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ANTI-PERIODIC SOLUTIONS FOR A KIND OF HIGH ORDER DIFFERENTIAL EQUATIONS WITH MULTI-DELAY

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Abstract

In this paper, using the Leray-Schauder degree theory, the new results on the existence and uniqueness of anti-periodic solutions are established for a kind of nonlinear high order differential equations with multiple deviating arguments of the form

$$x^{(n)}(t)+f(t,x^{(n-1)}(t))+\sum_{i=1}^{n}g_i(t,x(t-\tau_i(t)))=e(t)$$

Finally, an example is also given to demonstrate the obtaining results.

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Keywords: nth-order differential equations; Deviating argument; Anti-periodic solution; Existence and uniqueness; Leray-Schauder degree.

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1 Introduction

Consider the nonlinear nth-order differential equations with multiple deviating arguments of the form

$$x^{(n)}(t)+f(t,x^{(n-1)}(t))+\sum_{i=1}^{m}g_i(t,x(t-\tau_i(t)))=e(t) \tag{1.1}$$

where τ_i, $e:\ \mathbb{R}\to\mathbb{R}$ are continuous functions and T-periodic, f, $g_i:\ \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are continuous functions and T-periodic in their first arguments, $n\geq 2$ is an integer, $T>0$ and $i=1,2,\cdots,m$. Clearly, when $n=2$ and $f(t,x(t))=f(x(t))$, Eq. (1.1) reduces to which has been known as the delayed Rayleigh equation with multiple deviating arguments. Therefore, Eq. (1.1) is also considered as a high-order Rayleigh equation with multiple deviating arguments.

In applied sciences, some practical problems associated with the Rayleigh equation can be found in the literature. For example, an excess voltage of ferro-resonance known as some kind of nonlinear resonance having long duration arises from the magnetic saturation of inductance in an oscillating circuit of a power system, and a boosted excess voltage can give rise to some problems in relay protection. To probe this mechanism, a mathematical model was proposed in [1, 2], which is a special case of the Rayleigh equation with multiple delays. This implies that Eq. (1.1) with $n=2$ can represent analog voltage transmission. In a mechanical problem, f usually represents a damping or friction term, g_i $(i=1,2,\cdots,m)$ represent a series of the restoring forces, e is an externally applied force and τ_i is the time lag of the restoring force [3]. Some other examples in practical problems concerning physics and engineering technique fields can be found in [4, 5].

In such applications, it is well known that periodic phenomena and anti-periodic phenomena are widespread, and that the existence of anti-periodic solutions play a key role in characterizing the behavior of nonlinear differential equations[6, 7, 8, 9]. Hence, they have been the object of intensive analysis by numerous authors[10, 11, 12, 13, 14, 15, 16, 17, 18, 19]. The literature [20] considered the anti-periodic solutions for Eq. (1.1) with $n=2$ and $m=2$, the literature [21] considered the anti-periodic solutions for Eq. (1.1) with $m=1$. They obtained some sufficient conditions for the existence and uniqueness of anti-periodic solutions of the equation.

Inspired by the above-mentioned literatures, this paper is to establish sufficient conditions for the existence and uniqueness of anti-periodic solutions of Eq. (1.1). The obtaining results are different from those of the references listed above. As application, an example is also given to illustrate the effectiveness of the obtaining results.

2 Preliminary results

For convenience, one introduces a continuation theorem [22] as follows.

Lemma 2.1. *Let Ω be open bounded in a linear normal space X. Suppose that F is a complete continuous field on $\overline{\Omega}$. Moreover, assume that the Leray-Schauder degree*

$$deg\{F,\Omega,p\}\neq 0, for\ p\in X\setminus F(\partial\Omega).$$

Then equation $F(x)=p$ has at least one solution in Ω.

