Communications in **M**athematical **A**nalysis
Volume 11, Number 2, pp. 1–22 (2011) www.math-res-pub.org/cma
ISSN 1938-9787

CHERN-SIMONS PATH INTEGRAL ON R^3 USING ABSTRACT WIENER MEASURE

ADRIAN P. C. LIM*
Mathematics Research Unit
University of Luxembourg

(Communicated by Hitoshi Kitada)

Abstract

Instead of using white noise analysis, we use abstract Wiener measure to define the Chern-Simons path integral over $\mathbb{R}^3$. One rigorous and the other, not so rigorous, definitions will be given. The latter will be used to compute the Wilson Loop observable in the abelian case, which gives us the linking number of a link.

AMS Subject Classification: 81T08; 81T13; 60H99

Keywords: Chern-Simons, path integral, linking number, abstract wiener measure

1 Introduction

Witten in his paper [4] showed that the Chern-Simons path integral can be used to obtain knot polynomials in a 3-manifold. The authors in [1] used white noise analysis to make sense of the path integral Equation (2.2) on $\mathbb{R}^3$. They began with a Hilbert space, consisting of L^2 functions. Using Minlos theorem, they complete this space, using a sequence of inner products, to define a (Gaussian) probability space $\mathcal{E}^*$ consisting of distributions and $L^2(\mathcal{E}^*,\mu)$. The Chern-Simons path integral is now realized as a distribution acting on test functions contained in $L^2(\mathcal{E}^*,\mu)$.

A similar approach was adapted by Hahn in [3]. Furthermore, he computed the Wilson Loop observables Equation (3.1) for the abelian and non-abelian gauge group. When the gauge group is abelian, the theory agrees very well with the known knot literature. For the non-abelian gauge group, there is a slight discrepancy.

More recently, the authors in [9] defined the Chern-Simons integral on a 3-manifold using Wiener measure.

Before we begin on the math proper, we will use the following notation, to make the writing easier. This is helpful when the reader encounters unfamiliar notation used in this article.

---

*E-mail address: adrian.lim@uni.lu

Notation 1.1. *(Set Notation.) For any vector space S, $\vec{S}$ will denote the direct product $S \times S$. $\vec{s}$ will denote an element in $\vec{S}$, written as (s_+, s_-). Addition will be component-wise addition. Define a product, $\vec{s} \times \vec{u} := (s_+ u^+, s_- u^-)$, $\vec{s} \in \mathbb{R}^2$ and $\vec{u} \in \vec{S}$. $\pm$ as superscript or subscript do not make any significant difference.*

Given a Hilbert space $(H, \langle \cdot, \cdot \rangle)$, consider the direct product $\vec{H} := H \times H$. An element in H is usually denoted by u and an element in $H \times H$ is always denoted by a 2- component vector, $\vec{u}$. Let $P^+ : H \times H \to H$ ($P^- : H \times H \to H$) denote the projection onto the first (second) component.

Let $P : H \times H \to H \times H$ be an orthogonal projection. Let $Q_\pm = P^\pm P : H \times H \to H$ denote the composition of 2 projections and let $A_\pm = Q_\pm(H \times H)$ denote the range of $Q_\pm$. Note that $A_\pm$ is a subspace in H. Define for each orthogonal projection P, an orthogonal projection $P^\sharp$ whose range is equal to $(A_+ \oplus A_-) \times (A_+ \oplus A_-)$, a subspace inside $H \times H$.

The inner product $\langle \cdot, \cdot \rangle$ is extended to an inner product, $\langle\langle \cdot, \cdot \rangle\rangle$ on the direct product by

$$\langle\langle (A_1 + B_1), A_2 + B_2 \rangle\rangle := \langle A_1, A_2 \rangle + \langle B_1, B_2 \rangle.$$

Given $\vec{u} = (u_+, u_-) \in H \times H$, denote $\langle \vec{u} \rangle_\sharp := \langle u_+, u_- \rangle$.

The standard coordinates on $\mathbb{R}^3$ will be denoted by $x = (x_0, x_1, x_2)$.

$M_N(\mathbb{C})$ will denote the vector space of $N \times N$ matrices with complex entries. A^* will denote its hermitian conjugate of A. A^T will denote the transpose. Tr *will denote the matrix trace.*

ϕ_κ will always denote a 1,2 or 3 dimensional Gaussian density function with mean 0 and variance $1/\kappa^2$ and $\varsigma_1 := \kappa/(8\pi)$, $\varsigma_0 := \kappa/2\pi$.

Remark 1.1. There is a dependence on κ in most of the definitions in this article. To simplify the notation, reference to κ is omitted in most of the definitions, but the reader should bear in mind that there is a κ dependence in the definitions used.

2 Chern-Simons Path Integral

Only for this section of this article, we will set $\kappa = \sqrt{2}$.

Let E be a trivial bundle over $\mathbb{R}^3$ and G a (compact) connected Lie subgroup of $U(N)$, $N \in \mathbb{N}$. Denote the Lie algebra of G by $\mathfrak{g}$ and identify $\mathfrak{g}$ with the Lie subalgebra of the Lie algebra $\mathfrak{u}(N)$ of $U(N)$.

Remark 2.1. In general, given $A, B \in \mathfrak{g}$, $\langle A, B \rangle = -\mathrm{Tr}[AB]$ is degenerate. However, if $\mathfrak{g}$ is semisimple, then this bilinear form is non-degenerate.

Let $x = (x_0, x_1, x_2)$ be the usual coordinates on $\mathbb{R}^3$. The Chern-Simons action is given by

$$\mathrm{CS}(A) = \frac{\kappa}{4\pi} \int_{\mathbb{R}^3} \mathrm{Tr}[A \wedge dA + \frac{2}{3} A \wedge A \wedge A]\, d\mathrm{vol}_{\mathbb{R}^3},\ \kappa \neq 0. \tag{2.1}$$

A is a connection on $\mathbb{R}^3$, which decays to 0 fast enough for it to be integrable on $\mathbb{R}^3$.

Now, we want to make sense of an expression

$$Z_{\mathrm{CS}} := \int_{A \in \mathscr{A}} e^{i\mathrm{CS(A)}} DA. \tag{2.2}$$

Here, $\mathscr{A}$ is the space of $\mathfrak{g}$-valued connections on the bundle E and DA is some heuristic Lebesgue measure on $\mathscr{A}$.

Every connection over $\mathbb{R}^3$ can be gauge transformed into the form

$$A = a_0 dx_0 + a_1 dx_1,$$

where a_0 and a_1 are $\mathfrak{g}$-valued functions over $\mathbb{R}^3$ and

$$a_1(x_0,x_1,0) = 0,\ a_0(x_0,0,0) = 0. \tag{2.3}$$

This is called Axial gauge fixing in [3]. Assume a_0 and a_1 are smooth and decay fast enough for the integral to be defined.

With this gauge the cubic term in the action drops out and we are left with (Dot means matrix multiplication.)

$$2\varsigma_1 \int_{\mathbb{R}^3} \mathrm{Tr}[\partial_2 a_0 \cdot a_1 - a_0 \cdot \partial_2 a_1] d\mathrm{vol}_{\mathbb{R}^3},\ \varsigma_1 := \frac{\kappa}{8\pi} = \frac{\sqrt{2}}{8\pi}. \tag{2.4}$$

Consider the Schwartz space $\mathcal{S}(\mathbb{R}^3)$, with the Gaussian measure ϕ_κ, $\sqrt{\phi_\kappa}(x) = e^{-\kappa^2|x|^2/4}(\kappa^2/2\pi)^{3/4}$. $a_1(x) = x_2 p_-(x) \cdot \sqrt{\phi_\kappa}(x)$ will satisfy Equation (2.3). Write $a_0(x) = p_+(x) \cdot \sqrt{\phi_\kappa}(x)$. $p_\pm$ are $\mathfrak{g}$-valued, with each component being a polynomial over $\mathbb{R}^3$.

Let $\tau := x_2\partial_2$ and define an inner product

$$\langle \cdot,\cdot \rangle_{\mathfrak{g},\kappa} \equiv \langle \cdot,\cdot \rangle : A,B \in C^\infty(\mathbb{R}^3 \to \mathfrak{g}) \times C^\infty(\mathbb{R}^3 \to \mathfrak{g}) \to -\varsigma_1 \int_{\mathbb{R}^3} \mathrm{Tr}(AB)\phi_\kappa d\mathrm{vol}_{\mathbb{R}^3}.$$

Then Equation (2.4) becomes

$$\begin{aligned}&\langle x_2\partial_2 p_+, p_- \rangle - \langle p_+, p_- \rangle - \langle p_+, x_2\partial_2 p_- \rangle \\ &\quad = \langle \tau p_+ + \tau p_-, p_- - p_+ \rangle - \langle p_+, p_- \rangle + \langle \tau p_+, p_+ \rangle - \langle \tau p_-, p_- \rangle.\end{aligned} \tag{2.5}$$

Now, let $a^* = -\partial_2 + x_2$, $a = \partial_2 + x_2$. Apply $2xy = (x+y)^2 - x^2 - y^2 + [x,y]$, $2\tau = 2x_2\partial_2 = a^2 - \partial_2^2 - x_2^2 - 1 = a^2 - 2x_2^2 + a^*a$.

Now, $2\partial_2 = a - a^* := a^-$, $2x_2 = a + a^* := a^+$. Let

$$L = a^2 - \frac{a^{+,2}}{2},\ a^{+,2} = (a^+)^2.$$

Define

$$F(\vec{p}) := \langle 2\tau(p_+ + p_-), p_- - p_+ \rangle - 2\langle p_+, p_- \rangle + \langle Lp_+, p_+ \rangle - \langle Lp_-, p_- \rangle.$$

Then, Equation (2.5) becomes $F(\vec{p}) + \langle a^*ap_+, p_+ \rangle - \langle a^*ap_-, p_- \rangle$, which is equal to $-\mathrm{CS}(A)$.

The idea now is to split the expression into

$$\int_{\vec{p} \in \vec{\mathcal{S}}_{\mathfrak{g}}(\mathbb{R}^3)} e^{-iF(\vec{p})} e^{-\kappa_+\langle a^*ap_+,p_+\rangle - \kappa_-\langle a^*ap_-,p_-\rangle} Dp_+ Dp_-,$$

for $\kappa_i > 0$ and then do an analytic continuation, followed by $\kappa_+ \to i$ and $\kappa_- \to -i$.

For $\vec{\kappa} = (\kappa_+, \kappa_-)$,

$$\frac{1}{N_{\vec{\kappa}}} \exp[-\kappa_+ \langle a^* a p_+, p_+ \rangle - \kappa_- \langle a^* a p_-, p_- \rangle] Dp_+ Dp_-$$

should be interpreted as a product of 2 infinite dimensional Gaussian measure, with variances κ_+^{-1} and κ_-^{-1} respectively and $1/N_\kappa$ is some normalization constant. Such a measure does not exists on $C^\infty(\mathbb{R}^3)$; one has to complete the space to define a sensible Gaussian measure.

The (physicists) Hermite polynomials $\{h_i\}_{i\geq 0}$ form an orthogonal basis on $L^2(\mathbb{R}, \mu)$ with the Gaussian measure $d\mu(x) \equiv e^{-x^2} dx/\sqrt{\pi}$. The operators a^* and a are the familiar raising and lowering operators, i.e. $a^* h_n = h_{n+1}$ and $a h_n = h_{n-1}$. The operator $a^* a$ is the Hamiltonian operator, $a^* a h_n = n h_n$.

There is a unitary isomorphism from $L^2(\mathbb{R}, \mu) \rightarrow L^2(\mathbb{C}, e^{-|z|^2} dx dp/\pi)$, given by the standard Geometric Quantization of the Harmonic Oscillator, which is also the Segal-Barmann transform. Using Kahler polarization, the Quantum Hilbert space is the space of Holomorphic functions integrable with respect to $e^{-|z|^2} dx dp/\pi$, $z = x + ip$. The quantized Hamiltonian is $z\partial_z$ and the orthonormal basis is $\{z^n/\sqrt{n!}\}_{n=0}^\infty$. Note that $Q : h_n \sqrt{\phi_1}/\sqrt{2^n n!} \rightarrow z^n/\sqrt{n!}$ and $Q : a^* a \rightarrow z\partial_z$, is a unitary isomorphism.

