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On a Common Fixed Point Problem for Two Pairs of Maps Satisfying the Property (W.T)

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Abstract

We introduce a new property that we call the property (W.T). This property generalizes and unifies the concept of noncompatible mappings due to Jungck (1986), the property (E.A) of Aamri and Moutawakil (2002) and the concept of asymptotically regular maps due to Browder and Petryshyn (1966). We use this concept to prove a general common fixed point for two pairs of compatible maps under a contractive condition of Lipschitz type. We study also the well-posedness of the fixed point problem for these maps. Our main result extends, unifies and improves several recent results involving this Lipschitz type condition which is is not a contractive condition of the classical type. So our work provides some new contributions to the field of metric fixed point theory.

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Keywords: Common fixed point for four maps, weakly compatible maps, noncompatible maps, property (E.A), property (W.T), Meir-Keeler type contractive condition, Lipschitz type condition, well-posedness.

1 Introduction

Many recent papers of metric fixed point theory are devoted to the study of fixed points of a set of four self-mappings of a metric space satisfying some conditions.

Let (X,d) be a metric space and let A,B,S and T be four self-mappings of (X,d). To simplify notations, for all $x,y \in X$, we set

$$m(x,y) := \max\left\{d(Sx,Ty), d(Ax,Sx), d(By,Ty), \frac{d(Sx,By)+d(Ax,Ty)}{2}\right\} \tag{1.1}$$

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and

$$\sigma(x,y) := d(Sx,Ty) + d(Ax,Sx) + d(By,Ty) + d(Sx,By) + d(Ax,Ty). \tag{1.2}$$

The quantity (1.1) is involved in certain conditions of Meir-Keeler type.

A Meir-Keeler type (ε,δ)-contractive condition for the mappings A,B,S and T may be given in the form: given $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\varepsilon \leq m(x,y) < \varepsilon + \delta \Longrightarrow d(Ax,By) < \varepsilon. \tag{1.3}$$

In connection to the Meir-Keeler type (ε,δ)-contractive condition, we consider the following two conditions: given $\varepsilon > 0$ there exists $\delta > 0$ such that for all x,y in X

$$\varepsilon < m(x,y) < \varepsilon + \delta \Longrightarrow d(Ax,By) \leq \varepsilon, \tag{1.4}$$

and

$$d(Ax,By) < m(x,y), \quad \text{whenever} \quad m(x,y) > 0. \tag{1.5}$$

Jachymski [7] has shown that contractive condition (1.3) implies (1.4) but contractive condition (1.4) does not imply the contractive condition (1.3). Also, it is easy to see that the contractive condition (1.3) implies (1.5).

Condition (1.3) is not sufficient to ensure the existence of common fixed points of the maps A,B,S and T. Some kinds of commutativity or compatibility between the maps are always required. Also, other topological conditions on the maps or on their ranges are invoked.

Two self-mappings A and S of a metric space (X,d) are called compatible (see Jungck [10]) if,

$$\lim_{n\to\infty} d(ASx_n, SAx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t,$$

for some t in X.

This concept was of frequent use to establish existence theorems in common fixed point theory. The study on common fixed point theory for noncompatible mappings is also interesting. Work along these lines has been recently initiated by Pant [13], [14], [15].

In 2002, Aamri and Moutawakil [1] introduced a generalization of the concept of noncompatible mappings.

Definition 1.1. Let S and T be two self mappings of a metric space (X,d). We say that S and T satisfy property (E.A) if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = t$$

for some $t \in X$.

Remark 1.2. It is clear that two self-mappings of a metric space (X,d) will be noncompatible if there exists at least one sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = t$$

for some $t \in X$ but

$$\lim_{n\to\infty} d(STx_n, TSx_n)$$

is either non-zero or not exists.

Therefore two noncompatible self-mappings of a metric space (X,d) satisfy property (E.A).

Definition 1.3. [11]. Two self mappings S and T of a metric space (X,d) are said to be weakly compatible if $Tu = Su$, for some $u \in X$, then $STu = TSu$.

It is obvious that compatibility implies weak compatibility. Examples exist to show that the converse is not true.

