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REITERATED HOMOGENIZATION OF LINEAR EIGENVALUE PROBLEMS IN MULTISCALE PERFORATED DOMAINS BEYOND THE PERIODIC SETTING

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Abstract

Reiterated homogenization of linear elliptic Neuman eigenvalue problems in multiscale perforated domains is considered beyond the periodic setting. The classical periodicity hypothesis on the coefficients of the operator is here substituted on each microscale by an abstract hypothesis covering a large set of concrete behaviors such as the periodicity, the almost periodicity, the weakly almost periodicity and many more besides. Furthermore, the usual double periodicity is generalized by considering a type of structure where the perforations on each scale follow not only the periodic distribution but also more complicated but realistic ones. Our main tool is Nguetseng's Sigma convergence.

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1 Introduction

We are interested in the spectral asymptotics (as $\varepsilon \to 0$) of the linear elliptic eigenvalue problem

$$\begin{cases} -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}(x,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2})\frac{\partial u_\varepsilon}{\partial x_j}\right) = \lambda_\varepsilon u_\varepsilon \text{ in } \Omega^\varepsilon \\ \sum_{i,j=1}^{N} a_{ij}(x,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2})\frac{\partial u_\varepsilon}{\partial x_j}\nu_i = 0 \text{ on } \partial T^\varepsilon \\ u_\varepsilon = 0 \text{ on } \partial\Omega \\ \int_{\Omega^\varepsilon} |u_\varepsilon|^2 dx = 1, \end{cases} \tag{1.1}$$

where Ω is a bounded open set in $\mathbb{R}^N_x$(the numerical space of variables $x = (x_1,...,x_N)$, with integer $N \geq 2$) with Lipschitz boundary $\partial\Omega$, $a_{ij} \in \mathcal{C}(\overline{\Omega}; L^\infty(\mathbb{R}^N_y \times \mathbb{R}^N_z))$ with the symmetry condition $a_{ji} = \overline{a}_{ij}$ and the ellipticity condition: there exists $\alpha > 0$ such that for any $x \in \overline{\Omega}$

$$\mathrm{Re} \sum_{i,j=1}^{N} a_{ij}(x,y,z)\xi_j\overline{\xi_i} \geq \alpha|\xi|^2 \tag{1.2}$$

for all $\xi \in \mathbb{C}^N$ and for almost all $(y,z) \in \mathbb{R}^N_y \times \mathbb{R}^N_z$, where $|\xi|^2 = |\xi_1|^2 + \cdots + |\xi_N|^2$.

The set Ω^ε ($\varepsilon > 0$) is a domain perforated on two scales defined as follows. Let S_y (resp. S_z) be an infinite subset of $\mathbb{Z}^N$ and let T_y (resp. T_z) be a closed subset of the unit cube $Y = (-\frac{1}{2},\frac{1}{2})^N$ (resp. $Z = (-\frac{1}{2},\frac{1}{2})^N$) in $\mathbb{R}^N_y$ (resp. $\mathbb{R}^N_z$). For $\varepsilon > 0$, we define

$$t^\varepsilon_y = \{k \in S_y : \varepsilon(k+T_y) \subset \Omega\}, \;\; t^\varepsilon_z = \{k \in S_z : \varepsilon^2(k+T_z) \subset \Omega\},$$

$$T^\varepsilon_y = \bigcup_{k\in t^\varepsilon_y} \varepsilon(k+T_y), \;\; T^\varepsilon_z = \bigcup_{k\in t^\varepsilon_z} \varepsilon^2(k+T_z),$$

$$T^\varepsilon = T^\varepsilon_y \cup T^\varepsilon_z$$

and

$$\Omega^\varepsilon = \Omega \setminus T^\varepsilon.$$

In this setup, T_y, T_z are the reference holes whereas $\varepsilon(k+T_y)$ and $\varepsilon^2(k+T_z)$ are holes of size ε and ε^2 respectively and T^ε is the collection of the holes (obstacles, inclusions) of the perforated domain Ω^ε. Each of the families T^ε_y, T^ε_z is made up with a finite number of holes and can be empty (for fixed ε) since Ω is bounded. Finally, $\nu = (\nu_i)$ denotes the outer unit normal vector to ∂T^ε with respect to Ω^ε. Except where otherwise stated, the letter E denotes throughout this paper a sequence of strictly positive real numbers ($\varepsilon > 0$) verifying the following: zero is an accumulation point of E and for any $\varepsilon \in E$, Ω^ε is such that the tiny holes T^ε_z do not intersect the boundary of the big holes ∂T^ε_y . This assumption is for example satisfied if we pick $\varepsilon > 0$ such that the domain $\overline{Y}$ is exactly covered by a finite number of cells $\overline{\varepsilon Z}$ and suppose that T_y is approximately covered by a finite number of cells εZ (this is a restriction on the geometry of T_y), and then consider the family $E = \{\frac{\varepsilon}{2^n}\}_{n\in\mathbb{N}}$. This assumption is crucial for the construction of an appropriate extension operator (Proposition 3.1).

The spectral asymptotic problem under consideration is a reiterated homogenization problem in a domain perforated on two scales. But as opposed to what is usually done, we do not make any periodicity assumption on the behavior of the coefficients a_{ij} nor any double periodicity assumption on the inclusions. Our problem is therefore beyond the scope of periodic homogenization but still nonstochastic. Reiterated homogenization, porous media and asymptotic spectral problems have been extensively studied and it is beyond the scope of this article to provide extensive references on any of these topics. In the following paragraphs we direct the reader to some relevant papers in these topics.

Problem devoted to reiterated homogenization were first consider by Bruggeman[13] in the 30's. In 1978 Bensoussan, Lions and Papanicolaou[9] proved a result for linear operators which has been known later as the iterated homogenization theorem. That theorem was generalized by Allaire and Briane[2] by means of multiscale convergence method (which is a generalization of the two-scale convergence method introduced by Nguetseng[35] and further developed by Allaire[1]). The corresponding Gamma-convergence result was obtained by Braides and Lukkassen[12]. we refer to [4, 8, 23, 29, 32, 33, 34, 40, 45, 46, 51] for some recent developments in this theory.

Perforated media are nowadays widely used in various domains and have lot of applications in petroleum engineering and fluids dynamic in particular. Homogenization of partial differential equations in perforated domains has been attracting the attention of an increasing number of researchers since the pioneering work of Cioranescu and Saint Jean Paulin[19]. For a detailed bibliography we refer to [2, 21, 24, 28, 34, 49, 50] and the references therein. For one decade the inclusions were on one scale and were periodically distributed in all the works dealing with non stochastic homogenization in porous media. Later on, T. Levy[28] consider a kind of structure with a double periodicity (periodic perforation on two scales) to study the Stokes problem in a porous fissured rock. In that direction we also mention the works [14, 20, 21]. This type of structure was generalized to multiscale perforation by Allaire and Briane[2] but still the holes were periodically distributed on each scale.

Recently, Nguetseng[38] released homogenization in perforated domains from the classical periodic perforation hypothesis by considering a more general situation where the periodic perforation is replaced by an abstract hypothesis covering a great set of concrete behaviors such as the equiperforation (usually refereed to as periodic perforation), the periodic perforation, the almost periodic perforation and others. But so far, in all the works[38, 49, 50] in perforated domains à la Nguetseng, the inclusions are always in one scale. On the one hand, we generalize the concept of multiscale periodic perforation and on the other hand, Nguetseng's deterministic perforation is upgraded by considering a two-scale deterministic perforation. For example the tiny holes, T_z^ε, could be concentrated in a neighborhood of a point whereas the big ones, T_y^ε, are almost periodically distributed. We believe this is a true advance in the study of perforated domains.

The spectral asymptotics of eigenvalue is a very important problem and has been widely explored (see e.g, [3, 5, 7, 24, 25, 26, 27, 41, 47, 48] and the references therein). Homogenization of eigenvalue problems in a fixed domains goes back to Kesavan [25, 26]. In a perforated domains it was first studied by Vanninathan[48] where he considered the dirichlet, Neumann and Stekloof eigenvalue problems for the Laplace operator ($a_{ij} = \delta_{ij}$ (Kronecker symbol)) and combined asymptotic expansion with Tartar's energy method to prove an homogenization result for the said problems. We also mention the works [24, 27] on eigenvalue problems in perforated domains. We replace here the Laplace operator by an elliptic linear differential operator of order two in divergence form with variable coefficients depending on the macroscopic variable and two microscopic variables. On each microscopic scale, the behavior of the coefficients may not only be periodic but also almost periodic and many other including the weakly almost periodic one. It is worth noticing that the homogenization process carried out in [48] is quite fastidious to adapt to the rather easy case when the coefficients a_{ij} are periodic on each microscale and the domaine is double periodically perforated. We only deal with the Neumann eigenvalue problem in this paper. We obtain a very accurate, precise and concise homogenization result (Theorem 3.10).

Unless otherwise specified, vector spaces throughout are considered over the complex field, $\mathbb{C}$, and scalar functions are assumed to take complex values. Let us recall some basic notation. If X and F denote a locally compact space and a Banach space, respectively, then we write $\mathcal{C}(X;F)$ for the continuous mappings from X into F, and $\mathcal{B}(X;F)$ for those mappings in $\mathcal{C}(X;F)$ that are bounded. We shall assume $\mathcal{B}(X;F)$ to be equipped with the supremum norm $\|u\|_\infty = \sup_{x\in X}\|u(x)\|_F$. For shortness we will write $\mathcal{C}(X)$ for $\mathcal{C}(X;\mathbb{C})$ and $\mathcal{B}(X)$ for $\mathcal{B}(X;\mathbb{C})$. Likewise in the case when $F = \mathbb{C}$, the usual spaces $L^p(X,F)$ and $L^p_{loc}(X,F)$ (X provided with a positive Radon measure) will be denoted by $L^p(X)$ and $L^p_{loc}(X)$, respectively. Finally, the numerical space $\mathbb{R}^N$ and its open sets are provided with Lebesgue measure denoted by $dx = dx_1...dx_N$ and sometimes by $|\cdot|$.

The rest of the paper is organized as follows. In Section 2 we recall some facts about reiterated Sigma convergence. Section 3 deals with the homogenization of the abstract problem for (1.1) and some concrete problems are worked out in Section 4 by way of illustration.

2 Reiterated Σ-convergence

The Σ-convergence method is a combination of the generalized Besicovitch spaces[15, 16] (that are built on algebras with mean value[37, 52]) with the multiscale convergence method[2] (which is a generalization of the two-scale convergence method introduced by Nguetseng[35] and further developed by Allaire[1]). We start this section with fundamentals of algebras with mean value then we recall some facts about the generalized Besicovitch spaces and the reiterated Σ-convergence method. We give some examples of algebras with mean value eventually.

2.1 Fundamentals of algebras with mean value

The concept of algebra with mean value (algebra wmv) was introduced by Zhikov and Krivenko [52] and further developed by Nguetseng[37] to extend to more general classes of oscillatory functions (such as almost periodic functions and others) the theory of periodic homogenization.

Let m be a positive integer. Let $\mathcal{H} = (H_\varepsilon)_{\varepsilon>0}$ be either of the following actions of

$\mathbb{R}_+^*$(the multiplicative group of positive real numbers) on $\mathbb{R}^m$, defined as follows:

$$H_\varepsilon(x) = \frac{x}{\varepsilon} \qquad (x \in \mathbb{R}^m) \tag{2.1}$$

$$H_\varepsilon(x) = \frac{x}{\varepsilon^2} \qquad (x \in \mathbb{R}^m). \tag{2.2}$$

Given $\varepsilon > 0$, let

$$u^\varepsilon(x) = u(H_\varepsilon(x)) \qquad (x \in \mathbb{R}^m) \tag{2.3}$$

for $u \in L^1_{\text{loc}}(\mathbb{R}^m_y)$ (as usual, $\mathbb{R}^m_y$ denotes the numerical space $\mathbb{R}^m$ of variables $y = (y_1, ..., y_m)$), u^ε lies in $L^1_{\text{loc}}(\mathbb{R}^m_x)$. More generally, if u lies in $L^p_{\text{loc}}(\mathbb{R}^m)$ (resp. $L^p(\mathbb{R}^m)$), $1 \leq p < +\infty$, then so does u^ε.

