) \partial_x^\alpha \sigma_2(x,\xi) \in S^{a_1+a_2-N}(U).$$

In particular, $\sigma_1 \star \sigma_2 \in S^{a_1+a_2}(U)$.

We denote by $S^{-\infty}(U) := \bigcap_{a\in\mathbb{C}} S^a(U)$ the algebra of smoothing symbols on U, by $S(U) := \langle \bigcup_{a\in\mathbb{C}} S^a(U) \rangle$ the algebra generated by all symbols on U.

A symbol σ in $S^a(U)$ is called classical of complex order a if there is a smooth cut-off function $\chi \in C^\infty(\mathbb{R}^n)$ which vanishes for $|\xi| \leq \frac{1}{2}$ and such that $\chi(\xi) = 1$ for $|\xi| \geq 1$ such that

$$\sigma(x,\xi) \sim \sum_{j=0}^{\infty} \chi(\xi)\,\sigma_{a-j}(x,\xi) \tag{2.3}$$

i.e. if for any integer $N \geq 1$, we have

$$\sigma_{(N)}(x,\xi) := \sigma - \sum_{j=1}^{N-1} \chi(\xi)\sigma_{a-j}(x,\xi) \in S^{a-N}(U), \tag{2.4}$$

where $\sigma_{a-j}(x,\xi)$ is a positively homogeneous function on $U\times\mathbb{R}^n$ of degree $a-j$, i.e. $\sigma_{a-j}(x,t\xi)=t^{a-j}\sigma_{a-j}(x,\xi)$ for all $t\in\mathbb{R}^+$.

Let $CS^a(U)$ denote the subset of classical symbols of order a. The symbol product of two classical symbols is a classical symbol and we denote by

$$CS(U)=\left\langle\bigcup_{a\in\mathbb{C}}CS^a(U)\right\rangle$$

the algebra generated by all classical symbols on U.

The noncommutative residue of a symbol $\sigma\in CS(U)$ at a point x in U is defined by

$$\mathrm{res}_x(\sigma):=\int_{S_x^*U}(\sigma(x,\xi))_{-n}\,đ_S\xi, \tag{2.5}$$

where $S_x^*U\subset T_x^*U$ is the cotangent unit sphere at the point x in U, $đ_S\xi=\frac{1}{(2\pi)^n}\,d_S\xi$ is the normalised volume measure on the sphere induced by the canonical volume measure on $\mathbb{R}^n$ and where as before, $(\cdot)_{-n}$ denotes the positively homogeneous component of degree $-n$ of the symbol.

Given a symbol σ in $S(U)$, we can associate to it the continuous operator

$$Op(\sigma):C_c^\infty(U)\to C^\infty(U)$$

defined for $u\in C_c^\infty(U)$– the space of smooth compactly supported functions on U– by

$$(Op(\sigma)u)(x)=\int e^{ix.\xi}\sigma(x,\xi)\widehat{u}(\xi)đ\xi,$$

where $đ\xi:=\frac{1}{(2\pi)^n}\,d\xi$ with $d\xi$ the ordinary Lebesgue measure on $T_x^*M\simeq\mathbb{R}^n$ and $\widehat{u}(\xi)$ is the Fourier transform of u. Since

$$(Op(\sigma)u)(x)=\int\int e^{i(x-y).\xi}\sigma(x,\xi)u(y)đ\xi dy,$$

$Op(\sigma)$ is an operator with Schwartz kernel given by $k(x,y)=\int e^{i(x-y).\xi}\sigma(x,\xi)đ\xi$, which is smooth off the diagonal.

A pseudodifferential operator A on U is an operator which can be written in the form $A=Op(\sigma)+R$ where σ is a symbol in $S(U)$ with compact support in the variable x, and where R is a smoothing operator i.e. R has a smooth kernel. Its symbol $\sigma_A\sim\sigma$ is defined modulo smoothing symbols. If σ is a classical symbol of order a, then A is called a classical pseudodifferential operator of order a. The composition of two classical operators A_1 and A_2 with symbols σ_{A_1} and σ_{A_2} and orders a_1 and a_2 respectively, is a classical operator A_1A_2 of order a_1+a_2 with symbol $\sigma_{A_1A_2}\sim\sigma_{A_1}\star\sigma_{A_2}$.

More generally, let M be a smooth closed manifold of dimension n and $\pi:E\to M$ a smooth finite rank vector bundle over M modelled on a linear space V.
An operator $A:C^\infty(M,E)\to C^\infty(M,E)$ is a (resp. classical) pseudodifferential operator of order a if given a local trivialising chart (V,ϕ) on M, for any localisation

$$A_v=\chi_v^2A\chi_v^1:C_c^\infty(V)\to C_c^\infty(V)$$

of A where $\chi_\nu^i \in C_c^\infty(V)$, the operator $\phi_*(A_\nu) := \phi A_\nu \phi^{-1}$ from the space $C_c^\infty(\phi(V))$ into $C^\infty(\phi(V))$ is a (resp. classical) pseudodifferential operator of order a.
Let $\mathrm{C}\ell^a(M,E)$ denote the set of classical pseudodifferential operators of order a.
If $A_1 \in \mathrm{C}\ell^{a_1}(M,E), A_2 \in \mathrm{C}\ell^{a_2}(M,E)$, then the product A_1A_2 lies in $\mathrm{C}\ell^{a_1+a_2}(M,E)$ and we denote by

$$\mathrm{C}\ell(M,E) := \left\langle \bigcup_{a\in\mathbb{C}} \mathrm{C}\ell^a(M,E) \right\rangle$$

the algebra generated by all classical pseudodifferential operators acting on smooth sections of E. Let us also introduce the algebra

$$\mathrm{C}\ell^{-\infty}(M,E) := \bigcap_{a\in\mathbb{R}} \mathrm{C}\ell^a(M,E)$$

of smoothing operators.

The noncommutative residue is a linear form on $\mathrm{C}\ell(M,E)$ built from the noncommutative residue density at a point x in M defined (with the notation of (2.5)) by

$$\omega_{\mathrm{res}}(A)(x) := \mathrm{res}_x(\sigma_A)\, dx; \quad \text{with} \quad \mathrm{res}_x(\sigma_A) := \int_{S_x^*M} \mathrm{tr}_x\left((\sigma_A(x,\xi))_{-n}\right) đ_S\xi. \tag{2.6}$$

This turns out to be a globally defined density on the manifold and gives rise to the noncommutative residue of an operator $A \in \mathrm{C}\ell(M,E)$ (see [33, 34] and [10])

$$\mathrm{res}(A) := \int_M \omega_{\mathrm{res}}(A)(x) := \int_M \left(\int_{S_x^*M} \mathrm{tr}_x\left((\sigma_A(x,\xi))_{-n}\right) đ\xi \right) dx. \tag{2.7}$$

The noncommutative residue vanishes on operators of order $< -n$ and is local in the sense that it only depends on a finite number of positively homogeneous components of the symbol of the operator.
It was proved by Wodzicki [34] (see also [17], [11], [27])[4], that when the manifold M is connected and has dimension larger than one, any trace on the algebra $\mathrm{C}\ell(M,E)$ i.e., any linear form which vanishes on operator brackets, is proportional to the noncommutative residue. Consequently (see e.g. [17])

$$\forall A \in \mathrm{C}\ell(M,E) \quad (\mathrm{res}(A) = 0 \Longrightarrow A \in [\mathrm{C}\ell(M,E), \mathrm{C}\ell(M,E)]). \tag{2.8}$$

We shall henceforth assume that the manifold M is connected and has dimension larger than 1.

[4]The uniqueness was proved in Wodzicki's thesis written in Russian but the main ideas of the proof can be found in [17]. Guillemin further studied the uniqueness of the noncommutative residue in a broader context in [11]. An alternative proof of the uniqueness can be found in [27], which also encompasses the one dimensional case. It uses arguments similar in spirit to those underlying the proof in the boundary case carried out in [7] and those underlying the proof of the uniqueness of the canonical trace on non integer operators carried out in [20].

3 Logarithms of operators: $\log(AB) - \log A - \log B$

We review the construction and properties of logarithms of elliptic operators and prove (see Proposition 3.9) that the expression $\log(AB) - \log A - \log B$ is a finite sum of commutators of zero order classical pseudodifferential operators.

An operator $A \in C\ell(M,E)$ has principal angle θ if for every $(x,\xi) \in T^*M \setminus M \times \{0\}$, the leading symbol $(\sigma_A(x,\xi))^L$ has no eigenvalue on the ray $L_\theta = \{re^{i\theta}, r \geq 0\}$; in that case A is elliptic.

Definition 3.1. We call an operator $A \in C\ell(M,E)$ admissible with spectral cut θ if A has principal angle θ and the spectrum of A does not meet $\mathrm{L}_\theta = \{re^{i\theta}, r \geq 0\}$. In particular such an operator is invertible and elliptic. Since the spectrum of A does not meet L_θ, θ is called an Agmon angle of A.

Remark 3.2. In applications, an invertible operator A is often obtained from an essentially self-adjoint elliptic operator $B \in C\ell(M,E)$ by setting $A = B + \pi_B$ using the orthogonal projection π_B onto the kernel $\mathrm{Ker}(B)$ of B corresponding to the orthogonal splitting $L^2(M,E) = \mathrm{Ker}(B) \oplus \mathrm{R}(B)$ where $R(B)$ is the (closed) range of B. Here $L^2(M,E)$ denotes the closure of $C^\infty(M,E)$ with respect to a Hermitian structure on E combined with a Riemannian structure on M.

Let $A \in C\ell(M,E)$ be admissible with spectral cut θ and positive order a. For $\mathrm{Re}(z) < 0$, the complex power A_θ^z of A is defined by the Cauchy integral [30]

$$A_\theta^z = \frac{i}{2\pi} \int_{\Gamma_{r,\theta}} \lambda_\theta^z (A - \lambda)^{-1} \, d\lambda, \tag{3.1}$$

where $\lambda_\theta^z = |\lambda|^z e^{iz(\arg \lambda)}$ with $\theta \leq \arg \lambda < \theta + 2\pi$.
Here

$$\Gamma_{r,\theta} = \Gamma_{r,\theta}^1 \cup \Gamma_{r,\theta}^2 \cup \Gamma_{r,\theta}^3 \tag{3.2}$$

where

$$\Gamma_{r,\theta}^1 = \{\rho\, e^{i\theta}, \infty > \rho \geq r\},$$

$$\Gamma_{r,\theta}^2 = \{\rho\, e^{i(\theta - 2\pi)}, \infty > \rho \geq r\},$$

and

$$\Gamma_{r,\theta}^3 = \{r\, e^{it}, \theta - 2\pi \leq t \leq \theta\}$$

is a contour oriented clockwise along the ray L_θ around the non zero spectrum of A. The positive real number r is chosen small enough for the ball $B(0,r)$ centered at zero with radius r not to intersect the spectrum of A i.e., $B(0,r) \cap Sp(A) = \emptyset$.
The operator A_θ^z is a classical pseudodifferential operator of order az with homogeneous components of the symbol of A_θ^z given by

$$\sigma_{az-j}(A_\theta^z)(x,\xi) = \frac{i}{2\pi} \int_{\Gamma_{r,\theta}} \lambda_\theta^z \, b_{-a-j}(x,\xi,\lambda) \, d\lambda.$$

Note that the components b_{-a-j} are the positive homogeneous components of the resolvent $(A-\lambda I)^{-1}$ in $(\xi, \lambda^{\frac{1}{a}})$. In particular, its leading symbol is given by

$$\left(\sigma_{A^z_\theta}(x,\xi)\right)^L = \left((\sigma_A(x,\xi))^L\right)^z_\theta$$

and hence A^z_θ is elliptic.
The definition of complex powers can be extended to the whole complex plane by setting $A^z_\theta := A^k A^{z-k}_\theta$ for $k \in \mathbb{N}$ and $\mathrm{Re}(z) < k$; this definition is independent of the choice of k in $\mathbb{N}$ and preserves the usual properties, i.e. $A^{z_1}_\theta A^{z_2}_\theta = A^{z_1+z_2}_\theta$, $A^j_\theta = A^j$, for $j \in \mathbb{Z}$. In particular, for $j = 0$, we have $A^0_\theta = I$.