Definition 2.2. Let $u(t):\mathbb{R}\to\mathbb{R}$ be continuous in t. $u(t)$ is said to be anti-periodic on $\mathbb{R}$ if

$$u(t+T)=u(t), u\left(t+\frac{T}{2}\right)=-u(t), \quad \text{for all} \quad t\in\mathbb{R}.$$

For ease of exposition, throughout this paper one will adopt the following notations

$$C_T^k \equiv \{x\in C^k(\mathbb{R},\mathbb{R})\ x \text{ is T-periodic}\},\ k\in\{0,1,2,\cdots\};$$

$$|x|_q=\left(\int_0^T |x(t)|^q \mathrm{d}t\right)^{1/q}; \qquad |x^{(k)}|_\infty=\max_{t\in[0,T]}|x^{(k)}(t)|;$$

$$C_T^{k,\frac{1}{2}} \equiv \left\{x\in C_T^k,\ x\left(t+\frac{T}{2}\right)=-x(t) \text{ for all } t\in\mathbb{R}\right\},$$

which is a linear normal space endowed with the norm $\|\cdot\|$ defined by

$$\|x\|=\max_{t\in[0,T]}\{|x|_\infty,|x'|_\infty,\cdots,|x^{(k)}|_\infty\},\ \text{for all}\ x\in_T^{k,\frac{1}{2}}.$$

The following lemma will be useful for proving the main results in Section 3.

Lemma 2.3 (Wirtinger Inequality, See[23]). *If $x\in C^2(\mathbb{R},\mathbb{R})$ with $x(t+T)=x(t)$, then*

$$\left|x'(t)\right|_2 \le \frac{T}{2\pi}\left|x''(t)\right|_2. \tag{2.1}$$

3 Main results and their proof

In this section, some sufficient conditions for the existence and uniqueness of anti-periodic solutions are established for Eq. (1.1).

First, one considers the uniqueness of anti-periodic solutions for Eq. (1.1).

Theorem 3.1. *Assume that one of the following conditions is satisfied :*

(C1) Suppose that there exist a nonnegative constant L_1 such that for all t, x_1, $x_2\in\mathbb{R}$,

$$|f(t,x_1)-f(t,x_2)|\le L_1|x_1-x_2|$$

holds and there exists nonnegative constants N_i such that for all t, x_1, $x_2\in\mathbb{R}$,

$$L_1\frac{T}{2\pi}+\frac{\sum_{i=1}^m N_i}{2}\frac{T^n}{(2\pi)^{n-1}}<1 \quad \textit{and} \quad |g_i(t,x_1)-g_i(t,x_2)|\le N_i|x_1-x_2|$$

hold;

(C2) Suppose that there exist nonnegative a constant L_2 such that for all u, x_1, $x_2\in\mathbb{R}$,

$$f(t,u)=f(u), \quad L_2|x_1-x_2|^2\le(x_1-x_2)[f(x_1)-f(x_2)] \tag{3.1}$$

hold and there exists nonnegative constants N_i such that for all t, x_1, $x_2 \in \mathbb{R}$,

$$0 \le \sum_{i=1}^{m} N_i < \frac{2L_2(2\pi)^{n-2}}{T^{n-1}} \quad \text{and} \ |g_i(t,x_1) - g_i(t,x_2)| \le N_i|x_1 - x_2|, \quad i = 1,2,\cdots,m.$$

hold.

Then Eq.(1.1) has at most one anti-periodic solution.

Proof. Suppose that $x_1(t)$ and $x_2(t)$ are two anti-periodic solutions of Eq. (1.1). Then $Z(t) = x_1(t) - x_2(t)$ is a anti-periodic function on $\mathbb{R}$ and

$$\int_0^T Z(t)\mathrm{d}t = \int_0^{\frac{T}{2}} Z(t)\mathrm{d}t + \int_{\frac{T}{2}}^{T} Z(t)\mathrm{d}t = \int_0^{\frac{T}{2}} Z(t)\mathrm{d}t + \int_0^{\frac{T}{2}} Z\left(t + \frac{T}{2}\right)\mathrm{d}t = 0.$$

It follows that there exists a constant $\xi \in [0,T]$ such that

$$Z(\xi) = 0. \tag{3.2}$$

Then, one has

$$|Z(t)| = \left|Z(\xi) + \int_\xi^t Z'(s)\mathrm{d}s\right| \le \int_\xi^t |Z'(s)|\mathrm{d}s,\ t \in [\xi, \xi + T],$$

and

$$|Z(t)| = |Z(t-T)| = \left|Z(\xi) - \int_{t-T}^{\xi} Z'(s)\mathrm{d}s\right| \le \int_{t-T}^{\xi} Z'(s)\mathrm{d}s,\ t \in [\xi, \xi + T],$$