Notation 2.1. *For integers $i, j, k \geq 0$, p_r will denote the triple (i, j, k) with $i + j + k = r$. $p_r! := i!j!k!$ and $p_r!* := p_r!k$. (Note the extra k factor.) For $z = (z_0, z_1, z_2) \in \mathbb{C}^3$, $z^{p_r} := z_0^i z_1^j z_2^k$. $\mathcal{P}_r$ will denote the set of all such triples, i.e.*

$$\mathcal{P}_r = \{(i, j, k) |\ i + j + k = r\}.$$

Let $\mathcal{P} = \bigcup_{r=0}^\infty \mathcal{P}_r$.

There is an ordering which we will adopt in the rest of the article. We will write $p_r \leq p_{r'}$, if in the order of priority, $r \leq r'$, followed by $i \leq i'$, $j \leq j'$ and $k \leq k'$. H_{p_r} precedes before $H_{p_{r'}}$ if $p_r \leq p_{r'}$. H_{p_r} will be defined shortly.

Definition 2.2. Let $\mathcal{S}_\mathfrak{g}(\mathbb{R}^3) = \mathcal{S}(\mathbb{R}^3) \otimes \mathfrak{g}$. Define $\vec{\mathcal{S}}_\mathfrak{g}(\mathbb{R}^3) \cong \mathcal{S}_\mathfrak{g}(\mathbb{R}^3) \times \mathcal{S}_\mathfrak{g}(\mathbb{R}^3)$ to be the Schwartz space of all $\mathfrak{g} \times \mathfrak{g}$-valued functions over $\mathbb{R}^3$, and denote the extension of $\langle a^* a \cdot, \cdot \rangle_{\mathfrak{g},\kappa}$ to the direct sum as $\langle\langle \cdot, \cdot \rangle\rangle_1$, dropping the dependence on κ for ease of notation and let $\vec{H}^1$ be the Hilbert space using this inner product. The norm is denoted by $\| \cdot \|_1 := \sqrt{\langle\langle a^* a \cdot, \cdot \rangle\rangle_1}$. We will let H^1 denote the Hilbert space containing $\mathcal{S}_\mathfrak{g}(\mathbb{R}^3)$ using the inner product $\langle a^* a \cdot, \cdot \rangle_{\mathfrak{g},\kappa}$.

Remark 2.3. In order for $\langle\langle a^* a \cdot, \cdot \rangle\rangle_1$ to qualify to be an inner product on $\vec{H}^1$, we only consider polynomials which vanish at $x_2 = 0$.

Let $\{E_{ij}\}$ be an orthonormal basis for $\mathfrak{g} \times \mathfrak{g}$, using $-\mathrm{Tr}$ extended to the direct sum. Let $H_{p_r}(x) := h_i(x_0) h_j(x_1) h_k(x_2)$ be a product of Hermite polynomials and $\tilde{H}_{p_r}$ will denote $H_{p_r}/\sqrt{2^r p_r!^*}$. Here, $\tilde{H}_{p_r}$ is normalized. Consider the tensor product set

$$\bigcup_{r=0}^\infty \{\varsigma_1^{-1/2} \tilde{H}_{p_r} \sqrt{\phi_\kappa} : p_r \in \mathcal{P}_r\} \otimes \{E_{ij}\},$$

which forms an orthonormal basis for $\vec{H}^1$. Order this basis according to the ordering on p_r. Given $\vec{u} \in \vec{H}^1$, write

$$\vec{u} = \sum_{p_r} \sum_{ij} c_{p_r}^{ij} \varsigma_1^{-1/2} \tilde{H}_{p_r} \otimes E_{ij}$$

and
$$\langle\langle a^* a\vec{u}, \vec{u}\rangle\rangle_1 = \sum_{p_r}\sum_{ij} c_{p_r}^{ij,2},\ c_{p_r}^{ij,2} = (c_{p_r}^{ij})^2.$$

Definition 2.4.

1. Let $\mathcal{F}$ be a partial ordered set of finite dimensional orthogonal projections onto $\vec{H}^1$, i.e. $P > Q$ if $QH^1 \subseteq PH^1$. Similarly let $\mathcal{G}$ be a partial ordered set of finite dimensional orthogonal projections onto H^1.

2. Let $P \in \mathcal{G}$. Given any Borel subset $F \subseteq PH^1$, define for $\kappa > 0$,
$$\mu_\kappa\left(x \in P^{-1}(F)\right) = \left(\frac{\kappa}{2\pi}\right)^{l/2} \int_{y\in F} e^{-\kappa|y|^2/2} dy, \tag{2.6}$$
where l is the dimension of PH^1.

3. Let $P = P^+ \times P^- \in \mathcal{F}$. Let $F \in P\vec{H}^1$ be a measurable set, such that $F = F_+ \times F_-$, $F_\pm \subseteq PH^1$. Define for $\vec{\kappa}$,
$$\begin{aligned}\mu_{\vec{\kappa}}\left(x \in P^{-1}(F)\right) &= \mu_{\kappa_+}\left(x \in P^{-1}(F_+)\right) \times \mu_{\kappa_-}\left(x \in P^{-1}(F_-)\right)\\ &= \left(\frac{\kappa_+}{2\pi}\right)^{l_+/2}\left(\frac{\kappa_-}{2\pi}\right)^{l_-/2} \int_{y_+\in F_+, y_-\in F_-} e^{-(\kappa_+|\vec{y}_1|^2 + \kappa_-|\vec{y}_2|^2)/2} dy_+ dy_-,\end{aligned} \tag{2.7}$$
where $l_\pm$ is the dimension of $P^\pm H^1$.

4. Let $\mathcal{D}^\pm$ denote the Borel σ-algebra in $P^\pm H^1$, $P^\pm \in \mathcal{G}$. Let $P = P^+ \times P^-$ and $F \subseteq P\vec{H}^1$ be in the σ-algebra generated by $\mathcal{D}^+ \times \mathcal{D}^-$. Extend $\mu_{\vec{\kappa}}$ to be defined on the measurable set F.

5. A semi-norm $\|\cdot\|$ in $\vec{H}^1$ is called measurable if for every $\varepsilon > 0$, there exists a $P_0 \in \mathcal{F}$ such that
$$\mu_{\vec{\kappa}}(\| Px \| > \varepsilon) < \varepsilon$$
for all $P \perp P_0$ and $P \in \mathcal{F}$.

As explained in [2], this does not define a measure on $\vec{H}^1$. To define a Gaussian measure, one has to complete $\vec{H}^1$ into a Banach space B, using any measurable norm $\|\cdot\|$, defined as follows. We will use $Q : \varsigma_1^{-1/2}\tilde{H}_{p_r} \mapsto z^{p_r}/\sqrt{p_r!*}$. For any $\vec{u} = \sum_{p_r\in\mathcal{P}}\sum_{ij} c_{p_r}^{ij}\varsigma_1^{-1/2}\tilde{H}_{p_r} \otimes E_{ij} \in \vec{H}^1$, Q maps it to a $\mathfrak{g}_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}}$-valued function on $\mathbb{C}^3$ by replacing each $\varsigma_1^{-1/2}\tilde{H}_{p_r}$ by $z^{p_r}/\sqrt{p_r!*}$. Note that $z^{p_r} = z_0^i z_1^j z_2^k$, $p_r = (i,j,k)$. The sesquilinear complex inner product on $\mathfrak{g}_{\mathbb{C}}$-valued functions over $\mathbb{C}^3$ is given by
$$\begin{aligned}&\langle z^{p_r} \otimes E_{ij}, z^{p_{r'}} \otimes E_{i'j'}\rangle\\ &\quad = -\operatorname{Tr}[E_{ij}E_{i'j'}]\frac{1}{\pi^3}\int_{\mathbb{C}^3} z_2\partial_{z_2} z^{p_r} \cdot \overline{z^{p_{r'}}} e^{-|z_0^2| - |z_1|^2 - |z_2|^2} dx_0 dp_0 dx_1 dp_1 dx_2 dp_2.\end{aligned}$$

Note that $z_j = x_j + ip_j$ and $\overline{z_j}$ means complex conjugate. We will use the same symbol $\vec{H}^1$ to denote the Hilbert space of $\mathfrak{g}_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}}$-valued functions over $\mathbb{C}^3$.

Definition 2.5. (Measurable norm)
Let $\vec{u} = \sum_{ij}\sum_r\sum_{p_r} c^{ij}_{p_r} z^{p_r}/\sqrt{p_r!*}$. Introduce a norm by setting

$$\| \vec{u} \| = \sup_{ij} \sup_{z\in B(0,1/2)} \sum_r \sum_{p_r} |c^{ij}_{p_r}||z^{p_r}|.$$

$B(0,1/2)$ is the ball with radius $1/2$, center 0 in $\mathbb{C}^3$.

Proposition 2.6. *The norm $\|\cdot\|$ is measurable.*

Proof. Let $\varepsilon > 0$ and $\lambda = \wedge\{\kappa_+,\kappa_-\}$. Choose a N large such that $\sum_{n\geq N} n^2 2^{-n} < \lambda\varepsilon^2$ and let P_o be a finite dimensional orthogonal projection onto the span of $\{z^{p_r}\otimes E_{ij} : r \leq N\}_{ij}$. Let V be a finite dimensional subspace in the complement of the range of P_o and let $\{\beta_1,\ldots\beta_l\}$ be an orthonormal basis for V, using $\langle\langle\cdot,\cdot\rangle\rangle_1$. Let P denote the projection onto V. Now, there are at most n^2 triple (i,j,k) with $i+j+k=n$. It is possible to write each basis vector as

$$\beta_s = \sum_{p_r\geq q_s} a^{ij,s}_{p_r} z^{p_r}/\sqrt{p_r!*}\otimes E_{ij},$$

with $q_1 < q_2 < \ldots < q_l$ and each q_t is a triple (i,j,k) with $i+j+k\geq N$.

Any projected vector Px can be written as $Px = \sum_s c_s(x)\beta_s$. By definition (See Equation (2.7).), $c_s(\cdot) : \vec{H}^1 \to \mathbb{R}$ is a Gaussian random variable with variance $1/\kappa_\pm$. Let $\mathbb{E}$ denote the expectation of l independent standard Gaussian random variables. Then

$$\begin{aligned}\mu_{\vec{\kappa}}(\| Px \| > \varepsilon) =& \mu_{\vec{\kappa}}\left(\sup_{ij}\sup_{z\in B(0,1/2)}\sum_s\sum_{p_r\geq q_s} |c_s a^{ij,s}_{p_r}||z^{p_r}| > \varepsilon\right)\\ \leq& \frac{1}{\varepsilon}\sup_{ij}\sup_{z\in B(0,1/2)}\sum_s\sum_{p_r\geq q_s}\mathbb{E}|c_s a^{ij,s}_{p_r}||z^{p_r}| \leq \frac{1}{\varepsilon}\sup_{z\in B(0,1/2)}\sum_s \mathbb{E}|c_s|^2 \sum_{p_r\geq q_s}|z^{p_r}|\\ \leq& \frac{1}{\lambda\varepsilon}\sum_{n\geq N} n^2 2^{-n} < \varepsilon.\end{aligned}$$

□

Complete $\vec{H}^1$ into a Banach space $\vec{\mathcal{B}}^1$, and forms a triple, an abstract Wiener space in the sense of Gross. Thus, $(i,\vec{H}^1,\vec{\mathcal{B}}^1)$ forms an abstract Wiener space. Identify $\vec{y}\in\vec{\mathcal{B}}^{1,*}\subseteq \vec{H}^1 \subseteq \vec{\mathcal{B}}^1$ with an element in $\vec{H}^1$ and denote the pairing $((\vec{u},\vec{y}))_1 = \vec{y}(\vec{u})$. Here, $\vec{y}\in\vec{\mathcal{B}}^{1,*}$.

