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Lower Bound for Sharp Constants of Brézis-Gallouët-Wainger Type Inequalities in Higher-Order Critical Sobolev Spaces on Bounded Domains

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Abstract

As an application of our previous paper, we give a sufficient condition that Brézis-Gallouët-Wainger type inequalities in higher order critical Sobolev spaces fails.

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1 Introduction and known results

We shall provide a lower bound for sharp constants of Brézis-Gallouët-Wainger type inequalities in Besov spaces and Triebel-Lizorkin spaces as well as fractional Sobolev spaces on a bounded domain $\Omega \subset \mathbb{R}^n$ as an application of our previous results. Throughout the present paper, we place ourselves in the setting of $\mathbb{R}^n$ with $n \geq 2$. We treat only real-valued functions.

Before we state the full version of our main theorem, we present it in a simpler form. Let $\omega_{n-1} = 2\pi^{n/2}/\Gamma(n/2)$ denote the surface area of $S^{n-1} = \{x \in \mathbb{R}^n; |x| = 1\}$.

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Theorem 1.1. *Let $n \geq 2$, $k \in \{1,2,\ldots,n-1\}$, $0 < \alpha < s$, and Ω be a bounded domain in $\mathbb{R}^n$. Define*

$$\ell_k = k! \sum_{\substack{j\in\mathbb{Z} \\ 0\leq j\leq k/2}} (k-2j)!\, j! \left(\sum_{\substack{l\in\mathbb{Z} \\ k/2\leq l\leq k-j}} 2^{2l-k+j} \frac{(-1)^l}{2l} \binom{l}{k-l} \binom{k-l}{j} \right)^2 \prod_{m=1}^{j} \left(\frac{n-3}{2} + m \right). \tag{1.1}$$

Here, we regard the empty product $\prod_{m=1}^{0} *$ as 1. Assume that either

$$\lambda_1 < \frac{1}{\ell_k^{n/(2(n-k))} \omega_{n-1}^{k/(n-k)} \alpha} \text{ and } \lambda_2 \in \mathbb{R}$$

$$\text{or} \quad \lambda_1 = \frac{1}{\ell_k^{n/(2(n-k))} \omega_{n-1}^{k/(n-k)} \alpha} \text{ and } \lambda_2 < \frac{k}{n \ell_k^{n/(2(n-k))} \omega_{n-1}^{k/(n-k)} \alpha}$$

holds. Then for any constant C, the inequality

$$\|u\|_{L^\infty(\Omega)}^{n/(n-k)} \leq \lambda_1 \log(1 + \|u\|_{W^{s,n/(s-\alpha)}(\Omega)}) + \lambda_2 \log(1 + \log(1 + \|u\|_{W^{s,n/(s-\alpha)}(\Omega)})) + C$$

fails for some $u \in C_c^\infty$ with $\|\nabla^k u\|_{L^{k,n/k}(\Omega)} = 1$, where

$$\|\nabla^k u\|_{L^{n/k}(\Omega)} = \| |\nabla^k u| \|_{L^{n/k}(\Omega)}, \ |\nabla^k u| = \left(\sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \cdots \sum_{i_k=1}^{n} \left(\frac{\partial}{\partial x_{i_1}} \frac{\partial}{\partial x_{i_2}} \cdots \frac{\partial}{\partial x_{i_k}} u \right)^2 \right)^{1/2}. \tag{1.2}$$

To consider what we learn from Theorem 1.1, we first recall the Sobolev embedding theorem in the critical case. For $1 < q < \infty$, it is well known that the embedding $W^{n/q,q}(\mathbb{R}^n) \hookrightarrow L^r(\mathbb{R}^n)$ holds for any $q \leq r < \infty$, and fails for $r = \infty$, i.e., one cannot estimate the L^∞-norm by the $W^{n/q,q}$-norm. However, the Brézis-Gallouët-Wainger inequality states that the L^∞-norm can be estimated by the $W^{n/q,q}$-norm with the partial aid of the $W^{s,p}$-norm with $s > n/p$ and $1 \leq p \leq \infty$ as follows:

$$\|u\|_{L^\infty(\mathbb{R}^n)}^{q/(q-1)} \leq \lambda(1 + \log(1 + \|u\|_{W^{s,p}(\mathbb{R}^n)})) \tag{1.3}$$

holds whenever $u \in W^{n/q,q}(\mathbb{R}^n) \cap W^{s,p}(\mathbb{R}^n)$ satisfies $\|u\|_{W^{n/q,q}(\mathbb{R}^n)} = 1$, where $1 \leq p \leq \infty$, $1 < q < \infty$ and $s > n/p$. The inequality (1.3) for the case $n = p = q = s = 2$ dates back to Brézis-Gallouët [1]. Later on, Brézis-Wainger [2] obtained (1.3) for the general case, and remarked that the power $q/(q-1)$ in (1.3) is maximal; (1.3) fails for any larger power. An attempt of replacing $\|u\|_{W^{s,p}(\mathbb{R}^n)}$ with the other norms has been made in several papers. We also mention that (1.3) was obtained in the Besov-Morrey spaces in [13].

In what follows, we concentrate on the case $n/q = k \in \{1,2,\ldots,n-1\}$, and replace the function space $W^{n/q,q}(\mathbb{R}^n)$ by $W^{k,n/k}(\Omega) \cap C_c(\Omega)$ with an arbitrary bounded domain Ω in $\mathbb{R}^n$. Note that the norm of $W^{k,n/k}(\Omega) \cap C_c(\Omega)$ is equivalent to $\|\nabla^k u\|_{L^{n/k}(\Omega)}$ because of the Poincaré inequality. When $k = 1$, the differential order $s = m$ is an integer with $1 \leq m \leq n$, and $n/m < p \leq n/(m-1)$, the authors together with T. Sato and H. Wadade [5, 7] generalized the inequality corresponding to (1.3) and discussed how optimal the constant λ is. To describe the sharpness of the constant λ, they made a formulation more precise. We set up the following problem in a fixed function space $X(\Omega)$, which is contained in $L^\infty(\Omega)$:

Fix a function space $X(\Omega)$. For given constants $\lambda_1 > 0$ and $\lambda_2 \in \mathbb{R}$, does there exist a constant C such that

$$\|u\|_{L^\infty(\Omega)}^{n/(n-k)} \le \lambda_1 \log(1+\|u\|_{X(\Omega)}) + \lambda_2 \log(1+\log(1+\|u\|_{X(\Omega)})) + C \tag{1.4}$$

holds for all $u \in W^{k,n/k}(\Omega) \cap X(\Omega) \cap C_c(\Omega)$ under the normalization $\|\nabla^k u\|_{L^{n/k}(\Omega)} = 1$?

We call $X(\Omega)$ *an auxiliary space* of (1.4). First we state the following proposition, which is an immediate consequence of an elementary inequality

$$\log(1+st) \le \log(s+st) = \log(1+t) + \log s \quad \text{for } t \ge 0,\ s \ge 1. \tag{1.5}$$

See [5, p4. Proposition 1.2] for the detailed proof of the next proposition.

Proposition 1.2. *Let Ω be a domain in $\mathbb{R}^n$, and $X_1(\Omega)$, $X_2(\Omega)$ be function spaces satisfying*

$$\|u\|_{X_1(\Omega)} \le M\|u\|_{X_2(\Omega)} \quad \text{for } u \in X_2(\Omega)$$

with some constant $M \ge 1$.

