Communications in Mathematical Analysis
Volume 12, Number 1, pp. 17–27 (2012) www.math-res-pub.org/cma
ISSN 1938-9787

Some Estimates For Hardy-Polya-Knopp Type Differences On Time Scales

Sabir Hussain *
Department of Aeronautics and Astronautics
Institute of Space Technology
Islamabad, 44000, Pakistan

(Communicated by Hitoshi Kitada)

Abstract

In this paper, an improvement and reverse of a strengthened Hardy-Knopp type inequality are obtained via log −convexity.

AMS Subject Classification: 2000

Keywords: log −convex function, Delta integral, Weights, Hardy inequality, Polya-Knopp inequality.

1 Introduction

In 1920, Hardy [3] proved the following integral inequality, if $p > 1$, $f(x) \geq 0$ and $F(x) = \int_0^x f(t)\,dt$, then

$$\int_0^\infty \left[\frac{F}{x}\right]^p dx \leq \left[\frac{p}{p-1}\right]^p \int_0^\infty f^p\,dx. \tag{1.1}$$

Its discrete version essentially takes the same form with sums instead of integrals [8]. After that (1.1) attracted the attention of researchers and mathematicians. One can find over two hundred papers, articles and notes by searching the words "Hardy"and "inequality"in the review journals Zentralblatt für Mathematik (or Mathematical reviews). Almost at the end of the 20th century, sufficient material related to (1.1) was available in different forms in generalizations, estimations and improvements both in discrete and continuous settings (see for instance [6]). There is a related inequality, the so called exponential integral inequality (or Polya-Knopp inequality) [9, 10]

$$\int_0^\infty \exp\left[\frac{1}{x}\int_0^x \log f(t)\,dt\right] dx \leq e \int_0^\infty f(x)\,dx, \tag{1.2}$$

*E-mail address: sabirhus@gmail.com

which holds for positive functions $f \in L^1(0,\infty)$. Inequalities (1.1) and (1.2) are closely related, since (1.2) can be obtained from (1.1) by replacing f with $f^{1/p}$ and letting $p \to \infty$. Therefore, Polya-Knopp's inequality may be considered as the limiting form of Hardy's inequality. In 2002, Kaijser et al. [8] pointed out that both (1.1) and (1.2) are special cases of the much more general Hardy-Knopp-type inequality for positive functions f

$$\int_0^\infty \Phi\left[\frac{1}{x}\int_0^x f(t)\,dt\right]\frac{dx}{x} \leq \int_0^\infty \Phi[f(x)]\frac{dx}{x}, \tag{1.3}$$

where, Φ is a convex function on $(0,\infty)$. One year later A. Čižmešija et al. [1] gave the weighted version of (1.1) generalizing (1.3). Hardy's inequality has many applications in different fields of both applied and pure mathematics. In 1988, Stefan Hilger [4] introduced the notion of time scale's calculus which unifies continuous and discrete analysis.

This paper is organized in the following way. After this Introduction, in Section 2 some basic facts and results are discussed. In Section 3, log−convexity of Hardy-Polya-Knopp type differences is proved and in Section 4, some improvements and reverses of relations (2.5) to (2.7) are given.

2 Preliminaries

A time scale (or measure chain) is a non-empty closed subset of the reals, **R**, together with the topology of subspace of **R** and we usually denote it by the symbol $\mathbb{T}$. The two most popular examples are $\mathbb{T} = \mathbf{R}$ and $\mathbb{T} = \mathbb{Z}$. For any interval I of **R** (open or closed) $I_{\mathbb{T}} = I \cap \mathbb{T}$ is called a time scale interval. We define the forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ by:

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\} \text{ and } \rho(t) = \sup\{s \in \mathbb{T} : s < t\}$$

(supplemented by $\inf\emptyset = \sup\mathbb{T}$ and $\sup\emptyset = \inf\mathbb{T}$, where $\emptyset$ denotes the empty set). If $\sigma(t) = t$, $t \in \mathbb{T}$, we say t is right dense. If $\rho(t) = t$, $t \in \mathbb{T}$, we say t is left dense. If $\sigma(t) > t$, $t \in \mathbb{T}$, we say t is right scattered. If $\rho(t) < t$, $t \in \mathbb{T}$, we say t is left scattered. The set $\mathbb{T}^k$ is defined to be $\mathbb{T}$ if $\mathbb{T}$ does not have a left scattered maximum; otherwise it is $\mathbb{T}$ without this left scattered maximum. The graininess $\mu : \mathbb{T} \to [0,\infty)$ is defined by:

$$\mu(t) = \sigma(t) - t.$$

Hence the graininess function is constant 0 if $\mathbb{T} = \mathbf{R}$ while it is constant 1 if $\mathbb{T} = \mathbb{Z}$. However, a time scale $\mathbb{T}$ could have nonconstant graininess. Let $f : \mathbb{T} \to \mathbf{R}$ be a function, then $f^\sigma : \mathbb{T} \to \mathbf{R}$ is defined by $f^\sigma(t) = f(\sigma(t))$ for $t \in \mathbb{T}$, where $\sigma(t)$ is defined above. We also, say that f is delta differentiable (or simply: differentiable) at $t \in \mathbb{T}^k$ provided there exists an α such that for all $\epsilon > 0$ there is a neighborhood $\aleph$ of t with

$$|f(\sigma(t)) - f(s) - \alpha(\sigma(t) - s)| \leq \epsilon|\sigma(t) - s| \quad \text{for all} \quad s \in \aleph.$$

In this case we denote the α by $f^\Delta(t)$, and if f is differentiable for every $t \in \mathbb{T}^k$, then f is said to be differentiable on $\mathbb{T}$ and f^Δ is a new function on $\mathbb{T}^k$. If f is differentiable at $t \in \mathbb{T}^k$, then it is easy to see that

$$f^{\Delta}(t) = \begin{cases} \lim_{s\to t(s\in\mathbb{T})} \frac{f(t)-f(s)}{t-s} & \text{if } \mu(t) = 0 \\ \frac{f(\sigma(t))-f(t)}{\mu(t)}, & \text{if } \mu(t) > 0. \end{cases}$$

Several useful delta derivative formulas can be recorded in [17, Lemma 1,2] (see also [16]).

A function $f : \mathbb{T} \to \mathbf{R}$ is said to be rd-continuous, provided it is continuous at every right-dense point and if the left sided limit exists at every left dense point. We denote by $C_{rd}(\mathbb{T},\mathbf{R})$ the set of all rd-continuous functions $f : \mathbb{T} \to \mathbf{R}$. The importance of rd-continuous functions is revealed by the following existence result by Hilger [5]: Every rd-continuous function possess an antiderivative. Here, F is called an antiderivative of a function f defined on $\mathbb{T}$ if $F^{\Delta} = f$ holds on $\mathbb{T}^k$. In this case we define an integral by:

$$\int_s^t f(\tau)\,\Delta\tau = F(t) - F(s), \quad \text{for } s,t \in \mathbb{T}.$$

Let $\mathbb{T}_1$ and $\mathbb{T}_2$ be two given time scales and set $\mathbb{T}_1\times\mathbb{T}_2 = \{(x,y) : x \in \mathbb{T}_1, y \in \mathbb{T}_2\}$. Then $\mathbb{T}_1\times\mathbb{T}_2$ is a complete metric space with metric d defined by:

$$d((x,y),(\acute{x},\acute{y})) = \sqrt{(x-\acute{x})^2 + (y-\acute{y})^2} \ \text{ for all } \ (x,y),(\acute{x},\acute{y}) \in \mathbb{T}_1\times\mathbb{T}_2.$$

Let f be a real-valued function on $\mathbb{T}_1\times\mathbb{T}_2$. The function f is called rd-continuous in t_2 if for every $\alpha_1 \in \mathbb{T}_1$, the function $f(\alpha_1,t_2)$ is rd-continuous on $\mathbb{T}_2$. The function f is called rd-continuous in t_1 if for every $\alpha_2 \in \mathbb{T}_2$, the function $f(t_1,\alpha_2)$ is rd-continuous on $\mathbb{T}_1$.

Let CC_{rd} denote the set of functions $f(t_1,t_2)$ on $\mathbb{T}_1\times\mathbb{T}_2$ with the properties:

- f is rd-continuous in t_1,
- f is rd-continuous in t_2,
- if $(x_1,x_2) \in \mathbb{T}_1\times\mathbb{T}_2$ with x_1,x_2 right-dense points, then f is continuous at (x_1,x_2),
- if x_1 and x_2 are both left-dense, then the limit of $f(t_1,t_2)$ exists as (t_1,t_2) approaches (x_1,x_2) along any path in the region

$$R(x_1,x_2) = \{(t_1,t_2) : t_1 \in [a,x_1]_{\mathbb{T}_1}, t_2 \in [c,x_2]_{\mathbb{T}_2}\}.$$