Definition 2.7. (Gaussian measure on Abstract Wiener Space)

1. Define $\tilde{\mu}_{\vec{\kappa}}$, a measure on $\vec{\mathcal{B}}^1$ with covariance $\vec{\kappa}$, by

$$\tilde{\mu}_{\vec{\kappa}}\left\{\vec{u}\in\vec{\mathcal{B}}^1 : \left(((\vec{u},\vec{y}_1))_1,\ldots,((\vec{u},\vec{y}_n))_1\right)\in F\right\} = \mu_{\vec{\kappa}}\left\{\vec{u}\in\vec{H}^1 : \left(\langle\langle\vec{u},\vec{y}_1\rangle\rangle_1,\ldots,\langle\langle\vec{u},\vec{y}_n\rangle\rangle_1\right)\in F\right\}.$$

   The $\vec{y}_j$'s are in $\vec{\mathcal{B}}^{1,*}$.

2. $\left\{\vec{u}\in\vec{\mathcal{B}}^1 : \left(((\vec{u},\vec{y}_1))_1,\ldots,((\vec{u},\vec{y}_n))_1\right)\in F\right\}$ is called a cylinder set in $\vec{\mathcal{B}}^1$. Let $\mathcal{R}_{\vec{\mathcal{B}}^1}$ be the collection of cylinder sets in $\vec{\mathcal{B}}^1$.

Remark 2.8. 1. It was shown by Gross that $\tilde{\mu}_{\vec{\kappa}}$ is σ-additive in the σ-field generated by $\mathcal{R}_{\vec{\mathcal{B}}^1}$.

2. Extend $\tilde{\mu}_{\vec{\kappa}}$ over the Borel field of $\vec{\mathcal{B}}^1$.

3. It can be shown that the σ-field generated by $\mathcal{R}_{\vec{\mathcal{B}}^1}$ is equal to the Borel field of $\vec{\mathcal{B}}^1$.

Now, any $\vec{u} \in \vec{\mathcal{B}}^1$ can be written as $\vec{u} = \sum_{ij}\sum_{p_r} c^{ij}_{p_r} z^{p_r}/\sqrt{p_r!*} \otimes E_{ij}$, convergence in the sense of $\|\cdot\|$. The space $\vec{\mathcal{B}}^1$ can be described explicitly. Let $\vec{H}^1_{\mathbb{C}} = \vec{H}^1 \otimes_{\mathbb{R}} \mathbb{C}$ and $\vec{\mathcal{B}}^{1,*}_{\mathbb{C}} = \vec{\mathcal{B}}^{1,*} \otimes_{\mathbb{R}} \mathbb{C}$.

Proposition 2.9. *For $w \in \mathbb{C}^3$, define $\chi(w) : \vec{u} \in \vec{\mathcal{B}}^1 \mapsto \vec{u}(w)$. Then $\chi(w)$ is in $\vec{\mathcal{B}}^{1,*}_{\mathbb{C}}$.*

Proof. Let $R \geq 2|w|$. Choose $M > 0$ such that for all $r > M$, $R^r \leq \sqrt{\lfloor (r/3) \rfloor !}$. Then,

$$\begin{aligned}
|\vec{u}(w)| \leq \sup_{ij}\left|\sum_{p_r} c^{ij}_{p_r} w^{p_r}/\sqrt{p_r!*}\right| \leq & R^M \sup_{ij}\sum_{r\leq M}\sum_{p_r}|c^{ij}_{p_r}|\left|\left(\frac{w}{R}\right)^{p_r}\right|\frac{1}{\sqrt{p_r!*}} \\
& + \sum_{r>M}\sup_{ij}\sum_{p_r}|c^{ij}_{p_r}|\left|\left(\frac{w}{R}\right)^{p_r}\right|\frac{R^r}{\sqrt{p_r!*}} \\
& < (R^M+1)\,\|\,\vec{u}\,\|\,. \qquad (2.8)
\end{aligned}$$

This shows that $\chi(w) : \vec{u} \to \vec{u}(w)$ is a bounded complex functional on $\vec{\mathcal{B}}^1$. □

Proposition 2.10. *The support of $\tilde{\mu}_{\vec{\kappa}}$ is on continuous $\mathfrak{g}_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}}$-valued functions on $\mathbb{C}^3$.*

Proof. Fix a $w_0 = (x_0, x_1, x_2)$. Now for any w, $\vec{u}(w_0 + w) - \vec{u}(w_0) = (\vec{u}, \chi(w_0 + w) - \chi(w_0))$ for $\vec{u} \in \vec{\mathcal{B}}^1$. From Equation (2.8), it is clear that $\chi(\cdot)$ is continuous at 0 using the operator norm.

Fix a $z_0 \neq 0$ and let $|z_0| \leq R$. Let $D = \{z \in \mathbb{C} : |z - z_0| < R/4\} \subseteq \mathbb{C}$ and γ be the boundary of D. By Cauchy integral formula,

$$z_0^n = \int_\gamma \frac{z^n}{z - z_0} dz.$$

Thus, for z_1 close to z_0,

$$|z_1^n - z_0^n| = \left|\int_\gamma \frac{z^n(z_1 - z_0)}{(z - z_1)(z - z_0)} dz\right| \leq 2\pi(2R)(2R+1)^n|z_1 - z_0|.$$

Thus, if write $p_r = (n_0, n_1, n_2)$ and $w + w_0 = (y_0, y_1, y_2)$, then

$$\begin{aligned}
\left|(w+w_0)^{p_r} - w_0^{p_r}\right| &= \left|(x_0^{n_0} - y_0^{n_0})x_1^{n_1}x_2^{n_2} + y_0^{n_0}(x_1^{n_1} - y_1^{n_1}) + y_0^{n_0}y_1^{n_1}(x_2^{n_2} - y_2^{n_2})\right| \\
&\leq 3(|w+w_0| + |w_0|)2^r(|w+w_0| + |w_0|)^r|w + w_0 - w_0|. \qquad (2.9)
\end{aligned}$$

Let $\vec{u} = \sum_{ij}\sum_{p_r}|c^{ij}_{p_r}|w^{p_r}/\sqrt{p_r!*} \otimes E_{ij}$. From Equation (2.9), choose $R > 0$ such that

$$|(w+w_0)^{p_r} - w_0^{p_r}| \leq R^r|w|$$

for all $w \in B(w_0, 2|w_0|)$. Then, if $\varepsilon < |w_0|$,

$$\begin{aligned}\sup_{w\in B(0,\varepsilon)} |\vec{u}(w+w_0) - \vec{u}(w_0)| \leq & \sup_{w\in B(0,\varepsilon)} N^2 \sup_{ij} \sum_{p_r} |c^{ij}_{p_r}| \frac{|(w+w_0)^{p_r} - w_0^{p_r}|}{\sqrt{p_r!*}} \\ \leq & \sup_{w\in B(0,\varepsilon)} N^2 \sup_{ij} \sum_{p_r} \frac{1}{2^r} |c^{ij}_{p_r}| \frac{2^r R^r |w|}{\sqrt{p_r!*}} \\ \leq & c(w_0) \cdot \varepsilon \parallel \vec{u} \parallel\end{aligned}$$

for some constant $c(w_0)$.

Thus for any $\varepsilon > 0$,

$$\begin{aligned}\tilde{\mu}_{\vec{\kappa}}\left(\sup_{w\in B(w_0,1/k)} |\vec{u}(w_0+w) - \vec{u}(w_0)| > \varepsilon \right) &= \tilde{\mu}_{\vec{\kappa}}\left(\sup_{w\in B(w_0,1/k)} |(\vec{u}, \chi(w_0+w) - \chi(w_0))| > \varepsilon \right) \\ &\leq \tilde{\mu}_{\vec{\kappa}}\left(\frac{c(w_0)}{k} \parallel \vec{u} \parallel > \varepsilon \right) \longrightarrow 0\end{aligned}$$

as k goes to infinity. Let

$$E_k = \{\vec{u} : \sup_{w\in B(w_0,1/k)} |(\vec{u}, \chi(w_0+w) - \chi(w_0))| > \varepsilon\}.$$

Choose an increasing subsequence $\{r_k\}_{k=1}^{\infty}$ in $\mathbb{N}$ such that $\sum_k \tilde{\mu}(E_{r_k}) < \infty$. Then, by the choice of r_k, $\sum_k \tilde{\mu}(E_{r_k}) < \infty$ and hence by the Borel Cantelli Lemma,

$$\bigcap_{q=1}^{\infty} \bigcup_{p=q}^{\infty} E_{r_p}$$

has probability 0. Hence with probability 1, for each $\vec{u}$, there exists a $r_k(\vec{u})$ such that

$$\sup_{w\in B(w_0,1/r_k)} |\vec{u}(w_0+w) - \vec{u}(w_0)| < \varepsilon.$$

□

Any $\vec{u} \in \vec{\mathcal{B}}^1$ in the support of $\tilde{\mu}_{\vec{\kappa}}$ is continuous. Since it is actually given by a power series, by Morera's Theorem, it is analytic in each of its variable. Hence, the support of $\tilde{\mu}_{\vec{\kappa}}$ in the Banach space $\vec{\mathcal{B}}^1$ is the space of entire ($(\mathfrak{g}_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}})$-valued) functions on $\mathbb{C}^3$.

Now, the problem is to interpret F on $\vec{\mathcal{B}}^1_{\kappa_+} \times \vec{\mathcal{B}}^1_{\kappa_-}$. Recall $Q : \varsigma_1^{-1/2} \tilde{H}_{p_r} \mapsto z^{p_r}/\sqrt{p_r!*}$. Under this map Q,

$$a \mapsto \partial_{z_2},\ a^* \mapsto z_2,\ \tau \mapsto \partial_{z_2}^2 - 2(z_2 + \partial_{z_2})^2 + z_2\partial_{z_2}$$

and

$$L \mapsto \partial_{z_2}^2 - \frac{(\partial_{z_2} + z_2)^2}{2}.$$

These operations make sense on holomorphic functions. The problem is that the holomorphic functions are not integrable with respect to Gaussian measure.

Write

$$\mathscr{H}(p,q) := (\| p+q \|^2 - \| p \|^2 - \| q \|^2)/2.$$

Then F is extended to $\tilde{F}$,

$$\tilde{F}(p) := \mathscr{H}(2\tau(p_\alpha + p_\beta), p_\beta - p_\alpha) - \mathscr{H}(p_\alpha, p_\beta) + \mathscr{H}(Lp_\alpha, p_\alpha) - \mathscr{H}(Lp_\beta, p_\beta),\ p = (p_\alpha, p_\beta).$$

Given a continuous function $G : \mathscr{A} \rightarrow \mathbb{R}$, suppose it has a continuous extension $\tilde{G} : \vec{\mathscr{B}}^1 \rightarrow \mathbb{R}$.

Definition 2.11. (First definition for Chern-Simons path integral)
The CS path integral, is now interpreted as

$$\frac{1}{Z_{\rm CS}} \int_{A \in \mathscr{A}} G(A) e^{i{\rm CS(A)}} DA = \lim_{\kappa_+ \rightarrow i,\ \kappa_- \rightarrow -i} \frac{1}{Z_{\vec{\kappa}}} \int_{\vec{u} \in \vec{\mathscr{B}}^1} \tilde{G}(\vec{u}) e^{-i\tilde{F}(\vec{u})} d\tilde{\mu}_{\vec{\kappa}}(\vec{u}). \tag{2.10}$$

$Z_{\vec{\kappa}}$ is defined as

$$Z_{\vec{\kappa}} := \lim_{\kappa_+ \rightarrow i,\ \kappa_- \rightarrow -i} \int_{\vec{u} \in \vec{\mathscr{B}}^1} e^{-i\tilde{F}(\vec{u})} d\tilde{\mu}_{\vec{\kappa}}(\vec{u})$$

Remark 2.12. It is not at all clear that such an analytic continuation exists in this definition. We will not address this issue, as in the rest of this article, we will not use this definition at all. Instead, we will give an alternative definition, based on the approach used in [1].

3 Wilson Loop Observables

Note that for the rest of this article, $\kappa > 0$ is allowed to vary.

