EXOTIC HEAT PDE'S

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Abstract

Exotic heat equations that allow to prove the Poincaré conjecture, some related problems and suitable generalizations too are considered. The methodology used is the PDE's algebraic topology, introduced by A. Prástaro in the geometry of PDE's, in order to characterize global solutions.[1]

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1 Introduction

"In a category $\mathcal{C}$ of manifolds
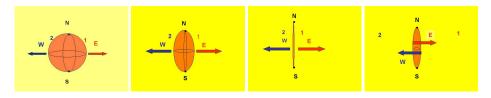*every homotopy sphere is homeomorphic to a sphere."*

Figure 1. Smale's paradox: turning a sphere $S^2 \subset \mathbb{R}^3$ inside out.

The famous Poincaré's conjecture is about n-dimensional manifolds, with $n = 3$, (R. S. Hamilton [24, 25, 26, 27, 28], G. Perelman [37, 38], A. Prástaro [53, 1]), but there

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[1]See also [55].

are also generalizations of this conjecture for higher dimension manifolds. For $n = 4$ the Poincaré conjecture has been proved by M. Freedman [19, 20]) and for $n \geq 5$, by S. Smale [60, 61]. More recently has been given a generalization for quantum supermanifolds by A. Prástaro, that has also proved it in [51]). Nowadays, one can state that a generalized Poincaré conjecture can be proved, or disproved, depending on the particular category C in which it is formulated. This problem aroses in the framework of the geometric topology, but in order to be solved it was necessary to go outside that framework and recast the problem in a theory of PDE's. But the more recent results by A. Prástaro [51, 53, 1]), have proved that it was necessary to return inside the algebraic topology framework, applied to the PDE's geometric theory. Really, it was soon evident that remaining in a pure algebraic topologic approach it was not enough to solve this conjecture. In fact, a fundamental idea to solve this problem is to ask whether it is possible find a smooth manifold, V, that without singular points bords a 3-dimensional compact, closed, smooth, simply connected manifold N with S^3, when N is homotopy equivalent to S^3. The bordism theory is able to state that a smooth manifold V such that $\partial V = N \bigcup S^3$, there exists, since the nonoriented and oriented 3-dimensional bordism groups Ω_3 and $^+\Omega_3$ respectively, are both trivial: $\Omega_3 = {}^+\Omega_3 = 0$. However, by simply looking to the above bordism groups it is impossible to state if V has singular points (i.e., has holes) or it is a cylinder. By the way, more informations can be obtained by the *h-cobordism theory*. More precisely the *h-cobordism theorem* in a category C of manifolds, states that if the compact manifold V has $\partial V = N_0 \sqcup N_1$, such that the inclusion maps $N_i \hookrightarrow V$, $i = 0, 1$, are homotopy equivalences, (i.e., V is a *h-cobordism*), and $\pi_1(N_i) = 0$, then $V \cong_C N_0 \times [0, 1]$. This theorem holds for $n \geq 5$ in the category of smooth manifolds (S. Smale) and for $n = 4$ in the category of topological manifolds (M. Freedman). But it does not work for $n = 3$!

A very important angular stone, in the long history about the solution of the Poincaré conjecture, has been the introduction, by R. S. Hamilton, of a new approach recasting the problem in to solving a PDE, the *Ricci flow equation*, and asking for nonsingular solutions there, that starting from a Riemannian manifold (N, γ) arrive to the 3-dimensional sphere S^3, respectively identified with initial and final Cauchy manifolds in the Ricci flow equation. In that occasion the Mathematical Analysis, or more precisely the Functional Analysis, entered in the Poincaré conjecture problem. This approach has had many improvements until the papers by G. Perelman. More recently, A. Prástaro, by using his algebraic topologic theory of PDE's was able to give a pure geometric proof of the Poincaré conjecture. Let us emphasize that the usual geometric methods for PDE's (Spencer, Cartan), were able to formulate for nonlinear PDE's, local existence theorems only, until the introduction, by A. Prástaro, of the algebraic topologic methods in the PDE's geometric theory. These give suitable tools to calculate integral bordism groups in PDE's, and to characterize global solutions. Then, on the ground of integral bordism groups, a new geometric theory of stability for PDE's and solutions of PDE's has been built. These general methodologies allowed to A. Prástaro to solve fundamental mathematical problems too, other than the Poincaré conjecture and some of its generalizations, like characterization of global smooth solutions for the Navier-Stokes equation and global smooth solutions with mass-gap for the quantum Yang-Mills superequation. (See [42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 54].[2])

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[2]See also Refs. [1, 2, 56], where interesting related applications of the PDE's Algebraic Topology are given.

The main purpose of this paper is to emphasize some problems related to *exotic heat PDE's* recently focused.[3]

Let us conclude this Introduction with a theorem, direct issue from results contained in [53].

Theorem 1.1. *Any* 3-*dimensional compact, closed simply connected smooth manifold, M, homotopy equivalent to* S^3, *is diffeomorphic to* S^3.[4]

Proof. In fact, the method followed by A. Prástaro to prove the Poincaré conjecture [53, 1], allows us to conclude that in dimension $n = 3$, under the hypotheses of the theorem, one can state that the manifold V, smooth solution of the Ricci flow equation, $(\partial t.g_{ij}) - \kappa R_{ij} = 0$, such that $\partial V = M \sqcup S^3$, has the following structure $V \cong M \times [0,1]$, hence must necessarily be $M \cong S^3$, (in the category of smooth manifolds). In other words the algebraic topology of PDE's, introduced by A. Prástaro, allows us to complete h-cobordism theorem in the category of 3 dimensional smooth manifolds. $\square$

2   RICCI FLOW EQUATION IN NON-COMPACT CASE

In this section we consider the problem of existence and uniqueness of smooth complete solutions of the Ricci flow equation, in the noncompact case. In particular we answer to a recent question put for the *n*-dimensional Euclidean case in [11]. Let us consider the Ricci flow equation, written in the form

$$(\partial t.g_{ij})(x,t) = -2R_{ij}(x,t), \quad (x,t) \in M \times \mathbb{R} \tag{2.1}$$

when M is a *n*-dimensional smooth, noncompact and complete Riemannian manifold, (M,γ). In particular the assumption that M is not compact and complete, is a substantial difference with respect to the usual way in which such equation has been considered. In fact, the Ricci flow equation has been principally introduced by R. S. Hamilton, to prove the Poincaré conjecture on compact closed 3-dimensional manifolds homotopic equivalent to S^3. (See [12, 13, 24, 25, 26, 27, 28, 37, 38] to follows the proof of the Poincaré conjecture in the approach by Hamilton-Perelmann. Furthermore, in [53, 1] the Poincaré conjecture has been proved by using a method different by the Hamilton-Perelmann's one.)

The main motivations to write this section are some recent works where the Ricci flow equation is implemented on a noncompact complete Riemannian manifold, and solutions

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[3]The Ricci flow equation can be considered a generalization of the classical Fourier's heat equation $u_t - \kappa u_{xx} = 0$. In this paper we call *exotic heat equations*, PDE's that, like the Ricci flow equation, are of the type $F^j \equiv u_t^j - f^j(u_k^i) = 0$, $1 \le i, j \le m$, with the length $|k|$ of the multi-index $k \in \{1, \cdots, n\}$, given by $0 \le |k| \le s \le r$, where $F^j : J^r(W) \to \mathbb{R}$ are analytic functions of order $r \ge 0$ on a fiber bundle $\pi : W \equiv \mathbb{R} \times E(M) \to \mathbb{R} \times M$, where $E(M)$ is a vector bundle over M, with M an analytic manifold of dimension n. Let us emphasize that the structure of exotic heat equation is the more suitable to use in order to prove (generalized) Poincaré conjectures. In fact, the idea to use PDE's to solve the Poincaré's conjecture, was the initial motivation to introduce and study the well-known *Yamabe equation* [68]. But that road did not turn out a lucky choice to prove the conjecture, even if the Yambe equation is a very important equation to study conformal problems in Riemannian geometry. (See, e.g., [1].)
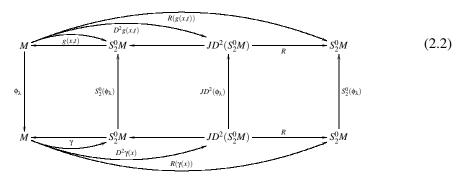
[4]This last result agrees with the Hauptvermuntung conjecture that was proved for $(n = 2,3)$-dimensional manifolds and disproved for $(n \ge 4)$-dimensional manifolds. (See [8, 32, 33, 34, 57, 62, 66].)

are request to identify complete metrics on the sectional noncompact manifolds. (See, e.g., [11, 59].) More specifically, it was very intriguing the recent paper by B.-L. Chen [11], where it is emphasized the uniqueness of such solutions in the dimensional 3, and the following question: "*Does the strong uniqueness of the Ricci flow hold on the Euclidean space $\mathbb{R}^n$, for $n \geq 4$?*".