Let CC_{rd}^1 be the set of all functions in CC_{rd} for which both the partial derivatives, Δ_i, $1 \le i \le 2$, with respect to first and second components of the function f respectively, exist and are in CC_{rd}. In 2005 P. Řehák [15] proved a time scale version of Hardy's inequality as:

$$\int_a^{\infty} \left[\frac{\int_a^{\sigma(t)} f(s)\,\Delta s}{\sigma(t)-a}\right]^p \Delta t \le \left[\frac{p}{p-1}\right]^p \int_a^{\infty} [f(t)]^p\,\Delta t, \tag{2.1}$$

where, $p > 1$ and f is a non-negative function. Let A_K be a general Hardy type operator on time scale defined by:

$$A_K f(t,s) = \frac{1}{K(t,s)} \int_a^t k(s,y)\, f(y)\,\Delta y, \tag{2.2}$$

where, $t,s>a$, $f\in C_{rd}([a,b),\mathbf{R})$ is a delta integrable function, $k(x,y)\in CC^1_{rd}([a,b)\times[c,d),\mathbf{R})$ is a non-negative delta integrable function and

$$K(t,s)=\int_a^t k(s,y)\,\Delta y.$$

Recently, Özkan et al. [12] proved a strengthened time scale Hardy-Knopp-type inequality as:

Theorem 2.1. *Let $u\in C_{rd}([a,b),\mathbf{R})$ be a non-negative function such that the delta integral $\int_t^b \frac{u(x)}{(x-a)(\sigma(x)-a)}\Delta x$ exists as a finite number. Consider the weight function v defined by:*

$$v(t)=(t-a)\int_t^b \frac{u(x)}{(x-a)(\sigma(x)-a)}\Delta x,\qquad t\in[a,b).$$

If $\Phi:(c,d)\to\mathbf{R}$ is continuous and convex, where $c,d\in\mathbf{R}$, then the inequality

$$\int_a^b u(x)\,\Phi\left[\frac{1}{\sigma(x)-a}\int_a^{\sigma(x)} f(t)\,\Delta t\right]\frac{\Delta x}{x-a}\leq\int_a^b v(x)\Phi[f(x)]\frac{\Delta x}{x-a},\tag{2.3}$$

holds for all delta integrable functions $f\in C_{rd}([a,b),\mathbf{R})$ such that $f(x)\in(c,d)$.

Moreover in [11], U. M. Özkan et al. extended (2.1) and (2.3) to a Hardy-Knopp-type integral inequality for several functions and a Hardy-Knopp-type integral inequality with a general kernel to arbitrary time scales as:

Theorem 2.2. *Let the conditions of Theorem 2.1 be satisfied and, let $f_1,\dots,f_k$ be non-negative delta integrable functions such that $F_k(x)=\frac{1}{x-a}\int_a^x f_k(s)\Delta s$, $1\leq k\leq n$, exists as a finite number, then*

$$\int_a^b u(x)\,\Phi\left[\left(\prod_{k=1}^n F_k^{\sigma}(x)\right)^{1/n}\right]\frac{\Delta x}{x-a}\leq\int_a^b v(x)\Phi\left[\frac{\sum_{k=1}^n f_k(x)}{n}\right]\frac{\Delta x}{x-a},\tag{2.4}$$

holds for all delta integrable functions $f_k\in C_{rd}([a,b),\mathbf{R})$ such that $f_k(x)\in(c,d)$.

Theorem 2.3. *Suppose $k(x,y)\in CC^1_{rd}([a,b)\times[c,d),\mathbf{R})$ and $u\in C_{rd}([a,b),\mathbf{R})$ are non-negative functions such that the delta integral $\int_y^b \frac{k(x,y)}{K^{\sigma}(x,x)}\,u(x)\,\frac{\Delta x}{x-a}$ exists as a finite number. Consider the weight function v defined by:*

$$v(y)=(y-a)\int_y^b \frac{k(x,y)}{K^{\sigma}(x,x)}\,u(x)\,\frac{\Delta x}{x-a},\quad y\in[a,b).$$

If $\Phi:(c,d)\to\mathbf{R}$ is continuous and convex, where $c,d\in\mathbf{R}$, then

$$\int_a^b u(x)\,\Phi\left[A_K f^{\sigma}(x,x)\right]\frac{\Delta x}{x-a}\leq\int_a^b v(x)\Phi\left[f(x)\right]\frac{\Delta x}{x-a},$$

holds for all delta integrable functions $f\in C_{rd}([a,b),\mathbf{R})$ such that $f(x)\in(c,d)$.