(i) *If the inequality* (1.4) *holds in $X(\Omega) = X_1(\Omega)$ with a constant C, then so does* (1.4) *in $X(\Omega) = X_2(\Omega)$ with another constant C,*

or equivalently,

(ii) *If the inequality* (1.4) *fails in $X(\Omega) = X_2(\Omega)$ with any constant C, then so does* (1.4) *in $X(\Omega) = X_1(\Omega)$ with any constant C.*

From the proposition, the sharp constants for λ_1 and λ_2 in (1.4) are independent of the choice of the equivalent norms of the auxiliary space $X(\Omega)$. On the other hand, note that these sharp constants may depend on the definition of $\|\nabla^k u\|_{L^{n/k}(\Omega)}$; there are several manners to define $\|\nabla^k u\|_{L^{n/k}(\Omega)}$. We choose (1.2) as the definition of $\|\nabla^k u\|_{L^{n/k}(\Omega)}$ throughout this paper.

In the present paper we shall include Besov spaces and Triebel-Lizorkin spaces as an auxiliary space $X(\Omega)$. We shall describe their definitions in Section 2.

In what follows, we denote

$$p_{\alpha,s} = \begin{cases} \dfrac{n}{s-\alpha} & \text{for } s > \alpha, \\ \infty & \text{for } s = \alpha. \end{cases}$$

We fix functions $\psi^0, \varphi^0 \in C_c^\infty(\mathbb{R}^n)$ which are supported in the ball $\overline{B_4}$, in the annulus $\overline{B_4} \setminus B_1$, respectively, and satisfying

$$\sum_{k=-\infty}^{\infty} \varphi_k^0(x) = \chi_{\mathbb{R}^n \setminus \{0\}}(x),\ \psi^0(x) = 1 - \sum_{k=0}^{\infty} \varphi_k^0(x) \quad \text{for } x \in \mathbb{R}^n, \tag{1.6}$$

where we set $\varphi_k^0 = \varphi^0(\cdot/2^k)$. Here, χ_E is the characteristic function of a set E, and $C_c^\infty(\Omega)$ denotes the class of compactly supported C^∞ functions on Ω. We also denote by $C_c(\Omega)$ the class of compactly supported continuous functions on Ω.

Definition 1.3. Take ψ^0, φ^0 satisfying (1.6) and let $u \in \mathcal{S}'(\mathbb{R}^n)$.

(i) Let $0 < s < \infty$, $0 < p \leq \infty$ and $0 < q \leq \infty$. The Besov space $B^{s,p,q}(\mathbb{R}^n)$ is normed by

$$\|u\|_{B^{s,p,q}(\mathbb{R}^n)} = \|\psi^0(D)u\|_{L^p(\mathbb{R}^n)} + \left(\sum_{k=0}^{\infty}\left(2^{sk}\|\varphi_k^0(D)u\|_{L^p(\mathbb{R}^n)}\right)^q\right)^{1/q} \tag{1.7}$$

with the obvious modification when $q = \infty$.

(ii) Let $0 < s < \infty$, $0 < p < \infty$ and $0 < q \leq \infty$. The Triebel-Lizorkin space $F^{s,p,q}(\mathbb{R}^n)$ is normed by

$$\|u\|_{F^{s,p,q}(\mathbb{R}^n)} = \|\psi^0(D)u\|_{L^p(\mathbb{R}^n)} + \left\|\left(\sum_{k=0}^{\infty}\left(2^{sk}|\varphi_k^0(D)u|\right)^q\right)^{1/q}\right\|_{L^p(\mathbb{R}^n)} \tag{1.8}$$

with the obvious modification when $q = \infty$; one excludes the case $p = \infty$.

Different choices of ψ^0 and φ^0 satisfying (1.6) yield equivalent norms in (1.7) and (1.8). We refer to [9] for exhaustive details of this fact. Here and below, we denote by $A^{s,p,q}$ the spaces $B^{s,p,q}$ with $0 < s < \infty$, $0 < p \leq \infty$, $0 < q \leq \infty$, or $F^{s,p,q}$ with $0 < s < \infty$, $0 < p < \infty$, $0 < q \leq \infty$. Unless otherwise stated, the letter A means the same scale throughout the statement. Also we need the following spaces to formulate our results. As in [9] and [11], we adopt a traditional method of defining function spaces on a domain $\Omega \subset \mathbb{R}^n$.

Definition 1.4. Let $0 < s < \infty$ and $0 < p, q \leq \infty$.

(i) The function space $A^{s,p,q}(\Omega)$ is defined as the subset of $\mathcal{D}'(\Omega)$ obtained by restricting elements in $A^{s,p,q}(\mathbb{R}^n)$ to Ω and the norm is given by

$$\|u\|_{A^{s,p,q}(\Omega)} = \inf\{\|v\|_{A^{s,p,q}(\mathbb{R}^n)};\, v \in A^{s,p,q}(\mathbb{R}^n),\, v|_\Omega = u \ \text{ in } \mathcal{D}'(\Omega)\}.$$

(ii) The potential space $H^{s,p}(\Omega)$ stands for $F^{s,p,2}(\Omega)$.

In the case $k = 1$, they proved the following theorem, which gives the sharp constants for λ_1 and λ_2 in (1.4). We shall restrict $0 < \alpha < 1$ in the affirmative assertion below for the sake of simplicity.

Theorem 1.5 ([5, Theorems 1.5–1.7])**.** *Let $n \geq 2$, $k = 1$, $0 < \alpha \leq s$, $0 < q \leq \infty$, Ω be a bounded domain in $\mathbb{R}^n$, and $X(\Omega) = A^{s,p_{\alpha,s},q}(\Omega)$.*

(i) *Let $0 < \alpha < 1$. Assume that either*

$$\text{(I) } \lambda_1 > \frac{1}{\omega_{n-1}^{1/(n-1)}\alpha} \text{ and } \lambda_2 \in \mathbb{R} \quad \text{or} \quad \text{(II) } \lambda_1 = \frac{1}{\omega_{n-1}^{1/(n-1)}\alpha} \text{ and } \lambda_2 \geq \frac{1}{n\omega_{n-1}^{1/(n-1)}\alpha}$$

holds. Then there exists a constant C such that the inequality (1.4) *holds for all $u \in W^{1,n}(\Omega) \cap A^{s,p_{\alpha,s},q}(\Omega) \cap C_c(\Omega)$ with $\|\nabla u\|_{L^n(\Omega)} = 1$.*

(ii) *Let $\alpha > 0$. Assume that either*

$$\text{(III) } \lambda_1 < \frac{1}{\omega_{n-1}^{1/(n-1)}\alpha} \text{ and } \lambda_2 \in \mathbb{R} \quad \text{or} \quad \text{(IV) } \lambda_1 = \frac{1}{\omega_{n-1}^{1/(n-1)}\alpha} \text{ and } \lambda_2 < \frac{1}{n\omega_{n-1}^{1/(n-1)}\alpha}$$

holds. Then for any constant C, the inequality (1.4) *fails for some $u \in C_c^\infty(\Omega)$ with* $\|\nabla u\|_{L^n(\Omega)} = 1$.

Furthermore, the assertions still hold true

if we replace $A^{s,p_{\alpha,s},q}(\Omega)$ by $H^{s,p_{\alpha,s}}(\Omega)$ or $A^{s,p_{\alpha,s},q}(\Omega) \cap C_c(\Omega)$.

We should mention that Ibrahim-Majdoub-Masmoudi [3, Theorems 1.3 and 1.4] have already obtained a similar result in the case $n = 2$ and $X(\Omega) = \dot{C}^{0,\alpha}(\Omega)$.

We shall investigate the inequality (1.4) for general $k \in \mathbb{N}$. The Brézis-Gallouët-Wainger inequality (1.3) shows that (1.4) holds for sufficiently large λ_1 in some cases as follows.