In fact we have the following theorem.

Theorem 2.1. *Let us consider the Ricci flow equation* (2.1) *on the Euclidean space* $(\mathbb{R}^n, \gamma)$. *Then, the unique solution of* (2.1) *for complete metrics is just* γ.

Proof. Let us also underline that the restriction to consider noncompact complete metrics does not permit flows with contractions, and neither simple homeomorphic ones. Isometric diffeomorphisms are instead permitted. Therefore, let us add to the equation (2.1) the condition that $g(x,t)$ is realized by means of a one parameter set of diffeomorphisms $\phi_t : M \to M$, by means of pull-back, i.e, put $g(x,t) = \phi_t^* \gamma$. One has for the Ricci tensor $R_{ij}(x)$ of $\gamma(x)$, the following induced deformation $R(x,t) = R(g(x,t)) = \phi_t^* R(\gamma(x))$. In fact, for the natural covariance of the Riemannian metric and its Ricci tensor, one has the following commutative diagram

$$(2.2)$$

Taking into account that in coordinates $(x^i)_{1 \leq i \leq n}$ on M, one has

$$
\begin{cases}
g_{ij}(x,t) = (\partial x_i.\phi_t^a)(\partial x_j.\phi_t^b)\gamma_{ab}(\phi_t(x)) \equiv (\phi_t)_i^a(x)(\phi_t)_j^b(x)\gamma_{ab}(\phi_t(x)) \\
(\partial t.g_{ij})(x,t) = (\dot{\phi}_t)_i^a(x)(\phi_t)_j^b(x)\gamma_{ab}(\phi_t(x)) + (\phi_t)_i^a(x)(\dot{\phi}_t)_j^b(x)\gamma_{ab}(\phi_t(x)) \\
\qquad\qquad + (\phi_t)_i^a(x)(\phi_t)_j^b(x)(\dot{\phi}_t)^\alpha(x)(\partial x_\alpha.\gamma_{ab})(\phi_t(x)) \\
R_{ij}(x,t) = (\phi_t)_i^a(x)(\phi_t)_j^b(x)R_{ab}(\phi_t(x))
\end{cases}
$$

$$(2.3)$$

we get that equation (2.1) can be written as follows

$$
\bullet(RF) \subset JD(W) : \begin{cases}
(\dot{\phi}_t)_i^a(x)(\phi_t)_j^b(x)\gamma_{ab}(\phi_t(x)) + (\phi_t)_i^a(x)(\dot{\phi}_t)_j^b(x)\gamma_{ab}(\phi_t(x)) \\
\qquad\qquad + (\phi_t)_i^a(x)(\phi_t)_j^b(x)(\dot{\phi}_t)^\alpha(x)(\partial x_\alpha.\gamma_{ab})(\phi_t(x)) \\
= -2(\phi_t)_i^a(x)(\phi_t)_j^b(x)R_{ab}(\phi_t(x))
\end{cases}
$$

$$(2.4)$$

where $\pi : W \equiv \mathbb{R} \times M^2 \to \mathbb{R} \times M$, $(t,x^i,y^j) \mapsto (t,x^i)$.

Let us first introduce an equivalence relation in the set $S_{ol}^\infty(\bullet(RF))$ of smooth solutions of $\bullet(RF)$.

Definition 2.2. Let (M,γ) be a Riemannian manifold of dimension n. We say that a diffeo-morphism $f : M \to M$, is *rigid* if $f^*\gamma = \gamma$, and $\det(j(f)) = 1$, where $j(f)$ is the Jacobian matrix of f. In other words $(j(f)_i^k) \in SO(n)$. We say that a flow, $\phi_t(x)$, solution of $_\bullet(RF)$, is *rigid*, if ϕ_t^*, is a rigid diffeomorphism, $\forall t$. Then we say, also, that $\phi_t(x)$ is a *rigid solution* of $_\bullet(RF)$.

Definition 2.3. We say that two flows $\phi_t(x)$ and $\psi_t(x)$, smooth solutions of $_\bullet(RF)$, are *rigid equivalent*, if there exists a rigid solution f_t of $_\bullet(RF)$ such that $\phi_t = \psi_t \circ f_t$.

Proposition 2.4. *The rigid equivalence is an equivalence relation in $\mathcal{S}_{ol}^\infty(_\bullet(RF))$, that we denote with $\sim_R$. Then for any representative $\psi_t \in [\phi_t]_R \in \mathcal{S}_{ol}^\infty(_\bullet(RF))/\sim_R \equiv [\mathcal{S}_{ol}^\infty(_\bullet(RF))]_R$, one has $(\psi_t)^*\gamma = (\phi_t)^*\gamma$, and $\psi_t^*\eta = \phi_t^*\eta$, where η is the canonical form associated to the metric γ.[5]*

Proof. In fact, if $\phi_t(x) \in _\bullet(RF)$ it follows that $\phi_t(x) \sim_R \phi_t(x)$, since $\phi_t = \phi_t \circ (id_M)_t$. If $\phi_t, \psi_t \in _\bullet(RF)$ and $\phi_t \sim_R \psi_t$ it follow that there exists a rigid solution $f_t \in _\bullet(RF)$ such that $\phi_t = \psi_t \circ f_t$. Since we can also write $\phi_t \circ f^{-1} = \psi_t$, and of course also $f^{-1} \in _\bullet(RF)$, it follows that $\psi_t \sim_R \phi_t$. Finally, if one has $\phi_t, \psi_t, \varphi_t \in _\bullet(RF)$ with $\phi_t \sim_R \psi_t \sim_R \varphi_t$, it follows that $\phi_t = \psi_t \circ (f_t)_1$, $\psi_t = \varphi_t \circ (f_t)_2$, for some rigid solutions $(f_t)_i$, $i = 1,2$. Then we get also $\phi_t = \varphi_t \circ (f_t)_2 \circ (f_t)_1 = \varphi_t \circ f_t$, where $f_t = (f_t)_2 \circ (f_t)_1$ is also a rigid solution of $_\bullet(RF)$. Therefore, $\phi_t \sim_R \varphi_t$. $\qquad\square$

Let us find solutions of equations(2.4) of the type
$\phi_t^a = e^{\omega t}h^a(x)$. Then one has that ω and $h^a(x)$ must satisfy the following equations:

$$h_i^a(x)h_j^b(x)\left\{\omega\left[2\gamma_{ab}(\phi_t(x)) + e^{\omega t}h^\alpha(x)(\partial x_\alpha.\gamma_{ab})(\phi_t(x))\right] + 2R_{ab}(\phi_t(x))\right\} = 0. \qquad (2.5)$$

In the case that $(M,\gamma) = (\mathbb{R}^n, \gamma_E)$, then $R_{ab} = 0$ and above equations (2.5) reduce to the following ones

$$h_i^a(x)h_j^b(x)\omega\left[2\gamma_{ab}(\phi_t(x)) + e^{\omega t}h^\alpha(x)(\partial x_\alpha.\gamma_{ab})(\phi_t(x))\right] = 0. \qquad (2.6)$$

Let us add the condition $\phi_0 = id_M$, then we get that necessarily $h^a(x) = x^a$, for $a = 1,\ldots,n$. Therefore, equations (2.6) reduce to the following ones:

$$\omega\left[2\gamma_{ij}(\phi_t(x)) + e^{\omega t}x^\alpha(\partial x_\alpha.\gamma_{ij})(\phi_t(x))\right] = 0. \qquad (2.7)$$

Of course we can take a cartesian coordinate system, so that $\gamma_{ij} = \delta_{ij}$. Therefore, equations (2.7) reduce to the following equations $2\omega\delta_{ij} = 0$. Thus we get the unique trivial solution for ω, i.e., $\omega = 0$ and the unique solution for $\phi_t(x)$, i.e., $\phi_t^a = x^a$, that gives the unique solution $g(x,t) = \gamma_E(x)$.