Combining the above two inequalities, one obtains

$$\begin{aligned} |Z|_\infty &= \max_{t\in[0,T]} |Z(t)| = \max_{t\in[\xi,\xi+T]} |Z(t)| \\ &\le \max_{t\in[\xi,\xi+T]} \left\{ \frac{1}{2}\left(\int_\xi^t |Z'(s)|\mathrm{d}s + \int_{t-T}^{\xi} |Z'(s)|\mathrm{d}s \right) \right\} \\ &\le \frac{1}{2}\int_0^T |Z'(s)|\mathrm{d}s \le \frac{1}{2}\sqrt{T}\,|Z'|_2 . \end{aligned} \tag{3.3}$$

On the other hand, one has

$$Z^{(n)}(t) + f(t, x_1^{(n-1)}(t)) - f(t, x_2^{(n-1)}(t)) + \sum_{i=1}^{m} [g_i(t, x_1(t - \tau_i(t))) - g_i(t, x_2(t - \tau_i(t)))] = 0. \tag{3.4}$$

Now suppose that (C1) (or (C2)) holds. One will consider two cases as follows.

Case (i) Suppose that (C1) holds. Multiplying both sides of (3.4) by $Z^{(n)}(t)$ and integrating them from 0 to T, one has

$$\begin{aligned}
\left|Z^{(n)}\right|_2^2 &= \int_0^T \left|Z^{(n)}(t)\right|^2 \mathrm{d}t \\
&= -\int_0^T \left[f\left(t,x_1^{(n-1)}(t)\right)-f\left(t,x_2^{(n-1)}(t)\right)\right]Z^{(n)}(t)\mathrm{d}t \\
&\quad -\sum_{i=1}^m \int_0^T [g_i(t,x_1(t-\tau_i(t)))-g_i(t,x_2(t-\tau_i(t)))]Z^{(n)}(t)\mathrm{d}t \\
&\le L_1\int_0^T \left|x_1^{(n-1)}(t)-x_2^{(n-1)}(t)\right||Z^{(n)}(t)|\mathrm{d}t \\
&\quad +\sum_{i=1}^m N_i\int_0^T \left|x_1^{(n-1)}(t-\tau_i(t))-x_2^{(n-1)}(t-\tau_i(t))\right|\left|Z^{(n)}(t)\right|\mathrm{d}t \qquad (3.5)
\end{aligned}$$

From (2.1), (3.3) and the Schwarz inequality, (3.5) implies that

$$\begin{aligned}
\left|Z^{(n)}\right|_2^2 &\le L_1\left[\int_0^T \left|x_1^{(n-1)}(t)-x_2^{(n-1)}(t)\right|^2\mathrm{d}t\right]^{\frac{1}{2}}\left[\int_0^T \left|Z^{(n)}(t)\right|^2\mathrm{d}t\right]^{\frac{1}{2}} \\
&\quad +\sum_{i=1}^m N_i|Z|_\infty\int_0^T 1\times\left|Z^{(n)}(t)\right|\mathrm{d}t \\
&\le L_1\left|Z^{(n-1)}\right|_2\left|Z^{(n)}\right|_2+\sum_{i=1}^m N_i|Z|_\infty\left[\int_0^T 1^2\mathrm{d}t\right]^{\frac{1}{2}}\left[\int_0^T\left|Z^{(n)}(t)\right|^2\mathrm{d}t\right]^{\frac{1}{2}} \\
&\le L_1\left|Z^{(n-1)}\right|_2\left|Z^{(n)}\right|_2+\sum_{i=1}^m N_i|Z|_\infty\sqrt{T}\left|Z^{(n)}\right|_2 \\
&\le L_1\frac{T}{2\pi}\left|Z^{(n)}\right|_2^2+\frac{\sum_{i=1}^m N_i}{2}\sqrt{T}|Z'|_2\sqrt{T}\left|Z^{(n)}\right|_2 \\
&\le \left[L_1\frac{T}{2\pi}+\frac{\sum_{i=1}^m N_i}{2}\frac{T^n}{(2\pi)^{n-1}}\right]\left|Z^{(n)}\right|_2^2
\end{aligned}$$

It follows from $L_1\frac{T}{2\pi}+\frac{\sum_{i=1}^m N_i}{2}\frac{T^n}{(2\pi)^{n-1}}<1$ that

$$Z^{(n)}(t)\equiv 0 \quad \text{for all } t\in\mathbb{R}. \tag{3.6}$$

Since $Z^{(n-2)}(0)=Z^{(n-2)}(T)$, there exists a constant $\xi_{n-1}\in[0,T]$ such that $Z^{(n-1)}(\xi_{n-1})=0$, in view of (3.6), one gets

$$Z^{(n-1)}(t)\equiv 0 \quad \text{for all } t\in\mathbb{R}. \tag{3.7}$$

By using a similar argument as in the proof of (3.7), in view of (3.2), one can show

$$Z(t)\equiv Z'(t)\equiv\cdots\equiv Z^{(n-2)}(t)\equiv 0 \quad \text{for all } t\in\mathbb{R}.$$

Thus, $x_1(t)\equiv x_2(t)$, for all $t\in\mathbb{R}$. Therefore, Eq.(1.1) has at most one anti-periodic solution.