Let G be a Lie group. Let $\{\rho_k\}$ be any set of finite dimensional representations of G. The interest in Chern-Simons path integrals is the evaluation of

$$Z(\mathbb{R}^3, C^i, \rho_i; q) := \frac{1}{Z_{\rm CS}} \int_{A \in \mathscr{A}} \prod_{k=1}^{l} W(C^k, \rho_k; q) e^{i{\rm CS}(A)} DA, \tag{3.1}$$

where $L = \{C^k\}_k$ is a link in $\mathbb{R}^3$ with non-intersecting (closed) curves C^k and

$$W(C^k, \rho_k; q)(A) := {\rm Tr}_{\rho_k} \mathcal{T} \exp\left[q \int_{C^k} A_i dx^i \right]. \tag{3.2}$$

Here, ${\rm Tr}_{\rho_k}$ is the matrix trace in the representation ρ_k and $\mathcal{T}$ is the time ordering operator. $W(C^k, \rho_k; q)(A)$ is the holonomy operator of A, computed along the loop C^k. The integral in Equation (3.1) will be known as the Wilson Loop observable (associated to the link L). q will be called the charge of the link. When L consists of only one curve, the link is termed a knot. We will write $Z(\mathbb{R}^3, L; q) \equiv Z(\mathbb{R}^3, C^i, \rho; q)$, when $\rho_k = \rho$ for some representation ρ.

When L is empty, then $Z(\mathbb{R}^3, \emptyset; q) = 1$. The completion of $\vec{H}^1$ is the space of 2-tuple entire ($\mathfrak{g}_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}}$-valued) functions on $\mathbb{C}^3$, which defines a space of 2-tuple $C^\infty(\mathbb{R}^3)$ ($\mathfrak{g}_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}}$-valued) functions, i.e.

$$A \in \vec{\mathscr{B}}^1 \mapsto A|_{\mathbb{R}^3} \in (C^\infty(\mathbb{R}^3) \otimes \mathfrak{g}_{\mathbb{C}}) \times (C^\infty(\mathbb{R}^3) \otimes \mathfrak{g}_{\mathbb{C}}).$$

Using the canonical embedding of $\mathbb{R}^3$ into $\mathbb{C}^3$, we could use Equation (3.2) to define a function on $\vec{\mathcal{B}}^1$ and this would give rigorous meaning to Equation (3.1).

There is a second approach to define $Z_{\rm CS}$, which was used in both [1] and [3]. Continue the same setting as from the previous section. From Equation (2.4), an integration by parts gives,

$$-\varsigma_0 \int_{\mathbb{R}^3} \mathrm{Tr}\ [a_0 \cdot \partial_2 a_1] d\mathrm{vol}_{\mathbb{R}^3},\ \varsigma_0 := \frac{\kappa}{2\pi}. \tag{3.3}$$

Define $\langle A,B\rangle_0 := -\varsigma_0 \int \mathrm{Tr}[AB] d\mathrm{vol}_{\mathbb{R}^3}$ and $(H^0, \langle\cdot,\cdot\rangle) \equiv H^0 := L^2_{\mathfrak{g}}(\mathbb{R}^3, \mathrm{vol}_{\mathbb{R}^3})$, the Hilbert space of $\mathfrak{g}$-valued, integrable functions. The Schwartz space $\mathcal{S}(\mathbb{R}^3)\otimes\mathfrak{g}$ is dense inside H^0 and thus $\{\varsigma_0^{-1/2}\hat{H}_{p_r}\sqrt{\phi_\kappa}\}_{p_r\in\mathcal{P}} \otimes \{E_{ij}\}$ forms an orthonormal basis.

Remark 3.1. Let $H_{p_r}/\sqrt{p_r!}$ be the normalized Hermite polynomials with respect to the Gaussian measure $e^{-(|x_0|^2+|x_1|^2+|x_2|^2)/2}/(2\pi)^{3/2}dx_0dx_1dx_2$. Thus,

$$\varsigma_0^{-1/2}\hat{H}_{p_r}(x_0,x_1,x_2)\sqrt{\phi_\kappa} = \varsigma_0^{-1/2}H_{p_r}(\kappa x_0,\kappa x_1,\kappa x_2)\sqrt{\phi_\kappa}/\sqrt{p_r!}.$$

Making use of Equation (2.3), let $f_1 = \partial_2 a_1$. If $\vec{u} = (a_0, f_1)$, then the Chern-Simons path integral is defined as

$$\lim_{\theta\to i}\int_{\vec{u}\in\vec{\mathcal{S}}_{\mathfrak{g}}(\mathbb{R}^3)} e^{i\langle\vec{u}\rangle_\sharp + \frac{i}{2}\langle\langle\vec{u},\vec{u}\rangle\rangle_0 - \frac{\theta}{2}\langle\langle\vec{u},\vec{u}\rangle\rangle_0} Da_0 Df_1. \tag{3.4}$$

Define $\vec{H}^0$, the completion of $\vec{\mathcal{S}}_{\mathfrak{g}}(\mathbb{R}^3)$ using $\langle\langle\cdot,\cdot\rangle\rangle_0$, using the same symbol to denote the space of $\mathfrak{g}_{\mathbb{C}}\times\mathfrak{g}_{\mathbb{C}}$-valued functions over $\mathbb{C}^3$. Define a Gaussian measure μ_θ on $P\vec{H}^0$ for any $P\in\mathcal{F}$ as in Equation (2.7). Define $\|\cdot\|_0 := \sqrt{\langle\langle\cdot,\cdot\rangle\rangle_0}$ and complete the space into $\vec{\mathcal{B}}^0$ using a measurable norm as before and define a Gaussian measure $\tilde{\mu}_\theta$ on $\vec{\mathcal{B}}^0$. Denote the paring $((\vec{y},\vec{u}))_0 := \vec{y}(\vec{u}) \in \mathbb{R}$ for $\vec{y}\in\vec{\mathcal{B}}^{0,*}$.

To make sense of $\langle\vec{u}\rangle_\sharp + \langle\langle\vec{u},\vec{u}\rangle\rangle_0/2$ on $\vec{\mathcal{B}}^0$, one can define it using polarization and hence define a measure on $\vec{\mathcal{B}}^0$. However, in this case, we will use a different approach.

Definition 3.2. Let $\theta > 0$.

1. Suppose $F : (0,\varepsilon) \to \mathbb{C}$. Do an analytic continuation on F and write $\tilde{F} : U \subseteq \mathbb{C} \to \mathbb{C}$ for some open and connected set U. Suppose $i \in U$. We will write

$$\lim_{\theta\to i} F(\theta) = \tilde{F}(i).$$

2. Let $P\in\mathcal{F}$, the set of finite dimensional orthogonal projections and let $\{\vec{e}_{+,k},\vec{e}_{-,k}\}_k$ be an orthonormal basis in $P^\sharp\vec{H}^0$. Let $y_k^\pm = \langle\langle\cdot,\vec{e}_{\pm,k}\rangle\rangle_0$ be the coordinates with respect to $\vec{e}_{\pm,k}$. Let the dimension of $P^\sharp\vec{H}^0$ be $2l$.

3. For each k, $\vec{y}_k = (y_k^+, y_k^-) \in \mathbb{R}^2$. Let $|\cdot|$ be the Euclidean distance norm in $\mathbb{R}^2$. Define for each $P\in\mathcal{F}$,

$$Z(P^\sharp,\theta) := \left(\frac{\theta}{2\pi}\right)^l \int_{\mathbb{R}^{2l}} e^{i\sum_{k=1}^l \langle\vec{y}_k\rangle_\sharp + \frac{1}{2}|\vec{y}_k|^2} e^{-\frac{\theta}{2}\sum_{k=1}^l |\vec{y}_k|^2} \prod_{k=1}^l dy_k^+ dy_k^-$$

and $Z(P^\sharp, i) = \lim_{\theta\to i} F(\theta)$.

4. For any continuous cylinder function $G \in C(P\vec{H}^0 \to \mathbb{R})$, define

$$G(\{\vec{y}_k\}_{k=1}^l) := G\left(\sum_{k=1}^l y_k^+ \vec{e}_{+,k} + y_k^- \vec{e}_{-,k}\right).$$

5. For any continuous cylinder function $G \in C(P\vec{H}^0 \to \mathbb{R})$, let

$$Z(G, P^\sharp, \theta)$$
$$:= \frac{1}{Z(P^\sharp, \theta)} \left(\frac{\theta}{2\pi}\right)^l \int_{\mathbb{R}^{2l}} G(\{\vec{y}_k\}_{k=1}^l) e^{i\sum_{k=1}^l \langle \vec{y}_k \rangle_\sharp + \frac{1}{2}|\vec{y}_k|^2} e^{-\frac{\theta}{2}\sum_{k=1}^l |\vec{y}_k|^2} \prod_{k=1}^l dy_k^+ dy_k^-. \quad (3.5)$$

Proposition 3.3. *Let $P^\sharp, Q^\sharp \in \mathcal{F}$ be orthogonal.*

1. $Z(P^\sharp + Q^\sharp, \theta) = Z(P^\sharp, \theta) Z(Q^\sharp, \theta)$.
2. *For each $\theta > 0$, $Z(P^\sharp, \theta) = (1/\sqrt{1 - 2i/\theta})^l$. l was defined in Item 2 in Definition 3.2.*
3. *For each $\theta > 0$, there exists a complex measure ν_θ such that $\int_{\vec{u} \in \vec{\mathcal{B}}^0} G(\vec{u}) d\nu_\theta(\vec{u})$ is given by Equation (3.5) for any cylinder function. Furthermore, $|\nu_\theta|$ is a probability measure on $\vec{\mathcal{B}}^0$.*
4. *Let $\vec{s} = (s^+, s^-) \in \mathbb{R}^2$ and $\vec{u} = (u^+, u^-) \in \vec{\mathcal{B}}^{0,*} \subseteq \vec{H}^0 \subseteq \vec{\mathcal{B}}^0$. Define a product $\vec{s} \times \vec{u} := (s^+u^+, s^-u^-)$. The Fourier transform of ν_θ is given by*

$$\begin{aligned} \mathfrak{F}\nu_\theta(s^+u^+, s^-u^-) &= \int_{\vec{\mathcal{B}}^0} e^{i((\cdot, \vec{s} \times \vec{u}))_0} d\nu_\theta \\ &= \exp\left(\frac{-i(s^+u^+ + s^-u^-)^2/2\theta^2}{1 - (2i/\theta)}\right) e^{-\frac{1}{2\theta}(s^{+,2}u^{+,2} + s^{-,2}u^{-,2})}. \end{aligned} \quad (3.6)$$

The RHS can be analytically extended to over $\mathbb{C}/\{0, 2i\}$ and thus

$$\lim_{\theta \to i} \mathfrak{F}\nu_\theta(\vec{s} \times \vec{u}) = e^{-i\langle \vec{s} \times \vec{u} \rangle_\sharp}. \quad (3.7)$$

Remark 3.4. 1. If one is interested in only the moment generating function, then

$$\int_{\vec{\mathcal{B}}^0} e^{((\cdot, \vec{s} \times \vec{u}))_0} d\nu_\theta = \exp\left(\frac{i(s^+u^+ + s^-u^-)^2/2\theta^2}{1 - 2i/\theta}\right) e^{\frac{1}{2\theta}(s^{+,2}u^{+,2} + s^{-,2}u^{-,2})}$$

and

$$\lim_{\theta \to i} \mathfrak{F}\nu_\theta(\vec{s} \times \vec{u}) = e^{i\langle \vec{s} \times \vec{u} \rangle_\sharp}. \quad (3.8)$$

2. Compare the result of [4] in Proposition 3.3 with Equation (2.5.3) in [1], it appears we are off by a minus sign. However, note that the L^2 inner product used in that article is negative of the one used here.

Proof. 1. It follows from $\langle \vec{u} + \vec{v} \rangle_\sharp = \langle \vec{u} \rangle_\sharp + \langle \vec{v} \rangle_\sharp$, $\vec{u} \in P^\sharp$ and $\vec{v} \in Q^\sharp$.

2. The integral

$$z := \frac{\theta}{2\pi}\int_{\mathbb{R}^2} e^{ixy+i(x^2+y^2)/2}e^{-\theta(x^2+y^2)/2}dxdy = \mathbb{E}[e^{iZ^2/\theta}],\ Z \sim N(0,1).$$

But $N^2 \sim \chi_1^2$, and the characteristic function $\mathbb{E}[e^{itN^2}] = 1/\sqrt{1-2it}$. Thus, $z = 1/\sqrt{1-2i/\theta}$ and hence the result follows from [1].

