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# TURNPIKE PROPERTIES OF APPROXIMATE SOLUTIONS FOR DISCRETE-TIME CONTROL SYSTEMS

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**Abstract**

We study the structure of approximate solutions of a discrete-time control system with a compact metric space of states which arises in economic dynamics. We are interested in turnpike properties of the approximate solutions which are independent of the length of the interval, for all sufficiently large intervals and are stable under perturbations of an objective function.

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## 1 Introduction

The study of the existence and the structure of solutions of optimal control problems defined on infinite intervals and on sufficiently large intervals has recently been a rapidly growing area of research. See, for example, [2, 6, 8, 10-15] and the references mentioned therein. In this paper we study the structure of solutions of a discrete-time optimal control system describing a general model of economic dynamics [3, 7, 9, 13-15].

Let $(X,\rho)$ be a compact metric space and $\Omega$ be a nonempty closed subset of $X\times X$.

A sequence $\{x_t\}_{t=0}^{\infty}\subset X$ is called a program if $(x_t,x_{t+1})\in\Omega$ for all integers $t\geq 0$. A sequence $\{x_t\}_{t=T_1}^{T_2}\subset X$ where integers $T_1,T_2$ satisfy $0\leq T_1<T_2$ is called a program if $(x_t,x_{t+1})\in\Omega$ for all integers $t\in[T_1,T_2-1]$.

In this paper we consider the problem

$$\sum_{i=0}^{T-1} v(x_i,x_{i+1})\to \max \qquad (P)$$

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$$\text{s. t. } \{(x_i, x_{i+1})\}_{i=0}^{T-1} \subset \Omega,\ x_0 = z_1,\ x_T = z_2,$$

where $T$ is a natural number, $z_1, z_2 \in X$ and $v : \Omega \to R^1$ is a bounded function. In models of economic growth the set $X$ is the space of states, $v$ is a utility function and $v(x_t, x_{t+1})$ evaluates consumption at moment $t$. The interest in discrete-time optimal problems of type (P) also stems from the study of various optimization problems which can be reduced to it, e.g., tracking problems in engineering [5], the study of Frenkel-Kontorova model related to dislocations in one-dimensional crystals [1] and the analysis of a long slender bar of a polymeric material under tension in [6]. Optimization problems of the type (P) with $\Omega = X \times X$ were considered in [10-12].

We are interested in a turnpike property of the approximate solutions of (P) which is independent of the length of the interval $T$, for all sufficiently large intervals. To have this property means, roughly speaking, that the approximate solutions of the optimal control problems are determined mainly by the cost function $v$, and are essentially independent of $T$, $z_1$ and $z_2$. Turnpike properties are well known in mathematical economics. The term was first coined by Samuelson in 1948 (see [9]) where he showed that an efficient expanding economy would spend most of the time in the vicinity of a balanced equilibrium path (also called a von Neumann path).

In the classical turnpike theory [3, 7, 9] the space $X$ is a compact convex subset of a finite-dimensional Euclidean space, the set $\Omega$ is convex and the function $v$ is strictly concave. Under these assumptions the turnpike property can be established and the turnpike $\bar{x}$ is a unique solution of the maximization problem $v(x,x) \to \max$, $(x,x) \in \Omega$. In this situation it is shown that for each program $\{x_t\}_{t=0}^{\infty}$ either the sequence $\{\sum_{t=0}^{T-1} v(x_t, x_{t+1}) - Tv(\bar{x},\bar{x})\}_{T=1}^{\infty}$ is bounded (in this case the program $\{x_t\}_{t=0}^{\infty}$ is called $(v)$-good) or it diverges to $-\infty$. Moreover, it is also established that any $(v)$-good program converges to the turnpike $\bar{x}$. In the sequel this property is called as the asymptotic turnpike property.

In [14] we showed that the turnpike property follows from the asymptotic turnpike property. More precisely, we assumed that any $(v)$-good program converges to a unique solution $\bar{x}$ of the problem $v(x,x) \to \max$, $(x,x) \in \Omega$ and showed that the turnpike property holds and $\bar{x}$ is the turnpike. Note that we do not use convexity (concavity) assumptions. It should be mentioned that in [13] analogous results were established for the problem

$$\sum_{i=0}^{T-1} v(x_i, x_{i+1}) \to \max,\ \{(x_i, x_{i+1})\}_{i=0}^{T-1} \subset \Omega,\ x_0 = z,$$

where $T$ is a natural number and $z \in X$.