---

[5]The deformation in Fig.1 cannot be rigid since it changes the orientation of the sphere S^2. In fact, this deformation can be realized by contracting the Est hemisphere, (resp. West hemisphere), endowed of a normal unitary vector E (resp. W), on the middle of its equatorial-line, on the oriented disk (D^2, E), (resp. (D^2, W)), with ∂D^2 passing for the poles N and S. Then, continuing to deform the two oriented disks in the same opposite directions. During this deformation the boundary ∂D^2 remains fixed, and both hemispheres conserve their orientations, but the resulting sphere has reversed orientation.

Above result does not depend on the particular assumption made on the type of solutions. In fact, if consider equation (2.4) in the case $M = \mathbb{R}^n$, with $\gamma = \delta_{ij}$, we get the following equations

$$
\begin{cases}
A^{rs}_{abij}(\partial t \partial x_r . \phi_t^a)(\partial x_s . \phi_t^b) = 0 \\
A^{rs}_{abij} \equiv \delta_{ab}(\delta^{rs}_{ij} + \delta^{rs}_{ji}).
\end{cases}
\tag{2.8}
$$

Then, one can easily see that the unique smooth solution of (2.8) that satisfies the initial condition $\phi_0^a = x^a$ is just $\phi_t^a = x^a$, for all $t \in \mathbb{R}$. (Of course this is up to rigid flows that do not produce any deformation.) In fact, let us write above equation for $1 \leq i = j \leq n$. We get the following equations:

$$
\begin{cases}
2\delta_{ab} f_{t1}^a f_1^b = 0 \\
\cdots \\
2\delta_{ab} f_{tn}^a f_n^b = 0
\end{cases}
\tag{2.9}
$$

Here, for simplicity, we have put $f \equiv \phi_t$. Then we can also write:

$$
\begin{cases}
\partial t . \left((\partial x_1 . f^1)^2 + \cdots + (\partial x_1 . f^n)^2\right) = 0 \\
\cdots \\
\partial t . \left((\partial x_n . f^1)^2 + \cdots + (\partial x_n . f^n)^2\right) = 0
\end{cases}
\tag{2.10}
$$

Therefore, we get the following first integrals:

$$
\begin{cases}
(\partial x_1 . f^1)^2 + \cdots + (\partial x_1 . f^n)^2 = c_1(x^1, \cdots, x^n) \\
\cdots \\
(\partial x_n . f^1)^2 + \cdots + (\partial x_n . f^n)^2 = c_n(x^1, \cdots, x^n)
\end{cases}
\tag{2.11}
$$

This means that the jacobian matrix $(\partial x_i . (\phi_t)^j) = (j_i^j(x^1, \cdots, x^n))$ is made by functions that depend only on the coordinates x^i.

$$
(\partial x_i . (\phi_t)^j) = j_i^j(x^1, \cdots, x^n), \quad i, j = 1, \cdots, n.
\tag{2.12}
$$

Let us integrate (2.12):

$$
(\phi_t)^j = \int j_i^j(x^1, \cdots, x^n) \, dx^i + c^j(t, x^1, \cdots, \hat{x}^i, \cdots, x^n),
\tag{2.13}
$$

where $c^j(t, x^1, \cdots, \hat{x}^i, \cdots, x^n)$ is any function that does not depend on x^i. Taking into account the initial condition $\phi_0^j = x^j$, we get

$$
x^j = \int j_i^j(x^1, \cdots, x^n) \, dx^i + c^j(0, x^1, \cdots, \hat{x}^i, \cdots, x^n).
\tag{2.14}
$$

Let us again derive (2.14) with respect to x^i, we get $\delta_i^j = j_i^j$. So we have $(\phi_t)^j = x^j + c^j(t, x^1, \cdots, \hat{x}^i, \cdots, x^n)$. Since this holds for any $1 \leq i \leq n$, we conclude that the arbitrary functions c^j can depend only on t. So we have the following functions:

$$
(\phi_t)^j = x^j + c^j(t).
\tag{2.15}
$$

The flows given in (2.15) are rigid flows, therefore do not produce any deformation. Since, of course, we aim find solutions up to rigid ones, we get that in the case of the n-dimensional Euclidean manifold $(\mathbb{R}^n, \gamma)$, the unique complete noncompact solution of Ricci flow equation is just $g(t, x) = \gamma$. $\qquad\square$

3   PINCHING PROBLEMS IN RICCI FLOW EQUATION

In the pinching problems connected to the Ricci flow equation one aims to characterize solutions of this equation on a connected compact Riemannian manifold (M, γ) with respect to the so-called *pinching constant* $\lambda = \min \sec / \max \sec$, i.e., by adding an upper as well as a lower bound on the sectional curvature (denoted sec). With respect to this scalar-characterization of Riemannian manifolds, there are some interesting informations on their global structure. Let us recall some ones.[6] (S. Brendle and R. Schoens [6]) If M is a compact n-dimensional Riemannian manifold with $\lambda \in (\frac{1}{4}, 1)$ (*strict pinching*), M is diffeomorphic to a spherical space form. If $\lambda \in [\frac{1}{4}, 1]$ M is diffeomorphic to a spherical space form or isometric to a locally symmetric space. (M. Berger [5] and W. Klingenberg [30]) If M is a compact simply connected manifold with $\lambda \geq \frac{1}{4}$ then M is either homeomorphic to S^n or isometric to $\mathbb{C}P^n$, $\mathbb{H}P^n$ or $Ca\mathbb{P}^2$, with their standard Fubini metric. [(J. L. Synge [63]) A manifold with positive curvature does not necessitate be simply connected. In fact if $\dim M = n = 2k$, one has $\pi_1(M) = 0$ if orientable and $\pi_1(M) = \mathbb{Z}_2$ if non-orentable. If $\dim M = n = 2k+1$ and positively curved M is orientable.] (J. Cheeger [9]) Given a constant $\varepsilon > 0$, there are only finitely many diffeomorphism types of compact simply connected $2n$-dimensional manifolds M with $\lambda \geq \varepsilon$. (F. Fang and X. Rong [18], A. Petrunin and W. Tuschmann [39, 66]) Given a constant $\varepsilon > 0$, there are only finitely many diffeomorphism types of compact $(2n+1)$-dimensional manifolds M with $\pi_1(M) = \pi_2(M) = 0$ and $\lambda \geq \varepsilon$.

Let us consider four-dimensional manifolds with λ-*pinched flag curvature*, $(0 \leq \lambda < 1)$, i.e., $R_u(v,v) \geq \lambda(x)R_u(w,w)$, where $R_u(.,.) = R(u,.,u,.)$ is the symmetric bilinear form, identified by the curvature tensor R and any nonzero vector $u \in T_x M$. Here $v, w \in T_x^{\perp} < \mathbb{R}u >$, $|v| = |w|$. In [3] B. Andrwes and H. Nguyen proved that the class of positively curved compact connected 4-dimensional manifolds, with λ-pinched flag curvature, $\lambda \geq \frac{1}{4}$, is invariant under Ricci-flow. Moreover, any such manifold is either diffeomorphic to a spherical space form or isometric to $\mathbb{C}P^2$ with Fubini-study metric (up to scaling). In that interesting paper, authors refer to a uniqueness existence theorem for local solutions of the Cauchy problem for the Ricci-flow equation. (A first proof has been given by R. S. Hamilton [24, 25]. Different proofs were obtained also by D. T. De Turk [15] and B. Chow and D. Knopp [10].) However, it is important to underline that such uniqueness is strictly related to the class of regular solutions considered there. In fact, by recasting the Cauchy problem in the geometric theory of PDE's, one can see that the Ricci-flow equation identifies an analytic submanifold $(RF) \subset JD^2(W)$ of the second jet-derivative space for sections of the following fiber bundle $\pi : W \equiv \mathbb{R} \times \widetilde{S_2^0(M)} \to \mathbb{R} \times M$, where $\widetilde{S_2^0(M)} \subset S_2^0(M)$ is the open subbundle of non-degenerate symmetric tensors of type $(0,2)$ on M. (See [53] to understand in which sense must be interpreted the statement about uniqueness of smooth solutions for the Ricci flow equation.) However, in order to describe singularities in the flow, it is useful to consider the embeddings $(RF) \subset JD^2(W) \subset J_{n+1}^2(W)$, where $J_{n+1}^2(W)$ is the 2-jet-space for $(n+1)$-dimensional submanifolds of W, $\dim M = n$. Since the Ricci-flow equation is formally integrable and completely integrable, with non-trivial symbol g_2, there

