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Existence of a Solution for Some Singular Quasilinear Problem with Variable Exponent and Containing Gradient Term

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Abstract

In this paper we study an elliptic equation involving variable exponents and containing a singular lower order terms with $p(x)-$growth in the gradient. Through an approximation approach, we prove the existence of a nonnegative distributional solution in the whole space $\mathbb{R}^N$.

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1 Introduction

Quasilinear equations containing a gradient term with different growth conditions without singularity, have been exhaustively studied in several papers in which many results of existence or nonexistence of solutions have been established. Among them are for instance [5-8, 18, 20, 21, 25]. Problems with terms having different kinds of singularities at the origin have known a great interest in the recent years and many papers dealing with this subject have been published. See for instance [3, 4, 15, 30]. In [3], the authors considered the problem

$$-\mathrm{div}(M(x,u)\nabla u)+g(x,u)\,|\nabla u|^2 = f \quad \text{in } \Omega$$

where Ω is a bounded domain of $\mathbb{R}^N$ and $g(\cdot,\cdot):\Omega\times(0,+\infty)\to\mathbb{R}$ is some nonnegative Carathéodory function having singularity at $s=0$. Under suitable conditions on $M(\cdot,\cdot)$ and on the data f, a result of existence of positive solution was found. This solution was obtained through a convergence process of some sequence of approximated solutions and, in

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order to overcome the main difficulty they had to face, which was the passage to the limit in the singular term, the authors established an important result (see [3, Proposition 2.3]) in which it was proved that the sequence of the approximated solutions is uniformly bounded from below in every compactly contained open subset of Ω. In [15], the authors investigated the problem of existence of distributional solution to the following elliptic equation

$$-\Delta u+\lambda u=\pm\frac{|\nabla u|^2}{|u|^k}+f \quad \text{in } \Omega$$

where $\lambda>0,\ k>0$ and $f\in L^\infty(\mathbb{R}^N)$ such that $f\geq 0$. In the case $0<k<1$, existence result has been proved independently of the singular term's sign, while, for the case $k\geq 1$, a positivity condition is needed. Using, as in [3], approximation approach but arguing differently concerning the passage to the limit in the singular term, Giachetti and Murat established the existence of a finite energy solution satisfying some properties. Inspired by their work, we have tried to extend their result to the case of anisotropic equations. In fact, quasilinear equations involving variable exponent is one of the most interesting topic in recent years. This great interest given for such type of equations could be explained by the many applications of such type of equations in modelling various physical phenomena as electrorheological fluids, image restoration and elastic mechanics. Concerning this type of problem with non-standard growth condition, we can refer, for example, to [12-14, 16, 19, 22, 23]. In the present work, we are concerned by a quasilinear equation modelling the motion of an incompressible fluid in a nonhomogeneous and anisotropic medium. Let us denote by $\vec{V}$ and p the velocity and the pressure of the fluid. In a homogeneous and isotropic medium, the incompressible fluid satisfies the continuity equation: $-\mathrm{div}\left(\vec{V}\right)=0$. With the Darcy law, i.e. $\vec{V}=-\xi|\nabla p|^{\lambda-2}\nabla p$ where λ and ξ are constants, the above continuity equation becomes: $-\mathrm{div}\left(|\nabla p|^{\lambda-2}\nabla p\right)=0$. Let now the medium be nonhomogeneous and anisotropic, i.e. its characteristics may vary in dependence on directions and points. So $\lambda\equiv\lambda(x)$ and $\xi\equiv\xi(x)$. In this case, if we also assume the existence of exterior forces, then the pression of the incompressible fluid satisfies the following quasilinear equation:

$$-\mathrm{div}\left(\xi(x)|\nabla p|^{\lambda(x)-2}\nabla p\right)=h(x,p,\nabla p).$$

If $h(x,p,\nabla p)$ contains a term of type $A(x,p)|\nabla p|^{\lambda(x)}$, this last term describes the diffusion of mass factor. For more details, see[1]. We are mainly interested by coefficient $A(\cdot,\cdot)$ having a singular behaviour which seems to represent a new topic. As in [15], for the case $0<k<1$, we establish an existence result of a nonnegative and nontrivial solution without specifying the sign of the singular term and a similar result is proved for the case $k\geq 1$ in the particular case of positive singular term. Furthermore, and in contrast of [15], by making modification in the approximating problem, we have been able to prove an existence result for the case $k=1$ when the singular term is negative.

2 Overview on Generalized Sobolev Spaces

Assume $\Omega\subset\mathbb{R}^N$ is an nonempty domain(bounded or unbounded).

Set $C_+(\overline{\Omega})=\left\{h\in C(\overline{\Omega})\cap L^\infty(\Omega),\ h(x)>1 \text{ for all } x\in\overline{\Omega}\right\}$.

For any $p\in C_+(\overline{\Omega})$, we define

$$p^+=\sup_{x\in\Omega}p(x) \quad \text{and} \quad p^-=\inf_{x\in\Omega}p(x).$$

For each $p \in C_+(\overline{\Omega})$, we define the variable exponent Lebesgue space

$$L^{p(\cdot)}(\Omega) = \left\{ u;\ u : \Omega \to \mathbb{R} \text{ mesurable such that } \int_\Omega |u(x)|^{p(x)}\, dx < +\infty \right\}.$$

This space becomes a Banach space with respect to the Luxemburg norm, that is

$$|u|_{L^{p(\cdot)}(\Omega)} = \inf\left\{ \mu > 0;\ \int_\Omega \left| \frac{u(x)}{\mu} \right|^{p(x)} dx \le 1 \right\}.$$

Moreover, $L^{p(\cdot)}(\Omega)$ is a reflexive space provided that $1 < p^- \le p^+ < +\infty$. Denoting by $L^{p'(\cdot)}(\Omega)$ the conjugate space of $L^{p(\cdot)}(\Omega)$ where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$; for any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$ we have the following Hölder type inequality

$$\begin{aligned} \left| \int_\Omega uv dx \right| &\le \left(\frac{1}{p^-} + \frac{1}{p'^-} \right) |u|_{L^{p(\cdot)}(\Omega)} |v|_{L^{p'(\cdot)}(\Omega)} \\ &\le 2 |u|_{L^{p(\cdot)}(\Omega)} |v|_{L^{p'(\cdot)}(\Omega)}. \end{aligned} \tag{2.1}$$

Similarly, if $\frac{1}{p_1(x)} + \frac{1}{p_2(x)} + \frac{1}{p_3(x)} = 1\ \forall x \in \overline{\Omega}$, then for any $u \in L^{p_1(\cdot)}(\Omega),\ v \in L^{p_2(\cdot)}(\Omega)$ and $w \in L^{p_3(\cdot)}(\Omega)$,

$$\begin{aligned} \left| \int_\Omega uvw\, dx \right| &\le \left(\frac{1}{p_1^-} + \frac{1}{p_2^-} + \frac{1}{p_3^-} \right) |u|_{L^{p_1}(\Omega)} |v|_{L^{p_2}(\Omega)} |w|_{L^{p_3}(\Omega)} \\ &\le 3 |u|_{L^{p_1}(\Omega)} |v|_{L^{p_2}(\Omega)} |w|_{L^{p_3}(\Omega)}. \end{aligned} \tag{2.2}$$

If $|\Omega| < +\infty$ and $p_1,\ p_2$ are variable exponents so that $p_1(x) \le p_2(x)$ for all $x \in \overline{\Omega}$, then there exists the continuous embedding

$$L^{p_2(\cdot)}(\Omega) \hookrightarrow L^{p_1(\cdot)}(\Omega).$$

We introduce now the modular of the Lebesgue-Sobolev space $L^{p(\cdot)}(\Omega)$, as the mapping

$$\rho_{p(\cdot)} : L^{p(\cdot)}(\Omega) \to \mathbb{R} \text{ defined by } \rho_{p(\cdot)}(u) = \int_\Omega |u|^{p(x)}\, dx.$$

We give here some relations which can be established between the Luxemburg norm and the modular. If $(u_n),\ u \in L^{p(\cdot)}(\Omega)$ and $1 \le p^- \le p^+ < +\infty$, then the following relations hold true

$$|u|_{L^{p(\cdot)}(\Omega)} > 1 \Rightarrow |u|^{p^-}_{L^{p(\cdot)}(\Omega)} \le \rho_{p(\cdot)}(u) \le |u|^{p^+}_{L^{p(\cdot)}(\Omega)}, \tag{2.3}$$

$$|u|_{L^{p(\cdot)}(\Omega)} < 1 \Rightarrow |u|^{p^+}_{L^{p(\cdot)}(\Omega)} \le \rho_{p(\cdot)}(u) \le |u|^{p^-}_{L^{p(\cdot)}(\Omega)}, \tag{2.4}$$

$$|u_n - u|_{L^{p(\cdot)}(\Omega)} \to 0 \Leftrightarrow \rho_{p(\cdot)}(u_n - u) \to 0. \tag{2.5}$$

Next, we define $W^{1,p(\cdot)}(\Omega)$ as the space

$$W^{1,p(\cdot)}(\Omega) = \left\{ u \in L^{p(\cdot)}(\Omega);\ |\nabla u| \in L^{p(\cdot)}(\Omega) \right\}$$

and it can be equipped with the norm

$$\|u\|_{1,p(\cdot)} = |u|_{L^{p(\cdot)}(\Omega)} + |\nabla u|_{L^{p(\cdot)}(\Omega)}.$$

The space $W^{1,p(\cdot)}(\Omega)$ is a Banach space which is reflexive under condition

$$1 < p^- \le p^+ < +\infty.$$

Let $p, q \in C_+(\overline{\Omega})$. If we have p Lipschitz continuous and

$$p(x) \le q(x) \le p^*(x) = \frac{Np(x)}{N - p(x)} \quad \forall\ x \in \overline{\Omega},$$

then there is a continuous embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$.
This last embedding is compact provided that Ω is bounded in $\mathbb{R}^N$ and that $q(x) < p^*(x) \quad \forall\ x \in \overline{\Omega}$. Finally, we denote by $W_0^{1,p(\cdot)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$.

In the present work, we look for solution in $W^{1,p(\cdot)}(\mathbb{R}^N)$ which is supposed equipped with the norm

$$||u|| = |\nabla u|_{L^{p(\cdot)}(\mathbb{R}^N)} + |u|_{L^{p(\cdot)}(\mathbb{R}^N)}.$$

For more properties of anisotropic variable exponent Lebesgue-Sobolev spaces, we refer to the book [24] and the papers [9-11, 26, 27].