To extend a result of K. Jha, R.P. Pant and S.L. Singh (see [8]), H. Bouhadjera and A. Djoudi (see [5]) have established the following result.

Theorem 1.4. *([5]) Let (A,S) and (B,T) be two weakly compatible pairs of self-mappings of a complete metric space (X,d) such that*

(a) *$AX \subseteq TX$ and $BX \subseteq SX$,*

(b) *one of AX, BX, SX or TX is closed,*

(c) *given $\varepsilon > 0$ there exists a $\delta > 0$ such that*

$$\varepsilon < M(x,y) < \varepsilon + \delta \Longrightarrow d(Ax,By) \leq \varepsilon, \quad \textit{and}$$

(c') *$x,y \in X$, $M(x,y) > 0 \Longrightarrow d(Ax,By) < M(x,y)$, where*

$$M(x,y) := \max\{d(Sx,Ty), d(Ax,Sx), d(By,Ty), [d(Sx,By) + d(Ax,Ty)]/2\}.$$

(d) *$d(Ax,By) \leq k[d(Sx,Ty) + d(Ax,Sx) + d(By,Ty) + d(Sx,By) + d(Ax,Ty)]$, for $0 \leq k < \frac{1}{3}$.*

Then A, B, S and T have a unique common fixed point.

A variant of Condition (d) together with other conditions of Meir-Keeler type are involved in the papers [16] and [17].

In this paper, we study the common fixed point problem for two pairs (A,S) and (B,T) of self-mappings of a complete metric space (X,d) which are satisfying the following Lipschitz type condition: there exists a constant $k \in [0,1]$ such that

$$d(Ax,By) \leq k[d(Sx,Ty) + d(Ax,Sx) + d(By,Ty) + d(Sx,By) + d(Ax,Ty)], \tag{1.6}$$

for all $x,y \in X$.

We discuss conditions on k and on the mappings A, B, S and T ensuring the existence of common fixed points, but without using conditions of Meir-Keeler type. Our main result (see Theorem 3.1) will make use of a new property called the property (W.T) which is weaker than the property (E.A) of Aamri and Moutawakil (see [1]). So our result will improve and generalize Thorem 1.1 of [5] and allow us to unify and generalize the results of [16] and [17] and other related papers.

In Section 2, we introduce the property (W.T). In Section 3, we establish our main result (see Theorem 3.1). In Section 4, we study the well-posedness of the fixed point problem for A, B, S and T satisfying the conditions of Theorem 3.1.

2 The property (W.T)

We introduce below the notion of weakly tangential mappings.

Definition 2.1. Let (X,d) be a metric space and $T,S:(X,d)\to(X,d)$ two self-mappings. S and T are said to be weakly tangential mappings if there exists a sequence $\{x_n\}$ of points in X such that

$$\lim_{n\to\infty} d(Sx_n,Tx_n)=0.$$

We say also that the pair (S,T) satisfies the property (W.T).

Remark 2.2. 1) We observe that if the mappings T and S satisfy the property (E.A), then S and T are weakly tangential.

2) We recall that Browder and Petryshyn (see [6]) have defined a selfmapping T on a metric space (X,d) to be asymptotically regular at a point x in X, if

$$\lim_{n\to\infty} d(T^n x, T^n Tx)=0, \tag{2.1}$$

where $T^n x$ denotes the n-th iterate of T at x.

Let $T:X\to X$ is a self-mapping of a metric space (X,d). For each point $x\in X$, we set $x_n:=T^n x$ for every non-negative integer n. We denote I the identity mapping. We observe that if T is asymptotically regular at the point x, then the mappings T and I are weakly tangential. Thus, the property (W.T) generalizes also the concept of asymptotically regular mappings.

To show that the notion of weakly tangential mappings is actually new, we give below an example.

Example 2.3. We set $X=[0,\infty)$ endowed with its usual metric. We consider the mappings $S,T:X\to X$ fefined by

$$Sx=x+\frac{2}{x+1}, \quad \text{and} \quad Tx=x+\frac{1}{x+1}, \quad \forall x\in[0,\infty).$$

Then we have the following observations.