The following consequences of these results were also discussed in [11, 12].

$$\int_a^b \left[\prod_{k=1}^n F_k^\sigma(x)\right]^{p/n} \frac{\Delta x}{x-a} \le \frac{1}{n^p(b-a)} \int_a^b (b-x)\left[\sum_{k=1}^n f_k(x)\right]^p \frac{\Delta x}{x-a}. \tag{2.5}$$

$$\int_a^b \left[\frac{1}{\sigma(x)-a}\int_a^{\sigma(x)} f(t)\,\Delta t\right]^p \frac{\Delta x}{x-a} \le \frac{1}{b-a}\int_a^b (b-x)[f(x)]^p \frac{\Delta x}{x-a}, \tag{2.6}$$

for $p > 1$ and $f \ge 0$, such that the delta integral $\int_a^b [f(x)]^p \frac{\Delta x}{x-a}$ exists as a finite number.

$$\int_a^b \exp\left[\frac{1}{\sigma(x)-a}\int_a^{\sigma(x)} \ln f(t)\,\Delta t\right] \frac{\Delta x}{x-a} \le \frac{1}{b-a}\int_a^b (b-x) f(x) \frac{\Delta x}{x-a}, \tag{2.7}$$

for $f > 0$, such that the delta integral $\int_a^b f(x) \frac{\Delta x}{x-a}$ exists as a finite number.

3 Log-Convexity Of Hardy-Polya-Knopp Type Differences

Definition 3.1. [2] A function $f : \mathbb{T} \to \mathbf{R}$ is called convex on $I_{\mathbb{T}}$, if

$$f(\lambda\, t + (1-\lambda)\, s) \le \lambda\, f(t) + (1-\lambda)\, f(s), \tag{3.1}$$

for all $s, t \in I_{\mathbb{T}}$ and all $\lambda \in [0,1]$ such that $\lambda t + (1-\lambda)s \in I_{\mathbb{T}}$. The function f is strictly convex on $I_{\mathbb{T}}$ if the inequality (3.1) is strict for distinct $s, t \in I_{\mathbb{T}}$ and $\lambda \in (0,1)$. The function f is concave (respectively, strictly concave) on $I_{\mathbb{T}}$, if $-f$ is convex (respectively, strictly convex).

Lemma 3.2. *[7] Consider the function:*

$$\varphi_s(x) = \begin{cases} \frac{x^s}{s(s-1)}, & s \ne 0, 1; \\ -\log x, & s = 0; \\ x \log x, & s = 1. \end{cases}$$

Then, $\varphi_s(x)$ is convex for $x > 0$.

Lemma 3.3. *[7] Consider another function:*

$$\psi_s(x) = \begin{cases} \frac{1}{s^2} e^{sx}, & s \ne 0; \\ \frac{1}{2} x^2, & s = 0. \end{cases}$$

Then, $\psi_s(x)$ is convex.

The following lemma is equivalent to the definition of convex function [14, p.2].

Lemma 3.4. *If ϕ is continuous and convex for all s_1, s_2, s_3 in an open interval I for which $s_1 < s_2 < s_3$, then*

$$\phi(s_1)(s_3 - s_2) + \phi(s_2)(s_1 - s_3) + \phi(s_3)(s_2 - s_1) \ge 0.$$

The following lemma from log −convexity theory, playing a key role, is given as:

Lemma 3.5. *[13] A positive function f is* log −*convex in Jensen sense on an open interval I, that is, for each $s, t \in I$*

$$f(s)\,f(t) \geq f^2\left(\frac{s+t}{2}\right)$$

if and only if the relation

$$m^2 f(s) + 2\,m\,w\,f\left(\frac{s+t}{2}\right) + w^2 f(t) \geq 0,$$

holds for each real m, w and s, $t \in I$.