3 Hypotheses and Main Results

In the present paper we are concerned by the problem of existence of nonnegative solution for the following equation:

$$-\mathrm{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) + |u|^{p(x)-2}u = A(x,u)|\nabla u|^{p(x)} + f(x,u) + h \quad \text{in } \mathbb{R}^N \tag{P}$$

where $p(\cdot) \in C_+(\mathbb{R}^N)$ is a Lipschitz continuous function such that $1 < p^- \le p^+ < N$, $N > 2$ and $A(\cdot,\cdot) : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. Our basic hypotheses are cited below:

(H_1) $\quad f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that

$$|f(x,s)| \le |g(x)||s|^{\beta(x)-1} \quad \text{a.e. } x \text{ in } \mathbb{R}^N \quad \text{and} \quad \text{for every } s \in \mathbb{R}$$

with $\beta(\cdot) \in C_+(\mathbb{R}^N)$, $\beta^+ < p^-$, $g \in L^{r(\cdot)}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ where
$r \in C_+(\mathbb{R}^N)$ and there exists $\mu \in C_+(\mathbb{R}^N)$ such that

$$p(x) \le \mu(x) \le p^*(x),\ \frac{1}{r(x)} + \frac{\beta(x)}{\mu(x)} = 1, \quad \forall x \in \mathbb{R}^N.$$

We also assume that $f(x,s) = 0 \quad$ a.e. x in $\mathbb{R}^N$ and for every $s \le 0$.

(H_2) $\quad h \in L^\infty(\mathbb{R}^N) \cap L^{p'(\cdot)}(\mathbb{R}^N)$, $h \ge 0$ and $h \ne 0$ where $p'(\cdot)$ denotes the conjugate of $p(\cdot)$.

By (H_1) and (H_2), we can easily find a positive real number $M > 1$ such that

$$\frac{p(x)-\beta(x)}{p(x)-1}|g(x)|^{\frac{p(x)-1}{p(x)-\beta(x)}} + h(x) - \frac{p(x)-\beta(x)}{p(x)-1}M^{p(x)-1} \le 0 \quad \text{a.e. } x \text{ in } \mathbb{R}^N. \tag{3.1}$$

We define, for $s \in \mathbb{R}$, the function

$$\Phi(s) = se^{\eta s^2}, \tag{3.2}$$

where η is a positive real number. Also, we introduce the following truncature function, defined for $s \geq 0$ and $t > 0$, by

$$T_t(s) = \begin{cases} t & \text{if } s \leq t \\ s & \text{if } s \geq t \end{cases} \tag{3.3}$$

From now on, we will denote by $C_0^\infty(\mathbb{R}^N)$ the space of all functions in $C^\infty(\mathbb{R}^N)$ with compact support.

Definition 3.1 We define a weak (or distributional) solution of the problem (P) as a function $u \in W_{loc}^{1,p(\cdot)}(\mathbb{R}^N)$ satisfying

$$\int_{\mathbb{R}^N} |\nabla u|^{p(x)-2} \nabla u \cdot \nabla v dx + \int_{\mathbb{R}^N} |u|^{p(x)-2} uv dx = \int_{\mathbb{R}^N} A(x,u) |\nabla u|^{p(x)} v dx$$

$$+ \int_{\mathbb{R}^N} f(x,u) v dx + \int_{\mathbb{R}^N} hv dx \quad \forall\, v \in C_0^\infty(\mathbb{R}^N).$$

If moreover $u \in W^{1,p(\cdot)}(\mathbb{R}^N)$, we say that u is a finite energy solution.

The main results of the present paper are cited in the following theorems :

Theorem 3.1 Assume that (H_1) and (H_2) hold true. If we also suppose that

(H_3) $|A(x,s)| \leq \psi(s)$ a.e $x \in \mathbb{R}^N$ and for every $s > 0$ where $\psi : (0,+\infty) \to (0,+\infty)$ is a continuous function such that

- $\sup\limits_{|s|\geq\delta} (\psi(s)) < +\infty \quad \forall\, \delta > 0$
- $s \longmapsto s\psi(s)$ is nondecreasing in $(0,1)$
- ψ is integrable in a neighborhood of zero.

then the problem (P) has at least one weak nontrivial and nonnegative solution $u \in W^{1,p(\cdot)}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Furthermore, the function u satisfies that $\left(|\nabla u|^{p(\cdot)} \psi(u) \chi_{\{u>0\}}\right) \in L^1_{loc}(\mathbb{R}^N)$.

Theorem 3.2 Assume that (H_1) and (H_2) hold true. If we also suppose that

(H_4) $\underline{\psi}(s) \leq A(x,s) \leq \overline{\psi}(s)$ a.e. x in $\mathbb{R}^N$ and for every $s > 0$ where $\underline{\psi}$, $\overline{\psi} : (0,+\infty) \to (0,+\infty)$ are two continuous functions such that

- $\sup\limits_{|s|\geq\delta} \overline{\psi}(s) < +\infty \quad \forall\, \delta > 0$
- $\underline{\psi}$ is nonincreasing in $(0,M)$ where M is defined by (3.1).

then the problem (P) has at least one weak nontrivial and nonnegative solution $u \in W_{loc}^{1,p(\cdot)}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Furthermore, the function u satisfies that $\left(|\nabla u|^{p(\cdot)} \underline{\psi}(u) \chi_{\{u>0\}}\right) \in L^1_{loc}(\mathbb{R}^N)$.

Theorem 3.3 Assume that (H_1) and (H_2) hold true. If we also suppose that

(H_5) there exists a positive constant $c_0 > 0$ such that

$$-\frac{c_0}{s} \leq A(x,s) \leq 0 \quad \text{a.e } x \in \mathbb{R}^N \quad \text{and} \quad \text{for every } s > 0$$

then the problem (P) has at least one weak nontrivial and nonnegative solution $u \in W^{1,p(\cdot)}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Furthermore, the function u satisfies that $\left(\frac{|\nabla u|^{p(\cdot)}}{u}\chi_{\{u>0\}}\right) \in L^1_{loc}(\mathbb{R}^N)$ and $\left(uA(\cdot,u)|\nabla u|^{p(\cdot)}\chi_{\{u>0\}}\right) \in L^1(\mathbb{R}^N)$.

In the next sections, we shall try to prove these three theorems. The keystone of the proofs is approximating (P) by a sequence of problems (P_n) for which we construct a sequence of weak solutions (u_n). We are essentially interested by the behaviour of this sequence. A uniform a priori estimates of (u_n) are proved. We point out that there is many common points in the proofs of Theorems 3.1, 3.2 and 3.3. We shall emphasize on the differences existing between these proofs.

4 Proof of Theorem 3.1

We begin by defining, for $s \geq 0$ and $n \geq 1$ an integer, the following functions

$$\psi_n(s) = \begin{cases} \psi(s) & \text{if} \quad s \geq \frac{1}{n} \\ \psi\left(\frac{1}{n}\right) & \text{if} \quad 0 \leq s \leq \frac{1}{n} \end{cases} \quad \text{and} \quad \gamma_n(s) = \int_0^s \psi_n(t)dt.$$

For $n \geq 1$ an integer, we consider the problem

$$-\text{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) + |u|^{p(x)-2}u = A_n(x,u)\frac{|\nabla u|^{p(x)}}{1+\frac{1}{n}|\nabla u|^{p(x)}} + f(x,u) + h(x) \quad \text{in } B_n \qquad (P_n)$$

where $B_n = \left\{x \in \mathbb{R}^N;\ |x| < n\right\}$ and $A_n(x,s) = \begin{cases} A(x,s) & \text{if} \quad s \geq \frac{1}{n} \\ nsA(x,s) & \text{if} \quad 0 < s \leq \frac{1}{n} \\ 0 & \text{if} \quad s \leq 0 \end{cases}$

We observe first that A_n satisfies, for a.e. x in $\mathbb{R}^N$ and for every $s > 0$,

$$\begin{cases} \lim\limits_{n\to+\infty} A_n(x,s) &= A(x,s) \\ |A_n(x,s)| &\leq |A(x,s)| \\ |A_n(x,s)| &\leq \psi_n(s) \end{cases}$$

Next, we note that there exists a positive constant $c_n > 0$ such that

$$|A_n(x,s)| \leq c_n \quad \text{a.e. } x \text{ in } \mathbb{R}^N \quad \text{and} \quad \text{for every } s \in \mathbb{R}.$$

Hence, as for constant coefficient $p(\cdot)$, we prove the existence of a solution $u_n \in W_0^{1,p(\cdot)}(B_n) \cap L^\infty(B_n)$ of the problem (P_n). Indeed, set $X_n = W_0^{1,p(\cdot)}(B_n)$ and X_n^* its dual and define the operator $L : X_n \to X_n^*$ by

$$L(u) = -\text{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) + |u|^{p(x)-2}u - A_n(x,u)\frac{|\nabla u|^{p(x)}}{1+\frac{1}{n}|\nabla u|^{p(x)}} - f(x,u).$$

Note first that

$$\frac{\langle L(u), u\rangle}{\|u\|_n} \to +\infty \quad \text{as} \quad \|u\|_n \to +\infty$$

where $\|u\|_n = |\nabla u|_{L^{p(\cdot)}(B_n)}$ is a norm on X_n. Thus L is coercive. We claim now that L is a pseudomonotone operator(see [29]). Let $(u_k) \subset X_n$ be such that $u_k \rightharpoonup u$ in X_n and

$$\limsup_{k\to+\infty} \langle L(u_k) - L(u), u_k - u\rangle \leq 0.$$

By the boundedness of the open set B_n, it follows that

$$0 \leq \limsup_{k\to+\infty} \int_{\Omega_n} \left(|\nabla u_k|^{p(x)-2}\,\nabla u_k - |\nabla u|^{p(x)-2}\,\nabla u\right)\nabla(u_k - u)dx \leq 0.$$

Then $u_k \to u$ strongly in X_n and therefore L is of (S_+) type. Observing also that L is demicontinuous, it yields that L is pseudomonotone. According to [29, Theorem 27.A], we deduce that the operator L is surjective. Moreover, this solution satisfies $u_n \geq 0$. Indeed, denoting, for $s \in \mathbb{R}$, $s^+ = \max(s,0)$ and $s^- = \min(s,0)$ and taking u_n^- as test function in (P_n), we get

$$\int_{B_n} \left|\nabla u_n^-\right|^{p(x)} dx + \int_{B_n} \left|u_n^-\right|^{p(x)} dx \leq 0$$

and therefore $u_n \geq 0 \quad$ a.e in B_n. Observe now that we can extend u_n by zero outside of B_n. We will continue denoting by u_n the zero-extension of u_n outside of B_n and it belongs now to $W^{1,p(\cdot)}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$.