3. Let $\Gamma : G \in L^2(\vec{\mathcal{B}}^0, \tilde{\mu}_\theta) \rightarrow Z(G, P^\sharp, \theta) \in \mathbb{R}$ for any bounded continuous cylinder function G. [1] says that for any cylinder function G, $Z(G, P^\sharp, \theta)$ is well defined for any $P\vec{H}^0$ such that G is defined on. Since cylinder functions are dense, this linear functional Γ can be extended to all of $L^2(\vec{\mathcal{B}}^0, \tilde{\mu}_\theta)$. From Equation (3.5), Γ has operator norm 1, using the L^2 norm on $L^2(\vec{\mathcal{B}}^0, \tilde{\mu}_\theta)$. By Riesz representation theorem, there exists a complex valued function γ_θ, $|\gamma_\theta| = 1$ such that $\Gamma(G) = \int_{\vec{\mathcal{B}}^0} G\gamma_\theta d\tilde{\mu}_\theta$.

4. Set $s^+ = s^- = 1$. Note that $e^{i\langle\langle\cdot,\vec{u}\rangle\rangle_0}$ is a cylinder function on $P^\sharp\vec{H}^0$, where P is projection onto $\vec{u}$. First assume that $P^\sharp = P$, i.e. $\vec{u} = (u^+e, u^-e)$, $u^+, u^- \in \mathbb{R}$ and e is a unit vector in H^0. Then, a straight forward computation gives

$$\int_{\vec{\mathcal{B}}^0} e^{i\langle\langle\cdot,\vec{u}\rangle\rangle_0}d\nu_\theta = \sqrt{1-2i/\theta}\,\mathbb{E}\left[e^{\frac{i}{\theta}\left(N+\frac{iu^+}{\sqrt{2\theta}}+\frac{iu^-}{\sqrt{2\theta}}\right)^2}\right]e^{-\left(\frac{1}{2\theta}u^{+,2}+\frac{1}{2\theta}u^{-,2}\right)},$$

N is the standard normal distribution. The quantity $Q = (N+\lambda)^2$ is a non-central χ^2 distribution and the characteristic function for Q is given by

$$\frac{\exp\left(\frac{i\lambda^2 t}{1-2it}\right)}{\sqrt{1-2it}}.$$

Plug in $\lambda = i(u^+ + u^-)/\sqrt{2\theta}$, $t = 1/\theta$,

$$\int_{\vec{\mathcal{B}}^0} e^{i\langle\langle\cdot,\vec{y}\rangle\rangle_0}d\nu_\theta = \exp\left(\frac{-i(u^+ + u^-)^2/2\theta^2}{1-2i/\theta}\right)e^{-\left(\frac{1}{2\theta}u^{+,2}+\frac{1}{2\theta}u^{-,2}\right)}.$$

Observe that $u^+u^- = \langle\vec{u}\rangle_\sharp$. Take the limit as θ go to i in the sense of Definition 3.2, we have the result.

Now assume that $\vec{u} = (u^1p, u^2p + tq)$, with p, q orthonormal vectors in H^0 and $u^\pm, t \in \mathbb{R}$. Associate $+$ with 1 and $-$ with 2 for convenience. $P^\sharp$ is now spanned by $\{p_j, q_j\}_{j=1,2}$, $p_1, p_2 \equiv p$ and $q_1, q_2 \equiv q$. Then $\vec{u} = u^1p_1 + u^2p_2 + tq_2$.

For any arbitrary vector $\sum_j \alpha_j p_j + \beta_j q_j = y \in P^\sharp$, $\langle\langle y, \vec{u}\rangle\rangle_0 = u^1\alpha_1 + u^2\alpha_2 + t\beta_2$. If $\vec{w} = (u^1, u^2)$, $\vec{\alpha} = (\alpha_1, \alpha_2)$ and $\vec{\beta} = (\beta_1, \beta_2)$, then the integral becomes

$$(1-2i/\theta)\left(\frac{\theta}{2\pi}\right)^2\int_{\mathbb{R}^2} e^{i(\vec{w}\cdot\vec{\alpha}+\langle\vec{\alpha}\rangle_\sharp+|\vec{\alpha}|^2/2)}e^{-\theta|\vec{\alpha}|^2/2}d\alpha_1 d\alpha_2\cdot\int_{\mathbb{R}^2} e^{i(t\beta_2+\langle\vec{\beta}\rangle_\sharp+|\vec{\beta}|^2/2)}e^{-\theta|\vec{\beta}|^2/2}d\beta_1 d\beta_2.$$

Note that $\langle\vec{u}\rangle_\sharp = u^1u^2$. Apply the calculations above, we get the result.

□

Definition 3.5. (Second definition for Chern-Simons path integral)
Suppose a bounded continuous function $G \in C(\vec{H}^0)$ can be extended to $\tilde{G} \in L^1(\vec{\mathcal{B}}^0, \tilde{\mu}_\theta)$. Define the Chern-Simons path integral as,

$$\frac{1}{Z_{\rm CS}} \int_{A \in \mathscr{A}} G(A) e^{i{\rm CS(A)}} DA := \lim_{\theta \to i} \int_{\vec{\mathcal{B}}^0} \tilde{G} d\nu_\theta. \tag{3.9}$$

The existence of ν_θ was proved in Proposition 3.3.

From Remark 3.4, we note that such an analytic continuation exists, at least for $\tilde{G}$ of the form $e^{i((\cdot,\vec{u}))_0}$, $\vec{u} \in \vec{\mathcal{B}}^{0,*} \subseteq \vec{H}^0 \subseteq \vec{\mathcal{B}}^0$. The Chern-Simons path integral is then interpreted as a linear functional acting on (a possibly subspace of) $L^1(\vec{\mathcal{B}}^0, \tilde{\nu}_\theta)$.

3.1 Some Important Comments

Definition 3.5 of the Chern-Simons path integral, is not exact. Write $\partial_2 = a - a^*$. In terms of Hermite polynomials, $\partial_2 : h_k \mapsto kh_{k-1} - h_{k+1}$. If one does a change of variables, $\partial_2 : a_1 \mapsto f_1 = \partial_2 a_1$, then Expression (3.4) should become

$$\frac{1}{\det \partial_2} \int_{\vec{u} \in \vec{S}_{\mathfrak{g}}(\mathbb{R}^3)} e^{i\langle \vec{u} \rangle_\sharp + \frac{i}{2}\langle\langle \vec{u}, \vec{u} \rangle\rangle_0 - \frac{\theta}{2}\langle\langle \vec{u}, \vec{u} \rangle\rangle_0} Da_0 Df_1. \tag{3.10}$$

Of course, $\det \partial_2$ does not make sense. It is either 0 or ∞.

Since we are only interested in defining a probability measure, one can just divide out this indeterminate. But the problem is that there is no way to rigorously justify this division. The bottom line is that in the second definition of the path integral is far from exact.

Because of this change of variables formula, the definition of the path integral should be

$$\frac{1}{Z_{\rm CS}} \int_{A \in \mathscr{A}} G(A) e^{i{\rm CS(A)}} DA := \lim_{\theta \to i} \int_{\vec{a} \in \vec{\mathcal{B}}^0} \tilde{G}((a_0, \partial_2^{-1} a_1)) d\nu_\theta(\vec{a}). \tag{3.11}$$

and the Fourier transform should yield $\exp[i\langle (u_0, \partial_2^{-1} u_1) \rangle_\sharp]$. (The adjoint of ∂_2^{-1} is $-\partial_2^{-1}$.) Unfortunately, given $u_1 \in H^0$, $\partial_2^{-1} u_1$ is not in H^0.

This change is necessary to get the right knot invariants later on. Actually, one is not interested in the measure. What one is really after is the Wilson Loop observable. As will be shown later, it is simply a Fourier transform. But Equation (3.6) is not well defined if we make this change.

Fortunately, Equations (3.7) and (3.8) are well defined if $\langle (u_0, \partial_2^{-1} u_1) \rangle_\sharp \equiv \langle u_0, \partial_2^{-1} u_1 \rangle$ is defined, even though $\partial_2^{-1} u_1$ is not in H^0. Let us be more precise about this statement. We mean that

$$\langle u_0 \otimes A, \partial_2^{-1} u_1 \otimes B \rangle := -{\rm Tr}[AB] \int_{\mathbb{C}^3} (u_0 \cdot \overline{\partial_2^{-1} u_1})_z e^{-|z|^2} \prod_{i=0}^{2} \frac{dx_i dp_i}{\pi} < \infty.$$

Here, $z = (z_0, z_1, z_2)$ and $z_j = x_j + ip_j$.

Therefore, we will make the following definition:

$$\mathbb{E}^{CS}\left[e^{((\cdot, \vec{s} \times \vec{u}))_0} \right] := e^{-i\langle s^+ u_0, s^- \partial_2^{-1} u_1 \rangle}, \tag{3.12}$$

for any $(u_0,u_1) \in \vec{\mathscr{B}}^{0,*}$ with $\langle u_0, \partial_2^{-1} u_1 \rangle < \infty$.

A word of caution: On the space of Hermite polynomials, $\partial_2 = a - a^*$ and $\partial_2^{-1} a_1 := \int_{-\infty}^{\cdot} a_1(x_2)dx_2 - \int_{\cdot}^{\infty} a_1(x_2)dx_2$. However, on the space of holomorphic functions, $\partial_2 \mapsto \partial_{z_2} - z_2$. The inverse of the operator is $(\partial_{z_2} - z_2)^{-1}$.

If $(u_0,u_1) \in \vec{B}^{0,*} \subseteq \vec{H}^0$, then $u_1 \in H^0$. Under the map Q^{-1}, $a_1 = Q^{-1}u_1$ is in $L^2(\mathbb{R}^3 \rightarrow \mathfrak{g})$. If a_1 is in $L^2(\mathbb{R}^3 \rightarrow \mathfrak{g})$, then

$$\partial_2^{-1} a_1 = \int_{-\infty}^{\cdot} a_1(x_2)dx_2 - \int_{\cdot}^{\infty} a_1(x_2)dx_2$$

is bounded, but unfortunately it is not L^2 integrable. But,

$$\langle a_0, \partial_2^{-1} a_1 \rangle = \int_{\mathbb{R}^3} a_0 \cdot \partial_2^{-1} a_1 d\mathrm{vol}_{\mathbb{R}^3} < \infty,$$

if a_0 is L^1 integrable.

So far, we considered the Lie algebra $\mathfrak{g}$ as a Lie subalgebra in $\mathfrak{su}(N)$. One can view it as a representation of $\mathfrak{g}$ in $\mathfrak{u}(N)$. In fact, our method can be generalized to any semisimple Lie algebra, if there is a representation $\hat{\rho}$ of $\mathfrak{g}$ such that the bilinear form $\langle A,B \rangle_{\hat{\rho}} := -\mathrm{Tr}[\hat{\rho}(A)\hat{\rho}(B)]$, Tr is the usual matrix trace, is positive and non-degenerate. In this representation, $\langle A,B \rangle := -\varsigma_0 \int \mathrm{Tr}[\hat{\rho}(A)\hat{\rho}(B)] d\mathrm{vol}_{\mathbb{R}^3}$, for $A,B : \mathbb{R}^3 \rightarrow \mathfrak{g}$.

Definition 3.6. (Third definition for Chern-Simons path integral)
Let G be any complex (gauge) Lie group for a trivial bundle E. Let $\hat{\rho}$ be any finite-dimensional representation of G, such that the bilinear form $\langle \cdot,\cdot \rangle_{\hat{\rho}}$ defined above is positive and non-degenerate. Define a bilinear form $\langle \cdot,\cdot \rangle$ on H^0 as above. If $\vec{\phi} : \vec{\mathscr{B}}^0 \rightarrow \mathbb{R}$ is a bounded linear functional, define

$$\mathbb{E}_{\kappa,\hat{\rho}}^{CS}\left[e^{((\cdot,\vec{s}\times\vec{\phi}))_0} \right] := e^{-i\langle s^+\phi_0, s^-\partial_2^{-1}\phi_1 \rangle}, \tag{3.13}$$

$\vec{s} = (s^+, s^-) \in \mathbb{R}^2$, if

$$\langle s^+\phi_0, s^-\partial_2^{-1}\phi_1 \rangle < \infty.$$

There is a κ dependence in the RHS of the definition, which is not obvious from the notation. The term ∂_2^{-1} is actually dependent on κ. When the representation is clear, $\mathbb{E}_{\kappa}^{CS}[\cdot] \equiv \mathbb{E}_{\kappa,\hat{\rho}}^{CS}[\cdot]$.