Proposition 1.6. *Let $n, k \in \mathbb{N}$, $0 < \alpha \le s$, Ω be a domain in $\mathbb{R}^n$, and $X(\Omega) = A^{s,n/(s-\alpha),q}(\Omega)$ be either $F^{s,n/(s-\alpha),q}(\Omega)$ with $0 < q \le 2$, or $B^{s,n/(s-\alpha),q}(\Omega)$ with $0 < q \le \min\{n/(s-\alpha), 2\}$. If $\lambda_1 > 0$ is sufficiently large, then there exists a constant C such that the inequality* (1.4) *holds for all $u \in W^{k,n/k}(\Omega) \cap A^{s,p_{\alpha,s},q}(\Omega) \cap C_c(\Omega)$ with $\|\nabla^k u\|_{L^{n/k}(\Omega)} = 1$.*

In the present paper, we generalize Theorem 1.5 (ii), that is, a lower bound of the sharp constants for λ_1 and λ_2.

We invoke the following result in our previous paper [4]. We will need to calculate the exact values of homogeneous Sobolev norms of the function $\log|\cdot|$ on annuli.

Theorem 1.7 ([4, Theorem 1.2])**.** *For any $n, k \in \mathbb{N}$, it holds*

$$(|x|^k|\nabla^k[\log|x|]|)^2 = \ell_k \ \text{ for } x \in \mathbb{R}^n \setminus \{0\},$$

where ℓ_k is given in (1.1).

Remark 1.8. We obtained the following precise values in [4].

(i) In the case $n = 2$, for any $k \in \mathbb{N}$, it holds

$$\ell_k = 2^{k-1}((k-1)!)^2.$$

(ii) For small k, we have calculated the concrete values of ℓ_k;

$$\ell_1 = 1,\ \ell_2 = n,\ \ell_3 = 4(3n-2),\ \ell_4 = 12(n^2+18n-16),\ \ell_5 = 192(5n^2+30n-32),$$
$$\ell_6 = 960(n^3+78n^2+224n-288),\ \ell_7 = 34560(7n^3+196n^2+308n-496),$$
$$\ell_8 = 241920(n^4+204n^3+3052n^2+2736n-5888).$$

We now state our main result.

Theorem 1.9. *Let $n \geq 2$, $k \in \{1, 2, \ldots, n-1\}$, $0 < \alpha \leq s$, $0 < q \leq \infty$, Ω be a bounded domain in $\mathbb{R}^n$, and $X(\Omega) = A^{s,p_{\alpha,s},q}(\Omega)$. Define ℓ_k as in* (1.1). *Assume that either*

$$\begin{aligned} &\text{(III) } \lambda_1 < \frac{1}{\ell_k^{n/(2(n-k))}\omega_{n-1}^{k/(n-k)}\alpha} \text{ and } \lambda_2 \in \mathbb{R} \\ \text{or} \quad &\text{(IV) } \lambda_1 = \frac{1}{\ell_k^{n/(2(n-k))}\omega_{n-1}^{k/(n-k)}\alpha} \text{ and } \lambda_2 < \frac{k}{n\ell_k^{n/(2(n-k))}\omega_{n-1}^{k/(n-k)}\alpha} \end{aligned}$$

holds. Then for any constant C, the inequality (1.4) *fails for some*

$$u \in W^{k,n/k}(\Omega) \cap A^{s,p_{\alpha,s},q}(\Omega) \cap C_c(\Omega) \text{ with } \|\nabla^k u\|_{L^{k,n/k}(\Omega)} = 1.$$

Furthermore, the assertions above still hold true if we replace $A^{s,p_{\alpha,s},q}(\Omega)$ by $H^{s,p_{\alpha,s}}(\Omega)$ or $A^{s,p_{\alpha,s},q}(\Omega) \cap C_c(\Omega)$.

Remark 1.10. The power $n/(n-k)$ on the left-hand side of (1.4) is optimal in the sense that $r = n/(n-k)$ is the largest power for which there exist λ_1 and C such that

$$\|u\|_{L^\infty(\Omega)}^r \leq \lambda_1 \log(1 + \|u\|_{X(\Omega)}) + C \tag{1.9}$$

can hold for all $u \in W^{k,n/k}(\Omega) \cap X(\Omega) \cap C_c(\Omega)$ with $\|\nabla^k u\|_{L^{n/k}(\Omega)} = 1$. Here, $X(\Omega)$ is as in Theorem 1.5. Indeed, if $r > n/(n-k)$, then for any $\lambda_1 > 0$ and any constant C, (1.9) fails for some $u \in W^{k,n/k}(\Omega) \cap X(\Omega) \cap C_c(\Omega)$ with $\|\nabla^k u\|_{L^{n/k}(\Omega)} = 1$, which is shown by carrying out a similar calculation to the proof of Theorem 1.9; see Remark 3.4 below for the details.

In what follows, C denotes a constant which may vary from line to line.

Finally let us describe the organization of the present paper. In Section 2, we introduce some notation of function spaces and state embedding theorems. Section 3 is devoted to proving Theorem 1.9.

2 Preliminaries

To describe the definition of Besov spaces and Triebel-Lizorkin spaces, we denote by B_R the open ball in $\mathbb{R}^n$ centered at the origin with radius $R > 0$, i.e., $B_R = \{x \in \mathbb{R}^n; |x| < R\}$. Define the Fourier transform $\mathcal{F}$ and its inverse $\mathcal{F}^{-1}$ by

$$\mathcal{F}u(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-\sqrt{-1}x\cdot\xi} u(x)dx, \ \mathcal{F}^{-1}u(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{\sqrt{-1}x\cdot\xi} u(\xi)d\xi$$

for $u \in \mathcal{S}(\mathbb{R}^n)$, respectively, and they are also extended on $\mathcal{S}'(\mathbb{R}^n)$ by duality. For $\varphi \in \mathcal{S}(\mathbb{R}^n)$, define $\varphi(D)$ by

$$\varphi(D)u = \mathcal{F}^{-1}(\varphi\mathcal{F}u) = \frac{1}{(2\pi)^{n/2}}(\mathcal{F}^{-1}\varphi) * u.$$

We recall the following fact on the Sobolev type embedding for Besov spaces and Triebel-Lizorkin spaces. See [5, Lemma 3.1] for its detailed proof.

Lemma 2.1. *Let* $0 < s < \infty$*,* $0 < p < \tilde{p} \le \infty$*,* $0 < q < \tilde{q} \le \infty$ *and* Ω *be a domain in* $\mathbb{R}^n$*. Then*

$$B^{s,p,q}(\Omega) \hookrightarrow B^{s,p,\tilde{q}}(\Omega),$$
$$B^{s,p,q}(\Omega) \hookrightarrow B^{s-n(1/p-1/\tilde{p}),\tilde{p},q}(\Omega),$$
$$B^{s,p,\min\{p,q\}}(\Omega) \hookrightarrow F^{s,p,q}(\Omega) \hookrightarrow B^{s,p,\max\{p,q\}}(\Omega)$$

in the sense of continuous embedding.

In view of Proposition 1.2 (i), Proposition 1.6 is a direct consequence of the Brézis-Gallouët-Wainger inequality (1.3) and Lemma 2.1.