Lemma 4.1 The sequence (u_n) is bounded in $L^\infty(\mathbb{R}^N)$.

Proof Since $u_n \in W_0^{1,p(\cdot)}(B_n) \cap L^\infty(B_n)$, then the function $e^{\gamma_n(u_n)}(u_n - M)^+ \in W_0^{1,p(\cdot)}(B_n) \cap L^\infty(B_n)$ (M is defined by (3.1)) and we can take it as test function in (P_n) getting

$$\begin{aligned}
&\int_{B_n} |\nabla u_n|^{p(x)-2}\,\nabla u_n \cdot \nabla(u_n - M)^+\, e^{\gamma_n(u_n)}dx + \int_{B_n} |\nabla u_n|^{p(x)}\,\psi_n(u_n)\,(u_n - M)^+\, e^{\gamma_n(u_n)}dx \\
&\quad + \int_{B_n} (u_n)^{p(x)-1}\,(u_n - M)^+\, e^{\gamma_n(u_n)}dx \leq \int_{B_n} \psi_n(u_n)\,|\nabla u_n|^{p(x)}\, e^{\gamma_n(u_n)}\,(u_n - M)^+\, dx \\
&\quad + \int_{B_n} |g(x)|\,|u_n|^{\beta(x)-1}\,(u_n - M)^+\, e^{\gamma_n(u_n)}dx + \int_{B_n} h(x)\,(u_n - M)^+\, e^{\gamma_n(u_n)}dx.
\end{aligned}$$

Cancelling identical terms and forgetting the nonnegative term

$$\int_{B_n} |\nabla u_n|^{p(x)-2}\,\nabla u_n \cdot \nabla(u_n - M)^+\, e^{\gamma_n(u_n)}dx = \int_{B_n} \left|\nabla(u_n - M)^+\right|^{p(x)} e^{\gamma_n(u_n)}dx \geq 0,$$

we obtain

$$\begin{aligned}
\int_{B_n} (u_n)^{p(x)-1}\,(u_n - M)^+\, e^{\gamma_n(u_n)}dx &\leq \int_{B_n} |g(x)|\,|u_n|^{\beta(x)-1}\,(u_n - M)^+\, e^{\gamma_n(u_n)}dx \\
&\quad + \int_{B_n} h(x)\,(u_n - M)^+\, e^{\gamma_n(u_n)}dx.
\end{aligned} \tag{4.1}$$

Observe now that by Young's inequality, we have

$$
\begin{aligned}
|g(x)|\,|u_n|^{\beta(x)-1}\,(u_n-M)^+\,e^{\gamma_n(u_n)} \le \frac{p(x)-\beta(x)}{p(x)-1}&\left(|g(x)|^{\frac{p(x)-1}{p(x)-\beta(x)}}\,(u_n-M)^+\,e^{\gamma_n(u_n)}\right)\\
&+\frac{\beta(x)-1}{p(x)-1}(u_n)^{p(x)-1}\,(u_n-M)^+\,e^{\gamma_n(u_n)}.
\end{aligned}
\tag{4.2}
$$

Using (4.2), it follows from (4.1) that

$$
\begin{aligned}
&\int_{B_n}\frac{p(x)-\beta(x)}{p(x)-1}(u_n)^{p(x)-1}\,(u_n-M)^+\,e^{\gamma_n(u_n)}dx\\
&\le \int_{B_n}\frac{p(x)-\beta(x)}{p(x)-1}\,|g(x)|^{\frac{p(x)-1}{p(x)-\beta(x)}}\,(u_n-M)^+\,e^{\gamma_n(u_n)}dx\\
&+\int_{B_n}h(x)\,(u_n-M)^+\,e^{\gamma_n(u_n)}dx.
\end{aligned}
\tag{4.3}
$$

Adding to both sides $\left(-\int_{B_n}\frac{p(x)-\beta(x)}{p(x)-1}M^{p(x)-1}\,(u_n-M)^+\,e^{\gamma_n(u_n)}dx\right)$, we get from (4.3)

$$
\begin{aligned}
&\int_{B_n}\frac{p(x)-\beta(x)}{p(x)-1}\left((u_n)^{p(x)-1}-M^{p(x)-1}\right)(u_n-M)^+\,e^{\gamma_n(u_n)}dx\\
&\le \int_{B_n}\left(\frac{p(x)-\beta(x)}{p(x)-1}\,|g(x)|^{\frac{p(x)-1}{p(x)-\beta(x)}}+h(x)-\frac{p(x)-\beta(x)}{p(x)-1}M^{p(x)-1}\right)(u_n-M)^+\,e^{\gamma_n(u_n)}dx.
\end{aligned}
$$

Taking into account (3.1), it yields

$$
\int_{B_n}\left((u_n)^{p(x)-1}-M^{p(x)-1}\right)(u_n-M)^+\,e^{\gamma_n(u_n)}\le 0.
\tag{4.4}
$$

Observe now that (see [13, 17, 23, 28]) we have the following strict monotonicity inequalities satisfied for ξ and η in $\mathbb{R}^N$

$$
\left[\left(|\xi|^{p-2}\xi-|\eta|^{p-2}\eta\right)(\xi-\eta)\right]^{\frac{p}{2}}\left(|\xi|^p+|\eta|^p\right)^{\frac{2-p}{2}}\ge (p-1)\,|\xi-\eta|^p
\tag{4.5}
$$

$$\text{for } 1<p<2$$

and

$$
\left(|\xi|^{p-2}\xi-|\eta|^{p-2}\eta\right)(\xi-\eta)\ge 2^{-p}\,|\xi-\eta|^p,\ p\ge 2.
\tag{4.6}
$$

Using now (4.5) and (4.6), we deduce from (4.4) that

$$0\le u_n(x)\le M\quad \text{a.e. } x \text{ in } B_n\quad \forall n\ge 1$$

and by the zero-extension of u_n outside of B_n, we finally get

$$
0\le u_n(x)\le M\quad \text{a.e. } x \text{ in } \mathbb{R}^N\quad \forall n\ge 1.
\tag{4.7}
$$

Lemma 4.2 The sequence (u_n) is bounded in $W^{1,p(\cdot)}(\mathbb{R}^N)$.

Proof Taking $u_n e^{\gamma_n(u_n)}$ as test function, we get

$$\int_{\mathbb{R}^N} |\nabla u_n|^{p(x)} e^{\gamma_n(u_n)} dx + \int_{\mathbb{R}^N} |\nabla u_n|^{p(x)} \psi_n(u_n) e^{\gamma_n(u_n)} dx + \int_{\mathbb{R}^N} (u_n)^{p(x)} e^{\gamma_n(u_n)} dx$$
$$\leq \int_{\mathbb{R}^N} \psi_n(u_n) |\nabla u_n|^{p(x)} e^{\gamma_n(u_n)} dx + \int_{\mathbb{R}^N} f(x,u_n) u_n e^{\gamma_n(u_n)} dx + \int_{\mathbb{R}^N} h u_n e^{\gamma_n(u_n)} dx.$$

Since $0 \leq u_n \leq M$ uniformly in n and $\gamma_n(u_n) \geq 0$, then by (2.1), (H_1) and (H_2)

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla u_n|^{p(x)} dx + \int_{\mathbb{R}^N} |u_n|^{p(x)} dx \\
&\leq c_1 |g|_{L^{r(\cdot)}(\mathbb{R}^N)} \left| |u_n|^{\beta(\cdot)} \right|_{L^{r'(\cdot)}(\mathbb{R}^N)} + c_1 |h|_{L^{p'(\cdot)}(\mathbb{R}^N)} |u_n|_{L^{p(\cdot)}(\mathbb{R}^N)} \\
&\leq c_1 |g|_{L^{r(\cdot)}(\mathbb{R}^N)} |u_n|^{\overline{\beta}}_{L^{\mu(\cdot)}(\mathbb{R}^N)} + c_1 |h|_{L^{p'(\cdot)}(\mathbb{R}^N)} |u_n|_{L^{p(\cdot)}(\mathbb{R}^N)}
\end{aligned}$$

where $\overline{\beta} \in [\beta^-, \beta^+]$ (see[11, Lemma 3.4]). It follows, by (2.3) and (2.4), that

$$\inf\left(\|u_n\|^{p^+}, \|u_n\|^{p^-}\right) \leq c_2 |g|_{L^{r(\cdot)}(\mathbb{R}^N)} \|u_n\|^{\overline{\beta}} + c_2 |h|_{L^{p'(\cdot)}(\mathbb{R}^N)} \|u_n\|.$$

Since $1 \leq \beta^- \leq \beta^+ < p^-$, then the sequence (u_n) is bounded in $W^{1,p(\cdot)}(\mathbb{R}^N)$. Hence, there exists $u \in W^{1,p(\cdot)}(\mathbb{R}^N)$ such that (u_n) is weakly convergent to u in $W^{1,p(\cdot)}(\mathbb{R}^N)$ as n tends to $+\infty$. Moreover, we deduce from (4.7) that $u \in L^\infty(\mathbb{R}^N)$ and $0 \leq u(x) \leq M$ a.e. x in $\mathbb{R}^N$.

Lemma 4.3 The sequence $\left(\psi_n(u_n) |\nabla u_n|^{p(\cdot)}\right)$ is bounded in $L^1_{loc}(\mathbb{R}^N)$.