In the present paper we improve the turnpike results established in [13, 14] and show that the turnpike property is stable under perturbations of the objective function $v$. Note that the stability of the turnpike property is crucial in practice. One reason is that in practice we deal with a problem which consists a perturbation of the problem we wish to consider. Another reason is that the computations introduce numerical errors.

Let $(X, \rho)$ be a compact metric space and $\Omega$ be a nonempty closed subset of $X \times X$. Denote by $\mathcal{M}$ the set of all bounded functions $u : \Omega \to R^1$. For each $w \in \mathcal{M}$ set

$$||w|| = \sup\{|w(x,y)| : (x,y) \in \Omega\}. \tag{1.1}$$

For each $x, y \in X$, each integer $T \geq 1$ and each $u \in \mathcal{M}$ set

$$\sigma(u,T,x) = \sup\{\sum_{i=0}^{T-1} u(x_i,x_{i+1}) : \{x_i\}_{i=0}^{T} \text{ is a program and } x_0 = x\}, \tag{1.2}$$

$$\sigma(u,T,x,y) = \sup\{\sum_{i=0}^{T-1} u(x_i,x_{i+1}) : \{x_i\}_{i=0}^{T} \text{ is a program and } x_0 = x,\ x_T = y\}, \tag{1.3}$$

$$\sigma(u,T) = \sup\{\sum_{i=0}^{T-1} u(x_i,x_{i+1}) : \{x_i\}_{i=0}^{T} \text{ is a program}\}. \tag{1.4}$$

(Here we use the convention that the supremum of an empty set is $-\infty$).

Assume that $v \in \mathcal{M}$ is an upper semicontinuous function. Since in [13, 14] we assume that objective functions are defined on the set $X \times X$ in order to apply their results we set $v(x,y) = -||v|| - 1$ for all $(x,y) \in (X \times X) \setminus \Omega$.

We suppose that there exist $\bar{x} \in X$ and a constant $\bar{c} > 0$ such that the following assumptions hold.

(A1) $(\bar{x},\bar{x})$ is an interior point of $\Omega$ (there is $\varepsilon > 0$ such that $\{(x,y) \in X \times X : \rho(x,\bar{x}),$ $\rho(y,\bar{x}) \leq \varepsilon\} \subset \Omega$) and $v$ is continuous at $(\bar{x},\bar{x})$.

(A2) $\sigma(v,T) \leq Tv(\bar{x},\bar{x}) + \bar{c}$ for all integers $T \geq 1$.

It is easy to see that for each natural number $T$ and each program $\{x_t\}_{t=0}^{T}$

$$\sum_{t=0}^{T-1} v(x_t,x_{t+1}) \leq \sigma(v,T) \leq Tv(\bar{x},\bar{x}) + \bar{c}. \tag{1.5}$$

Inequality (1.5) implies the following result.

**Proposition 1.1.** *For each program $\{x_t\}_{t=0}^{\infty}$ either the sequence*

$$\{\sum_{t=0}^{T-1} v(x_t,x_{t+1}) - Tv(\bar{x},\bar{x})\}_{T=1}^{\infty}$$

*is bounded or* $\lim_{T\to\infty}[\sum_{t=0}^{T-1} v(x_t,x_{t+1}) - Tv(\bar{x},\bar{x})] = -\infty$.

A program $\{x_t\}_{t=0}^{\infty}$ is called $(v)$-good if the sequence

$$\{\sum_{t=0}^{T-1} v(x_t,x_{t+1}) - Tv(\bar{x},\bar{x})\}_{T=1}^{\infty}$$

is bounded [3, 4, 12].

In this paper we suppose that the following assumption holds.

(A3) (the asymptotic turnpike property) For any $(v)$-good program $\{x_t\}_{t=0}^{\infty}$,

$$\lim_{t\to\infty} \rho(x_t,\bar{x}) = 0.$$

Note that (A3) holds for many important infinite horizon optimal control problems. See, for example, [13-15]. In particular, (A3) holds for a general model of economic dynamics.