3 Counterexample for the inequality

In this section, we give the proof of Theorem 1.9. Lemma 2.1 shows that

$$B^{s,p,\min\{p,q\}}(\Omega) \hookrightarrow F^{s,p,q}(\Omega),$$

and hence it suffices to consider the case $A^{s,p_{\alpha,s},q}(\Omega) = B^{s,p_{\alpha,s},q}(\Omega)$. Furthermore, another embedding theorem also shows that

$$B^{\tilde{s},p_{\alpha,\tilde{s}},\min\{p_{\alpha,\tilde{s}},q\}}(\Omega) \hookrightarrow B^{s,p_{\alpha,s},q}(\Omega) \ \text{ for } \tilde{s} > s,$$

and hence we have only to consider the case $0 < q \le p_{\alpha,s} = n/(s-\alpha) \le 1$. As we have verified in Lemma 2.1, $B^{s,p_{\alpha,s},\min\{p_{\alpha,s},2\}}(\Omega) \hookrightarrow H^{s,p_{\alpha,s}}(\Omega) \hookrightarrow B^{s,p_{\alpha,s},\max\{p_{\alpha,s},2\}}(\Omega)$. Hence we can transplant our results to the potential space $H^{s,p_{\alpha,s}}(\Omega)$. Therefore, it suffices to show the following theorem for the proof of Theorem 1.9.

Theorem 3.1. *Let* $n \ge 2,\ k \in \{1, 2, \ldots, n-1\},\ \alpha > 0,\ s \ge n+\alpha,\ 0 < q \le p_{\alpha,s} \le 1,\ \Omega$ *be a bounded domain in* $\mathbb{R}^n$*, and* $X(\Omega) = B^{s,p_{\alpha,s},q}(\Omega)$*. Assume that either* (III) *or* (IV) *holds. Then for any constant C, the inequality* (1.4) *fails for some* $u \in C_c^\infty(\Omega)$ *with* $\|\nabla^k u\|_{L^{n/k}(\Omega)} = 1$.

Let us observe that the inequality (1.4) with $X^{s,p_{\alpha,s},q}(\Omega) = B^{s,p_{\alpha,s},q}(\Omega)$ holds for all $u \in W^{k,n/k}(\Omega) \cap B^{s,p_{\alpha,s},q}(\Omega) \cap C_c(\Omega)$ if and only if there exists a constant C depending only upon the given parameters k, s, α, q and the fixed constants (λ_1, λ_2) such that $F_k^{\alpha,s,q}[u; \lambda_1, \lambda_2] \le C$ holds for all $u \in W^{k,n/k}(\Omega) \cap B^{s,p_{\alpha,s},q}(\Omega) \cap C_c(\Omega)$ with $\nabla^k u \ne 0$ in $L^{n/k}(\Omega)$, where

$$\begin{aligned} F_k^{\alpha,s,q}[u; \lambda_1, \lambda_2] = &\left(\frac{\|u\|_{L^\infty(\Omega)}}{\|\nabla^k u\|_{L^{n/k}(\Omega)}} \right)^{n/(n-k)} - \lambda_1 \log\left(1 + \frac{\|u\|_{B^{s,p_{\alpha,s},q}(\Omega)}}{\|\nabla^k u\|_{L^{n/k}(\Omega)}}\right) \\ &- \lambda_2 \log\left(1 + \left(1 + \frac{\|u\|_{B^{s,p_{\alpha,s},q}(\Omega)}}{\|\nabla^k u\|_{L^{n/k}(\Omega)}}\right)\right) \end{aligned}$$
$$\text{for } u \in W^{k,n/k}(\Omega) \cap B^{s,p_{\alpha,s},q}(\Omega) \cap C_c(\Omega) \ \text{ with } \nabla^k u \ne 0 \ \text{ in } L^{n/k}(\Omega).$$

For the proof of Theorem 3.1, we have to find a sequence $\{u_j\}_{j=1}^\infty \subset W^{k,n/k}(\Omega) \cap B^{s,p_{\alpha,s},q}(\Omega) \cap C_c(\Omega)$ such that $F_k^{\alpha,s,q}[u_j; \lambda_1, \lambda_2] \to \infty$ as $j \to \infty$ under the assumption (III) or (IV). In the case that $\Omega = \mathbb{R}^n$ and that all the functions are supported in B_1, we can choose such a sequence.

Lemma 3.2. *Let $n \geq 2$, $k \in \{1,2,\dots,n-1\}$, $\alpha > 0$, $s \geq n+\alpha$, $0 < q \leq p_{\alpha,s} \leq 1$ and $\Omega = \mathbb{R}^n$. Then there exists a family of functions $\{u_j\}_{j=1}^{\infty} \subset W^{k,n/k}(\mathbb{R}^n) \cap B^{s,p_{\alpha,s},q}(\mathbb{R}^n) \cap C_{\mathrm{c}}(\mathbb{R}^n) \setminus \{0\}$ with $\nabla^k u_j \neq 0$ in $L^{n/k}(\mathbb{R}^n)$ and $\operatorname{supp} u_j \subset B_1$ for all $j \in \mathbb{N}$ such that*

$$F_k^{\alpha,s,q}[u_j;\lambda_1,\lambda_2] \to \infty \quad \text{as } j \to \infty$$

under the assumption (III) *or* (IV) *of Theorem 1.9.*

We can prove Theorem 3.1 once we accept Lemma 3.2, by arguing similarly as in [5, Proof of Theorem 3.1]; we omit the details.

We now concentrate on the proof of Lemma 3.2. For $0 < \tau \leq 1$, define a 1-dimensional function $P_\tau^{m,\alpha}$ by

$$P_\tau^{m,\alpha}(r) = \begin{cases} \dfrac{1}{2}\log\dfrac{1}{\tau} + \displaystyle\sum_{l=1}^{m}\frac{1}{l}\left(1-\frac{r}{\tau}\right)^l + \left(\frac{1}{\alpha} - \sum_{l=1}^{m}\frac{1}{l}\right)\left(1-\frac{r}{\tau}\right)^{m+1} & \text{for } 0 \leq r \leq \tau, \\ \dfrac{1}{2}\log\dfrac{1}{r} & \text{for } r \geq \tau. \end{cases}$$

Fix a cut-off function $\rho \in C^\infty(\mathbb{R})$ satisfying $\rho(r) = 1$ for $0 \leq r \leq 1/2$ and $\rho(r) = 0$ for $r \geq 1$. Then define

$$u_\tau^{m,\alpha}(x) = \frac{\alpha}{\alpha\log(1/\tau)+1}\rho(|x|)P_{\tau^2}^{m,\alpha}(|x|^2) \quad \text{for } x \in \mathbb{R}^n.$$

We also use the following estimates to prove Lemma 3.2.

Proposition 3.3. *Let $n \geq 2$, $k \in \{1,2,\dots,n-1\}$, $m \in \mathbb{N}$ and $\alpha > 0$. Then the following hold for $0 < \tau \leq 1/2$:*

(i) $u_\tau^{m,\alpha} \in C_{\mathrm{c}}^m(\mathbb{R}^n)$.

(ii) *It holds* $\|u_\tau^{m,\alpha}\|_{L^\infty(\mathbb{R}^n)} \geq 1$.

(iii) *If $0 < \tau \leq 1/e^{1/\alpha}$, then there exist constants $K_{k,m,\alpha}, L_k > 0$ such that*

$$\frac{L_k}{(\log(1/\tau))^{n/k-1}} \leq \|\nabla^k u_\tau^{m,\alpha}\|_{L^{n/k}(\mathbb{R}^n)}^{n/k} \leq \frac{\ell_k^{n/(2k)}\omega_{n-1}}{(\log(1/\tau))^{n/k-1}} + \frac{K_{k,m,\alpha}}{(\log(1/\tau))^{n/k}}.$$

(iv) *If $s \geq n+\alpha$, $0 < q \leq p_{\alpha,s} \leq 1$ and $m \geq s+2$, then there exists a constant $M_{m,s,\alpha,q} > 0$ such that*

$$\|u_\tau^{m,\alpha}\|_{B^{s,p_{\alpha,s},q}(\mathbb{R}^n)} \leq \frac{M_{m,s,\alpha,q}}{\tau^\alpha\log(1/\tau)}.$$

In particular, $u_\tau^{m,\alpha} \in W^{k,n/k}(\mathbb{R}^n) \cap B^{s,p_{\alpha,s},q}(\mathbb{R}^n) \cap C_{\mathrm{c}}(\mathbb{R}^n)$ with $\nabla^k u \neq 0$ in $L^{n/k}(\mathbb{R}^n)$.