Proof Let $\varphi \in C_0^\infty(\mathbb{R}^N)$ be such that $\varphi \geq 0$. Taking $\left(e^{\gamma_n(u_n)} - 1\right)\varphi$ as test function, we get

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla u_n|^{p(x)-2} \nabla u_n \cdot \nabla\varphi \left(e^{\gamma_n(u_n)} - 1\right) dx \\
&+ \int_{\mathbb{R}^N} |\nabla u_n|^{p(x)} \psi_n(u_n) e^{\gamma_n(u_n)} \varphi dx \\
&+ \int_{\mathbb{R}^N} |u_n|^{p(x)-2} u_n \left(e^{\gamma_n(u_n)} - 1\right) \varphi dx \\
&\leq \int_{\mathbb{R}^N} \psi_n(u_n) |\nabla u_n|^{p(x)} \left(e^{\gamma_n(u_n)} - 1\right) \varphi dx \\
&+ \int_{\mathbb{R}^N} f(x,u_n) \left(e^{\gamma_n(u_n)} - 1\right) \varphi dx + \int_{\mathbb{R}^N} h \left(e^{\gamma_n(u_n)} - 1\right) \varphi dx.
\end{aligned}$$

Since (u_n) is bounded in $W^{1,p(\cdot)}(\mathbb{R}^N)$ and in $L^\infty(\mathbb{R}^N)$, we obtain the existence of a constant $c_3 > 0$ such that

$$\int_{\mathbb{R}^N} \psi_n(u_n) |\nabla u_n|^{p(x)} \varphi(x) dx \leq c_3 \quad \forall\, n \geq 1. \tag{4.8}$$

Let K be any compact of $\mathbb{R}^N$, there exists $\varphi \in C_0^\infty(\mathbb{R}^N)$ such that $\varphi \equiv 1$ on K. By (4.8), it yields that the sequence $\left(\int_K \psi_n(u_n) |\nabla u_n|^{p(x)} dx\right)$ is bounded.

We give here a convergence result of the nonlinearity term $f(\cdot,\cdot)$. The proof of this result can be found in [12, Lemma 3.2] and, for the convenience of the reader, we have included it in the appendix.

Lemma 4.4 Denoting by $W^{-1,p'(\cdot)}(\mathbb{R}^N)$ the dual space of $W^{1,p(\cdot)}(\mathbb{R}^N)$, then we have

$$f(\cdot,u_n)\to f(\cdot,u) \ \text{ strongly in } W^{-1,p'(\cdot)}(\mathbb{R}^N).$$

Lemma 4.5 The sequence (u_n) is strongly convergent to u in $W^{1,p(\cdot)}(\mathbb{R}^N)$.

Proof Taking $e^{\gamma_n(u_n)}(u_n-u)^+$ as test function, we get

$$\begin{aligned}
&\int_{\mathbb{R}^N}|\nabla u_n|^{p(x)-2}\nabla u_n\cdot\nabla(u_n-u)^+e^{\gamma_n(u_n)}dx\\
&+\int_{\mathbb{R}^N}\psi_n(u_n)|\nabla u_n|^{p(x)}(u_n-u)^+e^{\gamma_n(u_n)}dx\\
&+\int_{\mathbb{R}^N}|u_n|^{p(x)-2}u_n(u_n-u)^+e^{\gamma_n(u_n)}dx\\
&\le\int_{\mathbb{R}^N}\psi_n(u_n)|\nabla u_n|^{p(x)}(u_n-u)^+e^{\gamma_n(u_n)}dx\\
&+\int_{\mathbb{R}^N}f(x,u_n)(u_n-u)^+e^{\gamma_n(u_n)}dx+\int_{\mathbb{R}^N}h(u_n-u)^+e^{\gamma_n(u_n)}dx.
\end{aligned}$$

Cancelling identical terms, it yields

$$\begin{aligned}
&\int_{\mathbb{R}^N}\Big(|\nabla u_n|^{p(x)-2}\nabla u_n-|\nabla u|^{p(x)-2}\nabla u\Big)\cdot\nabla(u_n-u)^+e^{\gamma_n(u_n)}dx\\
&+\int_{\mathbb{R}^N}\Big(|u_n|^{p(x)-2}u_n-|u|^{p(x)-2}u\Big)(u_n-u)^+e^{\gamma_n(u_n)}dx\\
&\le\int_{\mathbb{R}^N}f(x,u_n)(u_n-u)^+e^{\gamma_n(u_n)}dx+\int_{\mathbb{R}^N}h(u_n-u)^+e^{\gamma_n(u_n)}dx \qquad (4.9)\\
&-\int_{\mathbb{R}^N}|\nabla u|^{p(x)-2}\nabla u\cdot\nabla(u_n-u)^+e^{\gamma_n(u_n)}dx\\
&-\int_{\mathbb{R}^N}|u|^{p(x)-2}u(u_n-u)^+e^{\gamma_n(u_n)}dx.
\end{aligned}$$

By the weak convergence of (u_n) to u in $W^{1,p(\cdot)}(\mathbb{R}^N)$ and the boundedness of $(e^{\gamma_n(u_n)})$, it follows that the following equalities hold true

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N}h(u_n-u)^+e^{\gamma_n(u_n)}dx=0, \tag{4.10}$$

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N}|\nabla u|^{p(x)-2}\nabla u\cdot\nabla(u_n-u)^+e^{\gamma_n(u_n)}dx=0, \tag{4.11}$$

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N}|u|^{p(x)-2}u(u_n-u)^+e^{\gamma_n(u_n)}dx=0. \tag{4.12}$$

On the other hand, in virtue of Lemma 4.4, we have

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N}f(x,u_n)(u_n-u)^+e^{\gamma_n(u_n)}dx=0. \tag{4.13}$$

Combining (4.10), (4.11), (4.12) and (4.13) and having in mind the nonnegativity of $(\gamma_n(u_n))$, we deduce from (4.9) that

$$\begin{aligned}&\lim_{n\to+\infty}\int_{\mathbb{R}^N}\Big(|\nabla u_n|^{p(x)-2}\nabla u_n-|\nabla u|^{p(x)-2}\nabla u\Big)\cdot\nabla(u_n-u)^+dx\\&+\lim_{n\to+\infty}\int_{\mathbb{R}^N}\Big(|u_n|^{p(x)-2}u_n-|u|^{p(x)-2}u\Big)(u_n-u)^+dx=0.\end{aligned}\tag{4.14}$$

Using the strict monotonicity conditions (4.5) and (4.6), we get from (2.5) that

$$(u_n-u)^+\to 0\quad\text{ strongly in } W^{1,p(\cdot)}(\mathbb{R}^N).$$

In a similar way, taking $(u_n-u)^-e^{-\gamma_n(u_n)}$ as test function and using the fact that

$$e^{-\gamma_n(u_n)}\geq e^{-\gamma_n(M)}\geq c_4>0\quad\forall n\geq 1,$$

we get

$$(u_n-u)^-\to 0\quad\text{ strongly in } W^{1,p(\cdot)}(\mathbb{R}^N).$$

Therefore, we conclude that

$$u_n\to u\quad\text{strongly in } W^{1,p(\cdot)}(\mathbb{R}^N).$$

In order to pass to the limit in the singular term, we need the following lemma:

Lemma 4.6 For every compact K in $\mathbb{R}^N$, we have

$$\lim_{t\to 0^+}\int_{K\cap\{u_n\leq t\}}\psi_n(u_n)|\nabla u_n|^{p(x)}dx=0\quad\text{uniformly in } n.$$

Proof For $\varphi\in C_0^\infty(\mathbb{R}^N)$ such that $\varphi\geq 0$ and $0<t<1$, we take $-\Big(e^{\gamma_n(t)-\gamma_n(u_n)}-1\Big)^+\varphi^{p^+}$ as test function, getting

$$\begin{aligned}&-p^+\int_{\{u_n\leq t\}}|\nabla u_n|^{p(x)-2}\nabla u_n\cdot\nabla\varphi\Big(e^{\gamma_n(t)-\gamma_n(u_n)}-1\Big)\varphi^{p^+-1}dx\\&+\int_{\{u_n\leq t\}}|\nabla u_n|^{p(x)}e^{\gamma_n(t)-\gamma_n(u_n)}\psi_n(u_n)\varphi^{p^+}dx\\&-\int_{\{u_n\leq t\}}|u_n|^{p(x)-2}u_n\Big(e^{\gamma_n(t)-\gamma_n(u_n)}-1\Big)\varphi^{p^+}dx\\&\leq\int_{\{u_n\leq t\}}|\nabla u_n|^{p(x)}\psi_n(u_n)\Big(e^{\gamma_n(t)-\gamma_n(u_n)}-1\Big)\varphi^{p^+}dx\\&+\int_{\{u_n\leq t\}}|g(x)|(u_n)^{\beta(x)-1}\Big(e^{\gamma_n(t)-\gamma_n(u_n)}-1\Big)\varphi^{p^+}dx\\&-\int_{\{u_n\leq t\}}h\Big(e^{\gamma_n(t)-\gamma_n(u_n)}-1\Big)\varphi^{p^+}dx.\end{aligned}$$

Since $h\geq 0$, then

$$\int_{\{u_n\leq t\}}\psi_n(u_n)|\nabla u_n|^{p(x)}\varphi^{p^+}dx\leq c_5\int_{\{u_n\leq t\}}|\nabla u_n|^{p(x)-1}|\nabla\varphi|\varphi^{p^+-1}dx$$

$$+c_5 \int_{\mathbb{R}^N} t^{\beta(x)-1} \varphi^{p^+} dx + c_5 \int_{\mathbb{R}^N} t^{p(x)-1} \varphi^{p^+} dx. \tag{4.15}$$

Taking now $(u_n - t)^- \varphi^{p^+}$ as test function, we obtain

$$\begin{aligned}
&p^+ \int_{\{u_n \le t\}} \varphi^{p^+-1} |\nabla u_n|^{p(x)-2} \nabla u_n \cdot \nabla \varphi (u_n - t) dx \\
&+ \int_{\{u_n \le t\}} |\nabla u_n|^{p(x)} \varphi^{p^+} dx + \int_{\{u_n \le t\}} |u_n|^{p(x)-2} u_n (u_n - t) \varphi^{p^+} dx \\
&\le \int_{\{u_n \le t\}} \psi_n(u_n) |\nabla u_n|^{p(x)} (u_n - t) \varphi^{p^+} dx + \int_{\{u_n \le t\}} f(x, u_n)(u_n - t) \varphi^{p^+} dx.
\end{aligned} \tag{4.16}$$

Observing that

$$\begin{aligned}
&\int_{\{u_n \le t\}} \psi_n(u_n) |\nabla u_n|^{p(x)} |u_n - t| \varphi^{p^+} dx \\
&\le t \int_{\mathbb{R}^N} \psi_n(u_n) |\nabla u_n|^{p(x)} \varphi^{p^+} dx \\
&\le c_6 t \quad \text{(by Lemma 4.3)}
\end{aligned}$$

and

$$\begin{aligned}
&\left| \int_{\{u_n \le t\}} |\nabla u_n|^{p(x)-2} \nabla u_n \cdot \nabla \varphi (u_n - t) dx \right| \\
&\le t \int_{\mathbb{R}^N} |\nabla u_n|^{p(x)-1} |\nabla \varphi| dx \\
&\le c_7 t \quad \Big(\text{by the boundedness of } (u_n) \text{ in } W^{1,p(\cdot)}(\mathbb{R}^N)\Big).
\end{aligned}$$

It follows from (4.16) that

$$\int_{\{u_n \le t\}} |\nabla u_n|^{p(x)} \varphi^{p^+} dx \le c_8 \int_{\mathbb{R}^N} \left(t^{p(x)} + t^{\beta(x)}\right) \varphi^{p^+} dx + c_8 t. \tag{4.17}$$

Using (4.17) with (4.15), we deduce

$$\lim_{t \to 0^+} \int_{\{u_n \le t\}} \psi_n(u_n) |\nabla u_n|^{p(x)} \varphi^{p^+} dx = 0 \quad \text{uniformly in } n.$$

Let K a compact on $\mathbb{R}^N$, choosing $\varphi \in C_0^\infty(\mathbb{R}^N)$ such that $\varphi \equiv 1$ on K, we get the claimed result.