Note that $e^{((\cdot,\vec{s}\times\vec{\phi}))_0}$ is in $L^1(\vec{\mathscr{B}}^0, \tilde{\nu}_\theta)$. Similar to Definition 3.5, $\mathbb{E}_{\kappa,\hat{\rho}}^{CS}$ is viewed as a linear functional, defined only on a strict subspace of $L^1(\vec{\mathscr{B}}^0, \tilde{\nu}_\theta)$. Since we only need Equation (3.13), we will not find the largest possible subspace in $L^1(\vec{\mathscr{B}}^0, \tilde{\nu}_\theta)$ for which $\mathbb{E}_{\kappa,\hat{\rho}}^{CS}$ can be defined on.

For the rest of this article, our Chern-Simons path integral will be computed using Definition 3.6.

4 Abelian Gauge Group

The next thing we want to compute is the Wilson Loop observables for an abelian gauge group using Definition 3.6 for Chern-Simons path integral. Write

$$\psi(z_0,z_1,z_2) = \exp[-(z_0^2 + z_1^2 + z_2^2)/2], \ (z_0,z_1,z_2) \in \mathbb{C}^3.$$

We will write

$$\langle u_0, \partial_2^{-1} u_1 \rangle = \int_{\mathbb{C}^3} (u_0 \cdot \overline{\partial_2^{-1} u_1})_z e^{-|z|^2} \prod_{i=0}^{2} \frac{dx_i dp_i}{\pi}$$

for $u_0, u_1 : \mathbb{C}^3 \rightarrow \mathbb{C}$. Here, $z_j = x_j + ip_j$. We will also write

$$\langle f, g \rangle = \int_{\mathbb{R}^k} f(x) g(x) d\mathrm{vol}_{\mathbb{R}^k}.$$

Here, $f, g : \mathbb{R}^k \rightarrow \mathbb{R}$, $k = 1, 2, 3$.

We will work out the case for $N = 1$ for an abelian gauge group G. Let $L = \{C^k\}_k$ be a link, and embed $\mathbb{R}^3$ into $\mathbb{C}^3$. We assume L is smooth and let $l^k : [0,1] \rightarrow \mathbb{R}^3$ be a parametrization of C^k. We need to do the following 2 scalings:

a The link L in $\mathbb{R}^3 \subseteq \mathbb{C}^3$ is scaled by a factor $\kappa/2$.

b Given $\vec{a} = (a_0, a_1) \in \vec{\mathscr{B}}^0$, we will scale it by $\sqrt{\varsigma_2}\psi$, $\varsigma_2 := \kappa/(4\sqrt{2\pi})$.

The reader might question why do we need to make these scaling. These factors are necessary in order to obtain the knot invariants. With these scaling factors, the holonomy operator of $\vec{a}$ computed along the L is given by

$$\prod_k W(C^k) = \exp\left[q\sqrt{\varsigma_2} \sum_k \int_{\kappa C^k/2} \psi \cdot (a_0 dx_0 + a_1 dx_1) \right], \ \vec{a} = (a_0, a_1) \in \vec{\mathscr{B}}^0.$$

The map

$$\vec{a} \mapsto \sqrt{\varsigma_2} \sum_k \int_{\kappa C^k/2} \psi \cdot (a_0 dx_0 + a_1 dx_1)$$

is linear. Suppose it can be written as $\vec{a} \mapsto \sqrt{\varsigma_2}((\vec{a}, \vec{\eta}(L)))_0$, whereby $\vec{\eta}(L) \in \vec{\mathscr{B}}^{0,*} \in \vec{H}^0$ and $((\cdot,\cdot))_0$ is a pairing.

Using Equation (3.13), the Wilson Loop observable, Equation (3.1) now becomes

$$Z(\mathbb{R}^3, L) = \mathbb{E}_\kappa^{CS}\left[e^{\sqrt{\varsigma_2}((\cdot, \vec{\eta}(L)))_0} \right] = \exp[-iq^2 \varsigma_2 \langle \eta(L)_0, \partial_2^{-1} \eta(L)_1 \rangle_0],$$

$\vec{\eta}(L) = (\eta(L)_0, \eta(L)_1)$. There is a κ dependence on the RHS. We will work out the RHS explicitly in the following calculations. To obtain the knot invariants, we will have to take the limit as κ goes to infinity.

Lemma 4.1. *Continue with the discussion as above. Then,*

$$Z(\mathbb{R}^3, L) = \exp\left[-\frac{\pi i q^2 \kappa^3}{16\pi\sqrt{2\pi}} \sum_{j \geq k} \int_0^1 ds \int_0^1 dt\ \delta_j^k \left[l_0^{j,\prime}(s) l_1^{k,\prime}(t) - l_0^{k,\prime}(t) l_1^{j,\prime}(s) \right] \right.$$
$$\left. \cdot h_j(s) h_k(t) \left\langle \chi(\bar{\kappa} l^j(s)), \partial_2^{-1} \chi(\bar{\kappa} l^k(t)) \right\rangle \right],$$

with $\delta_j^k = \delta_k^j = 1 - (\delta_{jk}/2)$ *and* $\bar{\kappa} = \kappa/2$.

Proof. Let $\{l^k \equiv (l_0^k, l_1^k, l_2^k)\}$ and $l^k : [0,1] \mapsto \mathbb{R}^3$ be a parametrization of C^k. Write $h_k^{\kappa}(s) \equiv h_k(s) = \exp[-\kappa^2 l^{k,2}(s)/8]$ and $\bar{\kappa} = \kappa/2$. Let $\{s_0 = 0 < s_1 < \ldots s_n = 1\}$ be a partition of $[0,1]$ and $\Delta_i s = s_i - s_{i-1}$. Let $\Delta = \sup_i \Delta_i s$. Then, doing a Riemannian sum approximation,

$$\begin{aligned}
&\bar{\kappa}\left[\int_0^1 h_k(s)a_0(\bar{\kappa}l^k(s))l_0^{k,\prime}(s)ds + h_k(t)a_1(\bar{\kappa}l^k(t))l_1^{k,\prime}(t)dt\right] \\
&= \lim_{\Delta\to 0}\bar{\kappa}\left[\sum_i h_k(s_i)a_0(\bar{\kappa}l^k(s_i))l_0^{k,\prime}(s_i)\Delta_i s + \sum_j h_k(t_j)a_1(\bar{\kappa}l^k(t_j))l_1^{k,\prime}(t_j)\Delta_j t\right] \\
&= \lim_{\Delta\to 0}\bar{\kappa}\left(\left(\vec{a}, \left(\sum_i h_k(s_i)\chi(\bar{\kappa}l^k(s_i))l_0^{k,\prime}(s_i)\Delta_i s, \sum_j h_k(t_j)\chi(\bar{\kappa}l^k(t_j))l_1^{k,\prime}(t_j)\Delta_j t\right)\right)\right)_0.
\end{aligned}$$

$\chi(l^k(s))$ was defined earlier, a linear functional $\chi(l^k(s)) : a \mapsto a(l^k(s))$, i.e. evaluate a at the point $l^k(s) \in \mathbb{R}^3$.

Define an operator F such that

$$F(\vec{a}) := \lim_{\Delta\to 0}\bar{\kappa}\left(\left(\vec{a}, \left(\sum_i h_k(s_i)\chi(\bar{\kappa}l^k(s_i))l_0^{k,\prime}(s_i)\Delta_i s, \sum_j h_k(t_j)\chi(\bar{\kappa}l^k(t_j))l_1^{k,\prime}(t_j)\Delta_j t\right)\right)\right)_0.$$

We need to show that F is bounded. Since $\vec{\mathcal{B}}^{0,*} \subseteq \vec{H}^0$, we can represent F as $F = (\cdot, \vec{\eta}(L))_0$, $\vec{\eta}(L) = (\eta(L)_0, \eta(L)_1)$, $\eta(L) \in \vec{\mathcal{B}}^{0,*}$.

From Equation (2.8), since the link L is bounded, thus we can find a M_1 such that $|\chi(t)| \leq M_2$ for all $t \in L$. Let M_2 be a constant such that $|l^{k,\prime}(s)| \leq M_2$. Hence,

$$\bar{\kappa}\left|\left(\left(\vec{a}, \left(\sum_i h_k(s_i)\chi(\bar{\kappa}l^k(s_i))l_0^{k,\prime}(s_i)\Delta_i s, \sum_j h_k(t_j)\chi(\bar{\kappa}l^k(t_j))l_1^{k,\prime}(t_j)\Delta_j t\right)\right)\right)_0\right| \leq 4\bar{\kappa}M_1M_2|\vec{a}|$$

for any partition. Thus, F is a bounded linear functional on $\vec{\mathcal{B}}^0$ and we can write

$$\bar{\kappa}\left[\int_0^1 h_k(s)a_0(\bar{\kappa}l^k(s))l_0^{k,\prime}(s)ds + h_k(t)a_1(\bar{\kappa}l^k(t))l_1^{k,\prime}(t)dt\right] = (\vec{a}, \vec{\eta}(L))_0.$$

Note that $\vec{\eta}(L) \in \vec{\mathcal{B}}^{0,*}$, which is embedded in a Hilbert space. Using the unitary map Q, we can represent $\eta(L)_0$ as a function in $L^2(\mathbb{R}^3, \mathrm{vol}_{\mathbb{R}^3})$. Likewise for $\eta(L)_1$. Recall that

$$\partial_2^{-1} f(x_0,x_1,x_2) = \int_{-\infty}^{x_2} f(x_0,x_1,y)dy - \int_{x_2}^{\infty} f(x_0,x_1,y)dy. \tag{4.1}$$

Let $w(l^k(s)) = Q^{-1}\chi(\bar{\kappa}l^k(s))$, a real-valued function on $\mathbb{R}^3$. Later on, we will show that $w(l^k(s))$ is in $L^1(\mathbb{R}^3, \mathrm{vol}_{\mathbb{R}^3})$. See the proof of Lemma 4.2. Under the map Q^{-1}, $\vec{\eta}(L)$ is represented as

$$\begin{aligned}
\eta(L)_0 &= \sum_k \int_0^1 h_k(s)w(l^k(s))l_0^{k,\prime}(s)\, ds, \\
\eta(L)_1 &= \sum_k \int_0^1 h_k(s)w(l^k(s))l_1^{k,\prime}(s)\, ds.
\end{aligned}$$

Using Equation (4.1), by Fubini's Theorem,

$$\partial_2^{-1}\int_0^1 h_k(s)w(l^k(s))l_1^{k,\prime}(s)\,ds = \int_0^1 h_k(s)\partial_2^{-1}w(l^k(s))\cdot l_1^{k,\prime}(s)\,ds.$$

By abuse of notation, using the same symbol ∂_2^{-1} to refer to the operator acting on the space of holomorphic functions, we will write

$$\int_0^1 h_k(s)\partial_2^{-1}w(l^k(s))\cdot l_1^{k,\prime}(s)\,ds = \int_0^1 h_k(s)\partial_2^{-1}\chi(\bar{\kappa}l^k(s))\cdot l_1^{k,\prime}(s)\,ds.$$

In a nutshell, we are justified in exchanging the operator ∂_2^{-1} with the integrals.