Proof. (i) First note that $P_\tau^{m,\alpha}$ is the unique polynomial of degree $m+1$ satisfying

$$\frac{\alpha}{\alpha\log(1/\tau)+1}P_\tau^{m,\alpha}(0) = 1, \; (P_\tau^{m,\alpha})^{(l)}(\tau) = \left(\frac{d}{dr}\right)^l\left[\log\frac{1}{r}\right]\Bigg|_{r=\tau} \quad \text{for } l \in \{1,2,\dots,m\}.$$

Hence, $P_\tau^{m,\alpha} \in C^m([0,1))$, and $u_\tau^{m,\alpha} \in C_{\mathrm{c}}^m(\mathbb{R}^n)$.

(ii) Since $u_\tau^{m,\alpha}$ is continuous on $\overline{B_1}$, we see that

$$\|u_\tau^{m,\alpha}\|_{L^\infty(\mathbb{R}^n)} \geq u_\tau^{m,\alpha}(0) = \frac{\alpha}{\alpha\log(1/\tau)+1}P_\tau^{m,\alpha}(0) = 1.$$

(iii) It follows from Theorem 1.7 that

$$|\nabla^k u_\tau^{m,\alpha}(x)| = \frac{\ell_k^{1/2}}{|x|^k} \quad \text{for } x \in B_{1/2} \setminus \overline{B_\tau}.$$

Since $0 < \tau \leq 1/e^{1/\alpha}$, then we can estimate the norm from below as follows:

$$\begin{aligned}
\|\nabla^k u_\tau^{m,\alpha}\|_{L^{n/k}(\mathbb{R}^n)}^{n/k} &\geq \|\nabla^k u_\tau^{m,\alpha}\|_{L^{n/k}(B_{1/2}\setminus\overline{B_\tau})}^{n/k} \\
&= \omega_{n-1}\left(\frac{\alpha\ell_k^{1/2}}{\alpha\log(1/\tau)+1}\right)^{n/k}\left(\log\frac{1}{\tau}+\log\frac{1}{2}\right) \\
&\geq \omega_{n-1}\left(\frac{\ell_k^{1/2}}{2}\right)^{n/k}\frac{1}{(\log(1/\tau))^{n/k-1}}.
\end{aligned}$$

We next estimate it from above. A direct calculation shows that

$$\begin{aligned}
\|\nabla^k u_\tau^{m,\alpha}\|_{L^{n/k}(B_{1/2}\setminus\overline{B_\tau})}^{n/k} &= \omega_{n-1}\left(\frac{\alpha\ell_k^{1/2}}{\alpha\log(1/\tau)+1}\right)^{n/k}\left(\log\frac{1}{\tau}+\log\frac{1}{2}\right) \\
&\leq \frac{\ell_k^{n/(2k)}\omega_{n-1}}{(\log(1/\tau))^{n/k-1}} + \frac{(\log(1/2))\ell_k^{n/(2k)}\omega_{n-1}}{(\log(1/\tau))^{n/k}}.
\end{aligned}$$

Since

$$P_{\tau^2}^{m,\alpha}(|x|^2) = P_1^{m,\alpha}\left(\frac{|x|^2}{\tau^2}\right) \quad \text{for } x \in B_\tau,$$

we have

$$\nabla^k u_\tau^{m,\alpha}(x) = \frac{1}{\alpha\log(1/\tau)+1}\nabla^k\left[u_1^{m,\alpha}\left(\frac{x}{\tau}\right)\right] = \frac{1}{\tau^k(\alpha\log(1/\tau)+1)}(\nabla^k u_1^{m,\alpha})\left(\frac{x}{\tau}\right)$$

for $x \in B_\tau$. Finally we have

$$\|\nabla^k u_\tau^{m,\alpha}\|_{L^{n/k}(B_\tau)} = \frac{\|\nabla^k u_1^{m,\alpha}\|_{L^{n/k}(B_1)}}{\alpha\log(1/\tau)+1} \leq \frac{\|\nabla^k u_1^{m,\alpha}\|_{L^{n/k}(B_1)}}{\alpha}\frac{1}{\log(1/\tau)}.$$

Similarly we have

$$\|\nabla^k u_\tau^{m,\alpha}\|_{L^{n/k}(\mathbb{R}^n\setminus\overline{B_{1/2}})} = \frac{\|\nabla^k u_1^{m,\alpha}\|_{L^{n/k}(\mathbb{R}^n\setminus\overline{B_{1/2}})}}{\alpha\log(1/\tau)+1} \leq \frac{\|\nabla^k u_1^{m,\alpha}\|_{L^{n/k}(\mathbb{R}^n\setminus\overline{B_{1/2}})}}{\alpha}\frac{1}{\log(1/\tau)}.$$

The proof of (iii) is therefore complete.

(iv) We first claim that for a multiindex β with $|\beta| \leq m$,

$$|\partial^\beta u_\tau^{m,\alpha}(x)| \leq \frac{C_{m,\alpha}}{\log(1/\tau)\max\{\tau,|x|\}^{|\beta|}}. \tag{3.1}$$

On one hand, since

$$\partial^\beta u_\tau^{m,\alpha}(x) = \frac{\alpha}{\alpha\log(1/\tau)+1} \sum_{\substack{l\in\mathbb{Z}\\ |\beta|/2\le l\le|\beta|}} \frac{1}{\tau^{2l}}(P_1^{m,\alpha})^{(l)}\left(\frac{|x|^2}{\tau^2}\right) \sum_{\substack{\gamma\le\beta\\ |\gamma|=2l-|\beta|}} c_{\beta,\gamma}x^\gamma \quad \text{for } x\in B_\tau,$$

we have

$$|\partial^\beta u_\tau^{m,\alpha}(x)| \le \frac{C_m}{\tau^{|\beta|}\log(1/\tau)} \sum_{\substack{l\in\mathbb{Z}\\ |\beta|/2\le l\le|\beta|}} \sup_{0\le r\le 1}|(P_1^{m,\alpha})^{(l)}(r)| \le \frac{C_{m,\alpha}}{\tau^{|\beta|}\log(1/\tau)} \quad \text{for } x\in B_\tau.$$

Meanwhile, since

$$\begin{aligned}\partial^\beta u_\tau^{m,\alpha}(x) = \frac{\alpha}{\alpha\log(1/\tau)+1}\Bigg(&\sum_{l=0}^{|\beta|-1}\rho^{(l)}(|x|)\sum_{\gamma\le\beta}c'_{\beta,\gamma}\frac{x^\gamma}{|x|^{|\beta|+|\gamma|-l}}\\ &+\sum_{l=1}^{|\beta|}\rho^{(l)}(|x|)\sum_{\gamma\le\beta}c''_{\beta,\gamma}\frac{x^\gamma}{|x|^{|\beta|+|\gamma|-l}}\log\frac{1}{|x|}\Bigg) \quad \text{for } x\in B_1\setminus\overline{B_\tau},\end{aligned}$$

we have

$$|\partial^\beta u_\tau^{m,\alpha}(x)| \le \frac{C_m}{\log(1/\tau)|x|^{|\beta|}}\left(1+|x|\log\frac{1}{|x|}\right) \le \frac{C_m}{\log(1/\tau)|x|^{|\beta|}} \quad \text{for } x\in B_1\setminus\overline{B_\tau}.$$