Remark 3.3. For a real number t, A and A^t_θ have spectral cuts θ and $t\theta$; for t close to one, $(A^t_\theta)^z_{t\theta} = (A^t_\theta)^z_\theta$ and hence, $(A^t_\theta)^z_{t\theta} = A^{tz}_\theta$ so that

$$\log_\theta(A^t) = \partial_z (A^t_\theta)^z_{t\theta|_{z=0}} = \partial_z (A^{tz}_\theta)_{|_{z=0}} = t \log_\theta A.$$

The complex powers of operators depend on the choice of spectral cut, namely we have

Proposition 3.4. *[32, 35, 26] Let θ and ϕ be two spectral cuts for an admissible operator A in $C\ell(M,E)$ such that $0 \le \theta < \phi < \theta + 2\pi$. The complex powers for these two spectral cuts are related by*

$$A^z_\theta - A^z_\phi = \left(1 - e^{2i\pi z}\right) \Pi_{\theta,\phi}(A) A^z_\theta, \tag{3.3}$$

where the sectorial projection of the operator A (see Section 3 in [26] and references therein) is defined by

$$\Pi_{\theta,\phi}(A) = A\left(\frac{1}{2i\pi}\int_{\Gamma_{r,\theta,\phi}} \lambda^{-1}(A-\lambda)^{-1}\, d\lambda\right)$$

where the contour

$$\Gamma_{r,\phi,\theta} = \Gamma^1_{r,\theta} \cup \{r\, e^{it}, \theta \le t \le \phi\} \cup \Gamma^1_{r,\phi}$$

(with $\Gamma^1_{r,\phi}$ defined as above replacing θ by ϕ) corresponds to the boundary of the set

$$\Lambda_{r,\theta,\phi} := \{\rho\, e^{it}, \infty > \rho \ge r, \quad \theta \le t \le \phi\}. \tag{3.4}$$

Remark 3.5. Formula (3.3) generalises to spectral cuts θ and ϕ such that $0 \le \theta < \phi + 2k\pi < \theta + (2k+1)\pi$ for some non negative integer k by

$$A^z_\theta - A^z_\phi = e^{2ik\pi z} I + \left(1 - e^{2i\pi z}\right) \Pi_{\theta,\phi}(A) A^z_\theta. \tag{3.5}$$

If the set $\Lambda_{r,\theta,\phi}$ defined by (3.4) delimited by the angles θ and ϕ does not intersect the spectrum of the leading symbol of A, it only contains a finite number of eigenvalues of A and $\Pi_{\theta,\phi}(A)$ is a finite rank projection and hence a smoothing operator. In general (see Propositions 3.1 and 3.2 in [26]), $\Pi_{\theta,\phi}(A)$, which is a pseudodifferential projection, is a zero order operator with leading symbol given by $\pi_{\theta,\phi}(\sigma^L(A))$ defined similarly to $\Pi_{\theta,\phi}$ replacing A by the leading symbol σ^L_A of A so that:

$$\sigma^L_{\Pi_{\theta,\phi}(A)} = \pi_{\theta,\phi}(\sigma^L_A) := \sigma^L_A \left(\frac{1}{2i\pi}\int_{\Gamma_{r,\theta,\phi}} \lambda^{-1}(\sigma^L_A - \lambda)^{-1}\, d\lambda\right),$$

where we have set $\sigma_B^L(x,\xi) = (\sigma_B(x,\xi))^L$ for any $(x,\xi) \in T^*M \setminus M \times \{0\}$ and any $B \in C\ell(M,E)$.

Let us now define the logarithm of an admissible operator A in $C\ell(M,E)$ of positive order a and with spectral cut θ. There are various ways of doing so, one of which is to first define $\log_\theta(A)A^{-1}$ as the bounded operator on any H^s-closure $H^s(M,E)$, $s \in \mathbb{R}$ of the space $C^\infty(M,E)$ of smooth sections of E (see e.g. [10]) by:

$$\log_\theta(A)A^{-1} = \frac{1}{2\pi i}\int_{\Gamma_{r,\theta}} \log_\theta(\lambda)\,\lambda^{-1}\,(\lambda - A)^{-1}\,d\lambda,$$

where the contour $\Gamma_{r,\theta}$ is defined in (3.1), and then to set:

$$\log_\theta(A) = \left(\log_\theta(A)A^{-1}\right)A.$$

Since the complex powers of the admissible operator A give rise to a holomorphic family $\left(A_\theta^z\right)_{\mathrm{Re}(z)<0}$ of bounded operators on $H^s(M,E)$, we have $\log_\theta A A^{-1} = \left(\partial_z A_\theta^z\right)_{z=-1}$, where the differentiation takes place in the algebra $\mathcal{B}(H^s(M,E))$ of bounded operators on $H^s(M,E)$. Consequently, for any u in the domain of A we have

$$\log_\theta A(u) = (\log_\theta(A)A^{-1}A)(u) = \left(\left(\partial_z A_\theta^z\right)_{z=-1} A\right)(u) = \left(\left(\partial_z A_\theta^z\right)_{z=0}\right)(u).$$

By construction, $\log_\theta A$ is a bounded linear operator from $H^s(M,E)$ to $H^{s-a}(M,E)$. Alternatively, for any positive ϵ, we observe that the map $z \mapsto A_\theta^{z-\epsilon}$ of order $a(z-\epsilon)$ defines a holomorphic function on the half plane $\mathrm{Re}(z) < \epsilon$ with values in $\mathcal{B}(H^s(M,E))$ for any real number s and we set:

$$\log_\theta A = A_\theta^\epsilon\left(\partial_z\left(A_\theta^{z-\epsilon}\right)\right)_{|z=0} = A_\theta^\epsilon\left(\partial_z\left(\frac{i}{2\pi}\int_{\Gamma_{r,\theta}} \lambda_\theta^{z-\epsilon}(A-\lambda)^{-1}\,d\lambda\right)\right)_{|z=0}. \tag{3.6}$$

For any positive ϵ the operator $\log_\theta(A)A^{-\epsilon} = A^{-\epsilon}\log_\theta(A)$ lies in $\mathcal{B}(H^s(M,E))$ for any real number s. It follows that $\log_\theta A$, which is clearly independent of the choice of $\epsilon > 0$, defines a bounded linear operator from $H^s(M,E)$ to $H^{s-a\epsilon}(M,E)$ for any positive ϵ.
The operator $\log_\theta A$ actually defines a pseudodifferential operator on $C^\infty(M,E)$, whose order is smaller than any positive number. It is not anymore classical, as we shall see below.

Just as complex powers, the logarithm depends on the choice of spectral cut [21]. Indeed, differentiating (3.3) with respect to z at $z = 0$ yields for spectral cuts θ,ϕ such that $0 \le \theta < \phi < 2\pi$ (compare with formula (1.4) in [21]):

$$\log_\theta A - \log_\phi A = -2i\pi\Pi_{\theta,\phi}(A). \tag{3.7}$$

Formula (3.7) generalises to spectral cuts θ and ϕ such that $0 \le \theta < \phi + 2k\pi < (2k+1)\pi$ for some non negative integer k by

$$\log_\theta A - \log_\phi A = 2ik\pi I - 2i\pi\Pi_{\theta,\phi}(A). \tag{3.8}$$

As a result of the above discussion and as already observed in [21], when the leading symbol σ_A^L has no eigenvalue inside the set $\Lambda_{r,\theta,\phi}$ delimited by $\Gamma_{r,\theta,\phi}$ then $\Pi_{\theta,\phi}$ which is a finite rank projection, is smoothing.

Logarithms of classical pseudodifferential operators are not classical since their symbols involve a logarithmic term $\log|\xi|$ as the following elementary result shows (see also Lemma 2.4 in [21]).

Proposition 3.6. *Let $A \in C\ell(M,E)$ be an admissible operator with spectral cut θ. In a local trivialisation, the symbol of $\log_\theta(A)$ reads:*

$$\sigma_{\log_\theta(A)}(x,\xi) = a \log|\xi| I + \sigma_0^A(x,\xi) \tag{3.9}$$

where a denotes the order of A and σ_0^A a symbol of order zero.
Moreover, the leading symbol of σ_0^A is given by

$$(\sigma_0^A)^L(x,\xi) = \log_\theta\left(\sigma_A^L\left(x, \frac{\xi}{|\xi|}\right)\right) \quad \forall (x,\xi) \in T^*M \setminus M \times \{0\}. \tag{3.10}$$

In particular, if σ_A has scalar leading symbol then so have σ_θ^A and $\sigma_{\Pi_{\theta,\phi}(A)}$ for any other spectral cut ϕ.

Proof. To simplify notation we drop the explicit mention of the spectral cut θ. Given a local trivialisation over some local chart, the symbol of A_θ^z has the formal expansion $\sigma_A^z \sim \sum_{j\geq 0} b_{az-j}^{(z)}$ where a is the order of A and $b_{az-j}^{(z)}$ is a positively homogeneous function of degree $az-j$. Since $\log_\theta A = A\left(\partial_z A_\theta^{z-1}\right)_{|_{z=0}}$, we have

$$\sigma_{\log_\theta A} \sim \sigma_A \star \sigma_{\left(\partial_z A_\theta^{z-1}\right)_{|_{z=0}}}.$$

Suppose that $\xi \neq 0$; using the positive homogeneity of the components, we have:

$$b_{az-a-j}^{(z-1)}(x,\xi) = |\xi|^{az-a-j} b_{az-a-j}^{(z-1)}\left(x, \frac{\xi}{|\xi|}\right)$$

and hence

$$\partial_z b_{az-a-j}^{(z-1)}(x,\xi) = a \log|\xi|\, |\xi|^{az-a-j} b_{az-a-j}^{(z-1)}\left(x, \frac{\xi}{|\xi|}\right) + |\xi|^{az-a-j} \partial_z b_{az-a-j}^{(z-1)}\left(x, \frac{\xi}{|\xi|}\right).$$

It follows that

$$\left(\partial_z b_{az-a-j}^{(z-1)}(x,\xi)\right)_{|_{z=0}} = a \log|\xi|\, b_{-a-j}^{(-1)}(x,\xi) + |\xi|^{-a-j}\left(\partial_z b_{az-a-j}^{(z-1)}\left(x, \frac{\xi}{|\xi|}\right)\right)_{|_{z=0}}.$$

Hence $\left(\partial_z A_\theta^{z-1}\right)_{|_{z=0}}$ has symbol $\left(\partial_z b^{(z-1)}(x,\xi)\right)_{|_{z=0}}$ of the form $a \log|\xi|\, \sigma_{A^{-1}}(x,\xi) + \tau_A(x,\xi)$ with τ_A a classical symbol of order $-a$ whose homogeneous component of degree $-a-j$ reads:

$$(\tau_A)_{-a-j}(x,\xi) = |\xi|^{-a-j}\left(\partial_z b_{az-a-j}^{(z-1)}\left(x, \frac{\xi}{|\xi|}\right)\right)_{|_{z=0}}.$$

Thus the operator $\log_\theta A = A\left(\partial_z A_\theta^{z-1}\right)_{|_{z=0}}$ has a symbol of the form $a \log|\xi| + \sigma_0(\log_\theta A)(x,\xi)$, where

$$\sigma_0(\log_\theta A)(x,\xi) \sim \sum_{k=0}^{\infty} \sum_{i+j+|\alpha|=k} \frac{(-i)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha (\sigma_A)_{a-i}\, \partial_x^\alpha (\tau_A)_{-a-j}(x,\xi)$$

is a classical symbol of order zero. Its leading symbol reads

$$\begin{aligned}
\sigma_0^L(\log_\theta A)(x,\xi) &= \sigma_A^L(x,\xi)\,|\xi|^{-a}\left(\partial_z b_{az-a}^{(z-1)}(x,\frac{\xi}{|\xi|}\right))_{|z=0} \\
&= \sigma_A^L(x,\xi)\left(\partial_z\left(\sigma_A^L\left(x,\frac{\xi}{|\xi|}\right)\right)_\theta^{z-1}\right)_{|z=0} \\
&= \log_\theta \sigma_A^L\left(x,\frac{\xi}{|\xi|}\right)
\end{aligned}$$

for any (x,ξ) in $T^*M \setminus M\times\{0\}$. □

This motivates the introduction of log-polyhomogeneous symbols (see e.g. [17]), to which the local noncommutative residue easily extends.

Definition 3.7. A symbol $\sigma \in S(U)$ is called log-polyhomogeneous of order a and type k for some non negative integer k if there is some smooth function χ on $\mathbb{R}^n$ which vanishes around zero and is identically one outside the unit ball, such that

$$\sigma(x,\xi) \sim \sum_{j=0}^{\infty} \chi(\xi)\,(\sigma(x,\xi))_{a-j}$$

where for any non negative integer j,

$$(\sigma_A(x,\xi))_{a-j} = \sum_{l=0}^{k} (\sigma_A(x,\xi))_{a-j,l}(x,\xi)\,\log^l|\xi| \quad \forall (x,\xi)\in T^*U,$$

with $\sigma_{a-j,l}$, $l=0,\cdots,k$ are positively homogeneous of degree $a-j$.
The local noncommutative residue at a point x in U defined in (2.5) extends to log-polyhomogeneous symbols by:

$$\mathrm{res}_x(\sigma) := \int_{S_x^*U} (\sigma(x,\xi))_{-n}\, đ_S\xi. \tag{3.11}$$

Powers of the logarithm of a given admissible operator combined with all classical pseudodifferential operators generate the algebra of log-polyhomogenous operators [17].