By (A3) $||v|| > 0$. For each $M > 0$ denote by $X_M$ the set of all $x \in X$ for which there exists a program $\{x_t\}_{t=0}^{\infty}$ such that $x_0 = x$ and that for all integers $T \geq 1$

$$\sum_{t=0}^{T-1} v(x_t, x_{t+1}) - Tv(\bar{x}, \bar{x}) \geq -M. \tag{1.6}$$

Clearly $\cup\{X_M : M \in (0, \infty)\}$ is the set of all $x \in X$ for which there exists a $(v)$-good program $\{x_t\}_{t=0}^{\infty}$ such that $x_0 = x$.

Let $T$ be a natural number. Denote by $Y_T$ the set of all $x \in X$ for which there exists a program $\{x_t\}_{t=0}^{T}$ such that $x_0 = \bar{x}$ and $x_T = x$.

Denote by $\mathrm{Card}(A)$ the cardinality of a set $A$.

The following two theorems which describe the structure of approximate solutions of our discrete-time control system are our main results.

**Theorem 1.2.** *Let $M_0$, $M_1$, $\varepsilon$ be positive numbers and let $L_0$ be a natural number. Then there exist $\delta > 0$ and a natural number $L_* > L_0$ such that for each $u \in \mathcal{M}$ satisfying $||u - v|| \leq \delta$, each integer $T > L_*$ and each program $\{x_t\}_{t=0}^{T}$ which satisfies*

$$x_0 \in X_{M_0},\ x_T \in Y_{L_0},$$

$$\sum_{t=0}^{T-1} u(x_t, x_{t+1}) \geq \sigma(u, T, x_0, x_T) - M_1$$

*the following inequality holds:*

$$Card(\{t \in \{0, \dots, T\} : \rho(x_t, \bar{x}) > \varepsilon\}) \leq L_*.$$

**Theorem 1.3.** *Let $M_0$, $M_1$, $\varepsilon$ be positive numbers. Then there exist $\delta > 0$ and a natural number $L_*$ such that for each $u \in \mathcal{M}$ satisfying $||u - v|| \leq \delta$, each integer $T > L_*$ and each program $\{x_t\}_{t=0}^{T}$ which satisfies*

$$x_0 \in X_{M_0},\ \sum_{t=0}^{T-1} u(x_t, x_{T+1}) \geq \sigma(u, T, x_0) - M_1$$

*the following inequality holds:*

$$Card(\{t \in \{0, \dots, T\} : \rho(x_t, \bar{x}) > \varepsilon\}) \leq L_*.$$

Theorems 1.2 and 1.3 establish the turnpike property for approximate solutions of the optimal control problems with an objective function $u$ which belongs to a small neighborhood of $v$. They extend the main results of [15] which were obtained in the case when $M_1$ is sufficiently small and depends of $M_0$ and $\varepsilon$.

Note that examples of pairs $(v, \Omega)$ for which the assumptions made in this paper hold are presented in [15].

The paper is organized as follows. Section 2 contains auxiliary results. Theorem 1.2 and 1.3 are proved in Section 3.

## 2 Auxiliary Results

By (A1) there exists $\bar{r} \in (0,1)$ such that

$$\{(x,y) \in X \times X : \ \rho(x,\bar{x}), \ \rho(y,\bar{x}) \le \bar{r}\} \subset \Omega. \tag{2.1}$$

Clearly, for each $w \in \mathcal{M}$, for each $x,y \in X$ satisfying $\rho(x,\bar{x}), \rho(y,\bar{x}) \le \bar{r}$ and any integer $T \ge 1$, $\sigma(w,T,x,y)$ is finite.

In order to prove our main results we need the following lemmas obtained in [15].

**Lemma 2.1** (15, Lemma 2.4)**.** . *Let* $\varepsilon > 0$. *Then there exists* $\delta \in (0,\bar{r})$ *such that for each* $w \in \mathcal{M}$ *satisfying* $||w - v|| \le \delta$, *each integer* $T \ge 1$ *and each program* $\{x_t\}_{t=0}^{T}$ *satisfying*

$$\rho(x_0,\bar{x}), \ \rho(x_T,\bar{x}) \le \delta, \ \sum_{t=0}^{T-1} w(x_t,x_{t+1}) \ge \sigma(w,T,x_0,x_T) - \delta$$

*the inequality* $\rho(x_t,\bar{x}) \le \varepsilon$ *holds for all* $t = 0,\dots,T$.