Case (ii) Suppose that (C2) holds. Multiplying both sides of (3.4) by $Z^{(n-1)}(t)$ and integrating them from 0 to T, together with (3.3), one has

$$\begin{aligned}
L_2\left|Z^{(n-1)}\right|_2^2 &= L_2\left[\int_0^T \left|x_1^{(n-1)}(t)-x_2^{(n-1)}(t)\right|^2 \mathrm{d}t\right] \\
&\le \int_0^T \left[f\left(x_1^{(n-1)}(t)\right)-f\left(x_2^{(n-1)}(t)\right)\right]\left(x_1^{(n-1)}(t)-x_2^{(n-1)}(t)\right)\mathrm{d}t \\
&= -\int_0^T Z^{(n)}(t)Z^{(n-1)}(t)\mathrm{d}t \\
&\quad -\sum_{i=1}^m \int_0^T [g_i(t,x_1(t-\tau_i(t)))-g_i(t,x_2(t-\tau_i(t)))]\,Z^{(n-1)}(t)\mathrm{d}t \\
&= -\sum_{i=1}^m \int_0^T [g_i(t,x_1(t-\tau_i(t)))-g_i(t,x_2(t-\tau_i(t)))]\,Z^{(n-1)}(t)\mathrm{d}t \\
&\le \sum_{i=1}^m N_i \int_0^T |x_1(t-\tau_i(t))-x_2(t-\tau_i(t))|\,|Z^{(n-1)}(t)|\mathrm{d}t \\
&= \sum_{i=1}^m N_i |Z|_\infty \sqrt{T}\left|Z^{(n-1)}\right|_2 \\
&\le \frac{\sum_{i=1}^m N_i}{2}\frac{T^{n-1}}{(2\pi)^{n-2}}\left|Z^{(n-1)}\right|_2^2 \qquad (3.8)
\end{aligned}$$

By using a similar argument as in the proof of Case (i), in view of (3.2), (C2) and (3.8), one obtains

$$Z(t)\equiv Z'(t)\equiv\cdots\equiv Z^{(n-2)}(t)\equiv 0 \quad \text{for all } t\in\mathbb{R}.$$

Thus, $x_1(t)\equiv x_2(t)$, for all $tt\in\mathbb{R}$. Therefore, Eq. (1.1) has at most one anti-periodic solution. The proof of Theorem 3.1 is now complete. □

Remark 3.2. If $f'(x)>L_2$ for all $t\in\mathbb{R}$, one can see that $f(x)$ satisfies the assumption (3.1).

Second, one considers the existence of anti-periodic solutions for Eq. (1.1).

Theorem 3.3. *Assume that for all t, $x\in\mathbb{R}$, $i=1,2,\cdots,m$,*

$$f\left(t+\frac{T}{2},-x\right)=-f(t,x),\quad g_i\left(t+\frac{T}{2},-x\right)=-g_i(t,x),$$
$$e\left(t+\frac{T}{2}\right)=-e(t),\quad \tau_i\left(t+\frac{T}{2}\right)=\tau_i(t)$$

hold and the condition (C1) or the condition (C2) is satisfied. Then Eq. (1.1) has a unique anti-periodic solution.

Proof. Consider the auxiliary equation of Eq. (1.1) as the following

$$\begin{aligned}
x^{(n)}(t) &= -\lambda f(t,x^{(n-1)}(t))-\lambda\sum_{i=1}^m g_i(t,x(t-\tau_i(t)))+\lambda e(t) \\
&= \lambda Q(t,x(t),x^{(n-1)}(t)),\quad \lambda\in(0,1]. \qquad (3.9)
\end{aligned}$$

By Theorem 3.1, together with (C1) and (C2), it is easy to see that Eq. (1.1) has at most one anti-periodic solution. Thus, to prove Theorem 3.2, it suffices to show that Eq. (1.1) has at least one anti-periodic solution. To do this, one will apply Lemma 2.1.