A log-polyhomogenous operator A of type k is a pseudodifferential operator whose local symbol $\sigma_A(x,\xi)$ in any local trivialisation asymptotically is log-polyhomogeneous of type k.
Let us denote the set of such operators by $C\ell^{a,k}(M,E)$ and its union over all non negative integers k by $C\ell^{a,*}(M,E) = \cup_{k=0}^{\infty} C\ell^{a,k}(M,E)$. In particular, a classical pseudodifferential operator is a log-polyhomogeneous operator of log-type 0 and $C\ell^{a,0}(M,E) = C\ell^a(M,E)$. The product of a log-polyhomogeneous operator of type k and a log-polyhomogeneous operator of type l is log-polyhomogeneous operator of type $k+l$ so that, following [17], we can build the algebra $C\ell^{*,*}(M,E) = \langle \cup_{a\in\mathbb{C},k\in\mathbb{Z}_+} C\ell^{a,k}(M,E)\rangle$ generated by all log-polyhomogeneous operators.
For an operator A in $C\ell^{*,*}(M,E)$, one can define the local noncommutative residue at a point x in M similarly to the case of classical operators by:

$$\mathrm{res}_x(A) := \int_{S_x^*M} \mathrm{tr}_x\,(\sigma_A(x,\xi))_{-n}\, đ_S\xi.$$

However, unlike Lesch's extended noncommutative residue on log-polyhomogeneous operators [17], the locally defined residue density $\mathrm{res}_x(A)\,dx$ is not expected to patch up to a globally defined residue density.
However, it does for logarithms of any admissible operator A in $\mathrm{C}\ell(M,E)$ and we have [29]:

$$\mathrm{res}(\log A) = -a\,\zeta_A(0) \tag{3.12}$$

where $\zeta_A(0)$ is the constant term in the Laurent expansion of the unique meromorphic extension $\zeta_A(s)$ of the map $s \mapsto \mathrm{Tr}(A^{-s})$ given by the L^2-trace of A^{-s} defined on the domain of holomorphicity $\mathrm{Re}(as) > n$ [5].
In [29], Scott showed the multiplicativity of the associated residue determinant

$$\det_{\mathrm{res}}(A) := e^{\mathrm{res}(\log A)}.$$

He actually proved a stronger statement, namely that given two admissible operators A,B such that their product AB is also admissible, the following expression

$$L(A,B) := \log(AB) - \log A - \log B$$

has vanishing noncommutative residue.

Remark 3.8. Strictly speaking, one should specify the spectral cuts θ of A, ϕ of B and ψ of AB in the expression $L(A,B)$ setting instead

$$L^{\theta,\phi,\psi}(A,B) := \log_\psi(AB) - \log_\theta A - \log_\phi B.$$

Then by (3.7)

$$L^{\theta,\phi,\psi}(A,B) - L^{\theta',\phi',\psi'}(A,B) = -2i\pi\left(\Pi_{\psi,\psi'}(AB) - \Pi_{\theta,\theta'}(A) - \Pi_{\phi,\phi'}(B)\right).$$

Since the noncommutative residue vanishes on pseudodifferential projections by a result of Wodzicki ([32]; see also [2]), it follows that

$$\mathrm{res}(L(A,B)) = 0. \tag{3.13}$$

Up to a modification of the operators A and B, one can actually choose fixed spectral cuts θ and ϕ by the following argument of Okikiolu [21]:

$$L^{\theta,\phi,\psi}(A,B) = L^{\pi,\pi,\psi-(\theta+\phi)}(e^{i(\pi-\theta)}A, e^{i(\pi-\phi)}B).$$

Indeed, if A,B,AB have spectral cut θ,ϕ,ψ respectively, then $A' = e^{i(\pi-\theta)}A$ and $B' = e^{i(\pi-\phi)}B$ have spectral cut π and $A'B'$ has spectral cut $\psi+2\pi-\theta-\phi$. So we can assume that $\theta=\phi=\pi$ without loss of generality.
Keeping in mind these observations, in order to simplify notations we assume that A and B have spectral cuts π and drop the explicit mention of the spectral cuts.

It follows from (2.8) that

$$L(A,B) \in [\mathrm{C}\ell(M,E), \mathrm{C}\ell(M,E)], \tag{3.14}$$

so that $L(A,B)$ is a finite sum of commutators. The following proposition provides a refinement this statement.

[5] This actually is an instance in the case $A(z) = A^{-z}$ of the more general defect formula (4.6) derived in [24].

Proposition 3.9. *Let A, B be two admissible operators, which w.l.o.g. are assumed to have π as spectral cut (see Remark 3.8), such that their product AB is also admissible with spectral cut π. Then $L(A,B)$ is a finite sum of Lie brackets of operators in $C\ell^0(M,E)$:*

$$L(A,B) \in [C\ell^0(M,E), C\ell^0(M,E)].$$

Proof. Let us check that $L(A,B)$ lies in $C\ell^0(M,E)$. If has order a and B has order b then AB has order $a+b$, we have

$$\begin{aligned}
\sigma_{L(A,B)} &= \sigma_{\log AB}(x,\xi) - \sigma_{\log A}(x,\xi) - \sigma_{\log B}(x,\xi) \\
&= (a+b)\log|\xi| I + \sigma_0^{AB}(x,\xi) - a\log|\xi| I - \sigma_0^A(x,\xi) - b\log|\xi| I - \sigma_0^B(x,\xi) \\
&\sim \sigma_0^{AB}(x,\xi) - \sigma_0^A(x,\xi) - \sigma_0^B(x,\xi) \qquad (3.15)
\end{aligned}$$

so that the operator $L(A,B)$ is indeed classical of order 0 and by (3.10) it has leading symbol given for any (x,ξ) in $T^*M \setminus M \times \{0\}$ by

$$(\sigma_{L(A,B)}(x,\xi))_0 = \log\sigma^L_{AB}\left(x, \frac{\xi}{|\xi|}\right) - \log\sigma^L_A\left(x, \frac{\xi}{|\xi|}\right) - \log\sigma^L_B\left(x, \frac{\xi}{|\xi|}\right) =: L\left(\sigma^L_A, \sigma^L_B\right)\left(x, \frac{\xi}{|\xi|}\right).$$

Here as before, σ^L_C stands for the leading symbol of the operator C.
Applying the usual Campbell-Hausdorff formula to the matrices $\sigma^L_A\left(x, \frac{\xi}{|\xi|}\right)$ and $\sigma^L_B\left(x, \frac{\xi}{|\xi|}\right)$ and implementing the fibrewise trace tr_x yields:

$$\mathrm{tr}_x\left(\log\sigma^L_{AB}\left(x, \frac{\xi}{|\xi|}\right) - \log\sigma^L_A\left(x, \frac{\xi}{|\xi|}\right) - \log\sigma^L_B\left(x, \frac{\xi}{|\xi|}\right)\right) = \mathrm{tr}_x\left(L\left(\sigma^L_A, \sigma^L_B\right)\left(x, \frac{\xi}{|\xi|}\right)\right) = 0.$$

It follows that any leading symbol trace $\mathrm{Tr}_0^\Lambda(C) := \Lambda\left((\mathrm{tr}_x(\sigma_C))_0\right)$ (see e.g. [25]) on the algebra $C\ell^0(M,E)$, where Λ is a linear form on $C^\infty(S^*M)$ and the index 0 stands for the positively homogeneous component of degree 0, vanishes on $L(A,B)$:

$$\mathrm{Tr}^\Lambda(L(A,B)) = \Lambda\left(\mathrm{tr}_x(\sigma_{L(A,B)})_0\right) = 0.$$

Thus both the noncommutative residue and leading symbol traces vanish on $L(A,B)$. By the results of [18] (see the proof of Theorem 4 formula (16)), in dimension larger than one the zeroth Hochschild homology

$$HH_0(C\ell^0(M,E)) = C\ell^0(M,E)/[C\ell^0(M,E), C\ell^0(M,E)]$$

of $C\ell^0(M,E)$ is isomorphic to $\mathbb{C} \oplus C^\infty(S^*M)$ via the map $A \mapsto \left(\mathrm{res}(A), \sigma_0^A\right)$. An alternative proof was given in [27] Corollary 5.4. Hence, any operator in $C\ell^0(M,E)$ with vanishing residue and leading symbol traces lies in $[C\ell^0(M,E), C\ell^0(M,E)]$. It follows that $L(A,B)$ lies in $[C\ell^0(M,E), C\ell^0(M,E)]$. □

4 Properties of weighted traces

Since traces on $C\ell(M,E)$ are proportional to the noncommutative residue which vanishes on smoothing operators, the L^2-trace on smoothing operators does not extend to the whole

algebra $C\ell(M,E)$. Instead we use linear extensions which we call weighted traces, of the ordinary L^2-trace on smoothing operators to the whole algebra $C\ell(M,E)$. We review basic properties of weighted traces and prove (see Proposition 4.8) that the canonical and weighted traces as well as the noncommutative residue commute with differentiation on differentiable families of operators with constant order. Weighted traces arise as finite parts of canonical traces of holomorphic families of classical pseudodifferential operators.
Let us recall the notion of holomorphic family of classical pseudodifferential symbols taken from [24]. It leads to the same notion of (weak) holomorphic family of classical pseudodifferential operators (see Definition 4.3) as the one defined in [15] by means of their kernels (see also [12] for the related notion of gauged distributions).

Definition 4.1. Let Ω be a domain of $\mathbb{C}$ and U an open subset of $\mathbb{R}^n$. A family $(\sigma(z))_{z\in\Omega}$ is a holomorphic family of $\mathrm{End}(V)$-valued classical symbols on U parametrised by Ω when

1. the map $z \mapsto \alpha(z)$ with $\alpha(z)$ the order of $\sigma(z)$, is holomorphic in z,
2. $z \mapsto \sigma(z)$ is holomorphic as element of $C^\infty(U\times\mathbb{R}^n)\hat{\otimes}\mathrm{End}(V)$ (here $\hat{\otimes}$ denotes the Grothendieck completion) and for each $z\in\Omega$, $\sigma(z)\sim\sum_{j=0}^\infty \chi\,\sigma(z)_{\alpha(z)-j}$ (for some smooth function χ which is identically one outside the unit ball and vanishes in a neighborhood of 0) lies in $CS^{\alpha(z)}(U)\otimes\mathrm{End}(V)$,
3. for any positive integer N, the remainder term $\sigma_{(N)}(z)=\sigma(z)-\sum_{j=0}^{N-1}\sigma(z)_{\alpha(z)-j}$ is holomorphic in $z\in\Omega$ as an element of $C^\infty(U\times\mathbb{R}^n)\otimes\mathrm{End}(V)$ and its k-th derivative
$$(x,\xi)\mapsto \partial_z^k\sigma_{(N)}(z)(x,\xi):=\partial_z^k\left(\sigma_{(N)}(z)(x,\xi)\right)$$
lies in $S^{\alpha(z)-N+\epsilon}(U)\otimes\mathrm{End}(V)$ for all $\epsilon>0$ locally uniformly in z, i.e the k-th derivative $\partial_z^k\sigma_{(N)}(z)$ satisfies a uniform estimate (2.1) in z on compact subsets in Ω.

In particular, for any integer $j\geq 0$, the (positively) homogeneous component $\sigma_{\alpha(z)-j}(z)$ of degree $\alpha(z)-j$ of the symbol is holomorphic on Ω as an element of $C^\infty(U\times\mathbb{R}^n)\otimes\mathrm{End}(V)$.

It is important to observe that the derivative of a holomorphic family $\sigma(z)$ of classical symbols is not classical anymore since it yields a holomorphic family of symbols $\sigma'(z)$ of order $\alpha(z)$, the asymptotic expansion of which involves a logarithmic term and reads [24]:

$$\sigma'(z)(x,\xi)\sim\sum_{j=0}^\infty\chi(\xi)\left(\log|\xi|\,\sigma'_{\alpha(z)-j,1}(z)(x,\xi)+\sigma'_{\alpha(z)-j,0}(z)(x,\xi)\right)\quad\forall(x,\xi)\in T^*U\setminus U\times\{0\}$$

for some smooth cut-off function χ around the origin which is identically equal to 1 outside the open unit ball and positively homogeneous symbols

$$\sigma'_{\alpha(z)-j,0}(z)(x,\xi)=|\xi|^{\alpha(z)-j}\,\partial_z\left(\sigma_{\alpha(z)-j}(z)\left(x,\frac{\xi}{|\xi|}\right)\right),$$

$$\sigma'_{\alpha(z)-j,1}(z)(x,\xi)=\alpha'(z)\,\sigma_{\alpha(z)-j}(z)(x,\xi)$$

of degree $\alpha(z)-j$.

The regularised cut-off integral on symbols we are about to introduce is an essential

ingredient to build linear extensions of the L^2-trace.
The integral $\int_{B_x(0,R)} \mathrm{tr}(\sigma(x,\xi)) d\!\!{}^-\xi$ of the trace of a symbol $\sigma \in CS(U)\otimes \mathrm{End}(V)$ over the ball $B_x(0,R)$ of radius R centered at 0 in the cotangent space T_x^*U at a point $x \in U$, has an asymptotic expansion in decreasing powers of R which is polynomial in $\log R$ so that the cut-off integral which corresponds to the constant term in this expansion

$$-\!\!\!\!\!\!\int_{T_x^*U} \mathrm{tr}\,(\sigma(x,\xi))\, d\!\!{}^-\xi := \mathrm{fp}_{R\to\infty} \int_{B_x(0,R)} \mathrm{tr}\,(\sigma(x,\xi))\, d\!\!{}^-\xi$$

is well defined. It coincides with the ordinary integral whenever the latter converges.
We now recall the properties of cut-off integrals of holomorphic families of symbols.