Theorem 3.6. *Let the conditions of Theorem 2.2 be satisfied, and let φ_s be given by Lemma 3.2. Consider $F : \mathbf{R} \to \mathbf{R}_+$, defined by:*

$$F(s) = \int_a^b v(x)\,\varphi_s\left[\frac{\sum_{k=1}^n f_k(x)}{n}\right]\frac{\Delta x}{x-a} - \int_a^b u(x)\,\varphi_s\left[\left(\prod_{k=1}^n F_k^{\sigma}(x)\right)^{1/n}\right]\frac{\Delta x}{x-a}. \tag{3.2}$$

Then, F is log −*convex, that is, the following inequality is valid:*

$$[F(p)]^{r-s} \leq\; [F(r)]^{p-s}\,[F(s)]^{r-p} \quad for\;\; -\infty < s < p < r < \infty. \tag{3.3}$$

Proof. Let us consider the function Φ defined by

$$\Phi(x,m,w,r,s,p) \equiv \Phi(x) = m^2\varphi_s(x) + 2\,m\,w\,\varphi_r(x) + w^2\varphi_p(x), \;\text{ where }\; r = \frac{s+p}{2};\; m,w \in \mathbb{R}.$$

$$\Phi''(x) = m^2\,x^{s-2} + 2\,m\,w\,x^{r-2} + w^2 x^{p-2} = (m\,x^{\frac{s}{2}-1} + w\,x^{\frac{p}{2}-1})^2 \geq\; 0.$$

Φ is convex for $x \in \mathbf{R}_+$; therefore (3.2) is equivalent to

$$m^2 F(s) + 2\,m\,wF(r) + w^2 F(p) \geq 0,$$

i.e., by Lemma 3.5

$$[F(r)]^2 \leq F(s)\,F(p).$$

So F is log −convex in Jensen sense. Since

$$\lim_{s\to 0} F(s) = F(0) \;\text{ and }\; \lim_{s\to 1} F(s) = F(1),$$

F is continuous for $s \in \mathbf{R}$ and therefore $\log F$ is convex. Lemma 3.4 for $-\infty < s < p < r < \infty$ yields:

$$(r-s)\log F(p) \leq (r-p)\log F(s) + (p-s)\log F(r),$$

which is equivalent to (3.3). □

Theorem 3.7. *Let the conditions of Theorem 2.3 be satisfied, and let φ_s be given by Lemma 3.2. Consider $\widetilde{F} : \mathbf{R} \to \mathbf{R}_+$, defined by:*

$$\widetilde{F}(s) = \int_a^b v(x)\,\varphi_s\,[f(x)]\,\frac{\Delta x}{x-a} - \int_a^b u(x)\,\varphi_s\,[A_k f^{\sigma}(x,x)]\,\frac{\Delta x}{x-a}. \tag{3.4}$$

Then, $\widetilde{F}$ is log −*convex, that is, the following inequality is valid:*

$$[\widetilde{F}(p)]^{r-s} \leq\; [\widetilde{F}(r)]^{p-s}\,[\widetilde{F}(s)]^{r-p} \quad for\;\; -\infty < s < p < r < \infty. \tag{3.5}$$

If we use ψ_s for φ_s, we get the followings.

Theorem 3.8. *Let the conditions of Theorem 2.2 be satisfied, and let ψ_s be given by Lemma 3.3. Consider $G : \mathbf{R} \to \mathbf{R}_+$, defined by:*

$$G(s) = \int_a^b v(x)\, \psi_s\,[f(x)]\,\frac{\Delta x}{x-a} - \int_a^b u(x)\, \psi_s\,[A_k f^{\sigma}(x,x)]\,\frac{\Delta x}{x-a}. \tag{3.6}$$

Then, G is $\log-$*convex, that is, the following inequality is valid:*

$$[G(p)]^{r-s} \leq\; [G(r)]^{p-s}\,[G(s)]^{r-p} \quad \textit{for} \quad -\infty < s < p < r < \infty. \tag{3.7}$$

4 Improvements And Reverses of Hardy-Polya-Knopp Type Inequality

Theorem 4.1. *Let $p \in \mathbf{R}\backslash\{0,1\}$ and, let f_k, $1 \leq k \leq n$, be a family of positive delta integrable functions such that $F_k(x) = \frac{1}{x-a}\int_a^x f_k(s)\Delta s$ exists as a finite number, then*

$$\frac{1}{p\,(p-1)}\left\{\frac{1}{n^p(b-a)}\int_a^b (b-x)\left[\sum_{k=1}^n f_k(x)\right]^p\frac{\Delta x}{x-a} - \int_a^b\left[\prod_{k=1}^n F_k^{\sigma}(x)\right]^{p/n}\frac{\Delta x}{x-a}\right\}$$
$$\leq [H(s)]^{(p-r)/(s-r)}\,[H(r)]^{(s-p)/(s-r)} \quad \textit{for} \quad -\infty < s < p < r < \infty. \tag{4.1}$$