**Lemma 2.2** (15, Lemma 2.5)**.** *Let* $M_0, M_1, \varepsilon$ *be positive numbers and let* $L_0$ *be a natural number. Then there exist a natural number* $L_* > L_0 + 2$ *and* $\delta \in (0,\varepsilon)$ *such that for each* $w \in \mathcal{M}$ *satisfying* $||w - v|| \le \delta$, *each integer* $T \ge L_*$, *each program* $\{x_t\}_{t=0}^{T}$ *satisfying*

$$\min\{\rho(x_t,\bar{x}) : \ t = 1,\dots,T-1\} > \varepsilon,$$

*each* $z_0 \in X_{M_0}$ *and each* $z_1 \in Y_{L_0}$ *there exists a program* $\{y_t\}_{t=0}^{T}$ *such that*

$$y_0 = z_0, \ y_T = z_1, \ \sum_{t=0}^{T-1} w(y_t,y_{t+1}) \ge \sum_{t=0}^{T-1} w(x_t,x_{t+1}) + M_1.$$

## 3 Proof of Theorems 1.2 and 1.3

We prove Theorems 1.2 and 1.3 simultaneously. Let $\bar{r} \in (0,1)$ satisfy (2.1). We may assume that $M_0 > 2$ and that

$$|v(x,y) - v(\bar{x},\bar{x})| \le 1/4 \text{ for all } x,y \in X \text{ satisfying } \rho(x,\bar{x}), \rho(y,\bar{x}) \le \bar{r}. \tag{3.1}$$

By Lemma 2.1 there exists a positive number

$$\delta_1 < \min\{\varepsilon, \bar{r}\} \tag{3.2}$$

such that the following property holds:

(P1) for each $w \in \mathcal{M}$ satisfying $||w - v|| \le \delta_1$, each integer $T \ge 1$ and each program $\{x_t\}_{t=0}^{T}$ satisfying

$$\rho(x_0,\bar{x}), \ \rho(x_T,\bar{x}) \le \delta_1, \ \sum_{t=0}^{T-1} w(x_t,x_{t+1}) \ge \sigma(w,T,x_0,x_T) - \delta_1$$

the inequality $\rho(x_t,\bar{x}) \le \varepsilon$ holds for all $t = 0,\dots,T$.

In the case of Theorem 1.2 the natural number $L_0$ is given. In the case of Theorem 1.3 put $L_0 = 4$.

By Lemma 2.2 there exist a natural number $L_1 > L_0 + 2$ and $\delta \in (0, \delta_1)$ such that the following property holds:

(P2) for each $w \in \mathcal{M}$ satisfying $||w - v|| \le \delta$, each integer $T \ge L_1$, each program $\{x_t\}_{t=0}^T$ satisfying

$$\min\{\rho(x_t, \bar{x}) : t = 1, \dots, T-1\} > \delta_1,$$

each $z_0 \in X_{M_0}$ and each $z_1 \in Y_{L_0}$ there exists a program $\{y_t\}_{t=0}^T$ such that

$$y_0 = z_0,\ y_T = z_1,\ \sum_{t=0}^{T-1} w(y_t, y_{t+1}) \ge \sum_{t=0}^{T-1} w(x_t, x_{t+1}) + M_1 + 4.$$

By (2.1), the choice of $\bar{r}$ and (3.1)

$$\{z \in X : \rho(x, \bar{x}) \le \bar{r}\} \subset X_1 \cap Y_1 \subset X_{M_0} \cap Y_{L_0}. \tag{3.3}$$

Choose a natural number

$$L_2 > 4 + L_1 \tag{3.4}$$

and a natural number

$$L_* > 8(L_0 + L_1 + L_2 + 2) + L_2(2 + M_1 \delta_1^{-1}). \tag{3.5}$$

Assume that $u \in \mathcal{M}$ satisfies

$$||u - v|| \le \delta, \tag{3.6}$$

an integer $T > L_*$ and a program $\{x_t\}_{t=0}^T$ satisfies

$$x_0 \in X_{M_0},\ x_T \in Y_{L_0},$$

$$\sum_{t=0}^{T-1} u(x_t, x_{t+1}) \ge \sigma(u, T, x_0, x_T) - M_1 \tag{3.7}$$

in the case of Theorem 1.2 and

$$x_0 \in X_{M_0},\ \sum_{t=0}^{T-1} u(x_t, x_{t+1}) \ge \sigma(u, T, x_0) - M_1 \tag{3.8}$$

in the case of Theorem 1.3.