First, one will claim that the set of all possible anti-periodic solutions of Eq. (3.9) is bounded. Let $x(t) \in C_T^{k,\frac{1}{2}}$ be an arbitrary anti-periodic solution of Eq. (3.9). By using a similar argument as that in the proof of (3.3), one has

$$|x|_\infty \leq \frac{1}{2}\sqrt{T}|x'|_2 \tag{3.10}$$

In view of (C1) and (C2), one considers two cases as follows.

Case (i) Suppose that (C1) holds. Multiplying both sides of Eq. (3.9) by $x^{(n)}(t)$ and then integrating them from 0 to T, in view of (2.1), (3.10), (C1) and the inequality of Schwarz, one obtains

$$\begin{aligned}
\left|x^{(n)}\right|_2^2 &= \int_0^T \left|x^{(n)}\right|^2 \mathrm{d}t \\
&= -\lambda \int_0^T f\left(t, x^{(n-1)}(t)\right) x^{(n)}(t) \mathrm{d}t \\
&\quad -\lambda \int_0^T \sum_{i=1}^m g_i(t, x(t-\tau_i(t))) \mathrm{d}t + \lambda \int_0^T e(t) \mathrm{d}t \\
&\leq \int_0^T \left| f\left(t, x^{(n-1)}(t)\right) - f(t,0) + f(t,0) \right| \left| x^{(n)}(t) \right| \mathrm{d}t \\
&\quad + \sum_{i=1}^m \int_0^T |g_i(t, x(t-\tau_i(t))) - g_i(t,0) + g_i(t,0)| \left| x^{(n)}(t) \right| \mathrm{d}t \\
&\quad + \int_0^T |e(t)| \left| x^{(n)}(t) \right| \mathrm{d}t \\
&\leq L_1 | \left| x^{(n-1)} \right|_2 \left| x^{(n)} \right|_2 + \sum_{i=1}^m N_i \int_0^T |x(t-\tau_i(t))| \left| x^{(n)}(t) \right| \mathrm{d}t \\
&\quad + \int_0^T \left[|f(t,0)| + \sum_{i=1}^m |g_i(t,0)| \right] \left| x^{(n)}(t) \right| \mathrm{d}t + \int_0^T |e(t)| \left| x^{(n)}(t) \right| \mathrm{d}t \\
&\leq L_1 \frac{T}{2\pi} \left| x^{(n)} \right|_2^2 + \sum_{i=1}^m N_i |x|_\infty \sqrt{T} \left| x^{(n)} \right|_2 \\
&\quad + \left[\max_{t\in[0,T]} \left\{ |f(t,0)| + \sum_{i=1}^m |g_i(t,0)| \right\} + |e|_\infty \right] \sqrt{T} \left| x^{(n)} \right|_2 \\
&\leq L_1 \frac{T}{2\pi} \left| x^{(n)} \right|_2^2 + \frac{\sum_{i=1}^m N_i}{2} \sqrt{T} |x'|_2 \sqrt{T} \left| x^{(n)} \right|_2 \\
&\quad + \left[\max_{t\in[0,T]} \left\{ |f(t,0)| + \sum_{i=1}^m |g_i(t,0)| \right\} + |e|_\infty \right] \sqrt{T} \left| x^{(n)} \right|_2 \\
&\leq L_1 \frac{T}{2\pi} \left| x^{(n)} \right|_2^2 + \frac{\sum_{i=1}^m N_i}{2} \frac{T^n}{(2\pi)^{n-1}} \left| x^{(n)} \right|_2^2 \\
&\quad + \left[\max_{t\in[0,T]} \left\{ |f(t,0)| + \sum_{i=1}^m |g_i(t,0)| \right\} + |e|_\infty \right] \sqrt{T} \left| x^{(n)} \right|_2,
\end{aligned} \tag{3.11}$$

together with (C1), which implies that there exists a positive constant D_1

$$\left|x^{(j)}\right|_2 \leq \left(\frac{T}{2\pi}\right)^{n-j} \left|x^{(n)}\right|_2 < D_1, \quad j = 1, 2, \cdots, n. \tag{3.12}$$

Since $x^{(j)}(0) = x^{(j)}(T)\,(j = 0, 1, 2, \cdots, n-1)$, it follows that there exists a constant $\xi_j \in [0, T]$ such that

$$x^{(j+1)}(\xi_j) = 0$$

and

$$\left|x^{(j+1)}(t)\right| = \left|x^{(j+1)}(\xi_j) + \int_{\xi_j}^{t} x^{(j+2)}(s)\mathrm{d}s\right| \leq \int_0^T x^{(j+2)}(t)\mathrm{d}t \leq \sqrt{T}\left|x^{(j+2)}\right|_2, \tag{3.13}$$

where $j = 0, 1, 2, \cdots, n-2$, $t \in [0, T]$.