Proposition 4.2. *1. [15] The cut-off regularised integral $-\!\!\!\!\!\!\int_{T_x^*U} \mathrm{tr}_x\,(\sigma(z)(x,\xi))\, d\!\!{}^-\xi$ of a holomorphic family $\sigma(z)$ of classical pseudodifferential symbols on a neighborhood $U \subset M$ of holomorphic order $\alpha(z)$ is a meromorphic function in z with simple poles. The residue at a pole z_0 for which $\alpha'(z_0) \neq 0$ is given by:*

$$\mathrm{Res}_{z=z_0} -\!\!\!\!\!\!\int_{T_x^*U} \mathrm{tr}_x\,(\sigma(z)(x,\xi))\, d\!\!{}^-\xi = -\frac{1}{\alpha'(z_0)} \mathrm{res}(\sigma(z_0)). \tag{4.1}$$

2. [24] Furthermore, if its holomorphic order is affine and non constant [6], its Laurent expansion has constant term at z_0 given by

$$\mathrm{fp}_{z=z_0} -\!\!\!\!\!\!\int_{T_x^*U} \mathrm{tr}_x\,(\sigma(z)(x,\xi))\, d\!\!{}^-\xi = -\!\!\!\!\!\!\int_{T_x^*U} \mathrm{tr}_x\,(\sigma(z_0)(x,\xi))\, d\!\!{}^-\xi - \frac{1}{\alpha'(z_0)} \mathrm{res}_x\,(\sigma'(z_0)). \tag{4.2}$$

Here we use the local residue extended to log-polyhomogeneous symbols (see (3.11)) since the derivative $\sigma'(z_0)$ of a holomorphic family of classical symbols $\sigma(z)$ with order $\alpha(z)$ at a point z_0 is expected to be logarithmic with the same order.

Let us now carry out these constructions to the operator level.
For any $A \in C\ell(M,E)$, for any $x \in M$, the following expression defines a local density:

$$\omega_{KV}(A)(x) := \mathrm{TR}_x(A)\, dx := \left(-\!\!\!\!\!\!\int_{T_x^*M} \mathrm{tr}_x\,(\sigma_A(x,\xi))\, d\!\!{}^-\xi \right) dx. \tag{4.3}$$

It patches up to a global density on M whenever the operator A in $C\ell(M,E)$ has non integer order or has order $< -n$ so that the Kontsevich-Vishik canonical trace [15] (see also [17]):

$$\mathrm{TR}(A) := \int_M \omega_{KV}(A)(x) := \int_M \mathrm{TR}_x(A)\, dx \tag{4.4}$$

makes sense.
The canonical trace can be applied to holomorphic families of classical pseudodifferential operators with varying complex order.

Definition 4.3. Let $(A(z))_{z\in\Omega}$ be a family of classical pseudodifferential operators in $C\ell(M,E)$ with distribution kernels $(x,y) \mapsto K_{A(z)}(x,y)$. The family is holomorphic if

[6] Here and in what follows, we assume that the order of the holomorphic family is affine and non constant so that applying the fibrewise trace, formula (1.50) of [24] boils down to the following one.

1. the order $\alpha(z)$ of $A(z)$ is holomorphic in z,

2. in any local trivialisation of E, we can write $A(z)$ in the form $A(z) = Op(\sigma_z) + R(z)$, for some holomorphic family of $\mathrm{End}(V)$-valued symbols $(\sigma(z))_{z\in\Omega}$ where V is the model space of the fibres of E, and some holomorphic family $(R(z))_{z\in\Omega}$ of smoothing operators i.e. given by a holomorphic family of smooth Schwartz kernels,

3. the (smooth) restrictions of the distribution kernels $K_{A(z)}$ to the complement of the diagonal $\Delta \subset M\times M$, form a holomorphic family with respect to the topology given by the uniform convergence in all derivatives on compact subsets of $M\times M - \Delta$.

Example 4.4. Given an admissible operator $A \in C\ell(M,E)$ with spectral cut θ, a family $z \mapsto A_\theta^{-\frac{z}{1+\mu z}}$ for $\mu \in \mathbb{R}$ is a holomorphic family of ψDOs. In particular, $A(z) = A_\theta^{-z}$ is a holomorphic family. Note that the derivatives $A'(z) = -\log_\theta A\, A_\theta^{-z}$ are not classical operators.

Integrating the formulae in Proposition 4.2 along the manifold M yields the following result on the level of operators.

Proposition 4.5. *1. [15] The canonical trace* $\mathrm{TR}(A(z))$ *of a holomorphic family* $A(z)$ *of classical pseudodifferential operators in* $C\ell(M,E)$ *of holomorphic order* $\alpha(z)$ *is a meromorphic function in* z *with simple poles and residue at a pole* z_0 *for which* $\alpha'(z_0) \neq 0$ *is given by:*

$$\mathrm{Res}_{z=z_0}\mathrm{TR}(A(z)) = -\frac{1}{\alpha'(z_0)}\mathrm{res}(A(z_0)). \tag{4.5}$$

2. [24] Furthermore, if its holomorphic order is affine and non constant, its Laurent expansion has constant term at z_0 *given by*

$$\mathrm{fp}_{z=z_0}\mathrm{TR}(A(z)) = \int_M dx\left(\mathrm{TR}_x(A(z_0)) - \frac{1}{\alpha'(z_0)}\mathrm{res}_x(A'(z_0))\right). \tag{4.6}$$

Applying these results to a holomorphic family $A(z) := A\,Q^{-z}$ where A is any operator in $C\ell(M,E)$ and Q an admissible operator in $C\ell(M,E)$ with positive order and spectral cut [7] α, we infer (see e.g. [15], [23], [3] and references therein) that the map $z \mapsto \mathrm{TR}\left(A\,Q_\alpha^{-z}\right)$ is meromorphic with simple poles.

Definition 4.6. Given an admissible operator Q with positive order, which we call a weight, the Q-weighted trace of an operator A in $C\ell(M,E)$ is given by:

$$\mathrm{tr}_\alpha^Q(A) := \mathrm{fp}_{z=0}\mathrm{TR}\left(A\,Q_\alpha^{-z}\right) := \lim_{z\to 0}\left(\mathrm{TR}\left(A\,Q_\alpha^{-z}\right) - \mathrm{Res}_{z=0}\left(\frac{\mathrm{TR}\left(A\,Q_\alpha^{-z}\right)}{z}\right)\right),$$

where α is a spectral cut for Q.

Applying (4.6) to the family $A(z) = A\,Q_\alpha^{-z}$ yields the following "defect formula" [24]:

$$\mathrm{tr}_\alpha^Q(A) = \int_M\left(\mathrm{TR}_x(A) - \frac{\mathrm{res}_x\left(A\log_\alpha Q\right)}{q}\right)dx, \tag{4.7}$$

[7] For further use, we need to distinguish the spectral cut α of Q from the spectral cut θ of A.

where q stands for the order of Q. In particular, for $A = I$ we get back (3.12):

$$\zeta_{Q,\alpha}(0) := \mathrm{fp}_{z=0}\mathrm{TR}(Q_\alpha^{-z}) = -\frac{\mathrm{res}\log_\alpha Q}{q}.$$

Whereas weighted traces are not expected to be local in general since they involve the whole symbol of the operator, the difference of two weighted traces is local in so far as it involves a finite number of homogeneous components of the symbol via the noncommutative residue. Weighted traces depend on the choice of weight and are not cyclic in spite of their name.

Proposition 4.7. *([3], [19])*

1. *Given two weights Q_1 and Q_2 with common spectral cut α and positive orders q_1, q_2 we have*
$$\mathrm{tr}_\alpha^{Q_1}(A) - \mathrm{tr}_\alpha^{Q_2}(A) = \mathrm{res}\left(A\left(\frac{\log_\alpha Q_2}{q_2} - \frac{\log_\alpha Q_1}{q_1}\right)\right), \tag{4.8}$$
which is a local expression.

2. *For any weight Q in $C\ell(M,E)$ with order q and spectral cut α, the operators $[A, \log_\alpha Q]$ and $[B, \log_\alpha Q]$ lie in $C\ell(M,E)$ and*
$$\mathrm{tr}_\alpha^Q([A,B]) = -\frac{1}{q}\mathrm{res}(A[B,\log_\alpha Q]) = \frac{1}{q}\mathrm{res}(B[A,\log_\alpha Q]). \tag{4.9}$$
In particular, if $Q = A$ or $Q = B$, or if the sum of the orders of A and B has real part $< -n$, then $\mathrm{tr}_\alpha^Q([A,B]) = 0$.

The following technical proposition shows that the canonical and weighted traces as well as the noncommutative residue commute with differentiation on families of operators of constant order, a fact that we will use to derive the multiplicative anomaly of determinants. Differentiable families of symbols and operators are defined in the same way as were holomorphic families in Definitions 4.1 and 4.3 replacing "holomorphic in the parameter z" by "differentiable in the parameter t".

Proposition 4.8. *Let A_t be a differentiable family of $C\ell(M,E)$ of constant order a.*

1. *The noncommutative residue commutes with differentiation*
$$\frac{d}{dt}\mathrm{res}(A_t) = \mathrm{res}(\dot{A}_t), \tag{4.10}$$
where we have set $\dot{A}_t = \frac{d}{dt}A_t$.

2. *If the order a is non integer, the canonical trace commutes with differentiation*
$$\frac{d}{dt}\mathrm{TR}(A_t) = \mathrm{TR}(\dot{A}_t). \tag{4.11}$$

3. *For any weight Q with order q and spectral cut α,*

$$\frac{d}{dt}\mathrm{tr}_\alpha^Q(A_t) = \mathrm{tr}_\alpha^Q(\dot{A}_t). \tag{4.12}$$

Proof. Using (2.4) we write the symbol σ_{A_t} of A_t as follows:

$$\sigma_{A_t}(x,\xi) = \sum_{j=0}^{N-1} \chi(\xi)\,(\sigma_{A_t})_{a-j}(x,\xi) + (\sigma_{A_t})_{(N)}(x,\xi).$$

1. By assumption, the map $t \mapsto \mathrm{tr}_x\left((\sigma_{A_t}(x,\xi))_{-n}\right)$ is differentiable leading to a differentiable map $t \mapsto \int_{S_x^*M} \mathrm{tr}_x\left((\sigma_{A_t}(x,\xi))_{-n}\right) đ_S\xi$ after integration over the compact set S_x^*M with derivative: $t \mapsto \int_{S_x^*M} \mathrm{tr}_x(\dot{\sigma}_{A_t})_{-n}\, đ_S\xi$, where $\dot{\sigma}_{A_t} = \sigma_{\dot{A}_t}$ stands for the derivative of σ_{A_t} at t. Thus, the map $t \mapsto \mathrm{res}(A_t)$ is differentiable with derivative given by (4.10).

2. By (4.3) and (4.4), to prove formula (4.11) we need to check the differentiability of the map $t \mapsto ⨍_{T_x^*M} \mathrm{tr}_x\sigma_{A_t}(x,\xi)\, đ\xi$ and to show that

$$\frac{d}{dt}⨍_{T_x^*M} \mathrm{tr}_x\sigma_{A_t}(x,\xi)\, đ\xi = ⨍_{T_x^*M} \mathrm{tr}_x\dot{\sigma}_{A_t}(x,\xi)\, đ\xi.$$

   The cut-off integral involves the whole symbol which we denote by $\sigma_t := \sigma_{A_t}$ in order to simplify notations. Since the family σ_t has constant order, N can be chosen independently of t in the asymptotic expansion. The corresponding cut-off integral can be computed explicitly (see e.g [24]):

$$\begin{aligned} ⨍_{T_x^*M} \mathrm{tr}_x(\sigma_t(x,\xi))\, đ\xi &= \int_{T_x^*M} \mathrm{tr}_x\left((\sigma_t)_{(N)}(x,\xi)\right) đ\xi \\ &+ \sum_{j=0}^{N-1} \int_{|\xi|\le 1} \chi(\xi)\, \mathrm{tr}_x\left((\sigma_t)_{a-j}(x,\xi)\right) đ\xi \\ &- \sum_{j=0, a-j+n\neq 0}^{N-1} \frac{1}{a-j+n} \int_{|\omega|=1} \mathrm{tr}_x\left((\sigma_t)_{a-j}(x,\omega)\right) đ_S\omega. \end{aligned}$$

   The map $t \mapsto \int_{T_x^*M} \mathrm{tr}_x\left((\sigma_t)_{(N)}(x,\xi)\right) đ\xi$ is differentiable at any point t_0 since by assumption the maps $t \mapsto \mathrm{tr}_x\left((\sigma_t(x,\xi))_{(N)}\right)$ are differentiable with modulus bounded from above $\left|\mathrm{tr}_x\left((\dot{\sigma}_t(x,\xi))_{(N)}\right)\right| \le C|\xi|^{\mathrm{Re}(a)-N}$ by an L^1 function provided N is chosen large enough, where the constant C can be chosen independently of t in a compact neighborhood of t_0. Its derivative is given by $t \mapsto \int_{T_x^*M} \mathrm{tr}_x\left((\dot{\sigma}_t)_{(N)}(x,\xi)\right) đ\xi$. The remaining integrals

$$\int_{|\xi|\le 1} \chi(\xi)\, \mathrm{tr}_x\left((\sigma_t)_{a-j}(x,\xi)\right) đ\xi \quad \text{and} \quad \int_{|\omega|=1} \mathrm{tr}_x\left((\sigma_t)_{a-j}(x,\omega)\right) đ_S\omega$$

are also differentiable as integrals over compact sets of integrands involving differentiable maps $t \mapsto \mathrm{tr}_x\left((\sigma_t(x,\xi))_{a-j}\right)$. Their derivatives are given by

$$\int_{|\xi|\leq 1} \chi(\xi)\,\mathrm{tr}_x\left((\dot{\sigma}_t)_{a-j}(x,\xi)\right) đ\xi \quad \text{and} \quad \int_{|\omega|=1} \mathrm{tr}_x\left((\dot{\sigma}_t)_{a-j}(x,\omega)\right) đ_S\,\omega.$$

Thus, the map $t \mapsto \int\!\!\!\!-_{T_x^*M} \mathrm{tr}_x(\sigma_{A_t}(x,\xi))\,đ\xi$ is differentiable with derivative given by $\int\!\!\!\!-_{T_x^*M} \mathrm{tr}_x(\dot{\sigma}_{A_t}(x,\xi))\,đ\xi$.