(a) The pair (S,T) does not satisfy the property (E.A). Indeed, suppose the contrary and let $\{x_n\}$ be a sequence in X such that

$$\lim_{n\to\infty}(x_n+\frac{2}{x_n+1})=\lim_{n\to\infty}(x_n+\frac{1}{x_n+1})=t, \tag{2.2}$$

for some $t\in[0,\infty)$. We deduce from (2.2) that

$$\lim_{n\to\infty}\frac{1}{x_n+1}=0, \tag{2.3}$$

which is possible, only if $\lim_{n\to\infty}x_n=\infty$. This, by virtue of (2.2), implies that $t=\infty$, which is a contradiction.

(b) The pair (S,T) is weakly tangential. To see this, we consider the sequence $\{x_n\}$, where $x_n = n$, for all non negative integer n. Then we have

$$\lim_{n\to\infty} |Sx_n - Tx_n| = \lim_{n\to\infty} \frac{1}{n+1} = 0. \tag{2.4}$$

Thus the pair (S,T) is weakly tangential without satisfying the property (E.A).

Thus, the notion of weakly tangential mappings generalizes and unifies the notion of non compatible mappings, the property (E.A) for two mappings and the notion of asymptotic regularity of one mapping.

3 Main Result

We need the following assumptions:

(H.1) $AX \subseteq TX$ and $BX \subseteq SX$.

(H.2) One of AX, BX, SX or TX is a closed subspace of (X,d).

(H.3) $d(Ax,By) \le k\sigma(x,y)$, for all $x,y \in X$, where k is such that $0 \le k < \frac{1}{3}$.

The main result of this paper is given as follows:

Theorem 3.1. *Let (A,S) and (B,T) be two weakly compatible pairs of self-mappings of a complete metric space (X,d) and suppose* (H.1)-(H.2)-(H.3) *hold. If one of the pairs $\{A,S\}$ or $\{B,T\}$ satisfies the property (W.T), then A,B,S and T have a unique common fixed point.*

Proof. (I) Suppose that the pair $\{A,S\}$ satisfies the property (W.T). Then there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} d(Ax_n, Sx_n) = 0, \tag{3.1}$$

Since $AX \subseteq TX$, then for each integer n, there exists y_n in X such that

$$Ax_n = Ty_n. \tag{3.2}$$

By using $(H.3)$, we have

$$d(Ax_n,By_n) \le k[d(Sx_n,Ty_n) + d(Ax_n,Sx_n) + d(By_n,Ty_n) + d(Sx_n,By_n) + d(Ax_n,Ty_n)],$$

which implies

$$d(Ax_n,By_n) \le \frac{3k}{1-2k} d(Ax_n,Sx_n). \tag{3.3}$$

By letting n to infinity in (3.3), we obtain

$$\lim_{n\to\infty} d(Ax_n,By_n) = 0. \tag{3.4}$$

By (3.1) and (3.4), we get

$$\lim_{n\to\infty} d(Ax_n,Sx_n) = 0 = \lim_{n\to\infty} d(By_n,Ty_n). \tag{3.5}$$

(3.5) shows that both pairs $\{A,S\}$ and $\{B,T\}$ are satisfying the property (W.T).

We shall prove that the sequence $\{Ax_n\}$ is a Cauchy sequence.

By using the assumption (H.3), we have

$$\begin{aligned}
& d(Ax_m,Ax_n) \leq d(Ax_m,By_n) + d(By_n,Ax_n) \\
\leq\ & k[d(Sx_m,Ty_n) + d(Sx_m,Ax_m) + d(Ty_n,By_n) + d(Sx_m,By_n) + d(Ax_m,Ty_n)] \\
+\ & d(Ax_n,By_n) \\
\leq\ & k[d(Sx_m,Ax_m) + d(Ax_m,Ax_n) + d(Sx_m,Ax_m) + d(Ty_n,By_n) \\
+\ & d(Sx_m,Ax_m) + d(Ax_m,Ax_n) + d(Ax_n,By_n) + d(Ax_m,Ax_n)] + d(Ax_n,By_n) \\
=\ & 3kd(Sx_m,Ax_m) + (1+2k)d(Ax_n,By_n) + 3kd(Ax_m,Ax_n).
\end{aligned}$$