Together with (3.10) and (3.12), (3.13) implies that there exists a positive constant D_2 such that

$$\left|x^{(j)}\right|_\infty \leq \sqrt{T}\left|x^{(j+1)}\right|_2 < D_2, \quad j = 0, 1, 2, \cdots, n-1,$$

which implies that, for all possible anti-periodic solutions $x(t)$ of (3.9), there exists a constant M_1 such that

$$\max_{1 \leq j \leq n} \left|x^{(j)}\right|_\infty < M_1.$$

Case (ii) Suppose that (C2) holds. Multiplying both sides of Eq. (3.9) by $x^{(n)}(t)$ and then integrating them from 0 to T , by (C2), (3.10) and the inequality of Schwarz, one has

$$
\begin{aligned}
L_2\left|x^{(n-1)}\right|_2^2 &= L_2\int_0^T x^{(n-1)}(t)x^{(n-1)}(t)\mathrm{d}t \\
&\leq \int_0^T\left[f\left(x^{(n-1)}(t)\right)-f(0)\right]x^{(n-1)}(t)\mathrm{d}t \\
&= -\int_0^T\sum_{i=1}^m g_i(t,x(t-\tau_i(t)))x^{(n-1)}(t)\mathrm{d}t \\
&\quad +\int_0^T e(t)x^{(n-1)}(t)\mathrm{d}t-\int_0^T f(0)x^{(n-1)}(t)\mathrm{d}t \\
&\leq \int_0^T\sum_{i=1}^m |g_i(t,x(t-\tau_i(t)))-g_i(t,0)|\left|x^{(n-1)}(t)\right|\mathrm{d}t \\
&\quad +\int_0^T|e(t)|\left|x^{(n-1)}(t)\right|\mathrm{d}t+\int_0^T\left[|f(0)|+\sum_{i=1}^m|g_i(t,0)|\right]\left|x^{(n-1)}(t)\right|\mathrm{d}t \\
&\leq \sum_{i=1}^m N_i\int_0^T|x(t-\tau_i(t))|\left|x^{(n-1)}(t)\right|\mathrm{d}t \\
&\quad +\int_0^T|e(t)|\left|x^{(n-1)}(t)\right|\mathrm{d}t+\int_0^T\left[|f(0)|+\sum_{i=1}^m|g_i(t,0)|\right]\left|x^{(n-1)}(t)\right|\mathrm{d}t \\
&\leq \sum_{i=1}^m N_i|x|_\infty\sqrt{T}\left|x^{(n-1)}\right|_2+\left[\max_{t\in[0,T]}\left\{|f(0)|+\sum_{i=1}^m|g_i(t,0)|\right\}+|e|_\infty\right]\sqrt{T}\left|x^{(n-1)}\right|_2 \\
&\leq \frac{\sum_{i=1}^m N_i}{2}T|x'|_2\left|x^{(n-1)}\right|_2+\left[\max_{t\in[0,T]}\left\{|f(0)|+\sum_{i=1}^m|g_i(t,0)|\right\}+|e|_\infty\right]\sqrt{T}\left|x^{(n-1)}\right|_2 \\
&\leq \frac{\sum_{i=1}^m N_i}{2}\frac{T^n}{(2\pi)^{n-2}}\left|x^{(n-1)}\right|_2^2+\left[\max_{t\in[0,T]}\left\{|f(0)|+\sum_{i=1}^m|g_i(t,0)|\right\}+|e|_\infty\right]\sqrt{T}\left|x^{(n-1)}\right|_2
\end{aligned}
$$