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[6]These results are usually referred as *sphere theorems*, or *quarter-pinched sphere theorems*. In fact, they are generalizations of the *sphere theorem for 3-manifolds* that states that if M is an orientable 3-manifold with $\pi_2(M) \neq 0$, then there exists a non-zero element of $\pi_2(M)$ having a representative that is an *embedding* $S^2 \to M$.

are also singular solutions satisfying smooth Cauchy problems.[7] Then the characterization of global solutions is obtained by means of the singular integral bordism groups of (RF). (For details on the geometry of PDE's see [40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 53].) For example, in the particular case of 4-dimensional closed, compact, simply connected, smooth Riemannian manifolds M, one obtains $\Omega_4^{(RF)}/K_4^{(RF)} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \cong \Omega_4$, where $K_4^{(RF)}$ is the kernel of the canonical projection $p : \Omega_4^{(RF)} \to \Omega_4 \cong \mathbb{Z}_2 \bigoplus \mathbb{Z}_2$. With this respect, and taking into account the well-known theorems by A. Dold [16] and C. T. C. Wall [67] on the (oriented) cobordism ring $(^+\Omega_\bullet) \, \Omega_\bullet$, we can get from the results of this paper the following interesting theorem.

Theorem 3.1. *The* 4-*dimensional Riemannian manifolds preserving* λ-*pinched flag curvature,* $\lambda \geq \frac{1}{4}$, *in a Ricci-flow, belong to the cobordism classes in the image* $p^{-1}(r(^+\Omega_4 \cong \mathbb{Z})) \subset \Omega_4^{(RF)}$, *where* $r : {}^+\Omega_\bullet \to \Omega_\bullet$ *is the forgetting orientation natural mapping, i.e. one has the exact sequences* (3.1).

$$
\begin{array}{c}
\boxed{{}^+\Omega_4 \cong \mathbb{Z}} \\
\downarrow{\scriptstyle 2} \\
\boxed{{}^+\Omega_4 \cong \mathbb{Z}} \\
\downarrow{\scriptstyle r} \\
0 \longrightarrow K_4^{(RF)} \longrightarrow \Omega_4^{(RF)} \xrightarrow{\ p\ } \boxed{\Omega_4 \cong \mathbb{Z}_2 \bigoplus \mathbb{Z}_2} \longrightarrow 0
\end{array}
\qquad (3.1)
$$

4 NECKPINCHING PROBLEMS IN SINGULAR EXOTIC HEAT PDE

In a recent paper Z. Gang and I. M. Sigal [22] considered some particular solutions for the Ricci flow equation encoding the mean curvature flow of an initial hypersurface $M_0 \subset \mathbb{R}^{d+1}$, that is of revolution around the axis $x = x_{d+1}$. There $u(x,t)$ represents, at fixed t, the "distance function" of the revolution hypersurface from the revolution axis at $x \in \mathbb{R}$. More precisely they considered the following boundary value problem:

$$
\boxed{
\begin{array}{l}
\text{(mean curvature flow equation): } u_t = \dfrac{u_{xx}^2}{1+(u_x)^2} - \dfrac{d-1}{u} \\[2ex]
\text{(boundary conditions): }
\begin{cases}
u(x,0) = u_0(x) > 0, \, \forall x \in \mathbb{R} \\
\liminf_{|x|\to\infty} u_0(x) > 0 \\
\left|\left|\frac{1}{u(x,t)}\right|\right|_\infty < \infty, t < t^* \\
\left|\left|\frac{1}{u(x,t)}\right|\right|_\infty \to \infty, t \to t^*
\end{cases}
\end{array}
}
\qquad (4.1)
$$

---

[7]It is well known that every compact smooth manifold, (or compactifiable C^∞ manifold), can be considered analytic too. This is exentially the meaning of the famous results by J. Nash [36] and some relative improvements by A. Tognoli [65] and T. Kawakami [29].

They proved, by using classic method of functional analysis, existence of solutions that in finite time collapse on the revolution axis. They called such solutions *collapsing* (or *neckpinching*) at the time t^*.

It is interesting to emphasize that translating above problem in the framework of the geometry of PDE's, the mean curvature flow equation in (4.1), identifies a real analytic 7-dimensional submanifold E_2 of the 2-jet-derivative space

$$JD^2(W) \cong \mathbb{R}^8, \ (t,x,u,u_t,u_x,u_{tt},u_{tx},u_{xx}),$$

over the trivial vector bundle $\pi : W \equiv \mathbb{R}^3 \to \mathbb{R}^2$, $(t,x,u) \mapsto (t,x)$. One has $E_2 \cap S_2 = \emptyset$, where $S_2 \subset JD^2(W)$ is the analytic submanifold identified by the constraint $u = 0$. So, a solution $V_\bullet \subset E_2$, such that $u(x,t^*) = 0$, for some time t^*, cannot exist. However, this does not exclude that by considering the *singular equation* ${}^\bullet E_2 \equiv E_2 \bigcup S_2 \subset JD^2(W)$, we can find *asymptotic solutions* of the above problem. This is in fact the case when S_2 is endowed with the distribution obtained from the Cartan distribution of $JD^2(W)$, just restricted on S_2. Then one can see that one can find solutions of E_2 that approach integral manifolds of S_2, when $u \to 0$. The tangent space to such solutions have zero time component in the soldering points. This just means that such solutions stop (or collapse), in a finite time t^*, to the revolution axis. (For informations on this geometric approach to singular PDE's see [42, 53, 54, 2].)

The Cartan distribution $\mathbf{E}_2 \subset TE_2$ of E_2, is given by the following vector fields

$$\begin{aligned} \zeta \ =\ & X^t(\partial t + u_t \partial u + u_{tt} \partial u_t + u_{tx} \partial u^x) \\ & + X^x(\partial x + u_x \partial u + u_{xt} \partial u_t + u_{xx} \partial u^x) \\ & + Z_{xx} \partial u^{xx} + Z_{tt} \partial u^{tt} + Z_{tx} \partial u^{tx} \end{aligned} \tag{4.2}$$

such that

$$\left\{ \begin{aligned} & X^t \left[u_t(u_t(1+u_x^2) - u_{xx}^2) + u_{tt}(u(1+u_x^2)) + u_{tx}(2uu_t u_x + 2u_x(d-1)) \right] \\ & + X^x \left[u_x(u_t(1+u_x^2) - u_{xx}^2) + uu_{tx}(1+u_x^2) + u_{xx}(2uu_t u_x + 2u_x(d-1)) \right] \\ & + 2Z_{xx}uu_{xx} = 0 \end{aligned} \right\} \tag{4.3}$$

with

$$u_t = \frac{uu_{xx}^2 - (1+u_x^2)(d-1)}{u(1+u_x^2)}. \tag{4.4}$$

For $u \to 0$ equation (4.3) becomes $0 = X^t[(1+u_x^2)(d-1)]^2$, hence we get $X^t = 0$. Therefore, for $u \to 0$ we can write (4.2) in the form given in (4.5).

$$\zeta = X^x(\partial x + u_x \partial u + u_{xt} \partial u_t + u_{xx} \partial u^x) + Z_{xx} \partial u^{xx} + Z_{tt} \partial u^{tt} + Z_{tx} \partial u^{tx}. \tag{4.5}$$