Lemma 4.7 The function u is a weak solution of the problem (P). Moreover u satisfies

$$\int_{K \cap \{u > 0\}} |\nabla u|^{p(x)} \psi(u) dx < +\infty \qquad \text{for every compact } K \subset \mathbb{R}^N.$$

Proof Let K be a compact of $\mathbb{R}^N$ and E a measurable set such $E \subset K$. We have, for $0 < t < 1$,

$$\begin{aligned}
\int_E |A_n(x, u_n)| \frac{|\nabla u_n|^{p(x)}}{1 + \frac{1}{n} |\nabla u_n|^{p(x)}} dx \le& \int_{E \cap \{u_n \le t\}} |A_n(x, u_n)| |\nabla u_n|^{p(x)} dx \\
&+ \int_{E \cap \{u_n \ge t\}} |A_n(x, u_n)| |\nabla u_n|^{p(x)} dx.
\end{aligned}$$

From Lemma 4.6, we get that

$$\lim_{t\to 0^+}\int_{E\cap\{u_n\le t\}}|A_n(x,u_n)||\nabla u_n|^{p(x)}=0 \quad \text{uniformly in } n.$$

Hence, for every $\epsilon>0$, there exists $0<t_0<1$ such that

$$\int_{E\cap\{u_n\le t_0\}}|A_n(x,u_n)||\nabla u_n|^{p(x)}\,dx<\frac{\epsilon}{2} \quad \forall n\ge 1. \tag{4.18}$$

On the other hand, we have

$$\begin{aligned}\int_{E\cap\{u_n\ge t_0\}}|A_n(x,u_n)||\nabla u_n|^{p(x)}\,dx &\le \int_{E\cap\{u_n\ge t_0\}}\psi_n(u_n)|\nabla u_n|^{p(x)}\,dx\\ &\le c_9\int_E|\nabla u_n|^{p(x)}\,dx.\end{aligned}$$

By Lemma 4.5, we can choose mes(E) small enough that

$$c_9\int_E|\nabla u_n|^{p(x)}\,dx<\frac{\epsilon}{2}. \tag{4.19}$$

Combining (4.18) and (4.19), we get the equi-integrability of the sequence $\left(A_n(\cdot,u_n)\frac{|\nabla u_n|^{p(\cdot)}}{1+\frac{1}{n}|\nabla u_n|^{p(\cdot)}}\right)$. This, together with the convergence of $\left(A_n(x,u_n)\frac{|\nabla u_n|^{p(x)}}{1+\frac{1}{n}|\nabla u_n|^{p(x)}}\right)$ to $A(x,u)|\nabla u|^{p(x)}$ a.e. x in $\left\{x\in\mathbb{R}^N;\ u(x)>0\right\}$ implies by Vitali's theorem that

$$\lim_{n\to+\infty}\int_{\{u>0\}}A_n(x,u_n)\frac{|\nabla u_n|^{p(x)}}{1+\frac{1}{n}|\nabla u_n|^{p(x)}}\varphi dx=\int_{\{u>0\}}A(x,u)|\nabla u|^{p(x)}\varphi dx \quad \forall\varphi\in C_0^\infty(\mathbb{R}^N). \tag{4.20}$$

It remains to prove that

$$\lim_{n\to+\infty}\int_{\{u=0\}}A_n(x,u_n)\frac{|\nabla u_n|^{p(x)}}{1+\frac{1}{n}|\nabla u_n|^{p(x)}}\varphi dx=0 \quad \forall\ \varphi\in C_0^\infty(\mathbb{R}^N).$$

Observe first that by the equi-integrability of $\left(A_n(\cdot,u_n)\frac{|\nabla u_n|^{p(\cdot)}}{1+\frac{1}{n}|\nabla u_n|^{p(\cdot)}}\right)$, for every $\epsilon>0$, there exists $\delta_\epsilon>0$ such that

$$\forall\ E\subset\operatorname{supp}(\varphi),\ \operatorname{mes}(E)<\delta_\epsilon,\ \int_E|A_n(x,u_n)|\frac{|\nabla u_n|^{p(x)}}{1+\frac{1}{n}|\nabla u_n|^{p(x)}}<\frac{\epsilon}{2\|\varphi\|_\infty}. \tag{4.21}$$

Next, by the boundedness of $(\operatorname{supp}(\varphi))$ in $\mathbb{R}^N$ and in virtue of Egorov's theorem, we can divide it into two measurable sets: K^ϵ with mes$(K^\epsilon)<\delta_\epsilon$ and $(\operatorname{supp}(\varphi)\backslash K^\epsilon)$ in which the sequence (u_n) converges uniformly to u. By (4.21), we have

$$\begin{aligned}&\int_{K^\epsilon\cap\{u=0\}}|A_n(x,u_n)|\frac{|\nabla u_n|^{p(x)}}{1+\frac{1}{n}|\nabla u_n|^{p(x)}}|\varphi|\,dx\\ &\le\|\varphi\|_\infty\int_{K^\epsilon\cap\{u=0\}}|A_n(x,u_n)|\frac{|\nabla u_n|^{p(x)}}{1+\frac{1}{n}|\nabla u_n|^{p(x)}}dx\\ &\le\frac{\epsilon}{2}.\end{aligned} \tag{4.22}$$

On the other hand, there exists $t_\epsilon > 0$ small enough and $n_0(\epsilon)$ large enough such that, for all $n \geq n_0(\epsilon)$

$$\int_{(\text{supp}(\varphi)\backslash K^\epsilon)\cap\{u=0\}} |A_n(x,u_n)| \frac{|\nabla u_n|^{p(x)}}{1+\frac{1}{n}|\nabla u_n|^{p(x)}} |\varphi|\, dx$$
$$\leq \|\varphi\|_\infty \int_{(\text{supp}(\varphi)\backslash K^\epsilon)\cap\{u_n\leq t_\epsilon\}} |A_n(x,u_n)|\,|\nabla u_n|^{p(x)}$$

and it follows from Lemma 4.6 that

$$\int_{(\text{supp}(\varphi)\backslash K^\epsilon)\cap\{u=0\}} |A_n(x,u_n)|\,|\nabla u_n|^{p(x)}\,|\varphi|\, dx \leq \frac{\epsilon}{2} \quad \forall n \geq n_0(\epsilon). \tag{4.23}$$

Combining (4.22) and (4.23), we get

$$\lim_{n\to+\infty} \int_{\{u=0\}} A_n(x,u_n) \frac{|\nabla u_n|^{p(x)}}{1+\frac{1}{n}|\nabla u_n|^{p(x)}} \varphi dx = 0 \quad \forall \varphi \in C_0^\infty(\mathbb{R}^N). \tag{4.24}$$

By (4.20) and (4.24), we deduce that, for every $\varphi \in C_0^\infty(\mathbb{R}^N)$, we have

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} A_n(x,u_n) \frac{|\nabla u_n|^{p(x)}}{1+\frac{1}{n}|\nabla u_n|^{p(x)}} \varphi dx = \int_{\mathbb{R}^N} A(x,u)\,|\nabla u|^{p(x)} \chi_{\{u>0\}} \varphi dx. \tag{4.25}$$

In a similar way, we can easily establish that, for every $\varphi \in C_0^\infty(\mathbb{R}^N)$,

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} |\nabla u_n|^{p(x)-2}\,\nabla u_n \cdot \nabla\varphi dx = \int_{\mathbb{R}^N} |\nabla u|^{p(x)-2}\,\nabla u \cdot \nabla\varphi dx \tag{4.26}$$

and

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} |u_n|^{p(x)-2}\, u_n \varphi dx = \int_{\mathbb{R}^N} |u|^{p(x)-2}\, u\varphi dx. \tag{4.27}$$

On the other hand, by Lemma 4.4, we also get

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} f(x,u_n)\varphi dx = \int_{\mathbb{R}^N} f(x,u)\varphi dx. \tag{4.28}$$

(Note, that this result can be obtained directly by applying Lebesgue dominated convergence theorem because of the boundedness of $(f(x,u_n))$). Combining (4.25), (4.26), (4.27) and (4.28), we conclude that u is a weak solution of the problem (P) and since $h \neq 0$, then $u \neq 0$. By inequality (4.7), we get $0 \leq u(x) \leq M$ a.e. x in $\mathbb{R}^N$. Moreover, in virtue of Fatou's Lemma, we immediately deduce from Lemma 4.3 that

$$\left(\psi(u)\,|\nabla u|^{p(\cdot)} \chi_{\{u>0\}}\right) \in L^1_{loc}(\mathbb{R}^N).$$

This ends the proof of Theorem 3.1.