Thus, (3.1) was established. Take ψ^0,φ^0 satisfying (1.6). Decompose

$$u_\tau^{m,\alpha} = v_\tau^{m,\alpha} + w_\tau^{m,\alpha},$$

where

$$\begin{aligned} v_\tau^{m,\alpha} &= \psi^0\left(\frac{\cdot}{\tau}\right)u_\tau^{m,\alpha},\\ w_\tau^{m,\alpha} &= \sum_{j=1}^\infty w_\tau^{m,\alpha,j},\\ w_\tau^{m,\alpha,j} &= \varphi_j^0\left(\frac{\cdot}{\tau}\right)u_\tau^{m,\alpha}.\end{aligned}$$

Fix $L_s\in\mathbb{N}$ satisfying $s<2L_s\le s+2$. Note that $2L_s\le m$. We shall calculate the following norm.

$$\begin{aligned}&\|u_\tau^{m,\alpha}\|_{B^{s,p_{\alpha,s},q}(\mathbb{R}^n)}\\ &=\|K*u_\tau^{m,\alpha}\|_{L^{p_{\alpha,s}}(\mathbb{R}^n)}+\left(\sum_{k=0}^\infty(2^{ks}\|2^{kn}[\Delta_s^L K](2^k\cdot)*u_\tau^{m,\alpha}\|_{L^{p_{\alpha,s}}(\mathbb{R}^n)})^q\right)^{1/q},\end{aligned}$$

where K is a compactly supported smooth function such that $\chi_{B_1}\le K\le\chi_{B_2}$. It is known that $\|\cdot\|_{B^{s,p,q}(\mathbb{R}^n)}$ is equivalent to the usual Besov norm defined by using the Littlewood-Paley decomposition (see [10]). It is easy to see that

$$\|K*w_\tau^{m,\alpha}\|_{L^{p_{\alpha,s}}(\mathbb{R}^n)} \le C_{m,\alpha} \le \frac{C_{m,\alpha}}{\tau^\alpha\log(1/\tau)}.$$

because $u_\tau^{m,\alpha}$ is supported in B_1, $|u_\tau^{m,\alpha}| \le C_{m,\alpha}$ and $|K * w_\tau^{m,\alpha}(x)| \le C_{m,\alpha}\chi_{B_4}(x)$. Analogously, we have

$$\|K * v_\tau^{m,\alpha}\|_{L^{p\alpha,s}(\mathbb{R}^n)} \le C_{m,\alpha} \le \frac{C_{m,\alpha}}{\tau^\alpha \log(1/\tau)}.$$

Thus, it follows that

$$\|K * u_\tau^{m,\alpha}\|_{L^{p\alpha,s}(\mathbb{R}^n)} \le \frac{C_{m,\alpha}}{\tau^\alpha \log(1/\tau)}.$$

So, let us concentrate on estimating

$$\sum_{k=0}^{\infty} (2^{ks} \|2^{kn}[\Delta^{L_s} K](2^k\cdot) * u_\tau^{m,\alpha}\|_{L^{p\alpha,s}(\mathbb{R}^n)})^q,$$

or more precisely, we shall estimate

$$\sum_{k=0}^{\infty} (2^{ks} \|2^{kn}[\Delta^{L_s} K](2^k\cdot) * v_\tau^{m,\alpha}\|_{L^{p\alpha,s}(\mathbb{R}^n)})^q, \ \sum_{k=0}^{\infty} (2^{ks} \|2^{kn}[\Delta^{L_s} K](2^k\cdot) * w_\tau^{m,\alpha}\|_{L^{p\alpha,s}(\mathbb{R}^n)})^q$$

separately. Now we estimate

$$\|2^{kn}[\Delta^{L_s} K](2^k\cdot) * w_\tau^{m,\alpha,j}\|_{L^{p\alpha,s}(\mathbb{R}^n)}.$$

Since

$$\int_{\mathbb{R}^n} \chi_{B_{1/2^{k-1}}}(y)\chi_{B_{2^{j+2}\tau}}(x-y)dy = |B_{1/2^{k-1}} \cap (x + B_{2^{j+2}\tau})|,$$

this quantity is not zero only when $x \in B_{1/2^{k-1}+2^{j+2}\tau}$, and then $x \in B_{2\max\{1/2^{k-1},2^{j+2}\tau\}}$. Also, if this is not zero, we still have

$$\begin{aligned}\int_{\mathbb{R}^n} \chi_{B_{1/2^{k-1}}}(y)\chi_{B_{2^{j+2}\tau}}(x-y)dy &= |B_{1/2^{k-1}} \cap (x + B_{2^{j+2}\tau})| \\ &\le \min\{|B_{1/2^{k-1}}|, |x + B_{2^{j+2}\tau}|\} \\ &\le C\min\left\{\frac{1}{2^{kn}}, (2^j\tau)^n\right\}.\end{aligned}$$

Observe also that $\Delta^{L_s} w_\tau^{m,\alpha,j}$ can be written as a linear combination of the functions of the form

$$\partial^\beta[\varphi_j^0(\cdot/\tau)]\partial^\gamma[u_\tau^{m,\alpha}],$$

where $|\beta| + |\gamma| = 2L_s$. The functions being supported on $B_{2^{j+2}\tau} \setminus B_{2^j\tau}$, we have

$$\left|\partial^\beta\left[\varphi_j^0\left(\frac{x}{\tau}\right)\right]\right| \le \frac{C_\beta}{(2^j\tau)^{|\beta|}}.$$

Meanwhile in order that $w_\tau^{m,\alpha,j}(x) \ne 0$, $|x|$ needs to belong to $(2^j\tau, 2^{j+2}\tau)$. Thus, using (3.1), we obtain

$$|\partial^\gamma[u_\tau^{m,\alpha}](x)| \le \frac{C_{m,\alpha,\gamma}}{(2^j\tau)^{|\gamma|}\log(1/\tau)}$$

on $\mathrm{supp}(w_\tau^{m,\alpha,j})$. Accordingly, we deduce

$$|\Delta^{L_s} w_\tau^{m,\alpha,j}(x)| \le \frac{C_{m,\alpha,s}}{(2^j\tau)^{2L_s}\log(1/\tau)}\chi_{B_{2^{j+2}\tau}}(x).$$

With this observation in mind, we calculate

$$\begin{aligned}
&2^{kn}|[\Delta^{L_s}K](2^k\cdot)*w_\tau^{m,\alpha,j}(x)|\\
&= \frac{1}{2^{k(2L_s-n)}}|K(2^k\cdot)*[\Delta^{L_s}w_\tau^{m,\alpha,j}](x)|\\
&\le \frac{1}{2^{k(2L_s-n)}}\int_{\mathbb{R}^n}|K(2^k y)|\cdot|[\Delta^{L_s}w_\tau^{m,\alpha,j}](x-y)|dy\\
&\le \frac{C_{m,\alpha,s}}{2^{k(2L_s-n)}(2^j\tau)^{2L_s}\log(1/\tau)}\int_{\mathbb{R}^n}\chi_{B_{1/2^{k-1}}}(y)\chi_{B_{2^{j+2}\tau}}(x-y)dy\\
&\le \frac{C_{m,\alpha,s}}{2^{k(2L_s-n)}(2^j\tau)^{2L_s}\log(1/\tau)}\min\left\{\frac{1}{2^{kn}},(2^j\tau)^n\right\}\chi_{B_{2\max\{1/2^{k-1},2^{j+2}\tau\}}}(x).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\|2^{kn}[\Delta^{L_s}K](2^k\cdot)*w_\tau^{m,\alpha,j}\|_{L^{p_{\alpha,s}}(\mathbb{R}^n)}\\
&\le \frac{C_{m,\alpha,s}}{2^{k(2L_s-n)}(2^j\tau)^{2L_s}\log(1/\tau)}\min\left\{\frac{1}{2^{kn}},(2^j\tau)^n\right\}\max\left\{\frac{1}{2^{kn/p_{\alpha,s}}},(2^j\tau)^{n/p_{\alpha,s}}\right\}.
\end{aligned}$$