A function $u \in \mathcal{B}(\mathbb{R}^m_y)$ (space of bounded uniformly continuous complex functions on $\mathbb{R}^m_y$) is said to have a mean value for $\mathcal{H}$ if a complex number $M(u)$ exists such that $u^\varepsilon \to M(u)$ in $L^\infty(\mathbb{R}^m_x)$-weak $*$ as $\varepsilon \to 0$. The complex number $M(u)$ is called the mean value of u for $\mathcal{H}$. There is no difficulty in verifying that this define a positive linear form (on the space of $u \in \mathcal{B}(\mathbb{R}^m_y)$ with mean value), invariant by translation, attaining the value 1 on the constant function 1 and verifying the inequality $|M(u)| \leq \|u\|_\infty$ for all such u's. The mapping M is called the mean value on $\mathbb{R}^m$ for $\mathcal{H}$. Moreover we have

$$M(u) = \lim_{R \to +\infty} \frac{1}{|B_R|} \int_{B_R} u(y)dy \tag{2.4}$$

where B_R stands for the open ball in $\mathbb{R}^m$ with radius R, and $|B_R|$ denotes its Lebesgue measure. Indeed let R be a positive real number. Set either $\varepsilon = 1/R$ if $H_\varepsilon(x) = x/\varepsilon$ or $\varepsilon = 1/R^{1/2}$ if $H_\varepsilon(x) = x/\varepsilon^2$. Then, as $R \to +\infty$, $\varepsilon \to 0$. We assume without lost of generality that $H_\varepsilon(x) = x/\varepsilon$ so that $\varepsilon = 1/R$. With this, since $u^\varepsilon \to M(u)$ in $L^\infty(\mathbb{R}^m)$-weak $*$, we have $\int u^{1/R} \chi_{B_1} dx \to M(u) |B_1|$ as $R \to +\infty$, where B_1 denotes the unit open ball in $\mathbb{R}^m$ and χ_{B_1} the characteristic function of B_1. But $\int u^{1/R} \chi_{B_1} dx = \int_{B_1} u(Rx)dx$, and a change of variable $y = Rx$ gives

$$\frac{1}{|B_1|} \int_{B_1} u(Rx)dx = \frac{1}{R^m |B_1|} \int_{B_R} u(y)dy = \frac{1}{|B_R|} \int_{B_R} u(y)dy,$$

hence our claim is justified. Moreover, expression (2.4) holds for $u \in L^1_{loc}(\mathbb{R}^m)$ whenever the limit therein makes sense.

Definition 2.1. By an algebra with mean value (algebra wmv) on $\mathbb{R}^m$ for $\mathcal{H}$ is meant any Banach subalgebra of $\mathcal{B}(\mathbb{R}^m)$ which contains the constants, is translation invariant (i.e., for every $u \in A$ and every $a \in \mathbb{R}^m$, $\tau_a(u) \equiv u(\cdot - a) \in A$) and whose elements possess a mean value for $\mathcal{H}$.

Let A be an algebra wmv on $\mathbb{R}^m$ for $\mathcal{H}$. Clearly A (with the sup norm topology) is a commutative $\mathcal{C}^*$-algebra with identity (the involution is here the usual one of complex conjugation). We denote by $\Delta(A)$ the spectrum of A and by $\mathcal{G}$ the Gelfand transformation on A. We recall that $\Delta(A)$ (a subset of the topological dual A' of A) is the set of all nonzero multiplicative linear forms on A, and $\mathcal{G}$ is the mapping of A into $\mathcal{C}(\Delta(A))$ (the complex continuous functions on $\Delta(A)$) such that $\mathcal{G}(u)(s) = \langle s, u \rangle$ $(s \in \Delta(A))$, where $\langle\ ,\ \rangle$ denotes the duality pairing between A' and A. The topology on $\Delta(A)$ is the relative weak $*$ topology on A'. We recap in the following theorem the most important result about algebras wmv (see [37] for details).

Theorem 2.2. *Let A be an algebra wmv on $\mathbb{R}^m$. Then*

(i) The spectrum $\Delta(A)$ is a compact space and the Gelfand transformation $\mathcal{G}$ is an isometric isomorphism of the $\mathcal{C}^*$-algebra A onto the $\mathcal{C}^*$-algebra $\mathcal{C}(\Delta(A))$.

*(ii) The mean value M considered as defined on A is representable by some Radon probability measure β (called the M-*measure for *A) as follows:*

$$M(u) = \int_{\Delta(A)} \mathcal{G}(u)d\beta \text{ for any } u \in A.$$

The notion of the spectrum of an algebra wmv seems to be too abstract at a first glance but it is not. In the case when A is the periodic algebra wmv $\mathcal{C}_{\text{per}}(Y)$ of Y-periodic continuous functions on $\mathbb{R}^m_y$ ($Y = (-\frac{1}{2}, \frac{1}{2})^m$), $\Delta(A)$ can be identified with the m-dimensional torus $\mathbb{T}^m = \mathbb{R}^m/\mathbb{Z}^m$. Let $\mathcal{R}$ be any subgroup of $\mathbb{R}^m$ and let $AP_{\mathcal{R}}(\mathbb{R}^m_y)$ denote the algebra of functions on $\mathbb{R}^m_y$ that can be uniformly approximated by finite linear combinations of functions in the set $\{\gamma_k : k \in \mathcal{R}\}$ where γ_k is defined by $\gamma_k(y) = \exp(2i\pi k \cdot y)$ ($y \in \mathbb{R}^m$). It is known that $AP_{\mathcal{R}}(\mathbb{R}^m_y)$ is an algebra wmv [42, 51] and its spectrum $\Delta(AP_{\mathcal{R}}(\mathbb{R}^m_y))$ is a compact topological group homeomorphic to the dual group $\widehat{\mathcal{R}}$ of $\mathcal{R}$ consisting of the characters γ_k ($k \in \mathcal{R}$) of $\mathbb{R}^m$.

Next, the partial derivative of index i ($1 \leq i \leq m$) on $\Delta(A)$ is defined to be the mapping $\partial_i = \mathcal{G} \circ D_{y_i} \circ \mathcal{G}^{-1}$ (usual composition) of $\mathcal{D}^1(\Delta(A)) = \{\varphi \in \mathcal{C}(\Delta(A)) : \mathcal{G}^{-1}(\varphi) \in A^1\}$ into $\mathcal{C}(\Delta(A))$, where $A^1 = \{\psi \in \mathcal{C}^1(\mathbb{R}^m) : \psi, D_{y_i}\psi \in A \ (1 \leq i \leq m)\}$. Higher order derivatives are defined analogously. At the present time, let A^∞ be the space of $\psi \in \mathcal{C}^\infty(\mathbb{R}^m_y)$ such that $D^\alpha_y \psi = \frac{\partial^{|\alpha|}\psi}{\partial y_1^{\alpha_1} \cdots \partial y_m^{\alpha_m}} \in A$ for every $\alpha = (\alpha_1, ..., \alpha_m) \in \mathbb{N}^m$, and let $\mathcal{D}(\Delta(A)) = \{\varphi \in \mathcal{C}(\Delta(A)) : \mathcal{G}^{-1}(\varphi) \in A^\infty\}$. Endowed with a suitable locally convex topology (see [37]), A^∞ (resp. $\mathcal{D}(\Delta(A))$) is a Fréchet space and further, $\mathcal{G}$ viewed as defined on A^∞ is a topological isomorphism of A^∞ onto $\mathcal{D}(\Delta(A))$.

Aiming at defining Sobolev spaces on $\Delta(A)$, a distribution on $\Delta(A)$ is defined as expected to be a continuous linear form on $\mathcal{D}(\Delta(A))$. The space of all distributions on $\Delta(A)$ is the dual, $\mathcal{D}'(\Delta(A))$, of $\mathcal{D}(\Delta(A))$. We endow $\mathcal{D}'(\Delta(A))$ with the strong dual topology. The following result whose proof can be found in [50] allows us to view $L^p(\Delta(A))$ ($1 \leq p \leq \infty$) as a subspace of $\mathcal{D}'(\Delta(A))$, and helps to define Sobolev type spaces on the spectrum of an algebra wmv.

Proposition 2.3. *Let A be an algebra wmv on $\mathbb{R}^m$. Then the space A^∞ is dense in A.*

The above result amounts to $\mathcal{D}(\Delta(A))$ ($= \mathcal{G}(A^\infty)$) is dense in $\mathcal{C}(\Delta(A))$ ($= \mathcal{G}(A)$) so that $L^p(\Delta(A))$ is continuously embedded in $\mathcal{D}'(\Delta(A))$ since $\mathcal{C}(\Delta(A))$ is dense in $L^p(\Delta(A))$. Hence we define the Sobolev space

$$W^{1,p}(\Delta(A)) = \{u \in L^p(\Delta(A)) : \ \partial_i u \in L^p(\Delta(A)) \ (1 \leq i \leq m)\}$$

where the derivative $\partial_i u$ is taken in the distribution sense on $\Delta(A)$ (exactly as the Schwartz derivative is taken in the classical case). We equip $W^{1,p}(\Delta(A))$ with the norm

$$||u||_{W^{1,p}(\Delta(A))} = ||u||_{L^p(\Delta(A))} + \sum_{i=1}^{m} ||\partial_i u||_{L^p(\Delta(A))} \quad \left(u \in W^{1,p}(\Delta(A))\right),$$

which makes it a Banach space. However, in practice it proves necessary to consider the space

$$W^{1,p}(\Delta(A))/\mathbb{C} = \left\{ u \in W^{1,p}(\Delta(A)) : \int_{\Delta(A)} u(s)d\beta(s) = 0 \right\}$$

instead of $W^{1,p}(\Delta(A))$. This is clearly a closed vector subspace of $W^{1,p}(\Delta(A))$. Provided with the seminorm

$$||u||_{W^{1,p}(\Delta(A))/\mathbb{C}} = \sum_{i=1}^{m} ||\partial_i u||_{L^p(\Delta(A))} \qquad \left(u \in W^{1,p}(\Delta(A))/\mathbb{C}\right),$$

$W^{1,p}(\Delta(A))/\mathbb{C}$ is in general nonseparate and noncomplete. We denote by $W^{1,p}_{\#}(\Delta(A))$ the separated completion of $W^{1,p}(\Delta(A))/\mathbb{C}$ and by J the canonical mapping of $W^{1,p}(\Delta(A))/\mathbb{C}$ into its separated completion. $W^{1,p}_{\#}(\Delta(A))$ is a Banach space and $W^{1,2}_{\#}(\Delta(A))$ is a Hilbert space. Furthermore, as pointed out in [37], the distribution derivative ∂_i viewed as a mapping of $W^{1,p}(\Delta(A))/\mathbb{C}$ into $L^p(\Delta(A))$ extends to an unique continuous linear mapping, still denoted by ∂_i, of $W^{1,p}_{\#}(\Delta(A))$ into $L^p(\Delta(A))$ such that $\partial_i J(v) = \partial_i v$ for $v \in W^{1,p}(\Delta(A))/\mathbb{C}$ and

$$||u||_{W^{1,p}_{\#}(\Delta(A))} = \sum_{i=1}^{m} ||\partial_i u||_{L^p(\Delta(A))} \text{ for } u \in W^{1,p}_{\#}(\Delta(A)).$$

However, the notion of product of algebras wmv (see e.g., [50]) will be of great importance since the homogenization problems consider here fall within the framework of reiterated homogenization. We first define the product action of the preceding actions (2.1) and (2.2), by

$$\mathcal{H}^* = (H_\varepsilon)_{\varepsilon>0} \tag{2.5}$$

$$H^*_\varepsilon(x,x') = \left(\frac{x}{\varepsilon}, \frac{x'}{\varepsilon^2}\right) \quad ((x,x') \in \mathbb{R}^m \times \mathbb{R}^m). \tag{2.6}$$

In the sequel, action (2.2) will be denoted by $\mathcal{H}' = (H'_\varepsilon)_{\varepsilon>0}$, that is, $H'_\varepsilon(x) = x/\varepsilon^2$ ($x \in \mathbb{R}^m$).

This being so, if A_y (resp. A_z) is an algebra wmv on $\mathbb{R}^m_y$ (resp. $\mathbb{R}^m_z$) for $\mathcal{H}$ (resp. $\mathcal{H}'$), we define the product algebra wmv of A_y and A_z to be the closure in $\mathcal{B}(\mathbb{R}^m_y \times \mathbb{R}^m_z)$ of the tensor product $A_y \otimes A_z = \{\sum_{\text{finite}} u_i \otimes v_i : u_i \in A_y, v_i \in A_z\}$. This clearly defines an algebra wmv on $\mathbb{R}^m \times \mathbb{R}^m$ for $\mathcal{H}^*$ denoted by $A_y \odot A_z$.

The following result whose proof can be found in [40] enhances the comprehension of the notion of product of algebras wmv. We have

Theorem 2.4. *Let $A = A_y \odot A_z$ where A_y and A_z are as above. For $f \in \mathcal{B}(\mathbb{R}^{m+m}_{y,z})$, we define $f_y \in \mathcal{B}(\mathbb{R}^m_z)$ and $f^z \in \mathcal{B}(\mathbb{R}^m_y)$ by*

$$f_y(z) = f^z(y) = f(y,z) \text{ for } (y,z) \in \mathbb{R}^m_y \times \mathbb{R}^m_z$$

and put

$$A_f = \{f_y : y \in \mathbb{R}^N\},\ B_f = \{f^z : z \in \mathbb{R}^m\}.$$

Then $A_f \subset A_z$ and $B_f \subset A_y$ for every $f \in A$. Also for $f \in A$ both A_f and B_f are relatively compact in A_z and in A_y respectively (in the sup norm topology).