5 Proof of Theorem 3.2

Here, the function $A(\cdot,\cdot)$ is no longer assumed to be bounded from above by an integrable function in a neighborhood of zero. In this case and in order to get a solution for the problem

(P), we have to make changes concerning the coefficients $A_n(\cdot,\cdot)$ defined in section 4. For $n \geq 1$ an integer, we define, now,

$$A_n(x,s) = \begin{cases} A(x,s) & \text{if} \quad s \geq \frac{1}{n} \\ A\left(x, \frac{1}{n}\right) & \text{if} \quad s \leq \frac{1}{n} \end{cases} \tag{5.1}$$

We define, also, for $s \in \mathbb{R}$, the three following functions

$$\underline{\psi_n}(s) = \begin{cases} \underline{\psi}(s) & \text{if} \quad s \geq \frac{1}{n} \\ \underline{\psi}\left(\frac{1}{n}\right) & \text{if} \quad s \leq \frac{1}{n} \end{cases}, \quad \overline{\psi_n}(s) = \begin{cases} \overline{\psi}(s) & \text{if} \quad s \geq \frac{1}{n} \\ \overline{\psi}\left(\frac{1}{n}\right) & \text{if} \quad s \leq \frac{1}{n} \end{cases}, \quad \overline{\gamma_n}(s) = \int_M^s \overline{\psi_n}(t)dt$$

with M defined by (3.1). We consider the approximate problem

$$-\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) + |u|^{p(x)-2}u = A_n(x,u)\frac{|\nabla u|^{p(x)}}{1+\frac{1}{n}|\nabla u|^{p(x)}} + f(x,u) + h \quad \text{in } B_n \tag{P_n}$$

where $B_n = \left\{x \in \mathbb{R}^N;\ |x| < n\right\}$ and $A_n(\cdot,\cdot)$ is now defined by (5.1). The existence of a solution $u_n \in W_0^{1,p(\cdot)}(B_n) \cap L^\infty(B_n)$ of the problem (P_n) can be justified exactly as in section 4. Indeed, it is sufficient to notice that there exists a positive constant $c'_n > 0$ such that

$$|A_n(x,s)| \leq c'_n \quad \text{a.e. } x \text{ in } \mathbb{R}^N \quad \text{and} \quad \text{for every } s \in \mathbb{R}.$$

The nonnegativity of (u_n) is immediate. We mention again that u_n can be extended by zero outside of B_n getting that $u_n \in W^{1,p(\cdot)}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Taking now $v_n = e^{\overline{\gamma_n}(u_n)}(u_n - M)^+$ as test function in (P_n) and following the same steps as in Lemma 4.1, we can prove that the estimate (4.7) still holds true. The main difference here arises when we are searching to estimate the sequence (u_n) in $W^{1,p(\cdot)}(\mathbb{R}^N)$. Whereas, in section 4, we have been able to prove the uniform boundedness of the sequence (u_n) in $W^{1,p(\cdot)}(\mathbb{R}^N)$, for the present case and under hypothesis (H_4), we can only obtain a local estimate.

Lemma 5.1 The sequence (u_n) is bounded in $W^{1,p(\cdot)}_{loc}(\mathbb{R}^N)$. Moreover $\left(A_n(\cdot,u_n)|\nabla u_n|^{p(\cdot)}\right)$ is bounded in $L^1_{loc}(\mathbb{R}^N)$.

Proof Let $\varphi \in C_0^\infty(\mathbb{R}^N)$ be such that $\varphi \geq 0$. Taking $\left(e^{\overline{\gamma_n}(u_n)} - 1\right)\varphi^{p^+}$ as test function in (P_n) and observing that $\left(e^{\overline{\gamma_n}(u_n)} - 1\right) \leq 0$, we get

$$\begin{aligned}
&p^+ \int_{\mathbb{R}^N} |\nabla u_n|^{p(x)-2}\nabla u_n \cdot \nabla\varphi\varphi^{p^+-1}\left(e^{\overline{\gamma_n}(u_n)} - 1\right)dx \\
&+ \int_{\mathbb{R}^N} |\nabla u_n|^{p(x)}\overline{\psi_n}(u_n)e^{\overline{\gamma_n}(u_n)}\varphi^{p^+}dx \\
&+ \int_{\mathbb{R}^N} |u_n|^{p(x)-2}u_n\left(e^{\overline{\gamma_n}(u_n)} - 1\right)\varphi^{p^+}dx \\
&\leq \int_{\mathbb{R}^N} |\nabla u_n|^{p(x)}\overline{\psi_n}(u_n)e^{\overline{\gamma_n}(u_n)}\varphi^{p^+}dx \\
&- \int_{\mathbb{R}^N} A_n(x,u_n)|\nabla u_n|^{p(x)}\varphi^{p^+}dx \\
&+ \int_{\mathbb{R}^N} f(x,u_n)\left(e^{\overline{\gamma_n}(u_n)} - 1\right)\varphi^{p^+}dx.
\end{aligned}$$

Since (u_n) is bounded in $L^\infty(\mathbb{R}^N)$ (by relation (4.7)), it follows that

$$\int_{\mathbb{R}^N} A_n(x,u_n)|\nabla u_n|^{p(x)}\varphi^{p^+}dx \le c_{10}\int_{\mathbb{R}^N}\varphi^{p^+}dx + c_{10}\int_{\mathbb{R}^N}|\nabla u_n|^{p(x)-1}|\nabla\varphi|\varphi^{p^+-1}dx. \tag{5.2}$$

By (H_4), we deduce from (5.2) that

$$\int_{\mathbb{R}^N}\underline{\psi_n}(u_n)|\nabla u_n|^{p(x)}\varphi^{p^+}dx \le c_{11} + c_{11}\int_{\mathbb{R}^N}|\nabla u_n|^{p(x)-1}|\nabla\varphi|\varphi^{p^+-1}dx. \tag{5.3}$$

For $0<\epsilon<1$, writing $p^+-1 = p^+\left(1-\frac{1}{p(x)}\right)+\frac{p^+}{p(x)}-1$, it yields from Young inequality that

$$\begin{aligned}|\nabla u_n|^{p(x)-1}|\nabla\varphi|\varphi^{p^+-1} &\le \epsilon^{\frac{p(x)}{p(x)-1}}|\nabla u_n|^{p(x)}\varphi^{p^+} + c_{12}\frac{|\nabla\varphi|^{p(x)}}{\epsilon^{p(x)}}\\ &\le \epsilon\underline{\psi}(M)\underline{\psi_n}(u_n)|\nabla u_n|^{p(x)}\varphi^{p^+} + c_{12}\frac{|\nabla\varphi|^{p(x)}}{\epsilon^{p(x)}}.\end{aligned} \tag{5.4}$$

Using (5.4) and choosing ϵ such that $0<1-c_{11}\epsilon\underline{\psi}(M)$, we obtain by (5.3) that

$$\left(\int_{\mathbb{R}^N}\underline{\psi_n}(u_n)|\nabla u_n|^{p(x)}\varphi^{p^+}\right) \quad \text{is bounded.}$$

Observing that $\underline{\psi_n}(u_n) \ge \frac{1}{\underline{\psi}(M)}\quad \forall\, n\ge 1$, then we get immediately that $\left(|\nabla u_n|^{p(\cdot)}\right)$ is bounded in $L^1_{\text{loc}}(\mathbb{R}^N)$. Taking again into account that (u_n) is uniformly bounded in $L^\infty(\mathbb{R}^N)$, we conclude that (u_n) is bounded in $W^{1,p(\cdot)}_{loc}(\mathbb{R}^N)$. We denote by u the weak limit of (u_n).

Lemma 5.2 The sequence $(T_t(u_n))$ (where $T_t(\cdot)$ is defined by (3.3)) is strongly convergent to $T_t(u)$ in $W^{1,p(\cdot)}_{loc}(\mathbb{R}^N)$ for every $t>0$.

Proof For simplicity in notation, we will denote by $\epsilon_n^1,\ \epsilon_n^2,\cdots$ various sequences of real numbers converging to zero when n tends to $+\infty$. For $n\ge 1$ and $t>0$, we also denote by $w_{n,t} = T_t(u_n)-T_t(u)$ and $\gamma_{n,t} = \overline{\gamma_n}(u_n)-\overline{\gamma_n}(T_t(u_n))$. Let φ be a function in $C_0^\infty(\mathbb{R}^N)$ such that $\varphi\ge 0$ and Φ be as in (3.2). Taking $\Phi\left((w_{n,t})^+\right)e^{\gamma_{n,t}}\varphi$ as test function, we get

$$\begin{aligned}&\int_{\mathbb{R}^N}|\nabla u_n|^{p(x)-2}\nabla u_n\cdot\nabla(w_{n,t}^+)\Phi'(w_{n,t}^+)e^{\gamma_{n,t}}\varphi dx\\ &+\int_{\mathbb{R}^N}|\nabla u_n|^{p(x)}\overline{\psi_n}(u_n)e^{\gamma_{n,t}}\Phi(w_{n,t}^+)\varphi dx\\ &+\int_{\mathbb{R}^N}|\nabla u_n|^{p(x)-2}\nabla u_n\cdot\nabla\varphi\ \Phi\left((w_{n,t})^+\right)e^{\gamma_{n,t}}dx\\ &-\int_{\mathbb{R}^N}|\nabla u_n|^{p(x)-2}\nabla u_n\cdot\nabla\left(T_t(u_n)\right)\overline{\psi_n}\left(T_t(u_n)\right)e^{\gamma_{n,t}}\Phi(w_{n,t}^+)\varphi dx\\ &+\int_{\mathbb{R}^N}|u_n|^{p(x)-2}u_n\Phi(w_{n,t}^+)e^{\gamma_{n,t}}\varphi dx\\ &\le\int_{\mathbb{R}^N}\overline{\psi_n}(u_n)|\nabla u_n|^{p(x)}\Phi(w_{n,t}^+)e^{\gamma_{n,t}}\varphi dx\\ &+\int_{\mathbb{R}^N}f(x,u_n)\Phi(w_{n,t}^+)e^{\gamma_{n,t}}\varphi dx+\int_{\mathbb{R}^N}h\Phi(w_{n,t}^+)e^{\gamma_{n,t}}\varphi dx.\end{aligned} \tag{5.5}$$

Let $R>0$ be such that $\mathrm{supp}(\varphi)\subset B(0,R)$; since $\beta(x)<p^*(x) \quad \forall\, x\in\mathbb{R}^N$, then the embedding of $W^{1,p(\cdot)}(B(0,R))$ into $L^{\beta(\cdot)}(B(0,R))$ is compact. This fact together with the weak convergence of $(w_{n,t}^+)$ to zero and the boundedness of the sequence $(e^{\gamma_{n,t}})$ imply that

$$\int_{\mathbb{R}^N} f(x,u_n)\Phi(w_{n,t}^+)e^{\gamma_{n,t}}\varphi dx\to 0 \tag{5.6}$$

and

$$\int_{\mathbb{R}^N} h\Phi(w_{n,t}^+)e^{\gamma_{n,t}}\varphi dx\to 0. \tag{5.7}$$