$$\frac{1}{p\,(p-1)}\left\{\frac{1}{n^p(b-a)}\int_a^b (b-x)\left[\sum_{k=1}^n f_k(x)\right]^p\frac{\Delta x}{x-a} - \int_a^b\left[\prod_{k=1}^n F_k^{\sigma}(x)\right]^{p/n}\frac{\Delta x}{x-a}\right\}$$
$$\geq [H(s)]^{(p-r)/(s-r)}\,[H(r)]^{(s-p)/(s-r)} \tag{4.2}$$

for $-\infty < p < r < s < \infty$ and $-\infty < r < s < p < \infty$, where

$$H(r) = \frac{1}{b-a}\int_a^b (b-x)\,\varphi_r\left[\frac{\sum_{k=1}^n f_k(x)}{n}\right]\frac{\Delta x}{x-a} - \int_a^b \varphi_r\left[\left(\prod_{k=1}^n F_k^{\sigma}(x)\right)^{1/n}\right]\frac{\Delta x}{x-a}. \tag{4.3}$$

Proof. The proof follows from Theorem 3.6 by choosing the weight function $u(x) \equiv 1$, so that

$$v(x) = \begin{cases} (x-a)\int_x^b \frac{\Delta t}{(t-a)(\sigma(t)-a)} = \frac{b-x}{b-a}, & b < \infty; \\ 1, & b = \infty. \end{cases}$$

In this case (3.2) becomes

$$F(s) = \begin{cases} \frac{1}{(b-a)}\int_a^b (b-x)\,\varphi_s\left[\frac{\sum_{k=1}^n f_k(x)}{n}\right]\frac{\Delta x}{x-a} - \int_a^b \varphi_s\left[\left(\prod_{k=1}^n F_k^{\sigma}(x)\right)^{1/n}\right]\frac{\Delta x}{x-a}, & b < \infty; \\ \int_a^\infty \varphi_s\left[\frac{\sum_{k=1}^n f_k(x)}{n}\right]\frac{\Delta x}{x-a} - \int_a^\infty \varphi_s\left[\left(\prod_{k=1}^n F_k^{\sigma}(x)\right)^{1/n}\right]\frac{\Delta x}{x-a}. \end{cases}$$

Now for $b<\infty$ (3.3) becomes

$$\frac{1}{b-a}\int_a^b (b-x)\,\varphi_p\left[\frac{\sum_{k=1}^n f_k(x)}{n}\right]\frac{\Delta x}{x-a} - \int_a^b \varphi_p\left[\left(\prod_{k=1}^n F_k^\sigma(x)\right)^{1/n}\right]\frac{\Delta x}{x-a} \\ \leq [H(s)]^{(p-r)/(s-r)}\,[H(r)]^{(s-p)/(s-r)}. \quad (4.4)$$

For $p\in\mathbf{R}\backslash\{0,1\}$ and $b<\infty$ we get (4.1) and

$$\frac{1}{p\,(p-1)}\left\{n^{-p}\int_a^\infty\left[\sum_{k=1}^n f_k(x)\right]^p\frac{\Delta x}{x-a} - \int_a^\infty\left[\prod_{k=1}^n F_k^\sigma(x)\right]^{p/n}\frac{\Delta x}{x-a}\right\} \\ \leq [\widehat{H}(s)]^{(p-r)/(s-r)}\,[\widehat{H}(r)]^{(s-p)/(s-r)} \quad \text{for } -\infty<s<p<r<\infty,$$

where,

$$\widehat{H}(r)=\int_a^\infty \varphi_r\left[\frac{\sum_{k=1}^n f_k(x)}{n}\right]\frac{\Delta x}{x-a} - \int_a^\infty \varphi_r\left[\left(\prod_{k=1}^n F_k^\sigma(x)\right)^{1/n}\right]\frac{\Delta x}{x-a}.$$

If in (3.3) $s\to r,\ p\to s,\ r\to p$ and $s\to p,\ p\to r,\ r\to s$, then

$$\frac{1}{b-a}\int_a^b (b-x)\,\varphi_p\left[\frac{\sum_{k=1}^n f_k(x)}{n}\right]\frac{\Delta x}{x-a} - \int_a^b \varphi_p\left[\left(\prod_{k=1}^n F_k^\sigma(x)\right)^{1/n}\right]\frac{\Delta x}{x-a} \\ \geq [H(s)]^{(p-r)/(s-r)}\,[H(r)]^{(s-p)/(s-r)}. \quad (4.5)$$