Corollary 2.5. *Let $A_y = AP(\mathbb{R}^m_y)$ and $A_z = AP(\mathbb{R}^m_z)$ be two almost periodic algebras wmv. Then $A \equiv A_y \odot A_z = AP(\mathbb{R}^m_y \times \mathbb{R}^m_z)$.*

Proof. The result is a consequence of the following fact: A function $f \in AP(\mathbb{R}^m_y \times \mathbb{R}^m_z)$ is in A if and only if either A_f or B_f is relatively compact (in the sup norm topology). □

Corollary 2.6. *Let $A_y = AP_{\mathcal{R}_y}(\mathbb{R}^m_y)$ and $A_z = AP_{\mathcal{R}_z}(\mathbb{R}^m_z)$ be two almost periodic Algebras wmv, where $\mathcal{R}_y$ and $\mathcal{R}_z$ are two subgroups of $\mathbb{R}^m_y$ and $\mathbb{R}^m_z$, respectively. Then $A \equiv A_y \odot A_z = AP_{\mathcal{R}_y \times \mathcal{R}_z}(\mathbb{R}^m_y \times \mathbb{R}^m_z)$.*

Proof. Since $\Delta(A_\zeta)$ $(\zeta = y, z)$ can be identify with the dual group $\widehat{\mathcal{R}_\zeta}$ of $\mathcal{R}_\zeta$, the result is a direct consequence of the equality $\widehat{\mathcal{R}_y} \times \widehat{\mathcal{R}_z} = \widehat{\mathcal{R}_y \times \mathcal{R}_z}$. □

Before we can recall some facts about reiterated Σ -convergence, we need a further notion, that of

2.2 The generalized Besicovitch spaces.

Let A be an algebra wmv on $\mathbb{R}^m$ and let $1 \le p < \infty$. For $u \in A$ we have $|u|^p \in A$ with $\mathcal{G}(|u|^p) = |\mathcal{G}(u)|^p$ so that the limit $\lim_{R\to\infty} \frac{1}{|B_R|}\int_{B_R} |u(y)|^p dy$ exists and moreover we have

$$\lim_{R\to\infty} \frac{1}{|B_R|}\int_{B_R} |u(y)|^p dy = M(|u|^p) = \int_{\Delta(A)} |\mathcal{G}(u)|^p d\beta.$$

By putting $\|u\|_p = (M(|u|^p))^{\frac{1}{p}}$ for $u \in A$, we define a seminorm on A with which A is not complete. We denote by B^p_A the completion of A with respect to $\|\cdot\|_p$. B^p_A is a Frechet space and moreover [10], B^p_A is a complete subspace of $L^p_{loc}(\mathbb{R}^m)$. It is straightforward from the theory of completion that A is dense in B^p_A and if F is a Banach space, any continuous linear mapping l from A to F extends by continuity to a unique continuous linear mapping L of B^p_A into F.

Owing to the fact that $B^q_A \subset B^p_A$ for $1 \le p \le q < \infty$, We define B^∞_A as follow:

$$B^\infty_A = \left\{ f \in \bigcap_{1\le p<\infty} B^p_A : \sup_{1\le p<\infty} \|f\|_p < \infty \right\}.$$

We endow B^∞_A with the seminorm $[f] = \sup_{1\le p<\infty}\|f\|_p$ which makes it a Frechet space. The following properties are worth recalling(see e.g. [37, 43, 51]).

1. The Gelfand transformation $\mathcal{G} : A \to \mathcal{C}(\Delta(A))$ extends by continuity to a unique continuous linear mapping still denoted by $\mathcal{G}$, of B^p_A into $L^p(\Delta(A))$. Furthermore, if $u \in B^p_A \cap L^\infty(\mathbb{R}^m)$ then $\mathcal{G}(u) \in L^\infty(\Delta(A))$ and

$$\|\mathcal{G}(u)\|_{L^\infty(\Delta(A))} \le \|u\|_{L^\infty(\mathbb{R}^m)}.$$

2. The mean value view as defined on A, extends by continuity to a positive continuous linear form (still denoted by M) on B_A^p satisfying

$$M(u) = \int_{\Delta(A)} \mathcal{G}(u) d\beta \qquad (u \in B_A^p).$$

Furthermore, for each $u \in B_A^p$ and all $a \in \mathbb{R}^m$, $M(\tau_a u) = M(u)$ where $\tau_a u(y) = u(y-a)$ for almost all $y \in \mathbb{R}^m$.

3. Let $1 \leq p, q, r < \infty$ be such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} \leq 1$. The usual multiplication $A \times A \to A; (u, v) \mapsto uv$, extends by continuity to a bilinear form $B_A^p \times B_A^q \to B_A^r$ with

$$\|uv\|_r \leq \|u\|_p \|v\|_q \text{ for } (u, v) \in B_A^p \times B_A^q.$$

As a direct consequence of Proposition 2.3 one has the following

Proposition 2.7. *If A is an algebra wmv on $\mathbb{R}^m$ then the space A^∞ is dense in B_A^p.*

If $u \in B_A^p$ then $|u|^p \in B_A^1$ so that by the above properties, one has

$$M(|u|^p) = \int_{\Delta(A)} |\mathcal{G}(u)|^p d\beta = \|\mathcal{G}(u)\|^p_{L^p(\Delta(A))},$$

which implies that $\|u\|_p = 0$ if and only if $\mathcal{G}(u) = 0$. But the mapping $\mathcal{G}$ defined on B_A^p is not injective in general. Put $\mathcal{N} = ker\mathcal{G}$ (the kernel of $\mathcal{G}$) and let

$$\mathcal{B}_A^p = B_A^p / \mathcal{N}.$$

Endowed with the norm

$$\|u + \mathcal{N}\|_{\mathcal{B}_A^p} = \|u\|_p \qquad (u \in B_A^p),$$

$\mathcal{B}_A^p$ is a Banach space and the mapping $\mathcal{G} : B_A^p \to L^p(\Delta(A))$ induces[51, Theorem 3.5] an isometric isomorphism $\mathcal{G}_1$ of $\mathcal{B}_A^p$ onto $L^p(\Delta(A))$. We may then define the mean value of $u + \mathcal{N}$ (for $u \in B_A^p$) as follow

$$M_1(u + \mathcal{N}) = M(u) \quad \left(= \lim_{R \to +\infty} \frac{1}{|B_R|} \int_{B_R} u(y) dy \right).$$

The main properties of the spaces $\mathcal{B}_A^p$ are recaped in the following (see e.g, [15, 43, 51])

Proposition 2.8. *The following hold true:*

(i) The spaces $\mathcal{B}_A^p$ are reflexive for $1 < p < \infty$;

(ii) The topological dual of $\mathcal{B}_A^p$ $(1 \leq p < \infty)$ is $\mathcal{B}_A^{p'}$ $(p' = p/(p-1))$, the duality pairing being given by

$$\langle u + \mathcal{N}, v + \mathcal{N} \rangle_{\mathcal{B}_A^{p'}, \mathcal{B}_A^p} = M(uv) = \int_{\Delta(A)} \mathcal{G}_1(u + \mathcal{N}) \mathcal{G}_1(v + \mathcal{N}) d\beta$$

for $u \in \mathcal{B}_A^{p'}$ and $v \in \mathcal{B}_A^p$;

(iii) The space $\mathcal{B}_A^p$ $(1 \leq p < \infty)$ *is the separated completion of* B_A^p *and the canonical mapping of* B_A^p *into* $\mathcal{B}_A^p$ *is just the canonical surjection of* B_A^p *onto* $\mathcal{B}_A^p$.

We now discuss ergodic algebras wmv.

Definition 2.9. An algebra wmv A on $\mathbb{R}^m$ is termed ergodic if for every $u \in B_A^1$ such that $\|u - u(\cdot + a)\|_1 = 0$ for all $a \in \mathbb{R}^m$, we have $\|u - M(u)\|_1 = 0$.

The following characterization of ergodicity is due to Casado and Gayte [15].

Proposition 2.10. *An algebra wmv on* $\mathbb{R}^m$ *is ergodic if and only if*

$$\lim_{R\to+\infty} \left\| \frac{1}{|B_R|} \int_{B_R} u(\cdot + y)dy - M(u) \right\|_p = 0 \text{ for all } u \in B_A^p,\ 1 \leq p < \infty. \tag{2.7}$$

Meanwhile in practice the following lemma whose proof can be found in [40, 42] proves useful.

Lemma 2.11. *Let A be an algebra wmv on* $\mathbb{R}^m$ *with the following property:*

$$\lim_{R\to+\infty} \frac{1}{|B_R|} \int_{B_R} u(x+y)dx = M(u) \text{ uniformly with respect to } y. \tag{2.8}$$

Then A is ergodic.

For the sake of simplicity we denote in the sequel by the same letter u (if no confusion is possible) an element of B_A^p and its equivalence class $u + \mathcal{N}$. The symbol ρ will stand for the canonical mapping of B_A^p onto $\mathcal{B}_A^p$. Our goal here is to define the Besicovitch analogue of the space $H_\#^1(Y)$ of the periodic setting. Let $u \in L^p(\Delta(A))$, and let $1 \leq i \leq m$. We know that $\partial_i u \in \mathcal{D}'(\Delta(A))$ exists and is defined by

$$\langle \partial_i u, \varphi \rangle = -\langle u, \partial_i \varphi \rangle \text{ for any } \varphi \in \mathcal{D}(\Delta(A)).$$

If we assume further that $\partial_i u \in L^p(\Delta(A))$, then there exists a unique $u_i \in \mathcal{B}_A^p$ such that $\partial_i u = \mathcal{G}_1(u_i)$. We are led to the following

Definition 2.12. *By a* formal derivative *of index* $1 \leq i \leq m$, *of a function* $u \in \mathcal{B}_A^p$ *is meant the unique element* $\overline{\partial} u/\partial y_i$ *of* $\mathcal{B}_A^p$ *(if there exists) such that*

$$\mathcal{G}_1\left(\overline{\partial} u/\partial y_i\right) = \partial_i \mathcal{G}_1(u). \tag{2.9}$$

Before we proceed, let us clarify the just defined derivative. For $u \in B_A^{1,p}$ (that is the space of $u \in B_A^p$ such that $D_y u \in (B_A^p)^m$) we have

$$\mathcal{G}_1\left(\rho\left(\frac{\partial u}{\partial y_i}\right)\right) = \mathcal{G}\left(\frac{\partial u}{\partial y_i}\right) = \partial_i \mathcal{G}(u) = \partial_i \mathcal{G}_1(\rho(u)) = (\textit{by definition})\ \mathcal{G}_1\left(\frac{\overline{\partial}}{\partial y_i}(\rho(u))\right),$$

hence

$$\rho \circ \frac{\partial}{\partial y_i} = \frac{\overline{\partial}}{\partial y_i} \circ \rho \text{ on } B_A^{1,p}. \tag{2.10}$$

Now, for $1 \leq p < \infty$ set $\mathcal{B}_A^{1,p} = \left\{ u \in \mathcal{B}_A^p : \frac{\overline{\partial} u}{\partial y_i} \in \mathcal{B}_A^p, \text{ for } 1 \leq i \leq m \right\}$ and endow it with the norm

$$\|u\|_{\mathcal{B}_A^{1,p}} = \left[\|u\|_p^p + \sum_{i=1}^{m} \left\| \frac{\overline{\partial} u}{\partial y_i} \right\|_p^p \right]^{1/p} \quad (u \in \mathcal{B}_A^{1,p})$$

which makes it a Banach space with the interesting property that the restriction of $\mathcal{G}_1$ to $\mathcal{B}_A^{1,p}$ is an isometric isomorphism of $\mathcal{B}_A^{1,p}$ onto $W^{1,p}(\Delta(A))$. However in practice the subspace $\mathcal{B}_A^{1,p}/\mathbb{C}$ of $\mathcal{B}_A^{1,p}$ consisting of functions $u \in \mathcal{B}_A^{1,p}$ with $M_1(u) \equiv M(u) = 0$ is more adequate. Equipped with the seminorm

$$\|u\|_{\mathcal{B}_A^{1,p}/\mathbb{C}} = \left\| \overline{D}_y u \right\|_p := \left[\sum_{i=1}^{m} \left\| \frac{\overline{\partial} u}{\partial y_i} \right\|_p^p \right]^{1/p} \quad (u \in \mathcal{B}_A^{1,p}/\mathbb{C})$$