3. By the defect formula (4.7) we have

$$\mathrm{tr}_\alpha^Q(A_t) = \int_M dx\left(\int\!\!\!\!\!\!-_{T_x^*M} \mathrm{tr}_x\sigma_{A_t}(x,\xi)\,đ\xi - \frac{1}{q}\int_{S_x^*M} \mathrm{tr}_x\left(\sigma_{A_t\log_\alpha Q}(x,\xi)\right)_{-n} đ_S\,\xi\right)$$

which reduces the proof of the differentiability of $t \mapsto \mathrm{tr}_\alpha^Q(A_t)$ to that of the two maps

$$t \mapsto \int\!\!\!\!\!\!-_{T_x^*M} \mathrm{tr}_x\sigma_{A_t}(x,\xi),đ\xi \quad \text{and} \quad t \mapsto \int_{S_x^*M} \mathrm{tr}_x\left(\sigma_{A_t\log_\alpha Q}(x,\xi)\right)_{-n} đ_S\,\xi.$$

Differentiability of the first map was shown in the second item of the proof. Let us first investigate the second map. By (2.2) we have

$$\left(\sigma_{A_t\log_\alpha Q}\right)_{-n} = \sum_{|\alpha|+a-j-k=-n} \frac{(-i)^{|\alpha|}}{\alpha!}\partial_\xi^\alpha\,(\sigma_{A_t})_{a-j}\,\partial_x^\alpha\left(\sigma_{\log_\alpha Q}\right)_{-k}.$$

By assumption, the maps $t \mapsto (\sigma_{A_t})_{a-j}$ are differentiable so that the map

$$t \mapsto \int_{S_x^*M} \mathrm{tr}_x\left(\sigma_{A_t\log_\alpha Q}\right)_{-n} đ_S\,\xi$$

is differentiable with derivative

$$t \mapsto \int_{S_x^*M} \mathrm{tr}_x\left(\dot{\sigma}_{A_t\log_\alpha Q}\right)_{-n}(x,\xi)\,đ_S\,\xi = \int_{S_x^*M} \mathrm{tr}_x\left(\sigma_{\dot{A}_t\log_\alpha Q}\right)_{-n}(x,\xi)\,đ_S\,\xi.$$

Integrating over the compact manifold M then yields the differentiability of the map $t \mapsto \mathrm{tr}_\alpha^Q(A_t)$ with derivative given by

$$\int_M dx\left(\int\!\!\!\!\!\!-_{T_x^*M} \mathrm{tr}_x\sigma_{\dot{A}_t}(x,\xi)\,đ\xi - \frac{1}{q}\int_{S_x^*M} \mathrm{tr}_x\left(\sigma_{\dot{A}_t\log_\alpha Q}(x,\xi)\right)\right) đ_S\,\xi = \mathrm{tr}_\alpha^Q(\dot{A}_t).$$

□

5 Locality of weighted traces of $L(A,B)$

Combining (3.14) with (4.9) yields the locality of weighted traces $\mathrm{tr}^Q(L(A,B))$ as a finite sum of noncommutative residues, independently of the choice of spectral cut.

In this section we show that weighted traces of $L(A,B)$ only depend on a finite number

of homogeneous components of the operators A and B (see Theorem 5.3), a fact reminiscent of a similar property observed by Okikiolu in [21] in the case of operators with scalar leading symbols.
We know from the results of Okikiolu [21], that the resolvents $(A_t - \lambda)^{-1}$ form a differentiable family of operators with parameters. The following lemma which tells us that the logarithms $\log_\alpha A_t$ built from these resolvents, form a differentiable family therefore comes as no surprise.

Lemma 5.1. *Let A_t be a differentiable family of admissible operators in $\mathrm{C}\ell(M,E)$ with constant spectral cut θ.*
Their logarithms $\log_\theta A_t$ form a differentiable family of pseudodifferential operators and for any positive integer K we have

$$\frac{d}{dt}\log_\theta A_t = \sum_{k=0}^{K} \frac{(-1)^k}{k+1} \mathrm{ad}^k_{A_t}(\dot{A}_t) A_t^{-(k+1)} + R_K(A_t, \dot{A}_t) \tag{5.1}$$

where we have set $\dot{A}_t := \frac{d}{dt}A_t$ and

$$R_K(A_t,\dot{A}_t) := -\frac{i}{2\pi}\int_{\Gamma_\theta} \log_\theta\lambda \left[(\lambda - A_t)^{-1}, \mathrm{ad}^K_{A_t}(\dot{A}_t)\right](\lambda - A_t)^{-K-1}\, d\lambda. \tag{5.2}$$

Here Γ_θ is a contour around the spectrum as in (3.2).

Remark 5.2. If A_t commutes with $\dot{A}_t$ then $\frac{d}{dt}\log_\theta A_t = \dot{A}_t A_t^{-1}$.
If A_t has scalar leading symbol and order a_t, then for fixed t, the operators $\mathrm{ad}^k_{A_t}(\dot{A}_t)A_t^{-(k+1)}$ have decreasing order $\mathrm{ord}(\dot{A}_t) - a_t - k$ as k grows so that (5.1) yields an asymptotic expansion in operators of decreasing order.

Proof. Since the spectral cut is constant, we drop it in the notation setting $\log A_t = \log_\theta A_t$. By (4.3) in [21] we first observe that for any t in a compact neighborhood K_{t_0} of some real number t_0, one can bound the modulus of the order $\alpha(t)$ from above by some integer k, in which case

$$\exists C > 0, \quad \forall t \in K_{t_0}, \quad \|(A_t - \lambda)^{-1}\|_{s,s-k} \leq |\lambda^{-1}|,$$

where $\|\cdot\|_{s,s'}$ stands for the operator norm of bounded operators form the Sobolev closure $H^s(M,E)$ to the Sobolev closure $H^{s'}(M,E)$ of $C^\infty(M,E)$. Moreover, $(A_t - \lambda)^{-1}$ is differentiable at t_0 with derivative given by:

$$\frac{d}{dt}_{|t=t_0}(A_t - \lambda)^{-1} = -(A_{t_0} - \lambda)^{-1}\,\dot{A}_{t_0}\,(A_{t_0} - \lambda)^{-1}$$

where we have set $\dot{A}_{t_0} = \frac{d}{dt}_{|t=t_0} A_t$. This follows from the identity

$$(\lambda - A_t)^{-1} - (\lambda - A_{t_0})^{-1} = -(t - t_0)\,(\lambda - A_{t_0})^{-1}\,\Delta_t\,(\lambda - A_t)^{-1},$$

where we have set $\Delta_t := \frac{A_t - A_{t_0}}{t - t_0}$.
For operators A_t of zero order this leads to

$$\frac{d}{dt}_{|t=t_0} \log A_t = \frac{i}{2\pi}\int_{\Gamma_\theta} \log_\theta\lambda\, \frac{d}{dt}_{|t=t_0}(A_t - \lambda)^{-1}\, d\lambda = -\frac{i}{2\pi}\int_{\Gamma_\theta} \log_\theta\lambda\,(A_{t_0} - \lambda)^{-1}\,\dot{A}_{t_0}\,(A_{t_0} - \lambda)^{-1}\, d\lambda.$$

In order to generalise this to higher order operators, we need to consider the family (see (3.6)):

$$\log A_t A_t^{-1} = \frac{i}{2\pi}\int_{\Gamma_\theta} \log_\theta \lambda\, \lambda^{-1} (A_t-\lambda)^{-1}\, d\lambda$$

for which we can also write:

$$\begin{aligned}
\frac{d}{dt}_{|_{t=t_0}} \left(\log A_t A_t^{-1}\right) &= \frac{i}{2\pi}\int_{\Gamma_\theta} \log_\theta \lambda\, \lambda^{-1} \frac{d}{dt}_{|_{t=t_0}} (A_t-\lambda)^{-1}\, d\lambda\\
&= -\frac{i}{2\pi}\int_{\Gamma_\theta} \log_\theta \lambda\, \lambda^{-1} (A_{t_0}-\lambda)^{-1} \dot{A}_{t_0} (A_{t_0}-\lambda)^{-1}\, d\lambda.
\end{aligned}$$

This leads to the expected formula

$$\begin{aligned}
\frac{d}{dt}_{|_{t=t_0}} \log A_t &= \frac{d}{dt}_{|_{t=t_0}} \left(\log A_t A_t^{-1}\right) A_t + \left(\log A_t A_t^{-1}\right) \dot{A}_t\\
&= -\frac{i}{2\pi}\int_{\Gamma_\theta} \log_\theta \lambda\, \lambda^{-1} (A_{t_0}-\lambda)^{-1} \dot{A}_{t_0} (A_{t_0}-\lambda)^{-1} A_{t_0}\, d\lambda\\
&+ \frac{i}{2\pi}\int_{\Gamma_\theta} \log_\theta \lambda\, \lambda^{-1} (A_{t_0}-\lambda)^{-1} \dot{A}_{t_0}\, d\lambda\\
&= -\frac{i}{2\pi}\int_{\Gamma_\theta} \log_\theta \lambda\, \lambda^{-1} (A_{t_0}-\lambda)^{-1} \dot{A}_{t_0} (A_t-\lambda)^{-1} \left(A_{t_0}-(A_{t_0}-\lambda)\right) d\lambda\\
&= -\frac{i}{2\pi}\int_{\Gamma_\theta} \log_\theta \lambda\, (A_{t_0}-\lambda)^{-1} \dot{A}_{t_0} (A_{t_0}-\lambda)^{-1}\, d\lambda.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\left[(\lambda-A_{t_0})^{-1}, \dot{A}_{t_0}\right]\\
=\ & (\lambda-A_{t_0})^{-1} [A_{t_0}, \dot{A}_{t_0}] (\lambda-A_{t_0})^{-1}\\
=\ & [A_{t_0}, \dot{A}_{t_0}](\lambda-A_{t_0})^{-2} + \mathrm{ad}^2_{A_{t_0}}(\dot{A}_{t_0})(\lambda-A_{t_0})^{-3} + \left[(\lambda-A_{t_0})^{-1}, \mathrm{ad}^2_{A_{t_0}}(\dot{A}_{t_0})\right](\lambda-A_0)^{-2}\\
=\ & \sum_{k=1}^{K} \mathrm{ad}^k_{A_{t_0}}(\dot{A}_{t_0})(\lambda-A_{t_0})^{-(k+1)} + \left[(\lambda-A_{t_0})^{-1}, \mathrm{ad}^K_{A_{t_0}}(\dot{A}_{t_0})\right](\lambda-A_{t_0})^{-K}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\frac{d}{dt}_{|_{t=t_0}} \log A_t &= -\frac{i}{2\pi}\int_{\Gamma_\theta} \log_\theta \lambda\, [(A_{t_0}-\lambda)^{-1}, \dot{A}_{t_0}] (A_{t_0}-\lambda)^{-1}\, d\lambda\\
&- \frac{i}{2\pi}\int_{\Gamma_\theta} \log_\theta \lambda\, \dot{A}_{t_0} (A_{t_0}-\lambda)^{-2}\, d\lambda\\
&= \sum_{k=0}^{K} \mathrm{ad}^k_{A_{t_0}}(\dot{A}_{t_0}) \frac{i}{2\pi}\int_{\Gamma_\theta} \log_\theta \lambda\, (\lambda-A_{t_0})^{-(k+2)}\, d\lambda\\
&- \frac{i}{2\pi}\int_{\Gamma_\theta} \log_\theta \lambda \left[(\lambda-A_{t_0})^{-1}, \mathrm{ad}^K_{A_{t_0}}(\dot{A}_{t_0})\right](\lambda-A_{t_0})^{-K-1}\, d\lambda.
\end{aligned}$$

Iterated integrations by parts then lead to

$$\frac{d}{dt}_{|t=t_0}\log(A_t)=\sum_{k=0}^{K}\frac{(-1)^k}{k+1}\mathrm{ad}^k_{A_{t_0}}(\dot{A}_{t_0})A_{t_0}^{-(k+1)}+R_K(A_{t_0},\dot{A}_{t_0}),$$

with

$$R_K(A_{t_0},\dot{A}_{t_0}):=-\frac{i}{2\pi}\int_{\Gamma_\theta}\log_\theta\lambda\left[(\lambda-A_{t_0})^{-1},\mathrm{ad}^K_{A_{t_0}}(\dot{A}_{t_0})\right](\lambda-A_{t_0})^{-K-1}\,d\lambda,$$

which in turn yields (5.1). □

The following result is reminiscent of an observation made in [21] (see also [29]), namely that only the first n homogeneous components of the symbols come into play for the derivation of the Campbell-Hausdorff formula for operators with scalar leading symbols; the weighted trace of $L(A,B)$ presents a similar feature in our more general situation.