And from here for $p\in\mathbf{R}\backslash\{0,1\}$ we get (4.2) □

Theorem 4.2. *Let $p\in\mathbf{R}\backslash\{0,1\}$ and, let f be a non-negative delta integrable function such that the delta integral $\int_a^b (b-x)[f(x)]^p\frac{\Delta x}{x-a}$ exists as a finite number, then*

$$\frac{1}{p\,(p-1)}\left\{\frac{1}{b-a}\int_a^b (b-x)[f(x)]^p\frac{\Delta x}{x-a} - \int_a^b\left[\frac{1}{\sigma(x)-a}\int_a^{\sigma(x)} f(t)\Delta t\right]^p\frac{\Delta x}{x-a}\right\} \\ \leq [\widetilde{H}(s)]^{(p-r)/(s-r)}\,[\widetilde{H}(r)]^{(s-p)/(s-r)} \quad \textit{for} \quad \infty<s<p<r<\infty. \quad (4.6)$$

$$\frac{1}{p\,(p-1)}\left\{\frac{1}{b-a}\int_a^b (b-x)[f(x)]^p\frac{\Delta x}{x-a} - \int_a^b\left[\frac{1}{\sigma(x)-a}\int_a^{\sigma(x)} f(t)\Delta t\right]^p\frac{\Delta x}{x-a}\right\} \\ \geq [\widetilde{H}(s)]^{(p-r)/(s-r)}\,[\widetilde{H}(r)]^{(s-p)/(s-r)} \quad (4.7)$$

for $-\infty<p<r<s<\infty$ and $-\infty<r<s<p<\infty$, where

$$\widetilde{H}(r)=\frac{1}{b-a}\int_a^b (b-x)\varphi_r\,[f(x)]\,\frac{\Delta x}{x-a} - \int_a^b \varphi_r\left[\frac{1}{\sigma(x)-a}\int_a^{\sigma(x)} f(t)\,\Delta t\right]\frac{\Delta x}{x-a}. \quad (4.8)$$

Proof. The proof follows from Theorem 3.7 by choosing $k(x,y) \equiv 1 \equiv u(x)$, so that

$$A_K f^{\sigma}(x,x) = \frac{1}{\sigma(x)-a} \int_a^{\sigma(x)} f(t)\, \Delta t.$$

$$v(x) = \begin{cases} \frac{b-x}{b-a}, & b < \infty; \\ 1, & b = \infty. \end{cases}$$

In this case (3.4) becomes

$$\widetilde{F}(s) = \begin{cases} \frac{1}{b-a}\int_a^b (b-x)\, \varphi_s[f(x)]\frac{\Delta x}{x-a} - \int_a^b \varphi_s\left[\frac{1}{\sigma(x)-a}\int_a^{\sigma(x)} f(t)\, \Delta t\right]\frac{\Delta x}{x-a}, & b < \infty; \\ \int_a^{\infty} \varphi_s[f(x)]\frac{\Delta x}{x-a} - \int_a^{\infty} \varphi_s\left[\frac{1}{\sigma(x)-a}\int_a^{\sigma(x)} f(t)\, \Delta t\right]\frac{\Delta x}{x-a}. \end{cases}$$

Now for $b < \infty$ (3.5) becomes

$$\frac{1}{b-a}\int_a^b (b-x)\, \varphi_p[f(x)]\frac{\Delta x}{x-a} - \int_a^b \varphi_p\left[\frac{1}{\sigma(x)-a}\int_a^{\sigma(x)} f(t)\, \Delta t\right]\frac{\Delta x}{x-a} \\ \leq [\widetilde{H}(s)]^{(p-r)/(s-r)}\, [\widetilde{H}(r)]^{(s-p)/(s-r)}. \quad (4.9)$$

From here for $p \in \mathbf{R}\backslash\{0,1\}$ we get (4.6).
If in (3.5) $s \to r,\ p \to s,\ r \to p$ and $s \to p,\ p \to r,\ r \to s$, then

$$\frac{1}{b-a}\int_a^b (b-x)\, \varphi_p[f(x)]\frac{\Delta x}{x-a} - \int_a^b \varphi_p\left[\frac{1}{\sigma(x)-a}\int_a^{\sigma(x)} f(t)\, \Delta t\right]\frac{\Delta x}{x-a} \\ \geq [\widetilde{H}(s)]^{(p-r)/(s-r)}\, [\widetilde{H}(r)]^{(s-p)/(s-r)}. \quad (4.10)$$