where $\overline{D}_y = (\overline{\partial}/\partial y_i)_{1 \leq i \leq m}$, $\mathcal{B}_A^{1,p}/\mathbb{C}$ is a locally convex topological space which is in general nonseparate and noncomplete. We denote by $\mathcal{B}_{\#A}^{1,p}$ the separated completion of $\mathcal{B}_A^{1,p}/\mathbb{C}$ with respect to $\|\cdot\|_{\mathcal{B}_A^{1,p}/\mathbb{C}}$, and by J_1 the canonical mapping of $\mathcal{B}_A^{1,p}/\mathbb{C}$ into $\mathcal{B}_{\#A}^{1,p}$. By the theory of completion of the uniform spaces [6, Chapitre II] the mapping $\overline{\partial}/\partial y_i : \mathcal{B}_A^{1,p}/\mathbb{C} \to \mathcal{B}_A^p$ extends by continuity to a unique continuous linear mapping still denoted by $\overline{\partial}/\partial y_i : \mathcal{B}_{\#A}^{1,p} \to \mathcal{B}_A^p$ and satisfying

$$\frac{\overline{\partial}}{\partial y_i} \circ J_1 = \frac{\overline{\partial}}{\partial y_i} \text{ and } \|u\|_{\mathcal{B}_{\#A}^{1,p}} = \left\| \overline{D}_y u \right\|_p \quad (u \in \mathcal{B}_{\#A}^{1,p}) \tag{2.11}$$

where $\overline{D}_y = (\overline{\partial}/\partial y_i)_{1 \leq i \leq m}$. Since $\mathcal{G}_1$ is an isometric isomorphism of $\mathcal{B}_A^{1,p}$ onto $W^{1,p}(\Delta(A))$ we have by (2.9) that the restriction of $\mathcal{G}_1$ to $\mathcal{B}_A^{1,p}/\mathbb{C}$ sends isometrically and isomorphically $\mathcal{B}_A^{1,p}/\mathbb{C}$ onto $W^{1,p}(\Delta(A))/\mathbb{C}$. So by [6] there exists a unique isometric isomorphism $\overline{\mathcal{G}}_1 : \mathcal{B}_{\#A}^{1,p} \to W_{\#}^{1,p}(\Delta(A))$ such that

$$\overline{\mathcal{G}}_1 \circ J_1 = J \circ \mathcal{G}_1 \tag{2.12}$$

and

$$\partial_i \circ \overline{\mathcal{G}}_1 = \mathcal{G}_1 \circ \frac{\overline{\partial}}{\partial y_i} \quad (1 \leq i \leq m). \tag{2.13}$$

We recall that J is the canonical mapping of $W^{1,p}(\Delta(A))/\mathbb{C}$ into its separated completion $W_{\#}^{1,p}(\Delta(A))$ while J_1 is the canonical mapping of $\mathcal{B}_A^{1,p}/\mathbb{C}$ into $\mathcal{B}_{\#A}^{1,p}$. Furthermore, as $J_1(\mathcal{B}_A^{1,p}/\mathbb{C})$ is dense in $\mathcal{B}_{\#A}^{1,p}$ (this is classical), it follows that $(J_1 \circ \rho)(A^\infty/\mathbb{C})$ is dense in $\mathcal{B}_{\#A}^{1,p}$, where $A^\infty/\mathbb{C} = \{u \in A^\infty : M(u) = 0\}$, since A^∞ is dense in A. We are now in a position to introduce

2.3 The $R\Sigma$-convergence

Throughout this subsection Ω is an open subset of $\mathbb{R}^N$ (integer $N \geq 1$) and $A = A_y \odot A_z$ is an algebra wmv on $\mathbb{R}_y^N \times \mathbb{R}_z^N$ for the product action $\mathcal{H}$ defined by (2.3), A_y and A_z being algebras wmv on $\mathbb{R}_y^N$ and $\mathbb{R}_z^N$, respectively. We use the same letter $\mathcal{G}$ to denote the Gelfand transformation on A_y, A_z and A, as well when there is no danger of confusion, but keep in mind that $\mathcal{G} = \mathcal{G}_y \otimes \mathcal{G}_z$ (see [37, Corollary 3.1]). Points in $\Delta(A_y)$ (resp. $\Delta(A_z)$) are denoted by s (resp. r). We still denote by M the mean value on $\mathbb{R}^N$ for $\mathcal{H}$ and for $\mathcal{H}'$, and on $\mathbb{R}^{2N}$ for $\mathcal{H}^*$ as well. The compact space $\Delta(A_y)$ (resp. $\Delta(A_z)$) is equipped with the M-measure β_y (resp. β_z) for A_y (resp. A_z). It is fundamental to recall that we have $\Delta(A) = \Delta(A_y) \times \Delta(A_z)$ and further the M-measure for A, with which $\Delta(A)$ is equipped, is precisely the product measure $\beta = \beta_y \otimes \beta_z$ (see [37, Corollary 3.2]). We may now introduce the concept of $R\Sigma$-convergence which is a generalization of that of multiscale convergence [2].

Definition 2.13. A sequence $(u_\varepsilon)_{\varepsilon>0} \subset L^p(\Omega)$ $(1 \leq p < \infty)$ is said to :
(i) weakly $R\Sigma$-converge in $L^p(\Omega)$ to some $u_0 \in L^p(\Omega; \mathcal{B}_A^p)$ if as $\varepsilon \to 0$, we have

$$\int_\Omega u_\varepsilon(x) f\left(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}\right) dx \to \iint_{\Omega\times\Delta(A)} \widehat{u}_0(x,s,r)\widehat{f}(x,s,r)dxd\beta \tag{2.14}$$

for every $f \in L^{p'}(\Omega;A)$ $(1/p' = 1-1/p)$;
(ii) strongly $R\Sigma$-converge in $L^p(\Omega)$ to some $u_0 \in L^p(\Omega;\mathcal{B}_A^p)$ if the following condition is fulfilled:

> Given $\eta > 0$ and $f \in L^p(\Omega, A)$ with $\|\widehat{u}_0 - \widehat{f}\|_{L^p(\Delta(A))} \leq \frac{\eta}{2}$, there is some $\alpha > 0$ such that $\|u_\varepsilon - f^\varepsilon\|_{L^p(\Omega)} \leq \eta$ provided $\varepsilon \leq \alpha$;

where $\widehat{u}_0 = \mathcal{G}_1 \circ u_0$ and $\widehat{f} = \mathcal{G}_1 \circ (\rho \circ f) = \mathcal{G} \circ f$.

Notation. We express this by writing $u_\varepsilon \to u_0$ in $L^p(\Omega)$-weak $R\Sigma$ in case (i) and $u_\varepsilon \to u_0$ in $L^p(\Omega)$-strong $R\Sigma$ in case (ii), where the letter "R" stands for reiteratively.

Remark 2.14. Due to the equality $\mathcal{G}_1(\mathcal{B}_A^p) = L^p(\Delta(A))$ one immediately sees that the right-hand side of (2.14) is equal to

$$\int_\Omega M(u_0(x,\cdot,\cdot)f(x,\cdot,\cdot))dx,$$

and as usual $u_\varepsilon \to u_0$ in $L^p(\Omega)$-weak $R\Sigma$ implies $u_\varepsilon \to M(u_0(x,\cdot,\cdot))$ in $L^p(\Omega)$-weak. The uniqueness of the limit u_0 is ensured since the above definition is exactly the one given by Nguetseng[37], up to the previous equality. In particular when $A = \mathcal{C}_{per}(Y) \odot \mathcal{C}_{per}(Z)$ one is led at once to the convergence result

$$\int_\Omega u_\varepsilon(x) f\left(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}\right) dx \to \int_\Omega \int_Y \int_Z u_0(x,y,z) f(x,y,z) dzdydx$$

where $u_0 \in L^p(\Omega \times Y \times Z)$, which is the original definition of the multiscale convergence[2].

We now state the most important results of this section, we refer to [16, 40, 42, 43, 51] for the proofs. In the following three theorems the letter E denotes a fundamental sequence, that is, any ordinary sequence $E = (\varepsilon_n)_{n\in\mathbb{N}}$ with $0 < \varepsilon_n \leq 1$ and $\varepsilon_n \to 0$ as $n \to \infty$.

Theorem 2.15. *Any bounded sequence $(u_\varepsilon)_{\varepsilon\in E}$ in $L^p(\Omega)$ $(1 < p < \infty)$ admits a subsequence which is weakly $R\Sigma$-convergent in $L^p(\Omega)$.*

For $p = 1$ we have the following

Theorem 2.16. *Any uniformly integrable sequence $(u_\varepsilon)_{\varepsilon\in E}$ in $L^1(\Omega)$ admits a subsequence which is weakly $R\Sigma$-convergent in $L^1(\Omega)$.*

We recall that a sequence $(u_\varepsilon)_{\varepsilon>0}$ in $L^1(\Omega)$ is said to be uniformly integrable if $(u_\varepsilon)_{\varepsilon>0}$ is bounded in $L^1(\Omega)$ and further $\sup_{\varepsilon>0}\int_X |u_\varepsilon|\,dx \to 0$ as $|X|\to 0$ (X being an integrable set in Ω with $|X|$ denoting the Lebesgue measure of X).

The last and the most important of these compactness result is the following

Theorem 2.17. *Let $1<p<\infty$ and Ω be an open subset in $\mathbb{R}^N$. Let $A=A_y\odot A_z$ where A_y (resp. A_z) is an ergodic algebra wmv on $\mathbb{R}^N_y$ (resp. $\mathbb{R}^N_z$). Finally, let $(u_\varepsilon)_{\varepsilon\in E}$ be a bounded sequence in $W_0^{1,p}(\Omega)$. There exist a subsequence E' from E and a triple $\mathbf{u}=(u_0,u_1,u_2)\in W_0^{1,p}(\Omega)\times L^p(\Omega;\mathcal{B}^{1,p}_{\#A_y})\times L^p(\Omega;\mathcal{B}^p_{A_y}(\mathbb{R}^N_y;\mathcal{B}^{1,p}_{\#A_z}))$ such that, as $E'\ni\varepsilon\to 0$,*

$$u_\varepsilon \to u_0 \text{ in } W_0^{1,p}(\Omega)\text{-weak} \tag{2.15}$$

and

$$\frac{\partial u_\varepsilon}{\partial x_j} \to \frac{\partial u_0}{\partial x_j} + \frac{\overline{\partial} u_1}{\partial y_j} + \frac{\overline{\partial} u_2}{\partial z_j} \text{ in } L^p(\Omega)\text{-weak } R\Sigma\ (1\le j\le N). \tag{2.16}$$

Before giving a few examples of algebras wmv which satisfy hypotheses of Theorems 2.15, 2.16 and 2.17, it is to be noted that although being often used in the periodic setting, Theorem 2.17 has been rigourously proved for the first time in the general framework of H-algebra in [33].

2.3.1 The periodic algebra wmv

Let $A_y=\mathcal{C}_{\text{per}}(Y)$ ($Y=(-\frac{1}{2},\frac{1}{2})^N$) be the algebra of Y-periodic continuous functions on $\mathbb{R}^N_y$. It is classically known that A_y is an ergodic algebra wmv so that Theorems 2.15, 2.16 and 2.17 apply with $A=A_y\odot A_z$ ($A_y=A_z$). Bear in mind that $\mathcal{C}_{\text{per}}(Y)\odot\mathcal{C}_{\text{per}}(Y)=\mathcal{C}_{\text{per}}(Y\times Y)$.

2.3.2 The almost periodic algebra wmv

Let $AP(\mathbb{R}^N)$ be the algebra of Bohr continuous almost periodic functions $\mathbb{R}^N$. We recall that a function $u\in\mathcal{B}(\mathbb{R}^N)$ is in $AP(\mathbb{R}^N)$ if the set of translates $\{\tau_a u : a\in\mathbb{R}^N\}$ is relatively compact in $\mathcal{B}(\mathbb{R}^N)$. Equivalently [10], $u\in AP(\mathbb{R}^N)$ if and only if u may be uniformly approximated by finite linear combinations of functions in the set $\{\gamma_k : k\in\mathbb{R}^N\}$ where $\gamma_k(y)=\exp(2i\pi k\cdot y)$ ($y\in\mathbb{R}^N$). It is also a classical result that A_y is an ergodic algebra wmv(see e.g. [52]). Therefore Theorems 2.15, 2.16 and 2.17 apply with $A_y=AP(\mathbb{R}^N_y)$ and $A_z=AP(\mathbb{R}^N_z)$.

Now, let $\mathcal{R}$ be any subgroup of $\mathbb{R}^N$. We denote by $AP_{\mathcal{R}}(\mathbb{R}^N_y)$ the space of those functions in $AP(\mathbb{R}^N_y)$ that can be uniformly approximated by finite linear combinations in the set $\{\gamma_k : k\in\mathcal{R}\}$. Then $A_y=AP_{\mathcal{R}}(\mathbb{R}^N_y)$ is an ergodic algebra wmv[43] so that the conclusions of all the three preceding theorems still hold with $AP_{\mathcal{R}}(\mathbb{R}^N)$ in place of $AP(\mathbb{R}^N)$.