Let an integer

$$\tau \in [0, T - L_2]. \tag{3.9}$$

We show that

$$\min\{\rho(x_t, \bar{x}) : t = \tau + 1, \dots, \tau + L_2\} \le \delta_1. \tag{3.10}$$

Assume the contrary. Then

$$\rho(x_t, \bar{x}) > \delta_1,\ t = \tau + 1, \dots, \tau + L_2. \tag{3.11}$$

By (3.7) and (3.8) there is an integer $S_1$ such that

$$0 \le S_1 \le \tau,\ x_{S_1} \in X_{M_0}, \tag{3.12}$$

$$x_t \notin X_{M_0} \text{ for all integers } t \text{ satisfying } S_1 < t \le \tau.$$

By (3.2), (3.3) and (3.12) for all integers $t$ satisfying $S_1 < t \le \tau$

$$\rho(x_t, \bar{x}) > \bar{r} > \delta_1. \tag{3.13}$$

We show that there is an integer $S_2$ such that

$$\tau + L_2 \le S_2 \le T,\ x_{S_2} \in Y_{L_0}. \tag{3.14}$$

In the case of Theorem 1.2 the existence of an integer $S_2$ satisfying (3.14) follows from (3.7). Consider the case of Theorem 1.3 and show that in this case an integer $S_2$ satisfying (3.14) also exists.

Assume the contrary. Then

$$x_t \notin Y_{L_0},\ t = \tau + L_2, \dots, T$$

and in view of (3.2) and (3.3)

$$\rho(x_t, \bar{x}) > \bar{r} > \delta_1,\ t = \tau + L_2, \dots, T.$$

Combined with (3.13) and (3.11) this implies that

$$\rho(x_t, \bar{x}) > \delta_1,\ t = S_1 + 1, \dots, T. \tag{3.15}$$

By (3.4), (3.9) and (3.12)

$$T - S_1 \ge T - \tau \ge L_2 > L_1. \tag{3.16}$$

By (3.6), (3.12), (3.15), (3.16) and (P2) there exists a program $\{y_t\}_{t=S_1}^{T}$ such that

$$y_{S_1} = x_{S_1},\ y_T = \bar{x},\ \sum_{t=S_1}^{T-1} u(y_t, y_{t+1}) \ge \sum_{t=S_1}^{T-1} u(x_t, x_{t+1}) + M_1 + 4. \tag{3.17}$$

Put

$$y_t = x_t,\ t = 0, \dots, S_1.$$

Clearly, $\{y_t\}_{t=0}^{T}$ is a program and in view of (3.17) and the equation above

$$y_0 = x_0,$$

$$\sum_{t=0}^{T-1} u(y_t, y_{t+1}) - \sum_{t=0}^{T-1} u(x_t, x_{t+1}) = \sum_{t=S_1}^{T-1} u(y_t, y_{t+1}) - \sum_{t=S_1}^{T-1} u(x_t, x_{t+1}) \ge M_1 + 4.$$

This contradicts (3.8). The contradiction we have reached proves that there is an integer $S_2$ satisfying (3.14). Thus in the case of Theorem 1.2 and in the case of Theorem 1.3 there exists an integer $S_2$ such that (3.14) holds.

We may assume without loss of generality that for all integers $t$ satisfying $\tau+L_2 < t < S_2$

$$x_t \notin Y_{L_0}. \tag{3.18}$$

Together with (3.2) and (3.3) this implies that for all integers $t$ satisfying $\tau+L_2 < t < S_2$

$$\rho(x_t,\bar{x}) > \bar{r} > \delta_1. \tag{3.19}$$

By (3.14), (3.12), (3.11), (3.13) and (3.19)

$$S_2 - S_1 \geq L_2,\ x_{S_1} \in X_{M_0},\ x_{S_2} \in Y_{L_0},$$

$$\rho(x_t,\bar{x}) > \delta_1,\ t = S_1+1,\dots,S_2-1. \tag{3.20}$$

By (3.4), (3.6), (3.20) and property (P2) there exists a program $\{y_t\}_{t=S_1}^{S_2}$ such that

$$y_{S_1} = x_{S_1},\ y_{S_2} = x_{S_2},$$

$$\sum_{t=S_1}^{S_2-1} u(y_t,y_{t+1}) \geq \sum_{t=S_1}^{S_2-1} u(x_t,x_{t+1}) + M_1 + 4. \tag{3.21}$$