Theorem 5.3. *Given a weight Q and two admissible operators A and B in $C\ell(M,E)$ with non negative orders, the weighted trace $\mathrm{tr}^Q(L(A,B))$ is a local expression as a finite sum of noncommutative residues, which only depends on the first n homogeneous components of the symbols of A and B:*

$$\frac{d}{dt}\mathrm{tr}^Q(L(A(1+tS),B)=\frac{d}{dt}\mathrm{tr}^Q(L(A,B(1+tS))=0\quad\forall S\in C\ell^{<-n}(M,E),\qquad(5.3)$$

where $C\ell^{<-n}(M,E)=\cup_{\mathrm{Re}(a)<-n}C\ell^a(M,E)$ stands for the algebra of classical operators of order with real part smaller than $-n$.

Proof. • On the one hand we know that $L(A,B)$ is a finite sum of commutators of classical pseudodifferential operators $[P_j,Q_j]$. By (4.9), each weighted trace $\mathrm{tr}^Q([P_j,Q_j]$ is proportional to $\mathrm{res}\big(Q_j\,[P_j,\log_\alpha Q]\big)$ so that $\mathrm{tr}^Q(L(A,B))$ is indeed a finite sum of noncommutative residues.

• Let us check that requirement (5.3) is equivalent to the fact that $\mathrm{tr}^Q(L(A,B))$ only depends on the first n homogeneous components of the symbols of A and B.
Given an operator S in $C\ell(M,E)$ of order $<-n$ and an operator A in $C\ell(M,E)$ of order a, we first observe that in any local trivialisation the first n homogeneous components of the symbols of A and $A(1+S)$ coincide since AS has order with real part smaller than $-n$. Conversely, if the first n homogeneous components of the symbols of two classical operators A and B of orders a and b coincide, then $a=b$. If furthermore B is invertible, the first n homogeneous components of the symbol of B^{-1} defined inductively using (2.2) by:

$$(\sigma_{B^{-1}})_{-b}=((\sigma_B)_b)^{-1},$$

$$(\sigma_{B^{-1}})_{-b-j}=-((\sigma_B)_b)^{-1}\sum_{k+l+|\alpha|=j,l<j}\frac{(-i)^{|\alpha|}}{\alpha!}\partial_\xi^\alpha(\sigma_B)_{b-k}\,\partial_x^\alpha(\sigma_{B^{-1}})_{-b-l},$$

coincide with that of the symbol of A^{-1} since the terms corresponding to $j\le n$ only involve homogeneous components $(\sigma_B)_{b-k}=(\sigma_A)_{a-k}$ and $(\sigma_{B^{-1}})_{-b-l}$ with k and l no larger than n. Consequently, by (2.2) it follows that $S = A^{-1}B$ has order $< -n$. Thus, showing that the expression $\mathrm{tr}^Q(L(A,B))$ only depends on the first n homogeneous components of A amounts to showing that $\mathrm{tr}^Q(L(A+S,B)) = \mathrm{tr}^Q(L(A,B))$ for any classical operator S whose order has real part smaller than $-n$.

- We are therefore left to prove that $\frac{d}{dt}\mathrm{tr}^Q(L(A(1+tS),B) = 0$. A similar proof (which we omit here) would yield $\frac{d}{dt}\mathrm{tr}^Q(L(A(,B(1+tS)) = 0$.
  Applying (4.12) to the operator $A_t := L(A(1+tS),B)$ we have
  $$\frac{d}{dt}\mathrm{tr}^Q(L(A(1+tS),B) = \mathrm{tr}^Q\left(\frac{d}{dt}L(A(1+tS),B)\right).$$
  Since
  $$L(A(1+tS),B) - L(A,B) = \log(A(1+tS)B) - \log(AB) - \log(A(1+tS)) + \log A,$$
  we have
  $$\frac{d}{dt}L(A(1+tS),B) = \frac{d}{dt}\log(A(1+tS)B) - \frac{d}{dt}(\log(A(1+tS))).$$
  Proving (5.3) therefore amounts to showing that
  $$\frac{d}{dt}\mathrm{tr}^Q(\log(A(1+tS)B)) = \frac{d}{dt}\mathrm{tr}^Q(\log(A(1+tS))). \tag{5.4}$$
  We apply Lemma 5.1 to the family $A_t := A(1+tS)C$ whose derivative reads $\dot{A}_t = ASC$. We shall show that $\frac{d}{dt}\mathrm{tr}^Q(\log(A_t))$ is independent of the choice of the operator C, a fact which when applied to $C = B$ and $C = I$, yields (5.4).
  When t varies in a small compact neighborhood of some fixed point t_0, the operators A_t have a common spectral cut which we drop in the notation. Using (5.1) and implementing the weighted trace tr^Q yields
  $$\begin{aligned} & \frac{d}{dt}_{|t=t_0}\mathrm{tr}^Q(\log A_t) \qquad\qquad (5.5)\\ = \quad & \mathrm{tr}^Q(\dot{A}_{t_0}A_{t_0}^{-1}) + \sum_{k=1}^{K}\frac{(-1)^k}{k+1}\mathrm{tr}^Q(\mathrm{ad}^k_{A_{t_0}}(\dot{A}_{t_0})A_{t_0}^{-(k+1)}) + \mathrm{tr}^Q(R_K(A_{t_0},\dot{A}_{t_0}))\end{aligned}$$
  for arbitrary large K and with remainder term
  $$R_K(A_{t_0},\dot{A}_{t_0}) := -\frac{i}{2\pi}\int_{\Gamma_\alpha}\log\lambda\left[(\lambda - A_{t_0})^{-1}, \mathrm{ad}^K_{A_{t_0}}\left(\dot{A}_{t_0}\right)\right](\lambda - A_{t_0})^{-(K+1)}\,d\lambda. \tag{5.6}$$
  By (4.9), for any positive integer k we have
  $$\begin{aligned}\mathrm{tr}^Q(\mathrm{ad}^k_{A_{t_0}}(\dot{A}_{t_0})A_{t_0}^{-(k+1)}) &= \mathrm{tr}^Q\left(\mathrm{ad}_{A_{t_0}}(\mathrm{ad}^{k-1}_{A_{t_0}}(\dot{A}_{t_0}))A_{t_0}^{-(k+1)}\right)\\ &= \mathrm{tr}^Q\left(\left[A_{t_0}, \mathrm{ad}^{k-1}_{A_{t_0}}(\dot{A}_{t_0})A_{t_0}^{-(k+1)}\right]\right)\\ &= \frac{1}{q}\mathrm{res}\left(\mathrm{ad}^{k-1}_{A_{t_0}}(\dot{A}_{t_0})A_{t_0}^{-(k+1)}[A_{t_0},\log Q]\right)\\ &= 0.\end{aligned}$$

The vanishing of the residue follows from a simple order counting.
Indeed, since $A_{t_0} = A(1+t_0 S)C$ has order $a+c$ and $\dot{A}_{t_0} = ASC$ has order $a+s+c$ (here s is the order of S, a the order of A, c the order of C), it follows that the operator $\mathrm{ad}_{A_{t_0}}^{k-1}(\dot{A}_{t_0}) A_{t_0}^{-(k+1)} [A_{t_0}, \log Q]$ has order

$$(k-1)(a+c)+a+c+s-(k+1)(a+c)+a+c = s,$$

whose real part is smaller than $-n$ as a consequence of which its residue vanishes.
A similar order counting shows that

$$\begin{aligned} & \mathrm{tr}^Q\left(R_K(A_{t_0}, \dot{A}_{t_0})\right) \\ = & -\mathrm{tr}^Q\left(\left[A_{t_0}, \frac{i}{2\pi}\int_{\Gamma_\alpha} \log\lambda\, (\lambda - A_{t_0})^{-1}\, \mathrm{ad}_{A_{t_0}}^{K-1}\left(\dot{A}_{t_0}\right)(\lambda - A_{t_0})^{-(K+1)}\, d\lambda\right]\right) \\ = & -\frac{1}{q}\mathrm{res}\left(\left(\frac{i}{2\pi}\int_{\Gamma_\alpha} \log\lambda\, (\lambda - A_{t_0})^{-1}\, \mathrm{ad}_{A_{t_0}}^{K-1}\left(\dot{A}_{t_0}\right)(\lambda - A_{t_0})^{-(K+1)}\, d\lambda\right)[A_{t_0}, \log Q]\right) \\ = & \ 0. \end{aligned}$$

Hence, only the first term on the right hand side of (5.5) survives and we have

$$\begin{aligned} \frac{d}{dt}_{|t=t_0} \mathrm{tr}^Q(\log(A(1+tS)C)) &= \mathrm{tr}^Q\left(ASC\,(A(1+t_0S)C)^{-1}\right) \\ &= \mathrm{tr}^Q\left(AS\,(1+t_0S)^{-1}A^{-1}\right) \end{aligned}$$

independently of the choice of C. Setting back $C = B$ and $C = I$ therefore yields (5.3), thus ending the proof of the theorem.

□

6 A local formula for the weighted trace of $L(A,B)$

We derive an explicit local expression for the weighted traces $\mathrm{tr}^A(L(A,B))$ and $\mathrm{tr}^B(L(A,B))$ of $L(A,B)$ (see Theorem 6.2). Our approach is inspired by Okikiolu's proof of the Campbell-Hausdorff formula for operators with scalar leading symbols. In the case of operators with scalar leading symbols, as it was noticed and used by Okikiolu, as from a certain order in the Campbell-Hausdorff expansion, one can implement ordinary traces since the iterated brackets have decreasing order. In our more general situation, such a phenomenon does not occur so that we use weighted traces instead.

Proposition 6.1. *Let A and B be two admissible operators with positive orders a and b in $C\ell(M,E)$ such that their product AB is also admissible. We have the following identities for weighted traces:*

$$\frac{d}{dt}_{|t=0} \mathrm{tr}^B(L(A^t, B^\mu)) = 0, \qquad \frac{d}{dt}_{|t=0} \mathrm{tr}^A(L(A^t, B^\mu)) = 0$$

as well as for the noncommutative residue:

$$\frac{d}{dt}_{|t=0} \mathrm{res}(L(A^t, B^\mu)) = 0.$$

Proof. Let us prove the result for the B-weighted trace; a similar proof yields the result for the A-weighted trace. By Proposition 4.8, weighted traces and the residue commute with differentiation on constant order operators so that

$$\frac{d}{dt}_{|t=0}\mathrm{tr}^Q\left(L(A^t,B^\mu)\right)=\mathrm{tr}^Q\left(\frac{d}{dt}_{|t=0}L(A^t,B^\mu)\right)$$

resp.

$$\frac{d}{dt}_{|t=0}\mathrm{res}\left(L(A^t,B^\mu)\right)=\mathrm{res}\left(\frac{d}{dt}_{|t=0}(L(A^t,B^\mu)\right).$$

But

$$\frac{d}{dt}_{|t=0}L(A^t,B^\mu)=\frac{d}{dt}_{|t=0}\log(A^tB^\mu)-\frac{d}{dt}_{|t=0}\log A^t.$$

We therefore apply Lemma 5.1 to $A_t:=A^tB^\mu$ so that $A_0=B^\mu$, including the case $\mu=0$ for which $A_t=A^t$ and $A_0=I$. Since $\dot{A}_0=\log A\,B^\mu$ and $\dot{A}_0A_0^{-1}=\log A$, implementing the weighted trace tr^B yields

$$\begin{aligned}&\frac{d}{dt}_{|t=0}\mathrm{tr}^B\left(\log(A^tB^\mu)\right)\\ =\;&\mathrm{tr}^B(\log A)+\sum_{k=1}^K\frac{(-1)^k}{k+1}\mathrm{tr}^B(\mathrm{ad}^k_{B^\mu}(\log A\,B^\mu)\,B^{-\mu(k+1)})+\mathrm{tr}^B(R_K(B^\mu,\log A\,B^\mu))\end{aligned}$$

for arbitrary large K, with remainder term

$$\begin{aligned}R_K(B^\mu,\log A\,B^\mu)&=-\frac{i}{2\pi}\int_{\Gamma_\alpha}\log\lambda\left[(\lambda-B^\mu)^{-1},\mathrm{ad}^K_{B^\mu}(\log A\,B^\mu)\right](\lambda-B^\mu)^{-(K+1)}\,d\lambda\\&=-\mathrm{ad}^K_{B^\mu}\left(\frac{i}{2\pi}\int_{\Gamma_\alpha}\log\lambda\left[(\lambda-B^\mu)^{-1},\log A\,B^\mu\right](\lambda-B^\mu)^{-(K+1)}\,d\lambda\right),\end{aligned}$$

since $(B-\lambda)^{-1}$ commutes with B^μ.
For any positive integer k, by (4.9) we have

$$\begin{aligned}\mathrm{tr}^B(\mathrm{ad}^k_{B^\mu}(A\,B^\mu)\,B^{-\mu(k+1)}))&=\mathrm{tr}^B\left(\mathrm{ad}_{B^\mu}(\mathrm{ad}^{k-1}_{B^\mu}(A\,B^\mu))\,B^{-\mu(k+1)})\right)\\&=\mathrm{tr}^B\left(\mathrm{ad}_{B^\mu}\left(\mathrm{ad}^{k-1}_{B^\mu}(A\,B^\mu)\,B^{-\mu(k+1)}\right)\right)\\&=\frac{1}{b}\mathrm{res}\left(\mathrm{ad}^{k-1}_{B^\mu}(A\,B^\mu)\,B^{-\mu(k+1)}\,[B^\mu,\log B]\right)\\&=0,\end{aligned}$$

since B^μ commutes with $\log B$. A similar computation shows that $\mathrm{tr}^B(R_K(B^\mu,\log A\,B^\mu))=0$. Thus

$$\frac{d}{dt}_{|t=0}\mathrm{tr}^B\left(\log(A^tB^\mu)\right)=\mathrm{tr}^B(\log A).$$

It follows that $\frac{d}{dt}_{|t=0}\mathrm{tr}^B(\log(A^tB^\mu))=\mathrm{tr}^B(\log A)$ independently of μ so that

$$\frac{d}{dt}_{|t=0}\mathrm{tr}^B\left(L(A^t,B^\mu)\right)=0.$$

Similarly, replacing the weighted trace tr^B by the noncommutative residue res and using the cyclicity of the noncommutative residue, yields

$$\frac{d}{dt}_{|t=0} \mathrm{res}\Big(L(A^t, B^\mu)\Big) = 0.$$

□

The following statement provides a local formula for the multiplicative anomaly of the zeta determinant. It also shows that the residue of $L(A,B)$ vanishes and therefore yields back the multiplicativity of the residue determinant derived in [29].