On the other hand, using again the boundedness of $(e^{\gamma_{n,t}})$, by (2.1) we have

$$\begin{aligned}&\left|\int_{\mathbb{R}^N}|\nabla u_n|^{p(x)-2}\nabla u_n\cdot\nabla\varphi\;\Phi\Big((w_{n,t})^+\Big)e^{\gamma_{n,t}}dx\right|\\ &\le c_{13}\left||\nabla u_n|^{p(\cdot)-1}\right|_{L^{p'(\cdot)}(B(0,R))}\left|\,|\nabla\varphi|\left|\Phi(w_{n,t}^+)\right|\,\right|_{L^{p(\cdot)}(B(0,R))}\end{aligned}$$

Since (u_n) is bounded in $W_{loc}^{1,p(\cdot)}(\mathbb{R}^N)$ and $\Phi(w_{n,t}^+)$ converges to zero a.e. in $\mathbb{R}^N$, we immediately deduce that

$$\int_{\mathbb{R}^N}|\nabla u_n|^{p(x)-2}\nabla u_n\cdot\nabla\varphi\;\Phi\Big((w_{n,t})^+\Big)e^{\gamma_{n,t}}dx\to 0 \tag{5.8}$$

Observe now that

$$\begin{aligned}&\int_{\mathbb{R}^N}|u_n|^{p(x)-2}u_n\Phi(w_{n,t}^+)e^{\gamma_{n,t}}\varphi dx\\ &=\int_{\{u_n\ge t\}}|T_t(u_n)|^{p(x)-2}T_t(u_n)\Phi(w_{n,t}^+)\varphi dx\\ &=\int_{\mathbb{R}^N}\Big(|T_t(u_n)|^{p(x)-2}T_t(u_n)-|T_t(u)|^{p(x)-2}T_t(u)\Big)\Phi(w_{n,t}^+)\varphi dx\\ &+\int_{\mathbb{R}^N}|T_t(u)|^{p(x)-2}T_t(u)\Phi(w_{n,t}^+)\varphi dx.\end{aligned}$$

By the weak convergence of $(w_{n,t}^+)$ to zero, we get

$$\int_{\mathbb{R}^N}|T_t(u)|^{p(x)-2}T_t(u)\Phi(w_{n,t}^+)\varphi dx\to 0$$

and it follows that

$$\begin{aligned}&\int_{\mathbb{R}^N}|u_n|^{p(x)-2}u_n\Phi(w_{n,t}^+)e^{\gamma_{n,t}}\varphi dx\\ &=\int_{\mathbb{R}^N}\Big(|T_t(u_n)|^{p(x)-2}T_t(u_n)-|T_t(u)|^{p(x)-2}T_t(u)\Big)\Phi(w_{n,t}^+)\varphi dx+\epsilon_n^1.\end{aligned} \tag{5.9}$$

Next, we have

$$\begin{aligned}&\int_{\mathbb{R}^N}|\nabla u_n|^{p(x)-2}\nabla u_n\cdot\nabla\Big(w_{n,t}^+\Big)\Phi'(w_{n,t}^+)e^{\gamma_{n,t}}\varphi dx\\ &=\int_{\{u_n\ge t\}}|\nabla T_t(u_n)|^{p(x)-2}\nabla T_t(u_n)\cdot\nabla\Big(w_{n,t}^+\Big)\Phi'(w_{n,t}^+)\varphi dx\\ &=\int_{\mathbb{R}^N}\Big(|\nabla T_t(u_n)|^{p(x)-2}\nabla T_t(u_n)-|\nabla T_t(u)|^{p(x)-2}\nabla T_t(u)\Big)\cdot\nabla(w_{n,t}^+)\Phi'(w_{n,t}^+)\varphi dx\\ &+\int_{\mathbb{R}^N}|\nabla T_t(u)|^{p(x)-2}\nabla T_t(u)\cdot\nabla(w_{n,t}^+)\Phi'(w_{n,t}^+)\varphi dx.\end{aligned}$$

Again by the weak convergence of $(w_{n,t}^+)$ to zero, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla u_n|^{p(x)-2}\nabla u_n \cdot \nabla(w_{n,t}^+)\Phi'(w_{n,t}^+)e^{\gamma_{n,t}}\varphi dx\\
&= \int_{\mathbb{R}^N} \Big(|\nabla T_t(u_n)|^{p(x)-2}\nabla T_t(u_n) - |\nabla T_t(u)|^{p(x)-2}\nabla T_t(u)\Big)\cdot \nabla(w_{n,t}^+)\Phi'(w_{n,t}^+)\varphi dx\\
&+\epsilon_n^2.
\end{aligned} \tag{5.10}$$

In a similar way, for n large enough, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla u_n|^{p(x)-2}\nabla u_n \cdot \nabla(T_t(u_n))\overline{\psi_n}(T_t(u_n))e^{\gamma_{n,t}}\Phi(w_{n,t}^+)\varphi dx\\
&= \int_{\{u_n\geq t\}} |\nabla T_t(u_n)|^{p(x)}\overline{\psi_n}(u_n)\Phi(w_{n,t}^+)\varphi dx\\
&\leq c_{14}(t)\int_{\mathbb{R}^N} \Big(|\nabla T_t(u_n)|^{p(x)-2}\nabla T_t(u_n) - |\nabla T_t(u)|^{p(x)-2}\nabla T_t(u)\Big)\cdot \nabla(w_{n,t}^+)\Phi(w_{n,t}^+)\varphi dx\\
&+\epsilon_n^3
\end{aligned}$$

which implies that

$$\begin{aligned}
&-\int_{\mathbb{R}^N} |\nabla u_n|^{p(x)-2}\nabla u_n \cdot \nabla(T_t(u_n))\overline{\psi_n}(T_t(u_n))e^{\gamma_{n,t}}\Phi(w_{n,t}^+)\varphi dx\\
&\geq -c_{14}(t)\int_{\mathbb{R}^N} \Big(|\nabla T_t(u_n)|^{p(x)-2}\nabla T_t(u_n) - |\nabla T_t(u)|^{p(x)-2}\nabla T_t(u)\Big)\cdot \nabla(w_{n,t}^+)\Phi(w_{n,t}^+)\varphi dx\\
&-\epsilon_n^3.
\end{aligned} \tag{5.11}$$

Combining (5.6), (5.7), (5.8), (5.9), (5.10) and (5.11) with (5.5), we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N} \Big(|\nabla T_t(u_n)|^{p(x)-2}\nabla T_t(u_n) - |\nabla T_t(u)|^{p(x)-2}\nabla T_t(u)\Big)\cdot \nabla((T_t(u_n)-T_t(u))^+)\\
&\qquad \times\Big(\Phi'(w_{n,t}^+) - c_{13}(t)\Phi(w_{n,t}^+)\Big)\varphi dx\\
&+\int_{\mathbb{R}^N} \Big(|T_t(u_n)|^{p(x)-2}T_t(u_n) - |T_t(u)|^{p(x)-2}T_t(u)\Big)(T_t(u_n)-T_t(u))^+\varphi dx \leq \epsilon_n^4.
\end{aligned} \tag{5.12}$$

Notice now that(see [2, Lemma 1.2]) we can choose $\eta > 0$ such that

$$\Phi'(w_{n,t}^+) - c_{14}(t)\Phi(w_{n,t}^+) \geq \frac{1}{2} \quad \forall n \geq 1.$$

Using the strict monotonicity conditions (4.5) and (4.6), we deduce from (5.12) and (2.5) that

$$(T_t(u_n)-T_t(u))^+ \to 0 \quad \text{strongly in } W_{loc}^{1,p(\cdot)}(\mathbb{R}^N).$$

Similarly, taking $\Phi(w_{n,t}^-)e^{-\gamma_{n,t}}\varphi$ as test function, we get

$$(T_t(u_n)-T_t(u))^- \to 0 \quad \text{strongly in } W_{loc}^{1,p(\cdot)}(\mathbb{R}^N).$$

Therefore, we conclude that $T_t(u_n) \to T_t(u)$ strongly in $W_{loc}^{1,p(\cdot)}(\mathbb{R}^N)$.

Lemma 5.3 For every compact K in $\mathbb{R}^N$, we have

$$\lim_{t\to 0^+}\int_{K\cap\{u_n\le t\}} A_n(x,u_n)\frac{|\nabla u_n|^{p(x)}}{1+\frac{1}{n}|\nabla u_n|^{p(x)}}dx = 0 \quad \text{uniformly in } n.$$

Proof Let $\varphi \in C_0^\infty(\mathbb{R}^N)$ be such that $\varphi \ge 0$, for $0 < t < 1$, taking $\left(e^{\overline{\gamma_n}(u_n)-\overline{\gamma_n}(t)} - 1\right)^- \varphi^{p^+}$ as test function and observing that $\left(e^{\overline{\gamma_n}(u_n)-\overline{\gamma_n}(t)} - 1\right)^-$ is bounded, we get

$$\int_{\{u_n\le t\}} A_n(x,u_n)\frac{|\nabla u_n|^{p(x)}}{1+\frac{1}{n}|\nabla u_n|^{p(x)}}\varphi^{p^+}dx \le c_{15}\int_{\{u_n\le t\}}|\nabla u_n|^{p(x)-1}|\nabla\varphi|\varphi^{p^+-1}dx$$
$$+c_{15}\int_{\mathbb{R}^N} t^{\beta(x)-1}\varphi^{p^+}dx + c_{15}\int_{\mathbb{R}^N} t^{p(x)-1}\varphi^{p^+}dx.$$

Continuing as in the proof of Lemma 4.6, we reach the claimed result.

Using the previous results, the passage to the limit can be achieved by following the same steps as in Lemma 4.7. Therefore, we prove that u is a nontrivial and nonnegative weak solution of the problem (P) which ends the proof of Theorem 3.2.