The Cartan distribution on S_2, (i.e., vector fields (4.2) satisfying the condition $\zeta.u = 0$), is given in (4.6).

$$\left\{ \begin{aligned} \zeta \ =\ & X^x \left[-\frac{u_x}{u_t}\partial t + \partial x + (u_{tx} - \frac{u_x}{u_t}u_{tt})\partial u_t + (u_{xx} - \frac{u_x}{u_t}u_{tx})\partial u^x \right] \\ & + Z_{xx}\partial u^{xx} + Z_{tt}\partial u^{tt} + Z_{tx}\partial u^{tx}. \end{aligned} \right. \tag{4.6}$$

Therefore, when $q \in S_2$ approaches equation E_2, i.e., when u_t approaches the function given in (4.4), one has $\lim_{q \to E_2} \mathbf{E}_2(S_2)_q = <\zeta>$, where ζ are given by the vector fields in (4.7)

$$\zeta = X^x[\partial x + u_{tx}\partial u_t + u_{xx}\partial u_x] + Z_{xx}\partial u^{xx} + Z_{tt}\partial u^{tt} + Z_{tx}\partial u^{tx}. \tag{4.7}$$

This means that $\lim_{q \to E_2} \mathbf{E}_2(S_2)_q \subset \mathbf{E}_2$. This proves that solutions of E_2 can be prolonged to an integral manifold of S_2. Since, in the asymptotic limit, both distributions have not time components, then the solutions of E_2 approach S_2 to a fixed t^* and stop on the revolution axis.

By resuming, we get the following theorem, proved by using geometric methods only.

Theorem 4.1. *The singular boundary value problem* (4.1) *admits smooth solutions, i.e., smooth solutions of the regular equation* $E_2 \subset JD^2(W)$ *that collapse in a finite time approaching the set* $S_2 \subset JD^2(W)$ *of singular points.*

5 EXOTIC HEAT-SCRÖDINGER EQUATION

In [35] it is studied the following so-called *cubic nonlinear Schrödinger equation* (NLS):

$$\begin{cases} u_t + \gamma u + i(u_{xx} \mp |u|^2 u) = f \\ u(t,x) \in \mathbb{C}, \quad \forall (t,x) \in \Omega \equiv \mathbb{R}_+ \times \mathbb{T}, \ \mathbb{T} \equiv \mathbb{R}/2\pi\mathbb{Z} \end{cases} \tag{5.1}$$

where $\gamma > 0$ (*damping parameter*) and $f \in L^2(\mathbb{T})$ (*time-independent-forcing term*). There the main result is that the nonlinear group $S(\cdot)$, associated to (5.1), i.e., $S(t)(u_0) = u(t)$, $t \in \mathbb{R}$, where u is the solution of (5.1), for to the initial condition u_0, provides an infinite-dimensional dynamic system in $L^2(\mathbb{T})$ that has a global attractor $\mathcal{A} \subset H^2(\mathbb{T})$, that is a connected and compact set of $H^2(\mathbb{T})$, invariant (positively and negatively) by $S(\cdot)$ that attracts for the $L^2(\mathbb{T})$-metric all positive orbits uniformly with respect to bounded sets of initial data in $L^2(\mathbb{T})$.

In the framework of the algebraic topology of PDE's, the interesting problem considered in that paper, has an intriguing issue. With this respect, let us recast equation (5.1) into a real analytic PDE of second order over the following trivial vector fiber bundle $\pi : W \equiv M \times \mathbb{R}^2 \to M \equiv \mathbb{R}_+ \times \mathbb{T}$, $(t,x,v,w) \mapsto (t,x)$:

$$\boxed{E_2 \subset J\mathcal{D}^2(W) : \begin{cases} v_t + \gamma v - w_{xx} \mp w(v^2 + w^2) = \phi \\ w_t + \gamma w + v_{xx} \mp v(v^2 + w^2) = \psi \end{cases}} \tag{5.2}$$

where $u = v + iw$ and $f = \phi + i\psi$. Here we assume that ϕ and ψ are analytic functions. Then one can prove that E_2 is an involutive, formally integrable and completely integrable PDE. In fact, one has

$$\boxed{\dim(E_2)_{+1} = 16} = \boxed{\dim E_2 = 12} + \boxed{\dim(g_2)_{+1} = 4}. \tag{5.3}$$

The relation (5.3) proves that the canonical mapping $(E_2)_{+1} \to E_2$ is surjective. Furthermore, since $\dim(g_2)_{+1} = 4 = \dim g_2$, we get that g_2 is an involutive symbol. This facts are enough to state that E_2 is formally integrable, and since it is analytic it is completely integrable too. This means that in the neighborhood of any point $q \in E_2$, (*initial condition*), passes a regular solution (analytic solution). Furthermore, since E_2 satisfies some conditions of regularity, we can solve Cauchy problems for 1-dimensional integral manifolds $N \subset E_2$, diffeomorphically projected in W, by means of the canonical projection $\pi_{2,0} : E_2 \to W$. We call *admissible* such Cauchy manifolds. Note that solutions passing

through admissible Cauchy manifolds do not necessitate to be smooth, in fact, in general, are singular ones with respect to the embeddings $E_2 \subset JD^2(W) \subset J_2^2(W)$, where $J_2^2(W)$ denotes the 2-jet-space for 2-dimensional submanifolds of W.

Furthermore, one can see that the 1-dimensional integral singular bordism group of $E_2 \subset J_2^2(W)$ is $\Omega_{1,s}^{E_2} \cong \mathbb{Z}_2$. Then for any two space-like smooth (or analytic) 1-dimensional closed admissible Cauchy hypersurfaces $N_1 \subset (E_2)_{t_1}$ and $N_2 \subset (E_2)_{t_2}$, $t_1 \neq t_2$, where $(E_2)_t \equiv \bar{\pi}_2^{-1}(t)$, with $\bar{\pi}$ the natural projection $E_2 \to \mathbb{R}_+$, exists a singular solution $V \subset E_2$, such that $\partial V = N_1 \sqcup N_2$, iff $[N_1 \sqcup N_2] = [0] \in \Omega_{1,s}^{E_2}$. In order that this condition should be satisfied, it is enough that N_1 and N_2 have the same integral characteristic numbers of second order. (The solution V bording N_1 and N_2 is a smooth solution iff above condition holds for all the orders, i.e. for all the conservation laws of E_2.) So, in general, there is not solution unicity for any admissible, 1-dimensional closed smooth space-like Cauchy manifold, $N \subset E_2$, but all such solutions bord ones belonging to the same 0-class in $\Omega_{1,s}^{E_2} \cong \mathbb{Z}_2$. This agrees with the main result in [35]. By resuming we get the following theorem.[8]

Theorem 5.1. *Equation* (5.2)*, considered in the analytic case, has a* global attractor*, in the sense that all its (singular) solutions bord with a same integral bordism class.*

6   EXOTIC SINGULAR VECTOR HEAT EQUATION

In [14] it is studied the PDE reported in (6.1).

$$(\partial t.F^k) + \kappa v^k = 0 \tag{6.1}$$

where $F^k = F^k(t,u^a) \equiv F_t^k$, is a family of parametric equations for embeddings $F_t : \Sigma^2 \to \mathbb{R}^3$ of 2-dimensional, convex, star-shaped, compact surface Σ^2 into $\mathbb{R}^3$, with $(u^a)_{1 \leq a \leq 2}$ local coordinates on Σ^2 and $(x^k)_{1 \leq k \leq 3}$, coordinates in $\mathbb{R}^3$. $\kappa = \kappa(t,u^a)$ denotes the so-called *harmonic mean curvature* of $\Sigma_t \equiv F_t(\Sigma^2) \subset \mathbb{R}^3$, i.e., $\kappa =$ Gauss curvature/mean curvature . $v = v^k \partial x_k$ is the unitary normal vector field on Σ_t. In [14] it is assumed the mean curvature positive. The main result there concerns an existence of solutions Σ_t^ε for short time starting from a Σ^2 of class $C^{2,1}$ and a maximal time T_ε of existence of a smooth solution such that the mean curvature goes to zero as $t \to T_\varepsilon$ at some point of Σ_t^ε, or Σ_t^ε shrinks to a point as $t \to T_\varepsilon$. In the particular case where Σ^2 is a surface of revolution, the flow always exists up to the time when the surface shrinks to a point. It is interesting to recast this problem in the framework for geometric theory of PDE's, i.e., to implement this problem on a trivial vector fiber-bundle $\pi : W \equiv \mathbb{R}^6 \to \mathbb{R}^3$, $(t,u^a,x^k) \mapsto (t,u^a)$. Then the problem is encoded by the PDE reported in (6.2).