Theorem 6.2. *For two admissible operators $A, B \in C\ell(M,E)$ with positive orders a and b such that their product AB is also admissible, we have*

$$\mathrm{res}(L(A,B)) = 0. \tag{6.1}$$

Moreover, there is an operator

$$W(\tau)(A,B) := \frac{d}{dt}_{|t=0} L(A^t, A^\tau B) \tag{6.2}$$

in $C\ell^0(M,E)$ depending continuously on τ such that

$$\mathrm{tr}^Q(L(A,B)) = \int_0^1 \mathrm{res}\left(W(\tau)(A,B)\left(\frac{\log(A^\tau B)}{a\tau + b} - \frac{\log Q}{q}\right)\right) d\tau \tag{6.3}$$

where Q is any weight of order q.

Proof. By Proposition 6.1, we know that $\frac{d}{dt}_{|t=0}\mathrm{res}(L(A^t,B)) = \frac{d}{dt}_{|t=0}\mathrm{tr}^Q(L(A^t,B)) = 0$. We want to compute

$$\frac{d}{dt}_{|t=\tau}\mathrm{res}(L(A^t,B)) = \frac{d}{dt}_{|t=0}\mathrm{res}(L(A^{t+\tau},B)) \quad \text{and} \quad \frac{d}{dt}_{|t=\tau}\mathrm{tr}^Q(L(A^t,B)) = \frac{d}{dt}_{|t=0}\mathrm{tr}^Q(L(A^{t+\tau},B)).$$

For this we observe that

$$L(AB,D) - L(A,BD) = -\log(AB) - \log(D) + \log A + \log(BD) = L(B,D) - L(A,B)$$

Replacing A by A^t, B by A^τ and D by B, we get

$$L(A^{t+\tau},B) - L(A^t, A^\tau B) = L(A^\tau, B) - L(A^t, A^\tau) = L(A^\tau, B).$$

Implementing the noncommutative residue, by Proposition 6.1 we have:

$$\frac{d}{dt}_{|t=\tau}\mathrm{res}(L(A^t,B)) = \frac{d}{dt}_{|t=0}\mathrm{res}(L(A^{t+\tau},B)) = \frac{d}{dt}_{|t=0}\mathrm{res}(L(A^t, A^\tau B)) = 0.$$

Hence

$$\mathrm{res}(L(A,B)) = \int_0^1 \frac{d}{dt}_{|t=\tau}\mathrm{res}(L(A^t,B))\, d\tau + \mathrm{res}(L(I,B)) = 0, \tag{6.4}$$

since $L(I,B)=0$.
If instead we implement the weighted trace tr^Q, we have:

$$\frac{d}{dt}_{|t=\tau}\mathrm{tr}^Q(L(A^t,B)) = \frac{d}{dt}_{|t=0}\mathrm{tr}^Q(L(A^{t+\tau},B)) = \frac{d}{dt}_{|t=0}\mathrm{tr}^Q(L(A^t,A^\tau B)).$$

Since A and B have positive order so has $A^\tau B$, so that applying Proposition 6.1 with weighted traces $\mathrm{tr}^{A^\tau B}$ yields:

$$\begin{aligned}
\frac{d}{dt}_{|t=\tau}\mathrm{tr}^Q(L(A^tB) &= \frac{d}{dt}_{|t=0}\mathrm{tr}^Q(L(A^t,A^\tau B))\\
&= \frac{d}{dt}_{|t=0}\mathrm{tr}^{A^\tau B}(L(A^t,A^\tau B))\\
&+ \frac{d}{dt}_{|t=0}\left(\mathrm{tr}^Q(L(A^t,A^\tau B)) - \mathrm{tr}^{A^\tau B}(L(A^t,A^\tau B))\right)\\
&= \frac{d}{dt}_{|t=0}\left(\mathrm{tr}^Q(L(A^t,A^\tau B)) - \mathrm{tr}^{A^\tau B}(L(A^t,A^\tau B))\right).
\end{aligned}$$

Applying (4.8) to $Q_1 = Q$ and $Q_2 = A^\tau B$, we infer that

$$\begin{aligned}
&\frac{d}{dt}_{|t=0}\left(\mathrm{tr}^Q(L(A^t,A^\tau B)) - \mathrm{tr}^{A^\tau B}(L(A^t,A^\tau B))\right)\\
=\ &\frac{d}{dt}_{|t=0}\mathrm{res}\left(L(A^t,A^\tau B)\left(\frac{\log(A^\tau B)}{a\tau+b} - \frac{\log Q}{q}\right)\right)\\
=\ &\mathrm{res}\left(W(\tau)(A,B)\left(\frac{\log(A^\tau B)}{a\tau+b} - \frac{\log Q}{q}\right)\right),
\end{aligned}$$

where q is the order of Q and where we have set $W(\tau)(A,B) := \frac{d}{dt}_{|t=0}L(A^t,A^\tau B)$. Since $L(I,B)=0$, we finally find that

$$\begin{aligned}
&\mathrm{tr}^Q(L(A,B)) = \mathrm{tr}^Q(L(A^1,B)) - \mathrm{tr}^Q(L(A^0,B))\\
=\ &\int_0^1 \mathrm{res}\left(W(\tau)(A,B)\left(\frac{\log(A^\tau B)}{a\tau+b} - \frac{\log Q}{q}\right)\right)d\tau. \qquad (6.5)
\end{aligned}$$

□

7 Multiplicative anomaly for determinants revisited

We first observe that the multiplicative anomaly for weighted determinants studied in [4] has logarithm given by the weighted trace of $L(A,B)$, as a result of which it is local. We then derive an explicit local formula for the multiplicative anomaly of ζ determinants, using the local formula derived previously for weighted traces of $L(A,B)$.

An admissible operator A in $C\ell(M,E)$ with spectral cut θ and positive order has well defined Q-weighted determinant [4] (see also [9]) where Q in $C\ell(M,E)$ is a weight with spectral cut α:

$$\mathrm{det}^Q_\alpha(A) := e^{\mathrm{tr}^Q_\alpha(\log_\theta A)}.$$

Here the weighted trace has been extended to logarithms as before, picking out the constant term of the meromorphic map $z \mapsto \mathrm{TR}(\log_\theta A\, Q_\alpha^{-z})$ which can have double poles in contrast to the case of classical operators studied in Section 3.

Remark 7.1. The weighted determinant, as well as being dependent on the choice of spectral cut θ, also depends on the choice of spectral cut α.

Since the weighted trace restricts to the ordinary trace on trace-class operators, this determinant, as the ζ-determinant, extends the ordinary determinant on operators in the determinant class.

Lemma 7.2. *Let $0 \le \theta < \phi < 2\pi$ be two spectral cuts for the admissible operator A. If there is a set $\Lambda_{r,\theta,\phi}$ (see 3.4) which does not intersect the spectrum of the leading symbol of A then*

$$\det{}_\theta^Q(A) = \det{}_\phi^Q(A).$$

Proof. Under the assumptions of the lemma, the set $\Lambda_{r,\phi,\theta}$ defined as in Proposition 3.4, contains only a finite number of points in the spectrum of A so that $\log_\phi A - \log_\theta A = 2i\pi\Pi_{\theta,\phi}(A)$ is a finite rank operator and hence smoothing. Hence,

$$\begin{aligned}
\frac{\det{}_\phi^Q(A)}{\det{}_\theta^Q(A)} &= e^{\mathrm{tr}^Q\left(\log_\phi A - \log_\theta A\right)} = e^{\mathrm{tr}^Q\left(2i\pi\Pi_{\theta,\phi}(A)\right)} \\
&= e^{2i\pi\,\mathrm{tr}\left(\Pi_{\theta,\phi}(A)\right)} = e^{2i\pi\,\mathrm{rk}(\Pi_{\theta,\phi}(A))} \\
&= 1,
\end{aligned}$$

where rk stands for the rank. □

The multiplicative anomaly for Q-weighted determinants of two admissible operators A, B with spectral cuts θ, ϕ such that AB has spectral cut ψ is defined by:

$$\mathcal{M}^Q_{\theta,\phi,\psi}(A,B) := \frac{\det{}_\psi^Q(AB)}{\det{}_\theta^Q(A)\det{}_\phi^Q(B)},$$

which we write $\mathcal{M}^Q(A,B)$ for simplicity.

Proposition 7.3. *Let A and B be two admissible operators with spectral cuts θ and ϕ in $[0,2\pi[$ such that there is a cone delimited by the rays L_θ and L_ϕ which does not intersect the spectra of the leading symbols of A, B and AB. Then the product AB is admissible with a spectral cut ψ inside that cone and for any weight Q with spectral cut, dropping the explicit mention of the spectral cuts we have:*

$$\log \mathcal{M}^Q(A,B) = \int_0^1 \mathrm{res}\left(W(\tau)(A,B)\left(\frac{\log(A^\tau B)}{a\tau + b} - \frac{\log_\alpha Q}{q}\right)\right)d\tau. \tag{7.1}$$

Weighted determinants are multiplicative on commuting operators.

Proof. Since the leading symbol of the product AB has spectrum which does not intersect the cone delimited by L_θ and L_ϕ, the operator AB only has a finite number of eigenvalues inside that cone. We can therefore choose a ray ψ which avoids both the spectrum of the leading symbol of AB and the eigenvalues of AB. By the above lemma, the weighted determinants $\det_\theta^Q(A)$, $\det_\phi^Q(B)$ and $\det_\psi^Q(AB)$ do not depend on the choices of spectral cuts satisfying the requirements of the proposition.
Since

$$\log \mathcal{M}^Q(A,B) = \log\det{}^Q(AB) - \log\det{}^Q(A) - \log\det{}^Q(B) = \mathrm{tr}^Q(L(A,B)),$$

the logarithm of the multiplicative anomaly for weighted determinants is a local quantity (6.3) derived in Theorem 6.2.
To prove the second part of the statement we observe that

$$[A,B] = 0 \Longrightarrow L(A,B) = 0. \tag{7.2}$$

Indeed, let Γ be a contour as in formula (4.12) along a spectral ray around the spectrum of $A^{t_0}B$ for some fixed t_0, then

$$\begin{aligned}
\frac{d}{dt}_{|_{t=t_0}} \log(A^tB) &= \frac{i}{2\pi}\int_\Gamma \log\lambda \frac{d}{dt}_{|_{t=t_0}} (A^tB-\lambda)^{-1}\,d\lambda \\
&= \frac{i}{2\pi}\int_\Gamma \log\lambda\,(A^{t_0}B-\lambda)^{-1}\log A\,A^{t_0}B\,(A^{t_0}B-\lambda)^{-1}\,d\lambda \\
&= \log A\,A^{t_0}B\,\frac{i}{2\pi}\int_\Gamma \log\lambda\,(A^{t_0}B-\lambda)^{-2}\,d\lambda \quad \text{since} \quad [A,B]=0 \\
&= -\log A\,A^{t_0}B\,\frac{i}{2\pi}\int_\Gamma \lambda^{-1}(A^{t_0}B-\lambda)^{-1}\,d\lambda \quad \text{by integration by parts} \\
&= -\log A\,A^{t_0}B(A^{t_0}B)^{-1} \\
&= -\log A.
\end{aligned}$$

Similarly, we have $\frac{d}{dt}_{|_{t=t_0}}\log(A^t) = -\log A$ so that finally

$$\frac{d}{dt}_{|_{t=t_0}} L(A^t,B) = \frac{d}{dt}_{|_{t=t_0}}\log(A^tB) - \frac{d}{dt}_{|_{t=t_0}}\log(A^t)$$

vanishes. It follows that $L(A,B) = \int_0^1 \frac{d}{dt}_{|_{t=\tau}} L(A^t,B)\,d\tau = 0$.
Since $L(A,B)$ vanishes when A and B commute, weighted determinants are multiplicative on commuting operators. □

Let us now turn to the multiplicative anomaly for ζ-determinants, relating it to weighted traces of $L(A,B)$. An admissible operator $A \in C\ell(M,E)$ with spectral cut θ and positive order has well defined ζ-determinant:

$$\det{}_{\zeta,\theta}(A) := e^{-\zeta'_{A,\theta}(0)} = e^{\mathrm{tr}_\theta^A(\log_\theta A)}$$

since $\zeta_{A,\theta}(z) := \mathrm{TR}(A_\theta^{-z})$ is holomorphic at $z=0$. In the second equality, the weighted trace has been extended to logarithms as before, picking out the constant term of the meromorphic map $z \mapsto \mathrm{TR}(\log_\theta A\, Q^{-z})$ (which can have double poles) with the notations of section 2. Recall from [24] that

$$\log \det_{\zeta,\theta}(A) = \int_M \left[\mathrm{TR}_x(\log_\theta A) - \frac{1}{2a}\mathrm{res}_x(\log_\theta^2 A)\right] dx \tag{7.3}$$

where a is the order of A and where $\mathrm{TR}_x\, dx$ and $\mathrm{res}_x\, dx$ are respectively the canonical trace density (see (4.3)) and the noncommutative residue density (see (2.6)) defined previously, extended to log-polyhomogeneous operators. This expression corresponds to minus the coefficient in z of the Laurent expansion of $\mathrm{TR}(A^{-z})$.