Put

$$y_t = x_t \text{ for all integers } t \text{ satisfying } 0 \leq t < S_1 \tag{3.22}$$

$$\text{and for all integers } t \text{ satisfying } S_2 < t \leq T.$$

Clearly, $\{y_t\}_{t=0}^{T}$ is a program and

$$y_0 = x_0,\ y_T = x_T. \tag{3.23}$$

By (3.21) and (3.22)

$$\sum_{t=0}^{T-1} u(y_t,y_{t+1}) - \sum_{t=0}^{T-1} u(x_t,x_{t+1}) = \sum_{t=S_1}^{S_2-1} u(y_t,y_{t+1}) - \sum_{t=S_1}^{S_2-1} u(x_t,x_{t+1}) \geq M_1 + 4.$$

Together with (3.23) this contradicts (3.7). The contradiction we have reached proves (3.10).

Thus we have shown that the following property holds:

(P3) for each integer $\tau \in [0,\dots,T-L_2]$

$$\min\{\rho(x_t,\bar{x}) :\ t = \tau+1,\dots,\tau+L_2\} \leq \delta_1.$$

Using (P3) by induction we construct a sequence of natural numbers $\{S_i\}_{i=1}^{q}$ such that

$$S_1 \in [1,L_2],\ \text{ for each integer } i \text{ satisfying } 1 \leq i \leq q-1, \tag{3.24}$$

$$S_{i+1} - S_i \in [1,L_2[,\ 0 \leq T - S_q < L_2,$$

$$\rho(x_{S_i},\bar{x}) < \delta_1,\ i = 1,\dots,q. \tag{3.25}$$

By (3.5) and (3.24) $q \geq 6$. Set

$$E_1 = \{i \in \{1, \ldots, q-1\} : \sum_{t=S_i}^{S_{i+1}-1} u(x_i, x_{i+1}) \geq \sigma(u, S_{i+1} - S_i, x_{S_i}, x_{S_{i+1}}) - \delta_1\}, \tag{3.26}$$

$$E_2 = \{1, \ldots, q-1\} \setminus E_1. \tag{3.27}$$

By (3.6), (3.25), (3.26) and (P1) for each $i \in E_1$

$$\rho(x_t, \bar{x}) \leq \varepsilon,\ t = S_i, \ldots, S_{i+1}.$$

Together with (3.2), (3.24) and (3.27) this implies that

$$\{t \in \{0, \ldots, T\} :\ \rho(x_t, \bar{x}) > \varepsilon\}$$

$$\subset \{0, \ldots, S_1 - 1\} \cup \{t :\ t \text{ is an integer such that } S_q < t \leq T\}$$

$$\cup_{i \in E_2} \{t :\ t \text{ is an integer such that } S_i < t < S_{i+1}\}.$$

Combined with (3.24) this implies that

$$\mathrm{Card}(\{t \in \{0, \ldots, T\} :\ \rho(x_t, \bar{x}) > \varepsilon\}) \leq 2L_2 + L_2 \mathrm{Card}(E_2). \tag{3.28}$$

By (3.7), (3.8), (3.24), (3.26) and (3.27)

$$M_1 \geq \sigma(u, T, x_0, x_T) - \sum_{t=0}^{T-1} u(x_i, x_{i+1})$$

$$\geq \sum_{i \in E_2} [\sigma(u, S_{i+1} - S_i, x_{S_i}, x_{S_{i+1}}) - \sum_{t=S_i}^{S_{i+1}-1} u(x_i, x_{i+1})] \geq \delta_1 \mathrm{Card}(E_2)$$

and

$$\mathrm{Card}(E_2) \leq \delta_1^{-1} M_1.$$

Together with (3.5) and (3.28) this implies that

$$\mathrm{Card}(\{t \in \{0, \ldots, T\} :\ \rho(x_t, \bar{x}) > \varepsilon\}) \leq 2L_2 + L_2 M_1 \delta^{-1} < L_*.$$

This completes the proof of Theorems 1.2 and 1.3.

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