This implies that there exists a constant $\overline{D}_2>0$ such that

$$|x^{(j)}(t)|\leq\sqrt{T}|x^{(j+1)}|_2<\overline{D}_2. \tag{3.14}$$

Multiplying $x^{(n)}(t)$ and Eq. (3.9) and integrating it from 0 to T, by (C2), (3.10), (3.11), (3.14) and the inequality of Schwarz, one obtains

$$
\begin{aligned}
\left|x^{(n)}\right|_2^2 &= \int_0^T \left|x^{(n)}(t)\right|^2 \mathrm{d}t \\
&\leq \int_0^T \sum_{i=1}^m [|g_i(t,x(t-\tau_i(t)))-g_i(t,0)|+|g(t,0)|] \left|x^{(n)}(t)\right| \mathrm{d}t + \int_0^T |e(t)| \left|x^{(n)}(t)\right| \mathrm{d}t \\
&\leq \int_0^T \sum_{i=1}^m N_i |x(t-\tau_i(t))| \left|x^{(n)}(t)\right| \mathrm{d}t + \int_0^T \sum_{i=1}^m |g_i(t,0)| \left|x^{(n)}(t)\right| \mathrm{d}t + \int_0^T |e(t)| \left|x^{(n)}(t)\right| \mathrm{d}t \\
&\leq \frac{\sum_{i=1}^m N_i}{2} \sqrt{T} |x'|_2 \sqrt{T} \left|x^{(n)}\right|_2 + \left[\max_{t\in[0,T]} \left\{\sum_{i=1}^m |g_i(t,0)|\right\} + |e|_\infty\right] \sqrt{T} \left|x^{(n)}\right|_2 \\
&\leq \frac{\sum_{i=1}^m N_i}{2} T\overline{D}_2 \left|x^{(n)}\right|_2 + \left[\max_{t\in[0,T]} \left\{\sum_{i=1}^m |g_i(t,0)|\right\} + |e|_\infty\right] \sqrt{T} \left|x^{(n)}\right|_2,
\end{aligned}
$$

it follows from (3.13) that that there exists a positive constant $\overline{D}_1$ such that

$$\left|x^{(n-1)}(t)\right| \leq \sqrt{T} \left|x^{(n)}\right|_2 < \overline{D}_1. \tag{3.15}$$

Therefore, in view of (3.14) and (3.15), for all possible anti-periodic solutions $x(t)$ of (3.9), there exists a constant $\overline{M}_1$ such that

$$\max_{1\leq j\leq n-1} \left|x^{(j)}\right|_\infty < \overline{M}_1 \tag{3.16}$$

together with (3.16), which implies that

$$\max_{1\leq j\leq n-1} \left|x^{(j)}\right|_\infty < M_1 + \overline{M}_1 + 1 := M. \tag{3.17}$$

Set

$$\Omega = \left\{x \in C_T^{n-1,\frac{1}{2}} = X \mid \max_{1\leq j\leq n-1} \left|x^{(j)}\right|_\infty < M\right\}.$$

One knows that Eq. (3.9) has no anti-periodic solution on $\partial\Omega$ as $\lambda \in (0,1]$.

Now, one considers the Fourier series expansion of a function $x(t) \in C_T^{n-1,\frac{1}{2}}$. One has

$$x(t) = \sum_{i=0}^\infty \left[a_{2i+1} \cos\frac{2\pi(2i+1)t}{T} + b_{2i+1} \sin\frac{2\pi(2i+1)t}{T}\right].$$

Define a operator $L : C_T^{k,\frac{1}{2}} \to C_T^{k+1,\frac{1}{2}}$ by setting

$$
\begin{aligned}
(Lx)(t) &= \int_0^t x(s)\mathrm{d}s - \frac{T}{2\pi} \sum_{i=0}^\infty \frac{b_{2i+1}}{2i+1} \\
&= \frac{T}{2\pi} \sum_{i=0}^\infty \left[\frac{a_{2i+1}}{2i+1} \sin\frac{2\pi(2i+1)t}{T} - \frac{b_{2i+1}}{2i+1} \cos\frac{2\pi(2i+1)t}{T}\right].
\end{aligned}
$$

Then

$$\frac{\mathrm{d}(Lx)(t)}{\mathrm{d}t} = x(t),$$

and

$$\begin{aligned}|(Lx)(t)| &\leq \int_0^T |x(s)|\mathrm{d}s + \frac{T}{2\pi}\sum_{i=0}^{\infty}\frac{|b_{2i+1}|}{2i+1} \\ &\leq T\|T\| + \frac{T}{2\pi}\left(\sum_{i=0}^{\infty} b_{2i+1}^2\right)^{\frac{1}{2}}\left(\frac{1}{(2i+1)^2}\right)^{\frac{1}{2}}.\end{aligned}$$