And from here for $p \in \mathbf{R}\backslash\{0,1\}$ we get (4.7). □

Theorem 4.3. *Let f be a positive function such that the delta integral $\int_a^b (b-x)f(x) \times \frac{\Delta x}{x-a}$ exists as a finite number, then*

$$\frac{1}{b-a}\int_a^b (b-x)f(x)\frac{\Delta x}{x-a} - \int_a^b \exp\left[\frac{1}{\sigma(x)-a}\int_a^{\sigma(x)} \ln f(t)\, \Delta t\right]\frac{\Delta x}{x-a} \\ \leq [\widetilde{P}(s)]^{(1-r)/(s-r)}\, [\widetilde{P}(r)]^{(s-1)/(s-r)} \quad \text{for } -\infty < s < 1 < r < \infty. \quad (4.11)$$

$$\frac{1}{b-a}\int_a^b (b-x)f(x)\frac{\Delta x}{x-a} - \int_a^b \exp\left[\frac{1}{\sigma(x)-a}\int_a^{\sigma(x)} \ln f(t)\, \Delta t\right]\frac{\Delta x}{x-a} \\ \geq [\widetilde{P}(s)]^{(1-r)/(s-r)}\, [\widetilde{P}(r)]^{(s-1)/(s-r)} \quad (4.12)$$

for $-\infty < 1 < r < s < \infty$ and $-\infty < r < s < 1 < \infty$, where

$$\widetilde{P}(r) = \frac{1}{b-a}\int_a^b (b-x)\, \psi_r[\ln f(x)]\frac{\Delta x}{x-a} - \\ \int_a^b \psi_r\left[\frac{1}{\sigma(x)-a}\int_a^{\sigma(x)} \ln f(t)\Delta t\right]\frac{\Delta x}{x-a}. \quad (4.13)$$

Proof. The proof follows from Theorem 3.8 by choosing $k(x,y) \equiv 1 \equiv u(x)$, so that

$$G(s) = \begin{cases} \frac{1}{b-a}\int_a^b (b-x)\,\psi_s[f(x)]\frac{\Delta x}{x-a} - \int_a^b \psi_s\left[\frac{1}{\sigma(x)-a}\int_a^{\sigma(x)} f(t)\,\Delta t\right]\frac{\Delta x}{x-a}, & b<\infty; \\ \int_a^\infty \psi_s[f(x)]\frac{\Delta x}{x-a} - \int_a^\infty \psi_s\left[\frac{1}{\sigma(x)-a}\int_a^{\sigma(x)} f(t)\,\Delta t\right]\frac{\Delta x}{x-a}. & \end{cases}$$

From here for $b < \infty$ (3.7) becomes

$$\frac{1}{b-a}\int_a^b (b-x)\,\psi_p[f(x)]\frac{\Delta x}{x-a} - \int_a^b \psi_p\left[\frac{1}{\sigma(x)-a}\int_a^{\sigma(x)} f(t)\,\Delta t\right]\frac{\Delta x}{x-a} \leq [P(s)]^{(p-r)/(s-r)}\,[P(r)]^{(s-p)/(s-r)}, \quad (4.14)$$

where,

$$P(r) = \frac{1}{b-a}\int_a^b (b-x)\,\psi_r[f(x)]\frac{\Delta x}{x-a} - \int_a^b \psi_r\left[\frac{1}{\sigma(x)-a}\int_a^{\sigma(x)} f(t)\,\Delta t\right]\frac{\Delta x}{x-a}.$$

By setting $f(x) \mapsto \ln f(x)$ and $p = 1$ in (4.14) we get (4.11).
If in (3.7) $s \to r,\ p \to s,\ r \to p$ and $s \to p,\ p \to r,\ r \to s$, then

$$\frac{1}{b-a}\int_a^b (b-x)\,\psi_p[f(x)]\frac{\Delta x}{x-a} - \int_a^b \psi_p\left[\frac{1}{\sigma(x)-a}\int_a^{\sigma(x)} f(t)\,\Delta t\right]\frac{\Delta x}{x-a} \geq [P(s)]^{(p-r)/(s-r)}\,[P(r)]^{(s-p)/(s-r)}. \quad (4.15)$$

By setting $f(x) \mapsto \ln f(x)$ and $p = 1$ in (4.15) we get (4.12) □

Remark 4.4. In fact in this paper more general results have been proved. Namely (4.4), (4.9) and (4.14) are valid for $-\infty < s < p < r < \infty$. The inequalities (4.5), (4.10) and (4.15) are valid for $-\infty < r < s < p < \infty$ and $-\infty < p < r < s < \infty$.

Acknowledgments

The author thanks the referee for valuable comments and suggestions, which he has used to improve the final version of this paper.

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