$$E_3 \subset JD^3(W) : \begin{cases} A(x^i,x_a^i,x_{ab}^i)x_t^k + B^k(x^i,x_a^i,x_{ab}^i) = 0 \text{ (HMCF)} \\ b_{ab;c} - b_{ac;b} = 0 \text{ (Gauss-Codazzi equation)} \\ b_{ab} = [\varepsilon_{ijk}x_{ab}^i x_{\bar{a}}^j x_{\bar{b}}^k \delta_1^{\bar{a}} \delta_2^{\bar{b}}] / \sqrt{EG - F^2}. \end{cases} \tag{6.2}$$

In (6.2) A and B^k are known analytic functions of their arguments, b_{ab} are the components of the second fundamental form and E, F, G, are the usual Gauss symbols of the

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[8]Here we have explicitly considered only the analytic case. However, it is also possible to extend such result to singular PDE's, similarly to what made in the previous Section 4.

surface Σ_t. The semi-colon, in the Gauss-Codazzi equation denotes tensor derivative with respect to u^c. In general the PDE in (6.2) is a singular equation, however in the case considered in [14], i.e., positive mean curvature, one has $A \neq 0$. Then, in order to characterize local solutions, it is important to study the formal properties of such equation, and, by means the determination of its integral bordism groups, characterize global solutions also. In this way one could generalize the following well known result of differential geometry: *Given symmetric functions $\gamma_{ab} = \gamma_{ab}(u^c)$ and $b_{ab} = b_{ab}(u^c)$, $a, b, c = 1, 2$, such that the Gauss-Codazzi equations are satisfied, there exists a surface $x^i = x^i(u^c)$, uniquely determined up to rigid motions, such that γ and b are respectively the first fundamental form and the second fundamental form of such a surface.* (See, e.g. L. P. Eisenhart [17].) Of course this agrees with the structure of equation in (6.2), since there coordinates with time derivatives are determined by the other ones containing coordinates with space derivatives only. It is important to emphasize that equation (6.2) is not formally integrable, since there is not surjectivity between the first prolongation $(E_3)_{+1}$ and E_3. In fact,

$$\boxed{\dim((E_3)_{+1}) = 88} < \boxed{\dim(E_3) = 58} + \boxed{\dim((g_3)_{+1}) = 41}. \tag{6.3}$$

However, by using the geometric theory of PDE's, it is possible to identify a subequation $\widehat{(E_3)} \subset E_3$ that is formally integrable and completely integrable. For this it is enough to add to the equations in (6.2) also the first prolongations of (HMCF). More precisely, we shall use equation (6.4).

$$\boxed{\widehat{E_3} \subset JD^3(W) : \begin{cases} Ax_t^k + B^k = 0 \\ Ax_{tt}^k + [(\partial x_i^a.A)x_{at}^i + (\partial x_i^{ab}.A)x_{abt}^i]x_t^k \\ \quad + (\partial x_i^a.B^k)x_{at}^i + (\partial x_i^{ab}.B^k)x_{abt}^i = 0 \\ Ax_{tc}^k + [(\partial x_i^a.A)x_{ac}^i + (\partial x_i^{ab}.A)x_{abc}^i]x_t^k \\ \quad + (\partial x_i^a.B^k)x_{ac}^i + (\partial x_i^{ab}.B^k)x_{abc}^i = 0 \\ b_{ab;c} - b_{ac;b} = 0 \\ [b_{ab;c} - b_{ac;b}]_{,t} = 0 \\ [b_{ab;c} - b_{ac;b}]_{,d} = 0 \\ b_{ab} = [\varepsilon_{ijk}x_{ab}^i x_{\bar{a}}^j x_{\bar{b}}^k \delta_1^{\bar{a}} \delta_2^{\bar{b}}]/\sqrt{EG - F^2}. \end{cases}} \tag{6.4}$$

Then $\widehat{E_{3+1}} \to \widehat{E_3}$ is surjective as proved in (6.5).

$$\boxed{\dim((\widehat{E_3})_{+1}) = 70} = \boxed{\dim \widehat{E_3} = 49} + \boxed{\dim((\widehat{g_3})_{+1}) = 21}. \tag{6.5}$$

Furthermore, the symbol $\widehat{g_3}$ is involutive, since one has $\dim(\widehat{g_3}_{+1}) = 21 = \dim(\widehat{g_3})$. Therefore $\widehat{E_3}$ is formally integrable and completely integrable.

Then the characterization of Cauchy problems and global solutions, weak, singular and smooth, can be directly obtained by using the algebraic topologic methods in PDE's introduced by A. Prástaro. The asymptotic behaviour of solutions as mean curvature approaches the zero value, can be obtained by considering the singular PDE in (6.6)

$$\boxed{\widehat{(E_3)}^{(S)} \equiv \widehat{(E_3)} \bigcup S_3 \subset JD^3(W) : \begin{cases} \widehat{(E_3)} \quad (6.4) \\ S_3 \equiv \{\frac{1}{2}\gamma^{ab}b_{ab} = 0\} \end{cases}} \tag{6.6}$$

i.e., adding to equation (6.4) the submanifold S_3 of $JD^3(W)$, identified by the condition of zero mean-curvature. (See, e.g. Section 4 where similar problems are considered.)

7 COMPLEX RICCI FLOW EQUATION AS A QUANTUM EQUATION AND POINCARÉ CONJECTURE

In (7.1) it is reported the so-called *normalized Kähler-Ricci flow equation*

$$
\begin{cases}
\text{(normalized Kähler-Ricci flow equation)}: \partial_t.g(t) = -Ric(g(t)) + g(t) \\
g(t) \text{ is a Kähler metric for each } t; \\
\omega(t) = \omega_{g(t)}, \text{ the Kähler form associated to } g(t), \text{ satisfies the condition} \\
[\omega(t) = \frac{i}{2\pi}g(t)_{\alpha\bar{\beta}}dz^\alpha \wedge d\bar{z}\beta] = c_1(M) = [\frac{i}{2\pi}R_{\alpha\beta}(g(t))dz^\alpha \wedge d\bar{z}\beta = \rho(t)] > 0.
\end{cases}
\tag{7.1}
$$

Here M is a compact Kähler manifold, with the first Chern class $c_1(M) > 0$. (Positivity of the first Chern class is in the sense of K. Kodaira [31].) $\omega(t)$ denotes the Kähler form associated to $g(t)$ and $\rho(t)$ is the Ricci form that represents also the first Chern class of M. Some authors study solutions of (7.1) by adding also the global constraint in (7.2).

$$
\int_M |Rm(g(t))|^{n/2}dv_t \leq C, \dim_{\mathbb{C}} M = n \geq 3, C \in \mathbb{R}
\tag{7.2}
$$

There $Rm(g(t))$ denotes the curvature operator. (See, e.g., the work by P. Daskalopoulos and M. Sesum [14], and papers quoted there.) The main result in [14] is an improvement of a previous result by Sesum, where was also made the hypothesis of bounded Ricci curvature: $|R(g(t)| \leq C$. More precisely if $g(t), t \in [0,\infty)$ is a solution of the problem (7.2) that at $t = 0$ has curvature operator uniformly bounded in L^n-norm, the curvature operator will also be uniformly bounded along the flow. This is enough to state that the flow will converge along a subsequence to a Kähler-Ricci soliton, i.e., a compact Kähler manifold (N, h), such that its Kähler form ω_h satisfies the following equation: $Ric(\omega_h) - \omega_h = \mathcal{L}_\zeta \omega_h$, where ζ is a holomorphic vector field on N, and $\mathcal{L}_\zeta$ denotes Lie derivative with respect to ζ. It follows that the first Chern class $c_1(N)$ of N is positive and represented by ω_h. Since there are no Kähler-Einstein metrics on N if N admits a Kähler-Ricci soliton, (this is a result by A. Futaki [21]), it follows that the existence of Kähler-Ricci soliton is an obstruction to the existence of Kähler-Einstein metrics on compact Kähler manifolds with positive first Chern class. The condition $c_1(M) > 0$ is important, after the result by S.-T. Yau [69], on the existence of Kähler-Einstein metrics on Kähler manifolds with $c_1(M) \leq 0$, and by T. Aubin [4] for Kähler manifolds with $c_1(M) < 0$. (This was first conjectured by E. Calabi [7]. For complementary problems on complex manifolds see, e.g., [70].)