In view of

$$\left[\frac{1}{(2i+1)^2}\right]^{\frac{1}{2}} = \frac{\pi}{2\sqrt{2}},$$

and the Parseval equality

$$\int_0^T |x(s)|^2\mathrm{d}s = \frac{T}{2}\sum_{i=0}^{\infty}(a_{2i+1}^2 + b_{2i+1}^2),$$

one obtains

$$\begin{aligned}|(Lx)(t)| &\leq T\|x\| + \frac{T}{4\sqrt{2}}\left(\sum_{i=0}^{\infty}(a_{2i+1}^2 + b_{2i+1}^2)\right)^{\frac{1}{2}} \\ &\leq T\|x\| + \frac{T}{4\sqrt{2}}\left(\frac{2}{T}\int_0^T |x(s)|^2\mathrm{d}s\right)^{\frac{1}{2}} \\ &\leq \left(T + \frac{T}{4}\right)\|x\|, \ \forall\, t \in [0,T].\end{aligned}$$

Thus, $|(Lx)(t)| \leq \left(T + \frac{T}{4}\right)\|x\|$ and the operator L is continuous.

For all $t \in C_T^{n-1,\frac{1}{2}}$, from (C1), one gets

$$Q_1\left(t + \frac{T}{2}, x\left(t + \frac{T}{2}\right), x^{(n-1)}\left(t + \frac{T}{2}\right)\right) = -Q_1\left(t, x(t), x^{(n-1)}(t)\right).$$

Therefore, $Q_1(t, x(t), x'(t)) \in C_T^{0,\frac{1}{2}}$. Define a operator $F_\mu : \overline{\Omega} \to C_T^{n,\frac{1}{2}} \subset X$ by setting

$$F_\mu(x) = \mu L(\cdots L(L(Q_1(x)))) = \mu L^n(Q_1(x)), \mu \in [0,1].$$

It is easy to see from the Arzela-Ascoli Lemma that F_μ is a compact homotopy, and the fixed point of F_1 on $\overline{\Omega}$ is the anti-periodic solution of Eq. (1.1). Define the homotopic continuous field as follows

$$H_\mu(x) : \overline{\Omega} \times [0,1] \to C_T^{n-1,\frac{1}{2}},\ H_\mu(x) = x - F_\mu(x).$$

Together with (3.17), one has

$$H_\mu(\partial\Omega) \neq 0,\ \mu \in [0,1].$$

Hence, using the homotopy invariance theorem, we obtain

$$\deg\{x - F_1x, \Omega, 0\} = \deg\{x, \Omega, 0\} \neq 0.$$

By now one knows that satisfies all the requirement in Lemma 2.1, and then $x - F_1x = 0$ has at least one solution in the Ω, i.e., F_1 has a fixed point on $\overline{\Omega}$. So, one has proved that Eq. (1.1) has a unique anti-periodic solution. This completes the proof of Theorem 3.3. □

4 Example

In this section, one gives an example to demonstrate the results obtained in previous sections.

Example 4.1. Let $g_1(t,x) = g_2(t,x) = (1+\cos^4(t))\frac{1}{12\pi}\cos x$. Then the Rayleigh equation

$$\begin{aligned} &x^{(3)}(t) + \frac{1}{8}x''(t) + \frac{1}{8}e^{-|\cos t|}\cos x''(t) \\ &+ g_1(t, x(t-\cos^2 t)) + g_2(t, x(t-\sin^2 t)) = \frac{1}{6\pi}\sin t, \end{aligned} \tag{4.1}$$

has a unique anti-periodic solution with period 2π.

Proof. One has $f(t,x) = \frac{1}{8}x(t) + \frac{1}{8}e^{-|\cos t|}\cos x(t)$, then

$$|f(t,x_1) - f(t,x_2)| \leq \frac{1}{4}|x_1 - x_2|, \quad \text{for all } t, x_1, x_2 \in \mathbb{R}.$$

Thus, $N_1 = N_2 = \frac{1}{6\pi}$, $L_1 = \frac{1}{4}$, $\tau_1(t) = \cos^2 t$, $\tau_2(t) = \sin^2 t$ and $e(t) = \frac{1}{6\pi}\sin t$. It is obvious that the assumptions (C1) holds. Therefore, in view of Theorem 3.3, Eq.(4.1) has a unique anti-periodic solution with period 2π. □

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