The ζ-determinant generally depends on the choice of spectral cut. However, it is invariant under mild changes of spectral cut in the following sense.

Lemma 7.4. *Let $0 \le \theta < \phi < 2\pi$ be two spectral cuts for the admissible operator A. If there is a cone $\Lambda_{r,\theta,\phi}$ (see 3.4) which does not intersect the spectrum of the leading symbol of A then*

$$\det_{\zeta,\theta}(A) = \det_{\zeta,\phi}(A).$$

Proof. By (7.3), and since $\log_\phi A - \log_\theta A = 2i\pi\Pi_{\theta,\phi}(A)$ is a finite rank operator and hence smoothing under the assumptions of the proposition, we have

$$\begin{aligned}
\frac{\det_{\zeta,\phi}(A)}{\det_{\zeta,\theta}(A)} &= e^{\int_M\left[\mathrm{TR}_x(\log_\phi A) - \frac{1}{2a}\mathrm{res}_x(\log_\phi^2 A)\right]dx - \int_M\left[\mathrm{TR}_x(\log_\theta A) - \frac{1}{2a}\mathrm{res}_x(\log_\theta^2 A)\right]dx} \\
&= e^{\int_M\left[\mathrm{TR}_x(\log_\phi A - \log_\theta A) - \frac{1}{2a}\mathrm{res}_x(\log_\phi^2 A - \log_\theta^2 A)\right]dx} \\
&= e^{\int_M\left[\mathrm{TR}_x(2i\pi\Pi_{\theta,\phi}(A)) - \frac{1}{2a}\mathrm{res}_x\left((\log_\phi A + \log_\theta A)\, 2i\pi\Pi_{\theta,\phi}(A)\right)\right]dx} \\
&= e^{2i\pi\, \mathrm{tr}(\Pi_{\theta,\phi}(A)) - \frac{2i\pi}{2a}\mathrm{res}\left((\log_\phi A + \log_\theta A)\,\Pi_{\theta,\phi}(A)\right)} \\
&= e^{2i\pi\, \mathrm{rk}(\Pi_{\theta,\phi}(A))} \\
&= 1,
\end{aligned}$$

where we have used the fact that the noncommutative residue vanishes on smoothing operators on which the canonical trace coincides with the usual trace on smoothing operators. □

The ζ-determinant is not multiplicative [8]. Indeed, let A and B be two admissible operators with positive order and spectral cuts θ and ϕ and such that AB is also admissible with spectral cut ψ. The multiplicative anomaly

$$\mathcal{M}_\zeta^{\theta,\phi,\psi}(A,B) := \frac{\det_{\zeta,\psi}(AB)}{\det_{\zeta,\theta}(A)\det_{\zeta,\phi}(B)},$$

was proved to be local, independently by Okikiolu [22] for operators with scalar leading symbol and by Kontsevich and Vishik [15] for operators "close to identity" (see the introduction for a more detailed historical account).
For simplicity, we drop the explicit mention of θ,ϕ,ψ and write $\mathcal{M}_\zeta(A,B)$.

Even though the operator $\log_\theta^2 A$ is not classical we have the following useful property.

[8] It was shown in [18] that all multiplicative determinants on elliptic operators can be built from two basic types of determinants; they do not include the ζ-determinant.

Lemma 7.5. *Let A, B be admissible operators in $C\ell(M,E)$ with positive orders a, b and spectral cuts θ and ϕ respectively and such that AB (which is elliptic) is also admissible with spectral cut ψ. Then*

$$K(A,B) := \frac{1}{2(a+b)} \log^2_\psi AB - \frac{1}{2a}\log^2_\theta A - \frac{1}{2b}\log^2_\phi B$$

has a symbol of the form

$$\sigma_K \sim \ln|\xi|(\sigma_0^{AB} - \sigma_0^A - \sigma_0^B) + \sigma_0^K$$

for some zero order classical symbol σ_0^K, where we have written $\sigma_{\log A}(x,\xi) = a\ln|\xi| I + \sigma_0^A(x,\xi)$ for an admissible operator A of order a.
In particular, both operators $L(A,B)\,\frac{\log A}{a} - K(A,B)$ and $L(A,B)\,\frac{\log B}{b} - K(A,B)$ are classical operators of zero order.

Proof. By formula (3.7), another choice of spectral cut only changes the logarithms by adding an operator in $C\ell^0(M,E)$ so that it will not affect the statement. As usual, we drop the explicit mention of spectral cut assuming the operators have common spectral cuts.
An explicit computation on symbols shows the result. Indeed, since $\sigma_{\log A}(x,\xi) \sim a\ln|\xi| + \sigma_0^A(x,\xi)$, we have

$$\begin{aligned}\sigma_{\log^2 A}(x,\xi) &= \sigma_{\log A} \star \sigma_{\log A}(x,\xi)\\ &\sim a^2\ln^2|\xi| I + 2a\ln|\xi|\,\sigma_0^A(x,\xi) + \sigma_0^A(x,\xi)\cdot\sigma_0^A(x,\xi)\\ &\quad + \sum_{\alpha\neq 0}\frac{(-i)^{|\alpha|}}{\alpha!}\partial_\xi^\alpha \sigma_0^A(x,\xi)\,\partial_x^\alpha\sigma_0^A(x,\xi).\end{aligned}$$

This yields:

$$\begin{aligned}\sigma_K(x,\xi) \sim\ & \ln|\xi|\left(\sigma_0^{AB} - \sigma_0^A - \sigma_0^B\right)(x,\xi)\\ & + \frac{1}{2(a+b)}\sigma_0^{AB}(x,\xi)\sigma_0^{AB}(x,\xi) + \sum_{\alpha\neq 0}\frac{1}{\alpha!}\partial_\xi^\alpha\sigma_0^{AB}(x,\xi)D_x^\alpha\sigma_0^{AB}(x,\xi)\\ & - \frac{1}{2a}\sigma_0^A(x,\xi)\sigma_0^A(x,\xi) - \sum_{\alpha\neq 0}\frac{1}{\alpha!}\partial_\xi^\alpha\sigma_0^A(x,\xi)D_x^\alpha\sigma_0^A(x,\xi)\\ & - \frac{1}{2b}\sigma_0^B(x,\xi)\sigma_0^B(x,\xi) - \sum_{\alpha\neq 0}\frac{1}{\alpha!}\partial_\xi^\alpha\sigma_0^B(x,\xi)D_x^\alpha\sigma_0^B(x,\xi)\end{aligned}$$

from which the first part of the statement follows.
On the other hand, it follows from (3.15) combined with (3.9) that the operators $L(A,B)\,\frac{\log A}{a}$ and $L(A,B)\,\frac{\log B}{b}$ both have symbols which differ from the symbol $\ln|\xi|\left(\sigma_0^{AB} - \sigma_0^A - \sigma_0^B\right)(x,\xi)$ by a classical symbol of order zero, from which we infer the second part of the statement. □

The following theorem provides a local formula for the multiplicative anomaly independently of Okikiolu's assumption that the leading symbols be scalar.

Theorem 7.6. *Let A and B be two admissible operators in $C\ell(M,E)$ with positive orders a,b and with spectral cuts θ and ϕ in $[0,2\pi[$ such that there is a cone delimited by the rays L_θ and L_ϕ which does not intersect the spectra of the leading symbols of A, B and AB. Then the product AB is admissible with a spectral cut ψ inside that cone and the multiplicative anomaly $\mathcal{M}_\zeta^{\theta,\phi,\psi}(A,B)$ is local as a noncommutative residue, independently of the choices of θ,ϕ, and ψ satisfying the above requirements.*
Explicitly, and dropping the explicit mention of the spectral cuts, there is a classical operator $W(\tau)(A,B)$ given by (6.2) of order zero depending continuously on τ such that:

$$\begin{aligned}
&\log \mathcal{M}_\zeta(A,B)\\
=\ & \int_0^1 \mathrm{res}\left(W(\tau)(A,B)\left(\frac{\log(A^\tau B)}{a\tau+b}-\frac{\log B}{b}\right)\right)d\tau\\
+\ & \mathrm{res}\left(\frac{L(A,B)\log B}{b}-\frac{\log^2 AB}{2(a+b)}+\frac{\log^2 A}{2a}+\frac{\log^2 B}{2b}\right)\\
=\ & \int_0^1 \mathrm{res}\left(W(\tau)(A,B)\left(\frac{\log(A^\tau B)}{a\tau+b}-\frac{\log A}{a}\right)\right)d\tau\\
+\ & \mathrm{res}\left(\frac{L(A,B)\log A}{a}-\frac{\log^2 AB}{2(a+b)}+\frac{\log^2 A}{2a}+\frac{\log^2 B}{2b}\right)
\end{aligned}\tag{7.4}$$

When A and B commute the multiplicative anomaly reduces to:

$$\begin{aligned}
\log \mathcal{M}_\zeta(A,B) &= -\mathrm{res}\left(\frac{1}{2(a+b)}\log^2(AB)-\frac{1}{2a}\log^2 A-\frac{1}{2b}\log^2 B\right)\\
&= \frac{ab}{2(a+b)}\mathrm{res}\left[\left(\frac{\log A}{a}-\frac{\log B}{b}\right)^2\right].
\end{aligned}\tag{7.5}$$

Remark 7.7. For commuting operators, (7.5) gives back the results of Wodzicki as well as formula (III.3) in [4]:

$$\log \mathcal{M}_\zeta(A,B)=\frac{\mathrm{res}\left(\log^2(A^bB^{-a})\right)}{2ab(a+b)}.$$

Proof. As in the proof of the locality of the multiplicative anomaly for weighted determinants (see Proposition 7.3), the independence of the choice of spectral cuts satisfying the requirements of the theorem follows from Lemma 7.4.
Combining equations (7.3), the defect formula (4.7) applied to the operator $L(A,B)$ and weight B with equation (6.3) applied to $Q=B$ we write:

$$\begin{aligned}
& \log \mathcal{M}_\zeta(A,B) \\
= \ & \log \det_\zeta(AB) - \log \det_\zeta(A) - \log \det_\zeta(B) \\
= \ & \int_M [\mathrm{TR}_x(L(A,B)) \\
& -\left(\frac{1}{2(a+b)} \mathrm{res}_x(\log^2 AB) - \frac{1}{2a}\mathrm{res}_x(\log^2 A) - \frac{1}{2b}\mathrm{res}_x(\log^2 B)\right)\Big] \, dx \\
= \ & \mathrm{tr}^B(L(A,B)) + \int_M \left[\frac{1}{b} \mathrm{res}_x \left(L(A,B) \log B\right)\right. \qquad (7.6) \\
& \left.-\left(\frac{1}{2(a+b)} \mathrm{res}_x \left(\log^2 A\, B\right) - \frac{1}{2a}\mathrm{res}_x(\log^2 A) - \frac{1}{2b}\mathrm{res}_x(\log^2 B)\right)\right] dx \\
= \ & \int_0^1 \mathrm{res}\left(W(\tau)(A,B)\left(\frac{\log(A^\tau B)}{a\tau+b} - \frac{\log B}{b}\right)\right) d\tau \\
+ \ & \mathrm{res}\left(\frac{L(A,B)\log B}{b} - \frac{\log^2 A\, B}{2(a+b)} + \frac{\log^2 A}{2a} + \frac{\log^2 B}{2b}\right),
\end{aligned}$$

which proves the first equality in (7.4). The second one can be derived similarly exchanging the roles of A and B.
When A and B commute, by (7.2), the operator $L(A,B)$ vanishes so that (7.6) reduces to:

$$\begin{aligned}
\log \mathcal{M}_\zeta(A,B) &= \mathrm{tr}^B(L(A,B)) + \int_M \left[\frac{1}{b} \mathrm{res}_x \left(L(A,B) \log B\right)\right. \\
&- \left.\left(\frac{1}{2(a+b)} \mathrm{res}_x \left(\log^2 A\, B\right) - \frac{1}{2a}\mathrm{res}_x(\log^2 A) - \frac{1}{2b}\mathrm{res}_x(\log^2 B)\right)\right] dx \\
&= -\mathrm{res}\left(\frac{\log^2 A\, B}{2(a+b)} - \frac{\log^2 A}{2a} - \frac{\log^2 B}{2b}\right) \\
&= \frac{ab}{2(a+b)} \mathrm{res}\left[\left(\frac{\log A}{a} - \frac{\log B}{b}\right)^2\right].
\end{aligned}$$

□

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