Therefore, we have

$$d(Ax_m,Ax_n) \leq \frac{3k}{1-3k} d(Sx_m,Ax_m) + \frac{1+2k}{1-3k} d(Ax_n,By_n). \tag{3.6}$$

From (3.4), (3.5) and (3.6), we deduce that

$$\lim_{n,m\to\infty} d(Ax_m,Ax_n) = 0, \tag{3.7}$$

which implies that the sequence $\{Ax_n\}$ is a Cauchy sequence. Since X is complete, then there exists a point (say) z in X such that the sequence $\{Ax_n\}$ converges to z. By virtue of (3.2) and (3.5), we conclude that we have

$$z = \lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = \lim_{n\to\infty} By_n. \tag{3.8}$$

(1) Suppose that $A(X)$ is a closed subspace of (X,d). Then $z \in A(X)$. Since $AX \subseteq TX$, then there exists $u \in X$ such that $z = Tu$. By (H.3), we get

$$d(Ax_n,Bu) \leq k[d(Sx_n,Tu) + d(Ax_n,Sx_n) + d(Bu,Tu) + d(Sx_n,Bu) + d(Ax_n,Tu)],$$

which, by letting $n \to \infty$, implies that

$$d(z,Bu) \leq 2kd(z,Bu). \tag{3.9}$$

Since $\leq k < \frac{1}{3}$, then it follows from (3.9) that $z = Bu$. Thus, we have $z = Tu = Bu$.

Since $B(X) \subset S(X)$, then there exists $v \in X$ such that $Bu = Sv$. Then $z = Tu = Bu = Sv$. By applying the inequality (H.3), we get

$$\begin{aligned}
d(Av,Sv) &= d(Av,Bu) \\
&\leq k[d(Sv,Tu) + d(Av,Sv) + d(Bu,Tu) + d(Sv,Bu) + d(Av,Tu)] \\
&= 2kd(Av,Sv),
\end{aligned}$$

which implies that $Av = Sv$. Hence, we obtain

$$z = Tu = Bu = Sv = Av. \tag{3.10}$$

The conclusions in (3.10) will be obtained by similar arguments, if we suppose that $T(X)$, $B(X)$ or $S(X)$ is a closed subspace of X.

(2) Since $\{A,S\}$ and $\{B,T\}$ are weakly compatible, it follows

$$Bz = Tz \quad \text{and} \quad Az = Sz \tag{3.11}$$

Now, we show that $z = Az$. To this end, we start by observing that

$$\sigma(z,u) = d(Sz,Tu) + d(Az,Sz) + d(Bu,Tu) + d(Sz,Bu) + d(Az,Tu) = 3d(Az,z).$$

So, by virtue of the assumption (H.3), we get

$$d(Az,z) = d(Az,Bu) \leq k\sigma(z,u) = 3kd(Az,z),$$

which (since $k \in [0, \frac{1}{3})$ implies that $d(Az,z) = 0$. Thus we get $z = Az$. Hence, we obtain $z = Az = Sz$.

Now, we show that $z = Bz$. We observe that

$$\begin{aligned}\sigma(v,z) &= d(Sv,Tz) + d(Av,Sv) + d(Bz,Tz) + d(Sv,Bz) + d(Av,Tz)\\ &= d(z,Bz) + d(z,Bz) + d(z,Bz) = 3d(z,Bz).\end{aligned}$$

By virtue of the assumption (H.3), we get

$$d(z,Bz) = d(Av,Bz) \leq k\sigma(v,z) = 3kd(z,Bz),$$

which implies that $d(Bz,z) = 0$. Thus we have $z = Bz = Tz$. Hence, we have

$$z = Bz = Tz = Az = Sz.$$

We conclude that z is a common fixed point for A,B,S and T.

(II) If we suppose that the pair $\{B,T\}$ satisfies the property (W.T), then by similar arguments we obtain the same conclusions as in the part (I).