Now, Equation (3.13) gives

$$\begin{aligned}
&Z(\mathbb{R}^3,L) = \mathbb{E}_\kappa^{CS}\left[e^{((\cdot,\vec{\eta}(L)))_0}\right]\\
&=\exp\left[-\frac{\pi i q^2\kappa^3}{16\pi\sqrt{2\pi}}\left\langle\sum_k\int_0^1 h_k(s)\chi(\bar{\kappa}l^k(s))l_0^{k,\prime}(s)\,ds, \partial_2^{-1}\sum_k\int_0^1 h_k(s)\chi(\bar{\kappa}l^k(s))l_1^{k,\prime}(s)\,ds\right\rangle\right]\\
&=\exp\left[-\frac{\pi i q^2\kappa^3}{16\pi\sqrt{2\pi}}\left\langle\sum_k\int_0^1 h_k(s)\chi(\bar{\kappa}l^k(s))l_0^{k,\prime}(s)\,ds, \sum_k\int_0^1 h_k(s)\partial_2^{-1}\chi(\bar{\kappa}\cdot l^k(s))l_1^{k,\prime}(s)\,ds\right\rangle\right].
\end{aligned}$$

Further simplification gives

$$\begin{aligned}
\exp\Bigg[-\pi i q^2\frac{\kappa^3}{16\pi\sqrt{2\pi}}\sum_{j\geq k}\int_0^1 ds\int_0^1 dt\ \delta_j^k\left[l_0^{j,\prime}(s)l_1^{k,\prime}(t) - l_0^{k,\prime}(t)l_1^{j,\prime}(s)\right]\\
\cdot h_j(s)h_k(t)\left\langle\chi(\bar{\kappa}l^j(s)), \partial_2^{-1}\chi(\bar{\kappa}l^k(t))\right\rangle\Bigg],
\end{aligned}$$

with $\delta_j^k = \delta_k^j = 1-(\delta_{jk}/2)$ and $\bar{\kappa} = \kappa/2$. □

The term

$$\frac{\kappa^3}{16\pi\sqrt{2\pi}}\int_0^1 ds\int_0^1 dt\ \left[l_0^{j,\prime}(s)l_1^{k,\prime}(t) - l_0^{k,\prime}(t)l_1^{j,\prime}(s)\right]\left\langle\chi(\bar{\kappa}l^j(s)), \partial_2^{-1}\chi(\bar{\kappa}l^k(t))\right\rangle h_j(s)h_k(t) \tag{4.2}$$

should be the linking number between curves l^k and l^j, provided both curves do not intersect. To check this, we have to first compute $\langle\chi(\kappa l^j(s)/2), \partial_2^{-1}\chi(\kappa l^k(t)/2)\rangle e^{-\kappa^2(l^{j,2}(s)+l^{k,2}(t))/8}$.

This is where we need to digress a bit. Here, one has to be careful. The Banach space $\vec{\mathcal{B}}^0$ is the space of $C^\infty(\mathbb{C}^3,\mathbb{R})\otimes\mathfrak{g}_{\mathbb{C}}\times C^\infty(\mathbb{C}^3,\mathbb{R})\otimes\mathfrak{g}_{\mathbb{C}}$. We identify using $Q: L^2(\mathbb{R}^3,\mathrm{vol}_{\mathbb{R}^3})\rightarrow L^2(\mathbb{C}^3, e^{-|z|^2}d\omega/\pi^3)$, ω is the standard symplectic form on $\mathbb{R}^6$ and

$$Q: \varsigma_0^{-1/2}\frac{H_{p_r}(\kappa\cdot)}{\sqrt{p_r!}}\sqrt{\phi_\kappa}\otimes E_{ij}\mapsto\frac{z^{p_r}}{\sqrt{p_r!}}\otimes E_{ij}. \tag{4.3}$$

See Remark 3.1. Recall Q sends $a^*\mapsto z_2$ and $a\mapsto\partial_{z_2}$. ∂_2^{-1} as an operator on $\vec{\mathcal{B}}^0$ is not the same operator as on $L^2(\mathbb{R}^3,\mathrm{vol}_{\mathbb{R}^3})$.

Furthermore, the linear functional $a\in\mathbb{R}^3: f\in L^2(\mathbb{R}^3,\mathrm{vol}_{\mathbb{R}^3})\mapsto f(a)$ is not $\chi_a\in\vec{\mathcal{B}}^{0,*}$. In fact, the completion of $L^2(\mathbb{R}^3,\mathrm{vol}_{\mathbb{R}^3})$ will no longer be functions, but generalized functions, hence such a linear functional does not make sense on the completion of $L^2(\mathbb{R}^3, \mathrm{vol}_{\mathbb{R}^3})$. Recall we complete the Hilbert space into a Banach space using a weaker measurable norm.

However, one can do an approximation to this functional. When integrated with respect to the function $\phi_\kappa(\cdot - a) \in H^0$, for large enough κ, $\langle f, \phi_\kappa(\cdot - a)\rangle$ gives a value close to any function f evaluated at the point $a \in \mathbb{R}^3$. In fact, it tends to the Dirac function as κ approaches infinity. However, one has to check that it is in the dual of the Banach space containing $L^2(\mathbb{R}^3, \mathrm{vol}_{\mathbb{R}^3})$.

Now, $\chi_w = \sum_{r=0}^{\infty}\sum_{p_r} z^{p_r}\bar{w}^{p_r}/p_r!$. Since our links lie in $\mathbb{R}^3$, we will only consider w real. An intelligent guess will suggest that $\sqrt{\phi}(\cdot - a)$ should somehow be connected to χ. Thus, we need to find the corresponding function of χ_a in $L^2(\mathbb{R}^3, \mathrm{vol}_{\mathbb{R}^3})$.

Lemma 4.2. *Continue with the discussion as above. Then,*

$$\begin{aligned}\frac{\kappa^3}{16\pi\sqrt{2\pi}}&\langle \chi(\kappa l^j(s)/2), \partial_2^{-1}\chi(\kappa l^k(t)/2)\rangle e^{-\kappa^2(l^{j,2}(s)+l^{k,2}(t))/8}\\ &= \frac{\kappa^2}{8\pi}e^{-\kappa^2|Pl^j(s)-Pl^k(t)|^2/8}\left\langle \frac{\kappa}{\sqrt{4\pi}}e^{-\kappa^2(\cdot - l_2^j(s))^2/4}, 2\Phi_{l_2^k(t),\sqrt{2}/\kappa}(\cdot) - 1\right\rangle,\end{aligned}$$

where $\Phi_{x_2,\sqrt{2}/\kappa}$ is the cumulative distribution function of a normal distribution, mean x_2 and variance $2/\kappa^2$. P is the projection on $\mathbb{R}^2$.

Proof. Q^{-1} maps $\chi_{\kappa t/2}$ to

$$\varsigma_0^{-1/2}\sqrt{\phi_\kappa}(x)\sum_{r=0}^{\infty}\sum_{p_r} H_{p_r}(\kappa x)\frac{(\kappa t/2)^{p_r}}{p_r!} = \varsigma_0^{-1/2}\sqrt{\phi_\kappa}(x)e^{\kappa^2(2x\cdot t - t^2)/4}e^{\kappa^2 t^2/8}$$

for t real, which upon simplification gives $\varsigma_0^{-1/2}\sqrt{\phi_\kappa}(\cdot - t)e^{\kappa^2 t^2/8}$. Here, $x\cdot t$ is the usual scalar product in $\mathbb{R}^3$.

In other words, for each $t \in \mathbb{R}^3$,

$$\begin{aligned}Q: \varsigma_0^{-1/2}\sqrt{\phi_\kappa}(\cdot - t)e^{\kappa^2 t^2/8} =& \varsigma_0^{-1/2}\sqrt{\phi_\kappa}\sum_{r=0}^{\infty}\sum_{p_r} H_{p_r}(\kappa\cdot)\frac{\kappa^r t^{p_r}}{2^r\cdot p_r!}\\ &\mapsto \sum_{r=0}^{\infty}\sum_{p_r} z^{p_r}\frac{\kappa^r t^{p_r}}{2^r\cdot p_r!} = \chi_{\kappa t/2} \in \vec{\mathcal{B}}^{0,*}.\end{aligned}$$

This shows that $\varsigma_0^{-1/2}\sqrt{\phi_\kappa}(\cdot - t)e^{\kappa^2 t^2/8}$ is in the dual space of the completion of $L^2(\mathbb{R}^3, \mathrm{vol}_{\mathbb{R}^3})$.

Now $\partial_2^{-1} f := \int_{-\infty}^{\cdot} f(x_2)dx_2 - \int_{\cdot}^{\infty} f(x_2)dx_2$. Thus, Equation (4.2) simplifies to

$$\begin{aligned}&\frac{\kappa^3}{16\pi\sqrt{2\pi}}\langle \chi(\kappa l^j(s)/2), \partial_2^{-1}\chi(\kappa l^k(t)/2)\rangle e^{-\kappa^2(l^{j,2}(s)+l^{k,2}(t))/8}\\ =&\frac{\kappa^3}{16\pi\sqrt{2\pi}}\langle Q^{-1}\chi(\kappa l^j(s)/2), \partial_2^{-1}Q^{-1}\chi(\kappa l^k(t)/2)\rangle e^{-\kappa^2(l^{j,2}(s)+l^{k,2}(t))/8}\\ =&-\prod_{\alpha=0}^{1}\frac{\kappa}{2\sqrt{2\pi}}e^{-\kappa^2|l_\alpha^j(s)-l_\alpha^k(t)|^2/8}\left\langle \frac{\kappa}{\sqrt{4\pi}}e^{-\kappa^2(\cdot - l_2^j(s))^2/4}, \int_{-\infty}^{\cdot} - \int_{\cdot}^{\infty}\frac{\kappa}{\sqrt{4\pi}}e^{-\kappa^2|x_2 - l_2^k(t)|^2/4}dx_2\right\rangle\\ =&-\frac{\kappa^2}{8\pi}e^{-\kappa^2|Pl^j(s)-Pl^k(t)|^2/8}\left\langle \frac{\kappa}{\sqrt{4\pi}}e^{-\kappa^2(\cdot - l_2^j(s))^2/4}, 2\Phi_{l_2^k(t),\sqrt{2}/\kappa}(\cdot) - 1\right\rangle,\end{aligned}$$

where $\Phi_{x_2,\sqrt{2}/\kappa}$ is the cumulative distribution function of a normal distribution, mean x_2 and variance $2/\kappa^2$. P is the projection on $\mathbb{R}^2$. Note also that in the first equality, we made use of the fact that Q^{-1} preserves the inner product between the two Hilbert spaces. □

The term $\exp[-\kappa^2(x-y)/8]$ will vanish as κ goes to infinity unless $x=y$. If $l_2^j(t) > l_2^k(s)$, then the integral gives a $+1$; if $l_2^j(t) < l_2^k(s)$, it gives -1. This amounts to giving a $+$ sign to overcrossing and $-$ sign for undercrossing. Otherwise it is zero. Hence the integral along the arcs contribute negligibly to the double integral for large enough κ.

As κ goes to infinity, the integral in Equation (4.2) reduces to a finite sum, assigning +1 to overcrossings and -1 to the undercrossings. It computes the linking number between l^j and l^k.

However, there is more. One has to take into account of the sign of the orientation at each crossing. When projected down onto $\mathbb{R}^2$, $l_0^{j,\prime}$ and $l_1^{k,\prime}$ form a frame at p. The orientation is given by $\mathrm{sgn}(Pl^j \vec{\times} Pl^k)$, where $\vec{\times}$ is the vector cross product in $\mathbb{R}^2$.

Definition 4.3. (Link Diagrams)
Assume that a link $L = \{l^1, l^2, \ldots l^m\} \in \mathbb{R}^3$ is C^1 and the individual curves do not intersect one another, i.e. $l^j \cap l^k = \emptyset$ for any j, k. Parametrise each curve, $l^j = (l_0^j, l_1^j, l_2^j) : [0,1] \rightarrow \mathbb{R}^3$ such that $|l^{j,\prime}| \neq 0$. Note that in the following definitions, it applies if L is just a knot.

1. Let $\pi : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be the projection on the $x-y$ plane. Define a standard projection of the link onto $\mathbb{R}^2$ if the following conditions are satisfied:

   a for any point $p \in \mathbb{R}^2$, $\pi^{-1}(p)$ intersects at most 2 distinct arcs in L. p is called a crossing if $\pi^{-1}(p)$ intersects exactly 2 distinct arcs.

   b at each crossing $p = l^j(s_0) = l^k(t_0)$, there exists an $\varepsilon > 0$ such that for all $|s-s_0| < \varepsilon$ and $|t-t_0| < \varepsilon$, the vector cross product $\pi l^j(s) \vec{\times} \pi l^k(t) \neq 0$, i.e. the 2 arcs at the crossing p are linearly independent.