2.3.3 The algebra wmv of convergence at infinity

Let $\mathcal{B}_\infty(\mathbb{R}^N)$ denote the space of all continuous functions on $\mathbb{R}^N$ that converge finitely at infinity, that is the space of all $u \in \mathcal{B}(\mathbb{R}^N)$ such that $\lim_{|y|\to\infty} u(y) \in \mathbb{C}$. One can easily check as in [22] that the space $\mathcal{B}_\infty(\mathbb{R}^N)$ is an ergodic algebra wmv. Indeed, by [22] any $u \in \mathcal{B}_\infty(\mathbb{R}^N)$ is uniformly continuous and in addition, $\mathcal{B}_\infty(\mathbb{R}^N)$ is translation invariant and the mean value of a function u is given by $M(u) = \lim_{|y|\to\infty} u(y)$. Therefore we have the conclusions of Theorems 2.15, 2.16 and 2.17 with $A_y = A_z = \mathcal{B}_\infty(\mathbb{R}^N)$.

2.3.4 The weakly almost periodic algebra wmv

The concept of weakly almost periodic function is due to Eberlein[22]. A continuous function u on $\mathbb{R}^N$ is weakly almost periodic if the set of translates $\{\tau_a u : a \in \mathbb{R}^N\}$ is relatively weakly compact in $\mathcal{C}(\mathbb{R}^N)$. We denote by $WAP(\mathbb{R}^N_y)$ the set of all weakly almost periodic functions on $\mathbb{R}^N_y$ which is a vector space over $\mathbb{C}$. Endowed with the norm sup topology, $WAP(\mathbb{R}^N_y)$ is a Banach algebra with the usual multiplication. As examples of Eberlein's functions we have the continuous Bohr almost periodic functions, the continuous functions vanishing at infinity, the positive definite functions (hence Fourier-Stieltjes transforms). $WAP(\mathbb{R}^N_y)$ is a translation invariant $\mathcal{C}^*$-subalgebra of $\mathcal{C}(\mathbb{R}^N_y)$ whose elements are uniformly continuous, bounded and possess a mean value with

$$M(u) = \lim_{R\to+\infty} \frac{1}{|B_R|} \int_{B_R} u(y+a)dy, \qquad (u \in WAP(\mathbb{R}^N_y)),$$

the convergence being uniform in $a \in \mathbb{R}^N$. Moreover, every $u \in WAP(\mathbb{R}^N_y)$ admits the unique decomposition $u = v + w$, v being a Bohr almost periodic function and w a continuous function of quadratic mean value zero: $M(|w|^2) = 0$. Hence denoting the complete vector subspace of $WAP(\mathbb{R}^N_y)$ consisting of elements of $WAP(\mathbb{R}^N_y)$ with quadratic mean value zero, one has

$$WAP(\mathbb{R}^N_y) = AP(\mathbb{R}^N_y) \oplus W_0(\mathbb{R}^N_y).$$

With this in mind, let $\mathcal{R}$ be a subgroup of $\mathbb{R}^N$ and set

$$WAP_{\mathcal{R}}(\mathbb{R}^N_y) = AP_{\mathcal{R}}(\mathbb{R}^N_y) \oplus W_0(\mathbb{R}^N_y) \tag{2.17}$$

(bear in mind that $WAP_{\mathcal{R}}(\mathbb{R}^N_y) = WAP(\mathbb{R}^N_y)$ when $\mathcal{R} = \mathbb{R}^N$). Then $WAP_{\mathcal{R}}(\mathbb{R}^N_y)$ is an ergodic algebra wmv[43].

We have the same conclusion as in the preceding examples with $A_y = A_z = WAP_{\mathcal{R}}(\mathbb{R}^N)$. Also, since any algebra wmv of all the preceding examples is a subalgebra of $WAP(\mathbb{R}^N)$, the conclusions of Theorems 2.15, 2.16 and 2.17 follow if we take instead of $WAP_{\mathcal{R}}(\mathbb{R}^N)$, any of the algebras of the above examples. In particular, Theorem 2.17 holds with $A = WAP_{\mathcal{R}}(\mathbb{R}^N_y) \odot AP(\mathbb{R}^N_z)$.

3 Homogenization of the abstract problem.

We make use of the assumptions and notations introduced earlier in Section 1. Before we can state and solve our abstract problem, we need a few details.

3.1 Preliminaries

Let $\varepsilon \in E$ be arbitrarily fixed and define

$$V_\varepsilon = \{u \in H^1(\Omega^\varepsilon) : u = 0 \text{ on } \partial\Omega\}.$$

We equip V_ε with the $H^1(\Omega^\varepsilon)$-norm which makes it a Hilbert space. The following proposition provides us with an appropriate extension operator.

Proposition 3.1. *For each $\varepsilon \in E$ there exists an operator P_ε of V_ε into $H_0^1(\Omega)$ with the following properties:*

- *P_ε sends continuously and linearly V_ε into $H_0^1(\Omega)$.*
- *$(P_\varepsilon v)|_{\Omega^\varepsilon} = v$ for all $v \in V_\varepsilon$.*
- *$\|D(P_\varepsilon v)\|_{L^2(\Omega)^N} \leq c\|Dv\|_{L^2(\Omega^\varepsilon)^N}$ for all $v \in V_\varepsilon$, where c is a constant independent of ε and D denotes the usual gradient operator.*

Proof. According to [38, Lemma 2.3], there are two operators P'_ε of V_ε into $W_\varepsilon = \{u \in H^1(\Omega \setminus T_y^\varepsilon) : u = 0 \text{ on } \partial\Omega\}$ and P''_ε of W_ε into $H_0^1(\Omega)$ with similar properties to the required ones. But $P_\varepsilon = P''_\varepsilon \circ P'_\varepsilon$ works. □

It is a well known fact that under the hypotheses mentioned earlier in the introduction, the spectral problem

$$\begin{cases} \text{Find } (\lambda_\varepsilon, u_\varepsilon) \in \mathbb{C} \times V_\varepsilon \text{ such that} \\ -\displaystyle\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2})\frac{\partial u_\varepsilon}{\partial x_j}\right) = \lambda_\varepsilon u_\varepsilon \quad \text{in} \quad \Omega^\varepsilon \\ \displaystyle\sum_{i,j=1}^{N} a_{ij}(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2})\frac{\partial u_\varepsilon}{\partial x_j}\nu_i = 0 \text{ on } \partial T^\varepsilon \\ \displaystyle\int_{\Omega^\varepsilon} |u_\varepsilon|^2 dx = 1, \end{cases} \tag{3.1}$$

where we refer to [8, Lemma 4.3] for the existence of the trace, has an increasing sequence of eigenvalues $\{\lambda_\varepsilon^k\}_{k=1}^\infty$

$$0 < \lambda_\varepsilon^1 \leq \lambda_\varepsilon^2 \leq \lambda_\varepsilon^3 \leq \cdots \leq \lambda_\varepsilon^n,$$
$$\lambda_\varepsilon^n \to +\infty \quad \text{as} \quad n \to +\infty.$$

It is to be noted that if the coefficients a_{ij}^ε are real-valued, then the first eigenvalue λ_1^ε is isolated. Each eigenvalue λ_ε^k is attached to an eigenvector $u_\varepsilon^k \in V_\varepsilon$ and is of finite multiplicity for each k . Moreover, $\{u_\varepsilon^k\}_{k=1}^\infty$ form an orthonormal basis in $L^2(\Omega^\varepsilon)$. In the sequel, the couple $(\lambda_\varepsilon^k, u_\varepsilon^k)$ will be refereed to as eigencouple without further ado.

We finally recall the Courant-Fisher minimax principle which gives a useful (as will be seen later) characterization of the eigenvalues to problem 3.1. To this end, we introduce the Rayleigh quotient defined, for each $v \in V_\varepsilon \setminus \{0\}$, by

$$R^\varepsilon(v) = \frac{\int_{\Omega^\varepsilon}(A^\varepsilon Dv, Dv)dx}{\|v\|_{L^2(\Omega^\varepsilon)}^2},$$

where A^ε is the N^2-square matrix $(a_{ij}^\varepsilon)_{1\leq i,j\leq N}$ and D denotes the usual gradient. Denoting by E^k ($k\geq 0$) the collection of all subspaces of dimension k of V_ε, the minimax principle states as follows: For any $k\geq 1$, the k'th eigenvalue to (3.1) is given by

$$\lambda_\varepsilon^k = \min_{W\in E^k}\left(\max_{v\in W\setminus\{0\}} R^\varepsilon(v)\right) = \max_{W\in E^{k-1}}\left(\min_{v\in W^\perp\setminus\{0\}} R^\varepsilon(v)\right). \tag{3.2}$$

In particular, the first eigenvalue satisfies

$$\lambda_\varepsilon^1 = \min_{v\in V_\varepsilon\setminus\{0\}} R^\varepsilon(v)$$

and every minimum in (3.2) is an eigenvector associated with λ_ε^1.

We introduce the characteristic functions χ_{G_y} and χ_{G_z} of

$$G_y = \mathbb{R}_y^N \setminus \Theta_y \text{ and } G_z = \mathbb{R}_z^N \setminus \Theta_z$$

with

$$\Theta_y = \bigcup_{k\in S_y}(k+T_y) \text{ and } \Theta_z = \bigcup_{k\in S_z}(k+T_z)$$

that will be important tools in the statement and the homogenization process of our problem. It follows from the closeness of T_y (resp. T_z) that Θ_y (resp. Θ_z) is closed in $\mathbb{R}_y^N$ (resp. $\mathbb{R}_z^N$) so that G_y (resp. G_z) is an open subset of $\mathbb{R}_y^N$ (resp. $\mathbb{R}_z^N$).

3.2 Abstract homogenization problem for (1.1).

Let $A = A_y \odot A_z$ be an algebra wmv on $\mathbb{R}_y^N \times \mathbb{R}_z^N$ for $\mathcal{H}$, A_y and A_z being two ergodic algebras wmv. Put $G = G_y \times G_z$. Our main purpose in this section is to investigate for each $k\geq 1$ the asymptotic behavior as $E \ni \varepsilon \to 0$ of the eigencouple $(\lambda_\varepsilon^k, u_\varepsilon^k)$ to (1.1) under the following abstract hypothesis.

$$\chi_{G_y} \in B_{A_y}^2,\ \chi_{G_z} \in B_{A_z}^2 \tag{3.3}$$

$$M(\chi_G) > 0, \tag{3.4}$$

$$a_{ij}(x,\cdot,\cdot) \in B_A^2 \text{ for all } x\in\overline{\Omega} \qquad (1\leq i,j\leq N). \tag{3.5}$$

We first collect the basic tools and preliminary results we need.

Lemma 3.2. *Under hypothesis (3.3), there exist a β-mesurable set $\widehat{G}\subset\Delta(A)$ such that $\chi_{\widehat{G}} = \widehat{\chi}_G$ a.e. in $\Delta(A)$, where $\widehat{\chi}_G = \mathcal{G}(\chi_G)$, and $\chi_{\widehat{G}}$ denotes the characteristic function of $\widehat{G}$ in $\Delta(A_z)$. Moreover $\widehat{G} = \widehat{G}_y\times\widehat{G}_z$ where $\widehat{G}_y\subset\Delta(A_y)$ and $\widehat{G}_z\subset\Delta(A_z)$ are β_y-mesurable and β_z-mesurable respectively and verify $\widehat{\chi}_{G_y} = \chi_{\widehat{G}_y}$ and $\widehat{\chi}_{G_z} = \chi_{\widehat{G}_z}$.*

Proof. From (3.3) we have $\chi_G = \chi_{G_y}\otimes\chi_{G_z} \in B_{A_y}^2 \otimes B_{A_z}^2 \subset B_A^2$. Hence $\chi_G\in B_A^1$ since $B_A^2\subset B_A^1$. Therefore, following the same line of reasoning as in the proof of [38, Lemma 2.1], we get on the one hand the first part of the lemma and on the other hand the existence of $\widehat{G}_y\subset\Delta(A_y)$ and $\widehat{G}_z\subset\Delta(A_z)$ which are β_y-mesurable and β_z-mesurable, respectively, and verify $\widehat{\chi}_{G_y} = \chi_{\widehat{G}_y}$ and $\widehat{\chi}_{G_z} = \chi_{\widehat{G}_z}$. But $\chi_{\widehat{G}} = \widehat{\chi}_{G_y\times G_z} = \widehat{\chi_{G_y}\otimes\chi_{G_z}} = \widehat{\chi}_{G_y}\otimes\widehat{\chi}_{G_z} = \chi_{\widehat{G}_y}\otimes\chi_{\widehat{G}_z} = \chi_{\widehat{G}_y\times\widehat{G}_z}$. $\square$

Remark 3.3. In view of the preceding Lemma, we have $\chi_G^\varepsilon \to \rho(\chi_G)$ in $L^p(\Omega)$-weak $R\Sigma$ as $\varepsilon \to 0$ where $1 < p < \infty$, $\chi_G^\varepsilon = \chi_G(\frac{x}{\varepsilon}, \frac{x}{\varepsilon^2})$ $(x \in \Omega)$ and where ρ is the canonical mapping from B_A^p onto $\mathcal{B}_A^p$. Furthermore

$$\begin{aligned}\beta(\widehat{G}) = \beta_y(\widehat{G}_y)\beta_z(\widehat{G}_z) &= \left(\int_{\Delta(A_y)} \mathcal{G}_y(\chi_{G_y})d\beta_y\right)\left(\int_{\Delta(A_z)} \mathcal{G}_z(\chi_{G_z})d\beta_z\right) \\ &= \int_{\Delta(A)} \mathcal{G}(\chi_G)d\beta \\ &= M(\chi_G).\end{aligned}$$

Now, let $Q^\varepsilon = \Omega \setminus (\varepsilon\Theta_y \cup \varepsilon^2\Theta_z)$. This is an open set in $\mathbb{R}^N$ and $\Omega^\varepsilon \setminus Q^\varepsilon$ is the intersection of Ω with the collection of the holes crossing the boundary $\partial\Omega$. We have the following result which implies, as will be seen later, that the holes crossing the boundary $\partial\Omega$ are of no effect as regard the homogenization process since they are in arbitrary narrow stripe along the boundary.