It is interesting to emphasize that the problem considered in ([14]) can be recast in the geometric theory of PDE, directly working in the category of complex manifolds. This can be made, by using the geometric theory of quantum PDE's formulated by A. Prástaro. In fact the category of complex manifolds can be considered a little subcategory of the one for quantum manifolds, (in the sense introduced by A. Prástaro), where the quantum algebra coincides with the algebra of complex numbers and quantum differentiable functions are identified with holomorphic functions. Then the system (7.1) can be encoded with the

following second order PDE $\widetilde{E}_2$ in the category of complex manifolds (like quantum manifolds):

$$\widetilde{E}_2 \subset J\widetilde{\mathcal{D}}^2(\widetilde{W}) \subset \widetilde{J}^2_{n+1}(\widetilde{W}): \begin{cases} (1(a)) & : (\partial_t.g)_{\alpha\bar{\beta}}(t) = -Ric(g(t))_{\alpha\bar{\beta}} + g(t)_{\alpha\bar{\beta}} \\ (1(b)) & : R_{\alpha\bar{\beta}} = -(\partial z_\alpha \partial \bar{z}_\beta . \ln \det(g_{\delta\bar{\varepsilon}})) \\ (2(a)) & : g_{\alpha\bar{\beta}} = g_{\bar{\beta}\alpha} \\ (2(b)) & : \bar{g}_{\alpha\bar{\beta}} = g_{\bar{\alpha}\beta} \\ (3(a)) & : (\partial z_\gamma.g_{\alpha\bar{\beta}}) - (\partial z_\alpha.g_{\gamma\bar{\beta}}) = 0 \\ (3(b)) & : (\partial \bar{z}_\gamma.g_{\alpha\bar{\beta}}) - (\partial \bar{z}_\beta.g_{\alpha\bar{\gamma}}) = 0 \\ (4) & : g_{\alpha\bar{\beta}} - R_{\alpha\bar{\beta}} - (\partial z_\alpha \partial \bar{z}_\beta . f) = 0 \end{cases} \tag{7.3}$$

In (7.3) $\widetilde{W}$ is the following fiber bundle $\widetilde{\pi}: \widetilde{W} \equiv \mathbb{R} \times \widetilde{S^0_2 M} \times \mathbb{C} \to \mathbb{R} \times M$, $(t, z^\alpha, \bar{z}^\alpha, u_{\alpha\bar{\beta}}, \lambda) \mapsto (t, z^\alpha, \bar{z}^\alpha)$. (Local) sections of $\widetilde{\pi}$ are identified with the following (local) functions

$$\{g_{\alpha\bar{\beta}}(t, z^\alpha, \bar{z}^\alpha), f(t, z^\alpha, \bar{z}^\alpha)\}.$$

The "tilde" over the jet(-derivative) spaces denotes "holomorphic class of differentiability". Our geometric theory of quantum PDE's allows us to obtain solutions as $(n+1)$-dimensional complex integral submanifolds of $\widetilde{E}_2$. Then the characterization of global solutions is made by means of the integral bordism group $\Omega_n^{\widetilde{E}_2}$. With this respect one can generalize the Poincaré conjecture in this category of complex manifolds. (For details see [51] where a generalized version of the Poincaré conjecture is formulated, and proved too, in the category of quantum (super)manifolds.)

Acknowledgments

References

[1] R. P. Agarwal and A. Prástaro, Geometry of PDE's.III(I): Webs on PDE's and integral bordism groups. The general theory. *Adv. Math. Sci. Appl.* **17(1)**(2007), pp 239-266; Geometry of PDE's.III(II): Webs on PDE's and integral bordism groups. Applications to Riemannian geometry PDE's, *Adv. Math. Sci. Appl.* **17(1)**(2007), pp 267-281.

[2] R. P. Agarwal and A. Prástaro, Singular PDE's geometry and boundary value problems. *J. Nonlinear Conv. Anal.* **9(3)**(2008), pp 417-460; On singular PDE's geometry and boundary value problems. *Appl. Anal.* **88(8)**(2009), pp 1115-1131.

[3] B. Andrwes and H. Nguyen, Four-manifolds with $1/4$-pinched flag curvatures. *Asian J. Math.* **13(2)**(2009), pp 251-270.

[4] T. Aubin, Equations du type Monge-Ampére sur les variétés Kählériennes compactes. *Bull. Sci. Math., II Ser.* **102**(1998), pp 63-95.

[5] M. Berger, Les variétés riemaniennes $\frac{1}{4}$-pinces. *Ann. Scuola Norm. Sup. Pisa* **14**(1960), pp 161-170.

[6] S. Brendle and R. Schoens, Manifolds with 1/4-pinched curvature are space forms. *J. Amer. Math. Soc.* **22(1)**(2009), pp 287-307; Classification of manifolds with 1/4-pinched curvatures. *Acta Math.* **200(1)**(2008), pp 1-13.

[7] E. Calabi, The space of Kähler metrics. *Proc. Int. Congr. Math. Amsterdam* **2**(1954), pp 206-207.

[8] A. Casson, Fibrations over spheres. *Topology* **6**(1967), pp 489-499.

[9] J. Cheeger, Finiteness theorems for Riemannian manifolds. *Amer. J. Math.* **92**(1970), pp 61-74.

[10] B. Chow and D. Knopp, Ricci flow: an introduction. *Mathematical Surveys and Monographs*, vol. 119, Amer. Math. Soc., Providence, RI, 2004.

[11] B. L. Chen, Strong uniqueness of the Ricci flow. *J. Differential Geom.* **82**(2009), pp 363-382.

[12] B. Chow, S. C. Chu, D. Gliekenstein, C. Guenther, J. Isenberg, T. Ivey, D. Knopf, P. Lu, F. Luo and L. Ni, *The Ricci Flow: Techniques and Applications: Part I: Geometric Aspects*, Mathematical Surveys Monographs **135**, Amer. Math. Soc. 2008.

[13] B. Chow, S. C. Chu, D. Gliekenstein, C. Guenther, J. Isenberg, T. Ivey, D. Knopf, P. Lu, F. Luo and L. Ni, *The Ricci Flow: Techniques and Applications: Part II: Analytic Aspects*, Mathematical Surveys Monographs **144**, Amer. Math. Soc. 2008.

[14] P. Daskalopoulos and M. Sesum, The harmonic mean curvature flow of nonconvex surfaces in $\mathbb{R}^3$. *Calc. Var. Partial Differ. Equ.* **37(1-2)**(2010), pp 187-215.

[15] D. M. De Turk, Deforming metrics in the direction of their Ricci tensors. *J. Differential Geom.* **18(1)**(1983), pp 157-162.

[16] A. Dold, Erzewgende der Thomschen Algebra $\mathfrak{N}$. *Math. Zeitschr.* **65**(1956), pp 25-35.

[17] L. P. Eisenhart, *A Treatise on Differential Geometry of Curves and Surfaces*, Ginn and Company, Boston, 1909, xii+474 pages; 2nd edition Dover Publications, Inc., 1960, 474 pp. (Reviewed by G. A. Bliss, Bull. Amer. Math. Soc. **17(9)**(1911), pp 470-478.)

[18] F. Fang and X. Rong, Fixed point free circle actions and finiteness theorems. *Comm. Contemp. Math.* **2**(2000), pp 75-86.

[19] M. Freedman, The topology of four-dimensional manifolds. *J. Differential Geom.* **1(3)**(1982), pp 357-453.