6 Proof of Theorem 3.3

In order to prove Theorem 3.3, we have to consider a modified approximating problem. We introduce, here, the following approximate coefficients

$$A_n(x,s) = \begin{cases} A\left(x, s^{1-\frac{1}{n}}\right) & \text{if} \quad s \ge \frac{1}{n} \\ (ns)^{\frac{1}{n}} A\left(x, s^{1-\frac{1}{n}}\right) & \text{if} \quad 0 < s \le \frac{1}{n} \\ 0 & \text{if} \quad s \le 0 \end{cases} \tag{6.1}$$

and the corresponding approximate problem

$$-\mathrm{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) + |u|^{p(x)-2}u = A_n(x,u)\frac{|\nabla u|^{p(x)}}{1+\frac{1}{n}|\nabla u|^{p(x)}} + f(x,u) + h \quad \text{in } B_n \tag{P_n}$$

where B_n still denotes the set $B_n = \left\{x \in \mathbb{R}^N;\ |x| < n\right\}$. It is clear that, for every $n \ge 1$, the coefficient $A_n(\cdot,\cdot)$ defined by (6.1) satisfies

$$|A_n(x,s)| \le c_n'' \quad \text{a.e. } x \text{ in } \mathbb{R}^N \quad \text{and} \quad \text{for every } s \in \mathbb{R}$$

for some positive constant c_n''. It yields, as in sections 4 and 5, the existence of $u_n \in W_0^{1,p(\cdot)}(B_n) \cap L^\infty(B_n)$ which is a weak solution of the problem (P_n). By zero-extension outside of B_n, u_n may be assumed to belong to $W^{1,p(\cdot)}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$.

Putting $u_n^- = \min(u_n, 0)$ as test function in (P_n) and taking into account the nonpositivity of $A_n(\cdot,\cdot)$, we get immediately the nonnegativity of u_n. We prove now that relation (4.7) still holds true giving a uniform estimate of the sequence (u_n) in $L^\infty(\mathbb{R}^N)$.

Lemma 6.1 The sequence (u_n) is uniformly bounded in $L^\infty(\mathbb{R}^N)$.

Proof Let M be as in relation (3.1). Observing that $(u_n - M)^+ \in W_0^{1,p(\cdot)}(B_n)$ and taking it as test function, we get by the nonpositivity of $A_n(\cdot,\cdot)$ that

$$\int_{B_n} \left|\nabla(u_n - M)^+\right|^{p(x)} dx + \int_{B_n} (u_n)^{p(x)-1}(u_n - M)^+ dx$$

$$\leq \int_{B_n} |g(x)|\,(u_n)^{\beta(x)-1}(u_n - M)^+ dx + \int_{B_n} h(u_n - M)^+ dx.$$

Continuing exactly as in Lemma 4.1, we get that

$$0 \leq u_n(x) \leq M \quad \text{a.e. } x \text{ in } \mathbb{R}^N.$$

The boundedness of $(u_n)_n$ in $W^{1,p(\cdot)}(\mathbb{R}^N)$ is immediate. Indeed, taking u_n as test function in (P_n) and using again the nonpositivity of $A_n(\cdot,\cdot)$ and the nonnegativity of u_n, we obtain, so easily, that $(u_n)_n$ is bounded in $W^{1,p(\cdot)}(\mathbb{R}^N)$. It follows the existence of $u \in W^{1,p(\cdot)}(\mathbb{R}^N)$ such that (u_n) is weakly convergent to u in $W^{1,p(\cdot)}(\mathbb{R}^N)$. Moreover, by (4.7), $u \in L^\infty(\mathbb{R}^N)$ and $u \geq 0$.

Lemma 6.2 The sequence $\left(|A_n(\cdot,u_n)|\,\frac{|\nabla u_n|^{p(\cdot)}}{1+\frac{1}{n}|\nabla u_n|^{p(\cdot)}}\right)$ is bounded in $L^1_{loc}(\mathbb{R}^N)$.

Proof Let $\varphi \in C_0^\infty(\mathbb{R}^N)$ be such that $\varphi \geq 0$. Taking φ as test function, we get

$$\int_{\mathbb{R}^N} |\nabla u_n|^{p(x)-2}\,\nabla u_n \cdot \nabla\varphi dx + \int_{\mathbb{R}^N} (u_n)^{p(x)-1}\varphi dx + \int_{\mathbb{R}^N} |A_n(x,u_n)|\,\frac{|\nabla u_n|^{p(x)}}{1+\frac{1}{n}\,|\nabla u_n|^{p(x)}}\varphi dx$$

$$= \int_{\mathbb{R}^N} f(x,u_n)\varphi dx + \int_{\mathbb{R}^N} h\varphi dx.$$

By the boundedness of (u_n) in $W^{1,p(\cdot)}(\mathbb{R}^N)$, we obtain the claimed result.

We notice now that under minor modifications, we can prove, as in Lemma 5.2, that

$$T_t(u_n) \to T_t(u) \quad \text{strongly in } W^{1,p(\cdot)}_{loc}(\mathbb{R}^N) \quad \text{for every } t > 0.$$

Indeed, for $n \geq 1$ and $s \geq 0$, we define the following functions

$$\widetilde{\psi_n}(s) = \begin{cases} \frac{1}{s^{1-\frac{1}{n}}} & \text{if } s \geq \frac{1}{n} \\ \left(\frac{1}{n}\right)^{\frac{1}{n}-1} & \text{if } s \leq \frac{1}{n} \end{cases} \quad \text{and} \quad \widetilde{\gamma_n}(s) = \int_0^s \widetilde{\psi_n}(t)dt.$$

Reasoning as in Lemma 5.2 by taking successively, as test function, $\phi\left(w^+_{n,t}\right)\varphi e^{c_0\widetilde{\gamma_n}(u_n)-c_0\widetilde{\gamma_n}(T_t(u_n))}$ and $\phi\left(w^-_{n,t}\right)\varphi e^{c_0\widetilde{\gamma_n}(T_t(u_n))-c_0\widetilde{\gamma_n}(u_n)}$ where $\varphi \in C_0^\infty(\mathbb{R}^N)$ be such that $\varphi \geq 0$, we can easily get the strong convergence result. In same way, we can also establish that

$$\lim_{t\to 0^+} \int_{K\cap\{u_n\leq t\}} \widetilde{\psi_n}(u_n)\,|\nabla u_n|^{p(x)}\,dx = 0 \quad \text{uniformly in } n$$

for every compact $K \subset \mathbb{R}^N$. Passing now to the limit by proceeding as in Lemma 4.7, we obtain that u is a weak nontrivial and nonnegative solution of the problem (P). Furthermore, we have

$$\int_{\mathbb{R}^N} |A(x,u)||\nabla u|^{p(x)} \chi_{\{u>0\}} \varphi dx < +\infty$$

for every $\varphi \in C_0^\infty(\mathbb{R}^N)$ verifying that $\varphi \geq 0$. Taking now account of the boundedness of the sequence $\left(\int_{\mathbb{R}^N} |A_n(x,u_n)||\nabla u_n|^{p(x)} u_n dx \right)$, we get that

$$\int_{\mathbb{R}^N} |A(x,u)||\nabla u|^{p(x)} u \chi_{\{u>0\}} dx < +\infty.$$

So that, the proof of Theorem 3.3 is complete.

Appendix

Proof of Lemma 4.4 For $t > 0$, we denote by B_t the open ball in $\mathbb{R}^N$ of radius t, i.e. $B_t = \left\{ x \in \mathbb{R}^N;\ |x| < t \right\}$. Let now $t > 0$ and $v \in W^{1,p(\cdot)}(\mathbb{R}^N)$ such that $\|v\| \leq 1$, by (2.1) we have

$$\int_{B_t} |f(x,u_n) - f(x,u)||v| dx \leq 2 \left| \, |f(\cdot,u_n) - f(\cdot,u)| \, \right|_{L^{\beta'(\cdot)}(B_t)} |v|_{L^{\beta(\cdot)}(B_t)}$$

Since B_t is bounded in $\mathbb{R}^N$ and $\beta(x) < p^*(x)\ \forall\ x \in \mathbb{R}^N$, then

$$\left| \, |f(\cdot,u_n) - f(\cdot,u)| \, \right|_{L^{\beta'(\cdot)}(B_t)} \to 0 \quad \text{as} \quad n \to +\infty$$

which implies that

$$\lim_{n\to+\infty} \left(\sup_{\substack{v\in W^{1,p(\cdot)}(\mathbb{R}^N) \\ \|v\|\leq 1}} \left| \int_{B_t} (f(x,u_n) - f(x,u)) v dx \right| \right) = 0 \quad \forall t > 0. \tag{A.1}$$

Observing that

$$\frac{1}{\mu(x)} + \frac{\beta(x)-1}{\mu(x)} + \frac{1}{r(x)} = 1 \quad \forall\ x \in \mathbb{R}^N$$

and using (2.2), we get

$$\begin{aligned} &\int_{\mathbb{R}^N\setminus B_t} |f(x,u_n) - f(x,u)||v| dx \\ &\leq \int_{\mathbb{R}^N\setminus B_t} |g(x)| \left| |u_n|^{\beta(x)-1} + |u|^{\beta(x)-1} \right| |v| dx \\ &\leq 3 |g|_{L^{r(\cdot)}(\mathbb{R}^N\setminus B_t)} \left| |u_n|^{\beta(\cdot)-1} + |u|^{\beta(\cdot)-1} \right|_{L^{\frac{\mu(\cdot)}{\beta(\cdot)-1}}(\mathbb{R}^N\setminus B_t)} |v|_{L^{\mu(\cdot)}(\mathbb{R}^N\setminus B_t)} . \end{aligned} \tag{A.2}$$

Since (u_n) is bounded in $W^{1,p(\cdot)}(\mathbb{R}^N)$, from (A.2) we deduce that

$$\int_{\mathbb{R}^N\setminus B_t} |f(x,u_n) - f(x,u)||v| dx \leq c_{16} |g|_{L^{r(\cdot)}(\mathbb{R}^N\setminus B_t)} . \tag{A.3}$$

Now, since $g \in L^{r(\cdot)}(\mathbb{R}^N)$, then

$$|g|_{L^{r(\cdot)}(\mathbb{R}^N \backslash B_t)} \to 0 \quad \text{as} \quad t \to +\infty.$$

Thus for every $\epsilon > 0$, there exists $t_0 > 0$ large enough such that

$$c_{16}\, |g|_{L^{r(\cdot)}(\mathbb{R}^N \backslash B_{t_0})} < \frac{\epsilon}{2}.$$

By (A.3), we get

$$\int_{\mathbb{R}^N \backslash B_{t_0}} |f(x,u_n) - f(x,u)||v|\, dx < \frac{\epsilon}{2} \quad \forall n \geq 1. \tag{A.4}$$

Taking $t = t_0$ in (A.1) and combining (A.1) with (A.4), we obtain the claimed result.

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