Lemma 3.4. *[38] Let $K \subset \Omega$ be a compact set independent of ε. There is some $\varepsilon_0 > 0$ such that $\Omega^\varepsilon \setminus Q^\varepsilon \subset \Omega \setminus K$ for any $0 < \varepsilon \leq \varepsilon_0$.*

Next, let

$$\mathbb{F}_0^1 = H_0^1(\Omega) \times L^2\left(\Omega; \mathcal{B}_{\#A_y}^{1,2}\right) \times L^2\left(\Omega; \mathcal{B}_{A_y}^2\left(\mathbb{R}_y^N; \mathcal{B}_{\#A_z}^{1,2}\right)\right),$$

and for $u = (u_0, u_1, u_2) \in \mathbb{F}_0^1$, put $\mathbb{D}_i\mathbf{u} = \frac{\partial u_0}{\partial x_i} + \partial_{s_i}\hat{u}_1 + \partial_{r_i}\hat{u}_2$ $(1 \leq i \leq N)$ and $\mathbb{D}\mathbf{u} = Du_0 + \partial_s\hat{u}_1 + \partial_r\hat{u}_2 = (\mathbb{D}_i\mathbf{u})_{1\leq i\leq N}$ where $\partial_s\hat{u}_1 = (\partial_{s_i}\hat{u}_1)_{1\leq i\leq N}$, $\partial_r\hat{u}_2 = (\partial_{r_i}\hat{u}_2)_{1\leq i\leq N}$, $\partial_{s_i}\hat{u}_1 = \mathcal{G}_1\left(\frac{\overline{\partial} u_1}{\partial y_i}\right)$ and $\partial_{r_i}\hat{u}_2 = \mathcal{G}_1\left(\frac{\overline{\partial} u_2}{\partial z_i}\right)$. Endowed with the following norm

$$\|\mathbf{v}\|_{\mathbb{F}_0^1} = \left[\sum_{i=1}^N \|\mathbb{D}_i\mathbf{v}\|^2_{L^2(\Omega\times\Delta(A))}\right]^{\frac{1}{2}} \quad (\mathbf{v} \in \mathbb{F}_0^1),$$

$\mathbb{F}_0^1$ is an Hilbert space admitting $F_0^\infty = \mathcal{D}(\Omega) \times [\mathcal{D}(\Omega) \otimes (J_1^y \circ \rho_y)(A_y^\infty/\mathbb{C})] \times [\mathcal{D}(\Omega) \otimes \rho_y(A_y^\infty) \otimes (J_1^z \circ \rho_z)(A_z^\infty/\mathbb{C})]$ as a dense subspace where, for $\zeta = y, z$, J_1^ζ (resp. ρ_ζ) denotes the canonical mapping of $\mathcal{B}_{A_\zeta}^2/\mathbb{C}$ (resp. $B_{A_\zeta}^2$) into its separated completion $\mathcal{B}_{\#A_\zeta}^{1,2}$ (resp. $\mathcal{B}_{A_\zeta}^2$).

Now, let $B_A^{2,\infty} = B_A^2 \bigcap L^\infty(\mathbb{R}_y^N \times \mathbb{R}_z^N)$ be endowed with the $L^\infty(\mathbb{R}_y^N \times \mathbb{R}_z^N)$-norm. We know that for $u \in B_A^{2,\infty}$, we have $\mathcal{G}(u) \in L^\infty(\Delta(A))$ and $\|\mathcal{G}(u)\|_{L^\infty(\Delta(A))} \leq \|u\|_{L^\infty(\mathbb{R}_{y,z}^{2N})}$, $\mathcal{G}$ being the extension, of the usual $\mathcal{G}$, mapping B_A^2 into $L^2(\Delta(A))$. Thanks to (3.5), we have

$$a_{ij} \in \mathcal{C}(\overline{\Omega}, B_A^{2,\infty}) \quad (1 \leq i, j \leq N) \tag{3.6}$$

so that

$$\widehat{a}_{ij} = \mathcal{G}(a_{ij}) \in \mathcal{C}(\overline{\Omega}, L^\infty(\Delta(A))) \tag{3.7}$$

with

$$\widehat{a}_{ji} = \overline{\widehat{a}}_{ij}$$

and

$$\mathrm{Re}\sum_{i,j=1}^{N}\widehat{a}_{ij}(x,s,r)\xi_j\overline{\xi}_i \geq \alpha|\xi|^2$$

(same α as in (2.2)) for all $x\in\overline{\Omega}$ and all $\xi=(\xi_i)\in\mathbb{C}^N$, and for almost all $(s,r)\in\Delta(A)$. The preceding ellipticity condition follows from (1.2) exactly as in [44, Proposition 5.2]. This being so, for $(\mathbf{u},\mathbf{v})\in\mathbb{F}_0^1\times\mathbb{F}_0^1$, let

$$\widehat{a}_\Omega(\mathbf{u},\mathbf{v})=\sum_{i,j=1}^{N}\iint_{\Omega\times\widehat{G}}\widehat{a}_{ij}(x,s,r)\mathbb{D}_j\mathbf{u}(x,s,r)\overline{\mathbb{D}_i\mathbf{v}}(x,s,r)dxd\beta.$$

This define a hermitian, continuous sesquilinear form on $\mathbb{F}_0^1\times\mathbb{F}_0^1$. We will need the following

Lemma 3.5. *Let $(u_\varepsilon)_{\varepsilon\in E}\subset H_0^1(\Omega)$ and let Φ_ε be defined by*

$$\Phi_\varepsilon(x)=\psi_0(x)+\varepsilon\psi_1(x,\frac{x}{\varepsilon})+\varepsilon^2\psi_2(x,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2}) \tag{3.8}$$

where $\psi_0\in\mathcal{D}(\Omega)$, $\psi_1\in\mathcal{D}(\Omega)\otimes(A_y^\infty/\mathbb{C})$ and $\psi_2\in\mathcal{D}(\Omega)\otimes(A_y^\infty)\otimes(A_z^\infty/\mathbb{C})$. Suppose that, as $E\ni\varepsilon\to 0$, we have

$$\frac{\partial u_\varepsilon}{\partial x_i}\to\frac{\partial u_0}{\partial x_i}+\frac{\overline{\partial} u_1}{\partial y_i}+\frac{\overline{\partial} u_2}{\partial z_i}\quad in\quad L^2(\Omega)\text{-}weak R\Sigma\ \ (1\leq i\leq N) \tag{3.9}$$

where $\boldsymbol{u}=(u_0,u_1,u_2)\in\mathbb{F}_0^1$. Then

$$a^\varepsilon(u_\varepsilon,\Phi_\varepsilon)\to a_\Omega(\boldsymbol{u},\Phi)\quad as\quad E\ni\varepsilon\to 0,$$

where

$$a^\varepsilon(u_\varepsilon,\Phi_\varepsilon)=\sum_{i,j=1}^{N}\int_{\Omega^\varepsilon}a_{ij}^\varepsilon\frac{\partial u_\varepsilon}{\partial x_j}\overline{\frac{\partial\Phi_\varepsilon}{\partial x_i}}dx.$$

Proof. For $E\ni\varepsilon>0$, $\Phi_\varepsilon\in\mathcal{D}(\Omega)$ and all the functions Φ_ε $(\varepsilon\in E)$ have their supports contained in a fixed compact set $K\subset\Omega$. Thanks to Lemma 3.4, there is some $\varepsilon_0>0$ such that

$$\Phi_\varepsilon=0\quad\text{in}\quad\Omega^\varepsilon\setminus Q^\varepsilon\quad(E\ni\varepsilon\leq\varepsilon_0).$$

Using the decomposition $\Omega^\varepsilon=Q^\varepsilon\cup(\Omega^\varepsilon\setminus Q^\varepsilon)$ and the equality $Q^\varepsilon=\Omega\cap\varepsilon G_y\cap\varepsilon^2 G_z$, we get for $E \ni \varepsilon \leq \varepsilon_0$

$$\begin{aligned}
a^{\varepsilon}(u_{\varepsilon},\Phi_{\varepsilon}) &= \sum_{i,j=1}^{N}\int_{\Omega^{\varepsilon}} a_{ij}(x,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2})\frac{\partial u_{\varepsilon}}{\partial x_j}\frac{\overline{\partial \Phi_{\varepsilon}}}{\partial x_i}dx \\
&= \sum_{i,j=1}^{N}\int_{Q^{\varepsilon}} a_{ij}(x,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2})\frac{\partial u_{\varepsilon}}{\partial x_j}\frac{\overline{\partial \Phi_{\varepsilon}}}{\partial x_i}dx \\
&= \sum_{i,j=1}^{N}\int_{\Omega\cap\varepsilon G_y\cap\varepsilon^2 G_z} a_{ij}(x,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2})\frac{\partial u_{\varepsilon}}{\partial x_j}\frac{\overline{\partial \Phi_{\varepsilon}}}{\partial x_i}dx \\
&= \sum_{i,j=1}^{N}\int_{\Omega} a_{ij}(x,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2})\chi_{\varepsilon G_y}(x)\chi_{\varepsilon^2 G_z}(x)\frac{\partial u_{\varepsilon}}{\partial x_j}\frac{\overline{\partial \Phi_{\varepsilon}}}{\partial x_i}dx \\
&= \sum_{i,j=1}^{N}\int_{\Omega} a_{ij}(x,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2})\chi_{G_y}(\frac{x}{\varepsilon})\chi_{G_z}(\frac{x}{\varepsilon^2})\frac{\partial u_{\varepsilon}}{\partial x_j}\frac{\overline{\partial \Phi_{\varepsilon}}}{\partial x_i}dx \\
&= \sum_{i,j=1}^{N}\int_{\Omega} a_{ij}(x,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2})\chi_{G}(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2})\frac{\partial u_{\varepsilon}}{\partial x_j}\frac{\overline{\partial \Phi_{\varepsilon}}}{\partial x_i}dx.
\end{aligned}$$

On the other hand, the sequence $(\partial\Phi_{\varepsilon}/\partial x_i)_{\varepsilon\in E}$ being bounded in $L^{\infty}(\Omega)$ with

$$\frac{\partial \Phi_{\varepsilon}}{\partial x_i} \to \overline{D}_i\Phi = \frac{\partial \psi_0}{\partial x_i} + \rho_y\left(\frac{\partial \psi_1}{\partial y_i}\right) + \rho_z\left(\frac{\partial \psi_2}{\partial z_i}\right) \text{ in } L^2(\Omega)\text{-strong}\, R\Sigma$$

as $E \ni \varepsilon \to 0$ for each $1 \leq i \leq N$, Lemma 2.4 of [38] applies and leads to

$$\sum_{i,j=1}^{N}\frac{\partial u_{\varepsilon}}{\partial x_j}\frac{\overline{\partial \Phi_{\varepsilon}}}{\partial x_i} \to \sum_{i,j=1}^{N}\overline{D}_j u\overline{\overline{D}_i\Phi} \quad \text{in} \quad L^2(\Omega)\text{-weak}\, R\Sigma.$$

From (3.3) and (3.6) it is clear that $a_{ij}(x,y,z)\chi_G(y,z) \in \mathcal{C}(\overline{\Omega};B_A^{2,\infty})$ $(1 \leq i,j \leq N)$. But Property (2.14) in Definition 2.13 still holds for $f \in \mathcal{C}(\overline{\Omega};B_A^{2,\infty})$ instead of $L^2(\Omega;A)$ whenever the weak $R\Sigma$ convergence therein is ensured (see e.g., [37, Proposition 4.5]). Thus

$$\sum_{i,j=1}^{N}\int_{\Omega} a_{ij}(x,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2})\chi_G(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2})\frac{\partial u_{\varepsilon}}{\partial x_j}\frac{\overline{\partial \Phi_{\varepsilon}}}{\partial x_i}dx \to \sum_{i,j=1}^{N}\iint_{\Omega\times\Delta(A)}\widehat{a}_{ij}(x,s,r)\widehat{\chi}_G\mathbb{D}_j\mathbf{u}\overline{\mathbb{D}_i\mathbf{v}}dxd\beta$$

as $E \ni \varepsilon \to 0$, which completes the proof. □

We will also need the following

Proposition 3.6. *Let $(u_{\varepsilon})_{\varepsilon\in E} \subset L^2(\Omega)$. Suppose that $u_{\varepsilon} \to u$ in $L^2(\Omega)$ as $E \ni \varepsilon \to 0$. Then*

$$\int_{\Omega^{\varepsilon}}|u_{\varepsilon}|^2dx \to \beta(\widehat{G})\int_{\Omega}|u|^2dx$$

when $E \ni \varepsilon \to 0$.