   Denote the set of crossings between curves l^j and l^k by $\mathrm{DP}(l^j, l^k)$. $\mathrm{DP}(l^j) \equiv \mathrm{DP}(l^j, l^j)$ will denote the set of crossings in l^j. We will write $\mathrm{DP}(L)$ to denote the set of crossings of the standard projection of the link L.

2. For each curve l^j, write the interval $[0,1)$ as a union of intervals $\bigcup_{i=1}^{n(l^j)} A(l^j)^i$, where in each interval $A(l^j)^i$, $s \in A(l^j)^i \mapsto (l_0^j(s), l_1^j(s))$ is a bijection. Write $C(l^j)^i := (l_0^j(A(l^j)^i), l_1^j(A(l^j)^i)) \in \mathbb{R}^2$ be the image of the interval $A(l^j)^i$ under l^j. Without loss of generality, further assume each interval $C(l^j)_\alpha^i \equiv l_\alpha^j(A(l^j)^i)$ contains at most one crossing which is an interior point in $C(l^j)_\alpha^i$.

3. Given 2 arcs $C(l^j)^i, C(l^k)^{\hat{i}}$ which intersect at p, define $\mathrm{sgn}(J(C(l^j)^i, C(l^k)^{\hat{i}}))$ to be the sign of the determinant of the Jacobian $J(C(l^j)^i, C(l^k)^{\hat{i}}) = l_0^{j,\prime}(s) l_1^{k,\prime}(t) - l_0^{k,\prime}(s) l_1^{j,\prime}(t)$ at the crossing $p = (l_0^j(s), l_1^j(s)) = (l_0^k(t), l_1^k(t))$. Otherwise, define it to be zero if the 2 arcs do not intersect at all. We will also write $\mathrm{sgn}(p; l^j : l^k) \equiv \mathrm{sgn}(J(C(l^j)^i, C(l^k)^{\hat{i}}))$, $p = C(l^j)^i \cap C(l^k)^{\hat{i}}$ and call this the orientation of p.

4. Using the same notation as the previous item, for each crossing $p \in C(l^j)^i \cap C(l^k)^{\hat{i}}$, define
$$\operatorname{sgn}(C(l^j)^i : C(l^k)^{\hat{i}}) = \begin{cases} 1, & l_2^j > l_2^k; \\ -1, & l_2^j < l_2^k. \end{cases}$$
If the 2 arcs do not intersect, set it to be 0. We will also write $\operatorname{sgn}(p; l_2^j : l_2^k) \equiv \operatorname{sgn}(C(l^j)^i : C(l^k)^{\hat{i}})$ and call this the height of p.

Remark 4.4. The set DP(L) only makes sense for a link diagram projected on $\mathbb{R}^2$. Different link diagrams will give a different set of crossings. Thus it is not well defined for a link L, but rather on a link diagram.

Lemma 4.5.
$$\lim_{\kappa\to\infty} \frac{\kappa^3}{16\pi\sqrt{2\pi}} \int_{A(l^j)^i} ds \int_{A(l^k)^{\hat{i}}} dt \ \left[l_0^{j,\prime}(s) l_1^{k,\prime}(t) - l_0^{k,\prime}(t) l_1^{j,\prime}(s)\right] \left\langle \chi(l^j(s)), \partial_2^{-1}\chi(l^k(t)) \right\rangle \cdot e^{-\kappa^2(l^{j,2}(s)+l^{k,2}(t))/8}$$
$$= -\operatorname{sgn}(J(C(l^j)^i, C(l^k)^{\hat{i}}))\operatorname{sgn}(C(l^j)^i : C(l^k)^{\hat{i}})$$

Proof. Write $\Lambda_\kappa^{l_2^k(t)}(\cdot) = 2\Phi_{l_2^k(t),\sqrt{2}/\kappa}(\cdot) - 1$. Then the integral becomes
$$-\int_0^1 ds \int_0^1 dt \ \left[l_0^{j,\prime}(s) l_1^{k,\prime}(t) - l_0^{k,\prime}(t) l_1^{j,\prime}(s)\right] \phi_{\kappa/2}(Pl^j(s) - Pl^k(t)) \left\langle \phi_{\kappa/\sqrt{2}}(\cdot - l_2^j(s)), \Lambda_\kappa^{l_2^k(t)} \right\rangle.$$
Make a change of variables: $Pl^j : s \in A(l^j)^i \mapsto x_0 = (l_0^j(s), l_1^j(s)) \in \mathbb{R}^2$ and $y_j : x_0 \mapsto l_2^j(s)$. Similarly, $Pl^k : t \in A(l^k)^{\hat{i}} \mapsto x_1 = (l_0^k(t), l_1^k(t)) \in \mathbb{R}^2$ and $y_k : x_1 \mapsto l_2^k(t)$. Then the integral becomes ($d\omega = dx_0 \wedge dx_1$.)
$$\operatorname{sgn}(J(C(l^j)^i, C(l^k)^{\hat{i}})) \int_{C(l^j)^i \times C(l^k)^{\hat{i}}} \phi_{\kappa/2}(x_0 - x_1) \left\langle \phi_{\kappa/\sqrt{2}}(\cdot - y_j(x_0)), \Lambda_\kappa^{y_k(x_1)} \right\rangle \, d\omega$$
$$= \operatorname{sgn}(J(C(l^j)^i, C(l^k)^{\hat{i}})) \int_{C(l^j)^i \times C(l^k)^{\hat{i}} \times \mathbb{R}} \phi_{\kappa/2}(x_0 - x_1) \phi_{\kappa/\sqrt{2}}(x_2 - y_j(x_0)) \Lambda_\kappa^{y_k(x_1)}(x_2) \, d\omega dx_2. \tag{4.4}$$

When $C(l^j)^i \cap C(l^k)^{\hat{i}} = \emptyset$, the integral goes to 0. The only case is when $p \in C(l^j)^i \cap C(l^k)^{\hat{i}}$. We will only consider the case when $y_j(p) > y_k(p)$. The other case is similar. Then Equation (4.4) becomes
$$\operatorname{sgn}(J(C(l^j)^i, C(l^k)^{\hat{i}})) \int_{C(l^j)^i \times C(l^k)^{\hat{i}} \times \mathbb{R}} \phi_{\kappa/2}(x_0 - x_1) \phi_{\kappa/\sqrt{2}}(x_2 - y_j(x_0)) \, d\omega dx_2$$
$$+ \operatorname{sgn}(J(C(l^j)^i, C(l^k)^{\hat{i}})) \int_{C(l^j)^i \times C(l^k)^{\hat{i}} \times \mathbb{R}} \phi_{\kappa/2}(x_0 - x_1) \phi_{\kappa/\sqrt{2}}(x_2 - y_j(x_0)) \left[\Lambda_\kappa^{y_k(x_1)}(x_2) - 1\right] \, d\omega dx_2$$
$$\longrightarrow_{\kappa\to\infty} \operatorname{sgn}(J(C(l^j)^i, C(l^k)^{\hat{i}})).$$

The last step requires the following explanation: For any $|x_2 - y_j(x_0)| < \delta$, δ small enough, $x_2 > y_k(x_1)$ by assumption. Furthermore, there exists a $\sigma(\delta)$ such that for all $\kappa > \sigma$,
$$|\Phi_{y_k(x_1),\sqrt{2}/\sigma}(x_2) - 1| < \varepsilon$$
for any given ε. □

Definition 4.6. (Algebraic crossing number.)
For each crossing $p \in \mathrm{DP}(l^j, l^k)$, the quantity $\mathrm{sgn}(p; l^j : l^k)\mathrm{sgn}(p; l_2^j : l_2^k)$ is actually well defined on an oriented link diagram, independent of the parametrization used. Denote it by $\varepsilon(p) \in \{+1, -1\}$.

Corollary 4.7. *(Linking number between l^j and l^k.) For $j \neq k$,*

$$\lim_{\kappa\to\infty} \frac{\kappa^3}{16\pi\sqrt{2\pi}} \int_0^1 ds \int_0^1 dt \ \left[l_0^{j,\prime}(s) l_1^{k,\prime}(t) - l_0^{k,\prime}(t) l_1^{j,\prime}(s)\right] \left\langle \chi(l^j(s)), \partial_2^{-1}\chi(l^k(t)) \right\rangle e^{-\kappa^2(l^{j,2}(s)+l^{k,2}(t))/8}$$
$$= - \sum_{y_i \in \mathrm{DP}(l^j, l^k)} \mathrm{sgn}(y_i; l^j : l^k)\mathrm{sgn}(y_i; l_2^j : l_2^k) := -\mathrm{lk}(l^j, l^k),$$

the linking number between l^j and l^k.

There is a problem when $j = k$. In this case, one has to integrate over the arcs. Thus, the linking number between l^j and itself is ill-defined. The solution as explained in [4] would be to consider a framing v^j whereby $v^j(\cdot) \in \mathbb{R}^3$ is a normal vector field along the curve C^j that is nowhere tangent to C^j. Define $\hat{l}^{j,\varepsilon} := l^j + \varepsilon v^j$, ε is some small number. Essentially, $\hat{l}^{j,\varepsilon}$ is a small shift of l^j in a direction v^j. The framing v^j chosen depends only on the topological class of v^j. Then, one computes $\mathrm{lk}(l^j, \hat{l}^{j,\varepsilon})$ and take ε going down to zero.

Explicitly, the limit is computed as

$$\lim_{\varepsilon\to 0}\lim_{\kappa\to\infty} \frac{\kappa^3}{16\pi\sqrt{2\pi}} \int_0^1 ds \int_0^1 dt \ \left[l_0^{j,\prime}(s) \hat{l}_1^{j,\prime}(t) - \hat{l}_0^{j,\prime}(t) l_1^{j,\prime}(s)\right]$$
$$\cdot \left\langle \chi(l^j(s)), \partial_2^{-1}\chi(\hat{l}^j(t)) \right\rangle e^{-\kappa^2(l^{j,2}(s)+l^{k,2}(t))/8}.$$

We will define the above expression as the self-linking number of l^j, written as $\mathrm{lk}(l^j, v^j)$.

Remark 4.8. The self-linking number of l^j depends on the framing v^j.

Taking the sum over all pairs of curves, the Chern-Simons path integral, when $N = 1$, will give the linking number of the link for large enough κ.

Corollary 4.9. *For an abelian gauge group G with $N = 1$, the Wilson Loop observable Equation (3.1) for the limit as κ goes to infinity, is given by*

$$Z(\mathbb{R}^3, L; q) = \frac{1}{Z_{\mathrm{CS}}} \int_{A\in\mathscr{A}} \prod_{k=1}^{l} W(C^k; q) e^{i\mathrm{CS}(A)} DA$$
$$= \exp\left[\frac{\pi i q^2}{2} \sum_j \mathrm{lk}(l^j, v^j)\right] \exp\left[\pi i q^2 \sum_{j>k} \mathrm{lk}(l^j, l^k)\right].$$

References

[1] Albeverio, Sergio and Sengupta, Ambar, A mathematical construction of the non-abelian Chern-Simons functional integral. *Comm. Math. Phys.* **186** (1997), pp 563–579.

[2] Kuo, Hui Hsiung , Gaussian measures in Banach spaces. *Lecture Notes in Mathematics* **463** (1975).

[3] Hahn, Atle, The Wilson loop observables of Chern-Simons theory on $\mathbb{R}^3$ in axial gauge. *Comm. Math. Phys.* **248** (2004), pp 467–499.

[4] Witten, Edward, Quantum field theory and the Jones polynomial. *Comm. Math. Phys.* **121** (1989), pp 351–399.

[5] Kassel, Christian, Quantum groups. *Graduate Texts in Mathematics* **155** (1995).

[6] Fuchs, Jürgen and Schweigert, Christoph, Symmetries, Lie algebras and representations. *Cambridge Monographs on Mathematical Physics* (1997).

[7] Hall, Brian C., Lie groups, Lie algebras, and representations. *Graduate Texts in Mathematics* **222** (2003).

[8] Humphreys, James E., Introduction to Lie algebras and representation theory. *Graduate Texts in Mathematics* **9** (1978).

[9] Albeverio, Sergio and Mitoma, Itaru, Asymptotic expansion of perturbative Chern-Simons theory via Wiener space. *Bull. Sci. Math.* **133** (2009), pp 272–314.