Since $0<q\le p_{\alpha,s}\le 1$, we have

$$\begin{aligned}
&\sum_{k=0}^\infty(2^{ks}\|2^{kn}[\Delta^{L_s}K](2^k\cdot)*w_\tau^{m,\alpha}\|_{L^{p_{\alpha,s}}(\mathbb{R}^n)})^q\\
&\le \sum_{k=0}^\infty\sum_{j=1}^\infty(2^{ks}\|2^{kn}[\Delta^{L_s}K](2^k\cdot)*w_\tau^{m,\alpha,j}\|_{L^{p_{\alpha,s}}(\mathbb{R}^n)})^q\\
&\le \frac{C_{m,\alpha,s}}{(\log(1/\tau))^q}\sum_{k=0}^\infty\sum_{j=1}^\infty\frac{1}{2^{k(2L_s-s-n)q}(2^j\tau)^{2L_sq}}\\
&\qquad\times\min\left\{\frac{1}{2^{knq}},(2^j\tau)^{nq}\right\}\max\left\{\frac{1}{2^{knq/p_{\alpha,s}}},(2^j\tau)^{nq/p_{\alpha,s}}\right\}\\
&= \frac{C_{m,\alpha,s}}{(\log(1/\tau))^q}\sum_{k=0}^\infty 2^{k\alpha q}\sum_{j=1}^\infty\frac{1}{(2^{j+k}\tau)^{2L_sq}}\min\{1,(2^{j+k}\tau)^{nq}\}\max\{1,(2^{j+k}\tau)^{(s-\alpha)q}\}.
\end{aligned}$$

Since

$$\min\{1,t\}\max\{1,t^{1/p}\}\le\max\{t,t^{1/p}\}\quad\text{for } t>0,$$

we have

$$\begin{aligned}
&\sum_{k=0}^\infty(2^{ks}\|2^{kn}[\Delta^{L_s}K](2^k\cdot)*w_\tau^{m,\alpha}\|_{L^{p_{\alpha,s}}(\mathbb{R}^n)})^q\\
&\le \frac{C_{m,\alpha,s}}{(\log(1/\tau))^q}\sum_{k=0}^\infty 2^{k\alpha q}\sum_{j=1}^\infty\max\left\{\frac{1}{(2^{j+k}\tau)^{(2L_s-n)q}},\frac{1}{(2^{j+k}\tau)^{(2L_s-s+\alpha)q}}\right\}.
\end{aligned}$$

Decompose the summation with respect to k above into two parts $0 \le k < \log_2(1/\tau)$ and $k \ge \log_2(1/\tau)$ to obtain

$$\sum_{k=0}^{\infty}(2^{ks}\|2^{kn}[\Delta^{L_s}K](2^k\cdot) * w_\tau^{m,\alpha}\|_{L^{p_{\alpha,s}}(\mathbb{R}^n)})^q$$
$$\le \frac{C_{m,\alpha,s}}{(\log(1/\tau))^q}\sum_{j=1}^{\infty}\frac{1}{2^{j(2L_s-s+\alpha)q}}\left(\sum_{\substack{k\in\mathbb{Z}\\ 0\le k<\log_2(1/\tau)}}2^{k\alpha q}+\frac{1}{\tau^{\alpha q}}\sum_{\substack{k\in\mathbb{Z}\\ k\ge\log_2(1/\tau)}}\frac{1}{(2^k\tau)^{(2L_s-s)q}}\right).$$

Since $2L_s > s$, we have

$$\sum_{k=0}^{\infty}(2^{ks}\|2^{kn}[\Delta^{L_s}K](2^k\cdot) * w_\tau^{m,\alpha}\|_{L^{p_{\alpha,s}}(\mathbb{R}^n)})^q$$
$$\le \frac{C_{m,\alpha,s,q}}{(\log(1/\tau))^q}\left(\sum_{\substack{k\in\mathbb{Z}\\ 0\le k<\log_2(1/\tau)}}2^{k\alpha q}+\frac{1}{\tau^{\alpha q}}\sum_{\substack{k\in\mathbb{Z}\\ k\ge\log_2(1/\tau)}}\frac{1}{(2^k\tau)^{(2L_s-s)q}}\right)$$
$$\le \frac{C_{m,\alpha,s,q}}{(\tau^\alpha\log(1/\tau))^q}.$$

Thus, we deduce

$$\|w_\tau^{m,\alpha}\|_{B^{s,p_{\alpha,s},q}(\mathbb{R}^n)} \le \frac{C_{m,\alpha,s,q}}{\tau^\alpha\log(1/\tau)}.$$

The estimate of $v_\tau^{m,\alpha}$ is analogous. Note that

$$2^{kn}[\Delta^{L_s}K](2^k\cdot) * v_\tau^{m,\alpha} = \frac{1}{2^{k(2L_s-n)}}\Delta^{L_s}[K(2^k\cdot)] * v_\tau^{m,\alpha}.$$

Hence in view of the support condition, we obtain

$$|2^{kn}[\Delta^{L_s}K](2^k\cdot) * v_\tau^{m,\alpha}(x)| \le \frac{C_{m,\alpha,s}}{2^{k(2L_s-n)}\tau^{2L_s}}\chi_{B_4}(x).$$

Once this is obtained, we can go through the same argument as we did for

$$\sum_{k=0}^{\infty}(2^{ks}\|2^{kn}[\Delta^{L_s}K](2^k\cdot) * w_\tau^{m,\alpha}\|_{L^{p_{\alpha,s}}(\mathbb{R}^n)})^q$$

and we obtain the estimate

$$\sum_{k=0}^{\infty}(2^{ks}\|2^{kn}[\Delta^{L_s}K](2^k\cdot) * v_\tau^{m,\alpha}\|_{L^{p_{\alpha,s}}(\mathbb{R}^n)})^q \le \frac{C_{m,\alpha,s,q}}{\tau^\alpha\log(1/\tau)}.$$

The proof is now complete. □

Finally we prove Lemma 3.2.

Proof of Lemma 3.2. We may assume $\lambda_1, \lambda_2 \geq 0$. Let $0 < \tau \leq 1/e^{1/\alpha}$ be sufficiently small so that $\tau^\alpha(\log(1/\tau))^{k/n} \leq 1$. We estimate $F_k^{\alpha,s,q}[u_\tau^{m,\alpha}; \lambda_1, \lambda_2]$ from below. Since

$$
\begin{aligned}
&\frac{1}{\left(\dfrac{1}{s^{n/k-1}} + \dfrac{1}{t^{n/k-1}}\right)^{k/(n-k)}} \\
&= s - s\frac{(s^{n/k-1} + t^{n/k-1})^{k/(n-k)} - t}{(s^{n/k-1} + t^{n/k-1})^{k/(n-k)}} \\
&= s - \frac{k}{n-k}\frac{s^{n/k}}{(s^{n/k-1} + t^{n/k-1})^{k/(n-k)}}\int_0^1 (s^{n/k-1}\theta + t^{n/k-1})^{k/(n-k)-1}d\theta \\
&\geq \begin{cases} s - \dfrac{k}{n-k}\dfrac{s^{n/k}}{s^{n/k-1} + t^{n/k-1}} & \text{if } \ k \geq n/2, \\ s - \dfrac{k}{n-k}\dfrac{s^{n/k}}{t^{n/k-2}(s^{n/k-1} + t^{n/k-1})^{k/(n-k)}} & \text{if } \ k \leq n/2 \end{cases} \\
&\geq s - \frac{k}{n-k}\frac{s^{n/k}}{t^{n/k-1}} \quad \text{for } s, t > 0,
\end{aligned}
$$