[20] M. Freedman and F. Quinn, *Topology of 4-manifolds*, Princeton Math. Serier **39** Princeton Univ. Press, Princeton, N.J. 1990.

[21] A. Futaki, On compact Kähler manifolds of constant scalar curvature. *Proc. Japan Acad. ser. A Math. Sci.* **59**(1983), pp 401-402.

[22] Z. Gang and I. M. Sigal, Neck pinching dynamics under mean curvature flow. *J. Geom. Anal.* **19(1)**(2009), pp 36-80.

[23] H. Goldshmidt, Integrability criteria for systems of non-linear partial differential equations. *J. Differential Geom.* **1**(1967), pp 269-307.

[24] R. S. Hamilton, Three-manifolds with positive Ricci curvature. *J. Differential. Geom.* **17**(1982), pp 255-306.

[25] R. S. Hamilton, Four-manifolds with positive Ricci curvature operator. *J. Differential Geom.* **24**(1986), pp 153-179.

[26] R. S. Hamilton, Eternal solutions to the Ricci flow. *J. Differential Geom.* **38**(1993), pp 1-11.

[27] R. S. Hamilton, The formation of singularities in the Ricci flow. *Surveys in Differential Geometry* International Press, 1995, **2**(1995), pp 7–136.

[28] R. S. Hamilton, A compactness property for solutions of the Ricci flow on three-manifolds. *Comm. Anal. Geom.* **7**(1999), pp 695-729.

[29] T. Kawakami, Algebraic and Nash realizations of vector bundles and triviality of equivariant algebraic and Nash vector bundles. *Bull. Fac. Educ., Wakayama Univ., Nat. Sci.* **57**(2001), pp 1-10.

[30] W. Klingenberg, Neue ergebnisse über konvexe flächen. *Comment. Math. Helv.* **34**(1960), pp 17-36.

[31] K. Kodaira, The theorem of Riemann-roch for adjoint systems and Kählerian varieties. *Ann. Math. Stud.* **30**(1953), pp 247-264.

[32] J. Milnor, On manifolds homeomorphic to the 7-sphere. *Ann. of Math.* **64(2)**(1956), pp 399-405.

[33] E. Moise, Affine structures in 3-manifolds. V. The triangulation theorem and Hauptvermuntung. *Ann. of Math. Sec. Ser.* **56**(1952), pp 96-114.

[34] E. Moise, *Geometric topology in dimension* 2 *and* 3. Springer-Verlag, Berlin, 1977.

[35] L. Molinet, Global attractor and asymptotic smoothing effects for the wekly damped cubic Schrödinger equation in $L^2(\mathbb{T}^2)$. *Dyn. Part. Differ. Equ.* **6(2)**(2009), pp 15-34.

[36] J. Nash, Real algebraic manifolds. *Ann. of Math.* **56(2)**(1952), pp 405-421.

[37] G. Perelman, *The entropy formula for the Ricci flow and its geometry applications*, arXiv:math/0211159.

[38] G. Perelman, *Ricci flow with surgery on three-manifolds*, arXiv:math/0303109.

[39] A. Petrunin and W. Tuschmann, Diffeomorphism finiteness, positive pinching and second homotopy. *Geom. Funct. Anal.* **9(4)**(1999), pp 736-774.

[40] A. Prástaro, (Co)bordism groups in PDE's. *Acta Appl. Math.* **59(2)**(1999), pp 111-202.

[41] A. Prástaro, (Co)bordism groups in quantum PDE's. *Acta Appl. Math.* **64(2/3)**(2000), pp 111-217.

[42] A. Prástaro, *Quantized Partial Differential Equations*, World Scientific Publ., Singapore, 2004.

[43] A. Prástaro, Geometry of PDE's. I: Integral bordism groups in PDE's. *J. Math. Anal. Appl.* **319**(2006), pp 547-566.

[44] A. Prástaro, Geometry of PDE's. II: Variational PDE's and integral bordism groups. *J. Math. Anal. Appl.* **321**(2006), pp 930-948.

[45] A. Prástaro, Geometry of PDE's. IV: Navier-Stokes equation and integral bordism groups. *J. Math. Anal. Appl.* **338(2)**(2008), pp 1140-1151.

[46] A. Prástaro, (Un)stability and bordism groups in PDE's. *Banach J. Math. Anal.* **1(1)**(2007), pp 139-147.

[47] A. Prástaro, Extended crystal PDE's stability.I: The general theory. *Math. Comput. Modelling* **49(9-10)**(2009), pp 1759-1780.

[48] A. Prástaro, Extended crystal PDE's stability.II: The extended crystal MHD-PDE's. *Math. Comput. Modelling* **49(9-10)**(2009), pp 1781-1801.

[49] A. Prástaro, On the extended crystal PDE's stability.I: The *n*-d'Alembert extended crystal PDE's. *Appl. Math. Comput.* **204(1)**(2008), pp 63-69.

[50] A. Prástaro, On the extended crystal PDE's stability.II: Entropy-regular-solutions in MHD-PDE's. *Appl. Math. Comput.* **204(1)**(2008), pp 82-89.

[51] A. Prástaro, Surgery and bordism groups in quantum partial differential equations.I: The quantum Poincaré conjecture. *Nonlinear Anal. Theory Methods Appl.* **71(12)**(2009), pp 502-525.

[52] A. Prástaro, Surgery and bordism groups in quantum partial differential equations.II: Variational quantum PDE's. *Nonlinear Anal. Theory Methods Appl.* **71(12)**(2009), pp 526-549.

[53] A. Prástaro, *Extended crystal PDE's*, arXiv: 0811.3693[math.AT].

[54] A. Prástaro, *Quantum extended crystal super PDE's*, arXiv: 0906.1363[math.AT].

[55] A. Prástaro, *Exotic heat PDE's*, arXiv: 1006.4483[math.GT].

[56] A. Prástaro and Th. M. Rassias, Ulam stability in geometry of PDE's. *Nonlinear Funct. Anal. Appl.* **8(2)**(2003), pp 259-278.

[57] T. Radó, Uber den Begriff von Riemannsche Fläche. *Acta Sci. Math. (Szeged)* **2**(1924), pp 101-120.

[58] W. D. Ruan, Y. Zhang and Z. Zhang, Bounding sectional curvature along the Kähler-Ricci flow. *Commun. Contemp. Math.* **11(6)**(2009), pp 1067-1077.

[59] W. X. Shi, Deforming the metric on complete Riemannian manifold. *J. Differential Geom.* **30**(1989), pp 223-301.

[60] S. Smale, Generalized Poincaré conjecture in dimension greater than four. *Ann. of Math.* **74(2)**(1961), pp 391-406.

[61] S. Smale, Differentiable dynamical systems. *Bull. Amer. Math. Soc.* **73**(1967), pp 747-817.

[62] D. Sullivan, On the Hauptvermuntung for manifolds. *Bull. Amer. Math. Soc.* **73**(1967), pp 598-600.

[63] J. L. Synge, Connectivity of spaces of positive curvature. *Quart. J. Math. Oxf. Ser.* **7**(1936), pp 316-320.

[64] R. Thom, Quelques propriété globales des variétés différentieles. *Comm. Math. Helv.* **28**(1954), pp 17-86.

[65] A. Tognoli, Su una congettura di Nash. *Ann. Scuola Norm. Sup. Pisa* **27**(1973), pp 167-185.

[66] W. Tuschmann, Geometric diffeomorphism finiteness in low dimensions and homotopy group finiteness. *Math. Annalen* **322(2)**(2002), pp 413-420.

[67] C. T. C. Wall, Determination of the cobordism ring. *Ann. of Math.* **72**(1960), pp 292-311.

[68] H. Yamabe, On a deformation of Riemannian structures on compact manifolds. *Osaka Math. J.* **12**(1960), pp 21-37.

[69] S. T. Yau, Calabi's conjecture and some new results in algebraic geometry. *Proc. Natl. acad. sci. USA* **74(5)**(1977), pp 1798-1799.

[70] S. T. Yau, Metrics on complex manifolds. *Sci. China. Math.* **53(3)**(2010), pp 565-572.