(III) It remains to show the uniqueness of the fixed common fixed point z. Suppose that w is another common fixed point for the mappings A,B,S and T, such that $w \neq z$. Obviously we have $\sigma(w,z) = 3d(w,z) > 0$. Then, by applying the condition (H 3), we obtain

$$d(w,z) = d(Aw,Bz) \leq k\sigma(w,z) = 3kd(w,z),$$

which is a contradiction. So the mappings A,B,S and T have a unique common fixed point. This completes the proof. □

As a consequence, we have the following.

Corollary 3.2. *Let (A,S) and (B,T) be two weakly compatible pairs of self-mappings of a complete metric space (X,d) and suppose* (H.1)-(H.2)-(H.3) *hold. If one of the following four conditions is satisfied.*

(i) A and S are noncompatible, or

(ii) the pair (A,S) satisfies the property (E.A), or

(iii) B and T are noncompatible, or

(iv) the pair (B,T) satisfies the property (E.A).

Then the mappings A,B,S and T have a unique common fixed point.

4 Well-posedness

Several authors have studied the notion of well-posednes of a fixed point problem for a mapping (see for example [20], [4], [12], [18], [19], [2] and [3]).

Definition 4.1. Let (X,d) be a metric space and $T:(X,d)\to(X,d)$ be a mapping. The fixed point problem of T is said to be well posed if:

(i) T has a unique fixed point z in X,

(ii) for any sequence $\{x_n\}$ of points in X such that $\lim_{n\to\infty} d(Tx_n,x_n)=0$, we have $\lim_{n\to\infty} d(x_n,z)=0$.

The following definition is a natural generalization of 4.1.

Definition 4.2. Let (X,d) be a metric space and let $\mathcal{A}$ be a set of sel-fmappings $T:X\to X$. The fixed point problem of the collection $\mathcal{A}$ is said to be well-posed if:

(i) the set $\mathcal{A}$ has a unique strict fixed point z in X,

(ii) for any sequence $\{x_n\}$ of points in X such that

$$\lim_{n\to\infty} d(Tx_n,x_n)=0, \quad \forall T\in\mathcal{A},$$

we have $\lim_{n\to\infty} d(x_n,z)=0$.

With respect to this definition, we establish the well-posedness of the common fixed point problem for the set of four selfmappings S,T,I,J of a metric space (X,d) satisfying the conditions of Theorem 3.1.

Theorem 4.3. *Let (A,S) and (B,T) be two weakly compatible pairs of self-mappings of a complete metric space (X,d) and suppose* (H.1)-(H.2)-(H.3) *hold. If one of the pairs $\{A,S\}$ or $\{B,T\}$ satisfies the property (W.T), then the fixed point problem of A,B,S and T is well-posed..*

Proof. By Theorem 3.1, the mappings A,S,B,T have a unique common fixed point z in X. Let $\{u_n\}$ be a sequence in X such that

$$\lim_{n\to\infty} d(Au_n,u_n)=\lim_{n\to\infty} d(Su_n,u_n)=\lim_{n\to\infty} d(Bu_n,u_n)=\lim_{n\to\infty} d(Tu_n,u_n)=0. \tag{5.1}$$

We have to show that $\lim_{n\to\infty} d(u_n,z)=0$. By using the inequality (H.3) and the triangle inequality, we have successively

$$\begin{aligned}
d(u_n,z) &\leq d(u_n,Au_n)+d(Au_n,Bz)\\
&\leq d(u_n,Au_n)+k[d(Su_n,z)+d(Su_n,Au_n)+0+d(Su_n,z)+d(Au_n,z)]\\
&\leq d(u_n,Au_n)+k[3d(Su_n,u_n)+3d(u_n,z)+2d(Au_n,u_n)]\\
&= 3kd(u_n,z)+3kd(Su_n,u_n)+(1+2k)d(Au_n,u_n),
\end{aligned}$$

from which we obtain

$$d(u_n,z) \leq \frac{3k}{1-3k} d(Su_n,u_n) + \frac{1+2k}{1-3k} d(Au_n,u_n).$$

Letting n go to infinity, we obtain

$$\lim_{n\to\infty} d(u_n,z) = 0,$$

which implies that the strict fixed point problem for the mappings A,B,S,T is well posed. □

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