we have from Proposition 3.3 (ii) and (iii) that

$$
\begin{aligned}
\left(\frac{\|u_\tau^{m,\alpha}\|_{L^\infty(\mathbb{R}^n)}}{\|\nabla^k u_\tau^{m,\alpha}\|_{L^{n/k}(\mathbb{R}^n)}}\right)^{n/(n-k)} &\geq \frac{1}{\left(\dfrac{\ell_k^{n/(2k)}\omega_{n-1}^{(n/k-1)/(n-1)}}{(\log(1/\tau))^{n/k-1}} + \dfrac{K_{k,m,\alpha}}{(\log(1/\tau))^{n/k}}\right)^{k/(n-k)}} \\
&\geq \frac{1}{\ell_k^{n/(2(n-k))}\omega_{n-1}^{k/(n-k)}}\log\frac{1}{\tau} - \frac{K_{k,m,\alpha}}{\ell_k^{n^2/(2k(n-k))}\omega_{n-1}^{n/(n-k)}}.
\end{aligned}
$$

Using the inequalities (1.5) and

$$
\log(1+s) \leq \log s + \log 2 \quad \text{for } s \geq 1,
$$

we have from Proposition 3.3 (iii) and (iv) that

$$
\begin{aligned}
\log\left(1 + \frac{\|u_\tau^{m,\alpha}\|_{B^{s,p_{\alpha,s,\alpha}}(\mathbb{R}^n)}}{\|\nabla^k u_\tau^{m,\alpha}\|_{L^{n/k}(\mathbb{R}^n)}}\right) &\leq \log\left(1 + \frac{M_{m,s,\alpha,q}}{L_k^{k/n}}\frac{1}{\tau^\alpha(\log(1/\tau))^{k/n}}\right) \\
&\leq \log\left(1 + \frac{1}{\tau^\alpha(\log(1/\tau))^{k/n}}\right) + \log\left(1 + \frac{M_{m,s,\alpha,q}}{L_k^{k/n}}\right) \\
&\leq \log\left(\frac{1}{\tau^\alpha(\log(1/\tau))^{k/n}}\right) + \log 2 + \log\left(1 + \frac{M_{m,s,\alpha,q}}{L_k^{k/n}}\right) \\
&= \alpha\log\frac{1}{\tau} - \frac{k}{n}\log\left(\log\frac{1}{\tau}\right) + C_{k,m,s,\alpha,q}
\end{aligned}
$$

and

$$\begin{aligned}\log\left(1+\log\left(1+\frac{\|u_\tau^{m,\alpha}\|_{B^{s,p_{\alpha,s,\alpha}}(\mathbb{R}^n)}}{\|\nabla^k u_\tau^{m,\alpha}\|_{L^{n/k}(\mathbb{R}^n)}}\right)\right) &\le \log\left(1+\alpha\log\frac{1}{\tau}-\frac{k}{n}\log\left(\log\frac{1}{\tau}\right)+C_{k,m,s,\alpha,q}\right)\\ &\le \log\left(1+\alpha\log\frac{1}{\tau}+C_{k,m,s,\alpha,q}\right)\\ &\le \log\left(1+\alpha\log\frac{1}{\tau}\right)+\log(1+C_{k,m,s,\alpha,q})\\ &\le \log\left(\alpha\log\frac{1}{\tau}\right)+\log 2+\log(1+C_{k,m,s,\alpha,q})\\ &= \log\left(\log\frac{1}{\tau}\right)+\log(2\alpha(1+C_{k,m,s,\alpha,q})).\end{aligned}$$

Therefore, we have

$$\begin{aligned}F_k^{\alpha,s,q}[u_\tau^{m,\alpha};\lambda_1,\lambda_2] \ge &\left(\frac{1}{\ell_k^{n/(2(n-k))}\omega_{n-1}^{k/(n-k)}}-\alpha\lambda_1\right)\log\frac{1}{\tau}+\left(\frac{k\lambda_1}{n}-\lambda_2\right)\log\left(\log\frac{1}{\tau}\right)\\ &-\frac{K_{k,m,\alpha}}{\ell_k^{n^2/(2k(n-k))}\omega_{n-1}^{n/(n-k)}}-\lambda_1 C_{k,m,s,\alpha,q}-\lambda_2\log(2\alpha(1+C_{k,m,s,\alpha,q}))\\ &\to\infty \quad \text{as } \tau\searrow 0\end{aligned}$$

under the assumption (III) or (IV). □

Remark 3.4. As we stated in Remark 1.10, if $r > n/(n-k)$, then for any $\lambda_1 > 0$ and any constant C, (1.9) fails for some $u \in W^{k,n/k}(\Omega)\cap X(\Omega)\cap C_c(\Omega)$ with $\|\nabla^k u\|_{L^{n/k}(\Omega)} = 1$. To see this, let

$$r = (1+\varepsilon)n/(n-k), \varepsilon > 0, X(\Omega) = B^{s,p_{\alpha,s},q}(\Omega)$$

and define

$$\begin{aligned}F_k^{\alpha,s,q,\varepsilon}[u;\lambda_1] = \left(\frac{\|u\|_{L^\infty(\Omega)}}{\|\nabla^k u\|_{L^{n/k}(\Omega)}}\right)^{(1+\varepsilon)n/(n-k)} - \lambda_1\log\left(1+\frac{\|u\|_{B^{s,p_{\alpha,s},q}(\Omega)}}{\|\nabla^k u\|_{L^{n/k}(\Omega)}}\right)\\ \text{for } u\in W^{k,n/k}(\Omega)\cap B^{s,p_{\alpha,s},q}(\Omega)\cap C_c(\Omega) \ \text{ with } \nabla^k u\ne 0 \ \text{ in } L^{n/k}(\Omega)\end{aligned}$$

instead of $F^{\alpha,s,q}[u;\lambda_1,\lambda_2]$. We argue as in the proof of Lemma 3.2 to obtain

$$\begin{aligned}F^{\alpha,s,q,\varepsilon}[u_j;\lambda_1] &\ge \left(\frac{1}{\ell_k^{n/(2(n-k))}\omega_{n-1}^{k/(n-k)}}\log\frac{1}{\tau}-\frac{K_{k,m,\alpha}}{\ell_k^{n^2/(2k(n-k))}\omega_{n-1}^{n/(n-k)}}\right)^{1+\varepsilon}-\alpha\lambda_1\log\frac{1}{\tau}\\ &\quad+\frac{k}{n}\lambda_1\log\left(\log\frac{1}{\tau}\right)-\lambda_1 C_{k,m,s,\alpha,q}\\ &\ge \left(\frac{1}{2\ell_k^{n/(2(n-k))}\omega_{n-1}^{k/(n-k)}}\log\frac{1}{\tau}\right)^{1+\varepsilon}-\alpha\lambda_1\log\frac{1}{\tau}\\ &\quad+\frac{k}{n}\lambda_1\log\left(\log\frac{1}{\tau}\right)-\lambda_1 C_{k,m,s,\alpha,q} \quad \text{if } \tau \text{ is sufficiently small}\\ &\to\infty \quad \text{as } \tau\searrow 0,\end{aligned}$$

which provides the assertion above.

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