Proof. For $E \ni \varepsilon > 0$ we have $\Omega^\varepsilon = (\Omega^\varepsilon \setminus Q^\varepsilon) \cup Q^\varepsilon$. When $E \ni \varepsilon \to 0$,

$$\int_{Q^\varepsilon} |u_\varepsilon|^2 dx \to \beta(\widehat{G}) \int_\Omega |u|^2 dx$$

since as $E \ni \varepsilon \to 0$, $\overline{u_\varepsilon}\chi_G^\varepsilon \to \beta(\widehat{G})\overline{u}$ in $L^2(\Omega)$-weak (see e.g., [50, Lemma 3.4]) and $u_\varepsilon \to u$ in $L^2(\Omega)$-strong. On the other hand, put $\xi_\varepsilon = \int_{\Omega^\varepsilon \setminus Q^\varepsilon} |u_\varepsilon|^2 dx$ ($\varepsilon \in E$). We now prove that $\xi_\varepsilon \to 0$ with ε by proving that each subsequence of $(\xi_\varepsilon)_{\varepsilon \in E}$ admits a further subsequence converging to 0. To this end, consider a subsequence (still denoted by E) such that $u_\varepsilon \to u$ a.e. in Ω and $|u_\varepsilon| \le h$ a.e. in Ω for all $\varepsilon \in E$ and for some function $h \in L^2(\Omega)$. Put now $f_\varepsilon = \chi_{\Omega^\varepsilon \setminus Q^\varepsilon} |u_\varepsilon|^2$ ($\varepsilon \in E$). Clearly, $|f_\varepsilon| \le h^2$ a.e. in Ω. Up to a subsequence (still denoted by E) $f_\varepsilon \to 0$ a.e in Ω since $f_\varepsilon \to 0$ in measure (this is a mere consequence of Lemma 3.2 and the inclusion $\{x \in \Omega : |f_\varepsilon(x) \ge \delta|\} \subset \Omega^\varepsilon \setminus Q^\varepsilon$ for any $\delta > 0$). We are led to the desired conclusion by means of the Lebesgue's dominated convergence theorem. □

Homogenized coefficients. We construct and point out the main properties of the so-called homogenized coefficients. Let $1 \le j \le N$ and $x \in \overline{\Omega}$ be fixed. Put

$$\widehat{a}_G(x; \mathbf{u}, \mathbf{v}) = \sum_{k,l=1}^N \int_{\widehat{G}} \widehat{a}_{kl}(x,s,r)(\partial_{s_l}\widehat{u}_1 + \partial_{r_l}\widehat{u}_2)(\overline{\partial_{s_k}\widehat{v}}_1 + \overline{\partial_{r_k}\widehat{v}}_2) d\beta \tag{3.10}$$

and

$$l_j(x, \mathbf{v}) = \sum_{k=1}^N \int_{\widehat{G}} \widehat{a}_{kj}(x,s,r)(\overline{\partial_{s_k}\widehat{v}}_1 + \overline{\partial_{r_k}\widehat{v}}_2) d\beta \tag{3.11}$$

for $\mathbf{u}, \mathbf{v} \in \mathcal{B}^{1,2}_{\#A_y} \times \mathcal{B}^2_{A_y}(\mathbb{R}^N_y; \mathcal{B}^{1,2}_{\#A_z}) \equiv \mathcal{H}$. Equipped with the seminorm

$$N_G(\mathbf{u}) = \|\partial_s \widehat{u}_1 + \partial_r \widehat{u}_2\|_{L^2(\widehat{G})^N} \quad (\mathbf{u} = (u_1, u_2) \in \mathcal{H}), \tag{3.12}$$

$\mathcal{H}$ is a pre-Hilbert space that is nonseparate and noncomplete. Let $\mathbb{H}$ be the separated completion of $\mathcal{H}$ with respect to the seminorm N_G and $\mathbf{i}$ the canonical mapping of $\mathcal{H}$ into its separated completion $\mathbb{H}$. We recall that

(i) $\mathbb{H}$ is a Hilbert space,

(ii) $\mathbf{i}$ is linear,

(iii) $\mathbf{i}(\mathcal{H})$ is dense in $\mathbb{H}$,

(iv) $\|\mathbf{i}(\mathbf{u})\|_{\mathbb{H}} = N_G(u)$ for every u in $\mathcal{H}$,

(v) If F is a Banach space and l a continuous linear mapping of $\mathcal{H}$ into F, then there exists a unique continuous linear mapping $L : \mathbb{H} \to F$ such that $l = L \circ \mathbf{i}$.

Proposition 3.7. *Let $j = 1, ..., N$ and fix x in $\overline{\Omega}$. The noncoercive meso-local variational problem*

$$\mathbf{u} = (u_1, u_2) \in \mathcal{H} \text{ and } \widehat{a}_G(x; \mathbf{u}, \mathbf{v}) = l_j(x, \mathbf{v}) \text{ for all } \mathbf{v} = (v_1, v_2) \in \mathcal{H} \tag{3.13}$$

admits at least one solution. Moreover, if $\chi^j(x) = (\chi^j_1(x), \chi^j_2(x))$ and $\theta^j(x) = (\theta^j_1(x), \theta^j_2(x))$ are two solutions, then

$$\partial_s \widehat{\chi}^j_1(x,s) + \partial_r \widehat{\chi}^j_2(x,s,r) = \partial_s \widehat{\theta}^j_1(x,s) + \partial_r \widehat{\theta}^j_2(x,s,r) \text{ a.e., in } \widehat{G}. \tag{3.14}$$

Proof. Proceeding as in the proof of [38, Lemma 2.5] we can prove that there exists a unique hermitian, coercive, continuous sesquilinear form $\widehat{A}_G(x;\cdot,\cdot)$ on $\mathbb{H}\times\mathbb{H}$ such that $\widehat{A}_G(x;\mathbf{i}(\mathbf{u}),\mathbf{i}(\mathbf{v})) = \widehat{a}_G(x;\mathbf{u},\mathbf{v})$ for all $\mathbf{u},\mathbf{v}\in\mathcal{H}$. Based on (v) above, we consider the antilinear form $\mathbf{l}_j(x,\cdot)$ on $\mathbb{H}$ such that $\mathbf{l}_j(x,\mathbf{i}(\mathbf{u})) = l_j(x,\mathbf{u})$ for any $\mathbf{u}\in\mathcal{H}$. Then $\chi^j(x) = (\chi_1^j(x),\chi_2^j(x))\in\mathcal{H}$ satisfies (3.13) if and only if $\mathbf{i}(\chi^j(x))$ satisfies

$$\mathbf{i}(\chi^j(x))\in\mathbb{H} \text{ and } \widehat{A}_G(x;\mathbf{i}(\chi^j(x)),V) = \mathbf{l}_j(x,V) \text{ for all } V\in\mathbb{H}. \tag{3.15}$$

But $\mathbf{i}(\chi^j(x))$ is uniquely determine by (3.15) (see e.g., [30, p. 216]). We deduce that (3.13) admits at least one solution and if $\chi^j(x) = (\chi_1^j(x),\chi_2^j(x))$ and $\theta^j(x) = (\theta_1^j(x),\theta_2^j(x))$ are two solutions, then $\mathbf{i}(\chi^j(x)) = \mathbf{i}(\theta^j(x))$, which means $\chi^j(x)$ and $\theta^j(x)$ have the same neighborhoods in $\mathcal{H}$ or equivalently $N_G(\chi^j(x)-\theta^j(x)) = 0$. Hence (3.14). □

Corollary 3.8. *Let $1\le i,j\le N$, x fixed in $\overline{\Omega}$ and let $\chi^j(x) = (\chi_1^j(x),\chi_2^j(x))\in\mathcal{H}$ be a solution to (3.13). The following homogenized coefficients*

$$q_{ij}(x) = \int_{\widehat{G}} \widehat{a}_{ij}(x,s,r)\,d\beta - \sum_{l=1}^{N}\int_{\widehat{G}} \widehat{a}_{il}(x,s,r)\left(\partial_{s_l}\widehat{\chi}_1^j(x,s) + \partial_{r_l}\widehat{\chi}_2^j(x,s,r)\right)d\beta, \tag{3.16}$$

are well defined in the sense that they do not depend on the solution to (3.13).

Lemma 3.9. *The following assertions are true:*
(i) $q_{ij}\in\mathcal{C}(\overline{\Omega})$.
(ii) $q_{ji} = \overline{q}_{ij}$.
(iii) There exists a constant $\alpha_0 > 0$ such that

$$Re\sum_{i,j=1}^{N} q_{ij}(x)\xi_j\overline{\xi}_i \ge \alpha_0|\xi|^2$$

for all $x\in\overline{\Omega}$ and all $\xi\in\mathbb{C}^N$.

Proof. It is an adaptation of that of [44, Lemma 5.3]. □

Homogenization of the abstract problem. We prove the global homogenization result for (1.1) in a general deterministic setting. E is still as specified in Section 1.

Theorem 3.10. *Assume that (3.3)-(3.5) hold and that A_y and A_z are ergodic algebras wmv on $\mathbb{R}^N$. For each $k\ge 1$ and each $\varepsilon\in E$, let $(\lambda_\varepsilon^k,u_\varepsilon^k)$ be the k'th eigencouple to (1.1). Then, there exists a subsequence E' of E such that*

$$\lambda_\varepsilon^k \to \lambda_0^k \text{ in } \mathbb{C} \text{ as } E\ni\varepsilon\to 0 \tag{3.17}$$

$$P_\varepsilon u_\varepsilon^k \to u_0^k \text{ in } H_0^1(\Omega)\text{-weak as } E'\ni\varepsilon\to 0 \tag{3.18}$$

$$P_\varepsilon u_\varepsilon^k \to u_0^k \text{ in } L^2(\Omega) \text{ as } E'\ni\varepsilon\to 0 \tag{3.19}$$

$$\frac{\partial P_\varepsilon u_\varepsilon^k}{\partial x_j} \to \frac{\partial u_0^k}{\partial x_j} + \frac{\overline{\partial} u_1^k}{\partial y_j} + \frac{\overline{\partial} u_2^k}{\partial z_j} \text{ in } L^2(\Omega)\text{-weak } R\Sigma \text{ as } E'\ni\varepsilon\to 0 (1\le j\le N) \tag{3.20}$$

where $(\lambda_0^k, u_0^k) \in \mathbb{C} \times H_0^1(\Omega)$ *is the k'th eigencouple to the spectral problem*

$$\begin{cases} -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(\frac{1}{\beta(\hat{G})} q_{ij}(x) \frac{\partial u_0}{\partial x_j}\right) = \lambda_0 u_0 \quad \text{in } \Omega \\ u_0 = 0 \quad \text{on } \partial\Omega \\ \int_\Omega |u_0|^2 dx = \frac{1}{\beta(\widehat{G})}, \end{cases} \tag{3.21}$$

and where $(u_1^k, u_2^k) \in L^2(\Omega; \mathcal{B}^{1,2}_{\#A_y} \times \mathcal{B}^2_{A_y}(\mathbb{R}^N_y; \mathcal{B}^{1,2}_{\#A_z}))$. *Moreover, for almost every* $x \in \Omega$ *the following hold true:*

(i) $\mathbf{u}^k(x) = (u_1^k(x), u_2^k(x))$ *is a solution to the noncoercive variational problem*

$$\begin{cases} \mathbf{u}^k(x) = (u_1^k(x), u_2^k(x)) \in \mathcal{H} \\ \widehat{a}_G(x; \mathbf{u}^k(x), \mathbf{v}) = -\sum_{i,j=1}^{N} \frac{\partial u_0^k}{\partial x_j} \int_{\hat{G}} \widehat{a}_{ij}(x,s,r)(\overline{\partial_{s_i}\widehat{v}_1}(s) + \overline{\partial_{r_i}\widehat{v}_2}(s,r))\, d\beta \\ \forall \mathbf{v} = (v_1, v_2) \in \mathcal{H}; \end{cases} \tag{3.22}$$

(ii) *We have*

$$\mathbf{i}(\mathbf{u}^k(x)) = \sum_{j=1}^{N} \frac{\partial u_0^k}{\partial x_j}(x)\mathbf{i}(\chi^j(x)) \tag{3.23}$$

where $\chi^j(x) = (\chi_1^j(x), \chi_2^j(x))$ *is any function in* $\mathcal{H}$ *defined by the meso-cell problem (3.13) and where* $\mathbf{i}$ *is the canonical mapping of* $\mathcal{H}$ *into its separated completion* $\mathbb{H}$.

Proof. Let us first recall that according to the properties of the coefficients q_{ij} (Lemma 3.9), the spectral problem (3.21) admits a sequence of eigencouples with similar properties to those of problem (1.1). However, this is also proved by our homogenization process.

Fix now $k \geq 1$. There exists a constant $0 < c_1 < \infty$ independent of ε such that $0 < \lambda_\varepsilon^k \leq c_1 \mu_\varepsilon^k$ for any $\varepsilon \in E$ where

$$\mu_\varepsilon^k = \min_{W \in E^k}\left(\max_{v \in W \setminus \{0\}} \frac{\int_{\Omega^\varepsilon} |Dv|^2 dx}{\|v\|^2_{L^2(\Omega^\varepsilon)}}\right),$$

E^k still being the collection of subspaces of dimension k of V_ε. The same lines of reasoning as in [48, Proposition 6.1] leads to the boundedness of μ_ε^k from above by a constant that does not depend on ε. Therefore the sequence $(\lambda_\varepsilon^k)_{\varepsilon \in E}$ is bounded in $\mathbb{C}$.

Clearly, for fixed $E \ni \varepsilon > 0$, u_ε^k lies in V_ε, and

$$\sum_{i,j=1}^{N} \int_{\Omega^\varepsilon} a_{ij}^\varepsilon \frac{\partial u_\varepsilon^k}{\partial x_j} \frac{\overline{\partial v}}{\partial x_i} dx = \lambda_\varepsilon^k \int_{\Omega^\varepsilon} u_\varepsilon^k \overline{v}\, dx \tag{3.24}$$

for any $v \in V_\varepsilon$. Bear in mind that $\|u_\varepsilon^k\|_{L^2(\Omega^\varepsilon)} = 1$ and chose $v = u_\varepsilon^k$ in (3.24). The boundedness of the sequence $(\lambda_\varepsilon^k)_{\varepsilon \in E}$ and the ellipticity assumption (1.2) implies at once by means of Proposition 3.1 that the sequence $(P_\varepsilon u_\varepsilon^k)_{\varepsilon \in E}$ is bounded in $H_0^1(\Omega)$. Theorem 2.17 applies and gives us $\mathbf{u}^k = (u_0^k, u_1^k, u_2^k) \in \mathbb{F}_0^1$ such that for some $\lambda_0^k \in \mathbb{C}$ and some subsequence $E' \subset E$

we have (3.17) (but as $E' \ni \varepsilon \to 0$) and (3.18)-(3.20), where (3.19) is a direct consequence of (3.18) by the Rellich-Kondrachov theorem.

For fixed $\varepsilon \in E'$, let $\Phi = (\psi_0, (J_1^y \circ \rho_y)(\psi_1), (J_1^z \circ \rho_z)(\psi_2)) \in F_0^\infty$ with $\psi_0 \in \mathcal{D}(\Omega)$, $\psi_1 \in \mathcal{D}(\Omega) \otimes (A_y^\infty/\mathbb{C})$ and $\psi_2 \in \mathcal{D}(\Omega) \otimes (A_y^\infty) \otimes (A_z^\infty/\mathbb{C})$, where $A_y^\infty/\mathbb{C} = \{\psi \in A_y^\infty : M(\psi) = 0\}$ (M the mean value on $\mathbb{R}_y^N$ for $\mathcal{H}$), $A_z^\infty/\mathbb{C} = \{\psi \in A_z^\infty : M(\psi) = 0\}$ (M the mean value on $\mathbb{R}_z^N$ for $\mathcal{H}'$). Define Φ_ε as in (3.8). Then $\Phi_\varepsilon \in \mathcal{D}(\Omega)$ and we have

$$\sum_{i,j=1}^{N} \int_{\Omega^\varepsilon} a_{ij}^\varepsilon \frac{\partial P_\varepsilon u_\varepsilon^k}{\partial x_j} \frac{\overline{\partial \Phi_\varepsilon}}{\partial x_i} dx = \lambda_\varepsilon^k \int_{\Omega^\varepsilon} P_\varepsilon u_\varepsilon^k \overline{\Phi}_\varepsilon \, dx. \tag{3.25}$$

Sending $\varepsilon \in E'$ to 0, keeping (3.17)-(3.20) and Lemma 3.5 in mind, we obtain

$$\sum_{i,j=1}^{N} \iint_{\Omega\times\hat{G}} \widehat{a}_{ij} \mathbb{D}_j \mathbf{u}^k \overline{\mathbb{D}_i \Phi} dx ds = \lambda_0^k \iint_{\Omega\times\hat{G}} u_0^k \overline{\psi}_0 \, dx. \tag{3.26}$$

Where the right hand side is obtained by the same routine as in the proof of Lemma 3.5. It is clear that $(\lambda_0^k, \mathbf{u}^k) \in \mathbb{C} \times \mathbb{F}_0^1$ solves the following *global homogenized spectral problem*:

$$\begin{cases} \text{Find } (\lambda, \mathbf{u}) \in \mathbb{C} \times \mathbb{F}_0^1 \text{ such that} \\ \displaystyle\sum_{i,j=1}^{N} \iint_{\Omega\times\hat{G}} \widehat{a}_{ij}(x,s,r) \mathbb{D}_j \mathbf{u} \overline{\mathbb{D}_i \Phi} dx d\beta = \lambda \beta(\widehat{G}) \int_\Omega u_0 \overline{\psi}_0 \, dx \\ \text{for all } \Phi \in \mathbb{F}_0^1. \end{cases} \tag{3.27}$$

To prove (i), choose $\Phi = (\psi_0, \psi_1, \psi_2)$ in (3.27) such that $\psi_0 = 0$, $\psi_1 = \varphi \otimes v_1$ and $\psi_2 = \varphi \otimes v_2$, where $\varphi \in \mathcal{D}(\Omega)$, $v_1 \in \mathcal{B}_{\#A_y}^{1,2}$, and $v_2 \in \mathcal{B}_{A_y}^2(\mathbb{R}_y^N; \mathcal{B}_{\#A_z}^{1,2})$, to get

$$\int_\Omega \varphi(x) \left[\sum_{i,j=1}^{N} \int_{\widehat{G}} \widehat{a}_{ij} \left(\frac{\partial u_0^k}{\partial x_j} + \partial_{s_j} \widehat{u}_1^k + \partial_{r_j} \widehat{u}_2^k \right) \left(\overline{\partial_{s_i} \widehat{v}_1} + \overline{\partial_{r_i} \widehat{v}_2} \right) d\beta \right] dx = 0.$$

Hence by the arbitrariness of φ, we have a.e. in Ω

$$\sum_{i,j=1}^{N} \int_{\widehat{G}} \widehat{a}_{ij} \left(\frac{\partial u_0^k}{\partial x_j} + \partial_{s_j} \widehat{u}_1^k + \partial_{r_j} \widehat{u}_2^k \right) \left(\overline{\partial_{s_i} \widehat{v}_1} + \overline{\partial_{r_i} \widehat{v}_2} \right) d\beta = 0$$

for any $\mathbf{v} = (v_1, v_2) \in \mathcal{H}$, which is nothing but (3.22).

As regard (ii), pick any $\chi^j(x) = (\chi_1^j(x), \chi_2^j(x))$ solution to the meso-cell problem (3.13) an put $z(x) = \sum_{j=1}^N \frac{\partial u_0^k}{\partial x_j}(x) \chi^j(x)$. On multiplying both sides of (3.13) by $-\frac{\partial u_0^k}{\partial x_j}(x)$ and then summing over $1 \leq j \leq N$, we see that $z(x)$ satisfies (3.22). Hence $\mathbf{i}(z(x)) = \mathbf{i}(u^k(x))$ by uniqueness of the solution to the coercive variational problem in $\mathbb{H}$ corresponding to the non-coercive variational problem (3.22) (see the proof of Proposition 3.7). Thus (3.23) since $\mathbf{i}$ is linear.

Now, by considering $\Phi = (\psi_0, \psi_1, \psi_2)$ in (3.27) such that $\psi_1 = 0$, $\psi_2 = 0$ and $\psi_0 \in \mathcal{D}(\Omega)$, we get

$$\sum_{i,j=1}^{N} \iint_{\Omega\times\hat{G}} \widehat{a}_{ij} \left(\frac{\partial u_0^k}{\partial x_j} + \partial_{s_j} \widehat{u}_1^k + \partial_{r_j} \widehat{u}_2^k \right) \frac{\overline{\partial \psi_0}}{\partial x_i} dx d\beta = \beta(\hat{G}) \lambda_0^k \int_\Omega u_0^k \overline{\psi}_0 \, dx.$$

As (3.23) is equivalent (see the proof of Proposition 3.7) to

$$\partial_s \widehat{u}_1^k(x) + \partial_r \widehat{u}_2^k(x) = \sum_{j=1}^N \frac{\partial u_0^k}{\partial x_j}(x)(\partial_s \widehat{\chi}_1^j(x) + \partial_r \widehat{\chi}_2^j(x)) \quad \text{a.e. in } \widehat{G},$$

we arrive at

$$\sum_{i,j=1}^N \int_\Omega \left[\int_{\hat{G}} \widehat{a}_{ij} \, d\beta - \sum_{l=1}^N \int_{\hat{G}} \widehat{a}_{il} \left(\partial_{s_l} \widehat{\chi}_1^j + \partial_{r_l} \widehat{\chi}_2^j \right) d\beta \right] \frac{\partial u_0^k}{\partial x_j} \frac{\overline{\partial \psi_0}}{\partial x_i} \, dx = \beta(\hat{G}) \lambda_0^k \int_\Omega u_0^k \overline{\psi}_0 \, dx,$$

i.e. (see (3.16))

$$\sum_{i,j=1}^N \int_\Omega \frac{1}{\beta(\hat{G})} q_{ij}(x) \frac{\partial u_0^k}{\partial x_j} \frac{\overline{\partial \psi_0}}{\partial x_i} \, dx = \lambda_0^k \int_\Omega u_0^k \overline{\psi}_0 \, dx.$$

Thanks to the arbitrariness of ψ_0 and the weak derivative formula, we conclude that (λ_0^k, u_0^k) is the k'th eigencouple to (3.21) and the whole sequence $(\lambda_\varepsilon^k)_{\varepsilon \in E}$ is found to converge.

The normalization condition in (3.21) is readily obtained by means of that in (1.1), Proposition 3.6 and (3.19). The proof of our theorem is therefore completed. □

Remark 3.11. **(1)** $\{u_0^k\}_{k=1}^\infty$ is an orthogonal basis in $L^2(\Omega)$. This follows from the orthogonality of $\{u_\varepsilon^k\}_{k=1}^\infty$ in $L^2(\Omega^\varepsilon)$ $(\varepsilon \in E)$ by an adaptation of the proof of Proposition 3.6.

(2) The spectral problem (3.21) is refereed to as the *macroscopic homogenized spectral problem* for (1.1) whereas (3.22) is the so-called *mesoscopic problem*. The behavior (as $\varepsilon \to 0$) of the eigenfunction u_ε^k has three fundamental aspects: the macroscopic behavior, the mesoscopic behavior and the microscopic behavior. The macroscopic behavior is described by u_0^k solution to the macroscopic problem. The mesoscopic behavior depends on the observation point in Ω and is described by $(u_1^k(x), u_2^k(x))$ solution to (3.22). The microscopic behavior depends on the observation point (x,y) and is described by $u_2^k(x,y)$ solution to the microscopic problem

$$\begin{cases} u_2^k(x,y) \in \mathcal{B}_{\# A_z}^{1,2} \\ \displaystyle\sum_{i,j=1}^N \int_{\widehat{G}_z} \widehat{a}_{ij} \partial_{s_j} \widehat{u}_2^k \overline{\partial_{r_j} \widehat{\omega}} d\beta_z = - \sum_{i,j=1}^N \left(\frac{\partial u_0^k}{\partial x_j} + \mathcal{G}_1 \left(\frac{\overline{\partial} u_1^k}{\partial y_j} \right) \right) \int_{\widehat{G}_z} \widehat{a}_{ij} \overline{\partial_{r_i} \widehat{\omega}} d\beta_z \\ \forall \omega \in \mathcal{B}_{\# A_z}^{1,2}. \end{cases} \tag{3.28}$$

4 Some concrete individual homogenization problems for (1.1).

We work out in this section some concrete homogenization problems for (1.1). Before we can do that we need a few preliminary results.

4.1 Preliminaries

The basic notations being those of Section 1, we begin by noting that for $\zeta = y, z$ the characteristic function χ_{Θ_ζ} of the set Θ_ζ is given by $\chi_{\Theta_\zeta} = \sum_{k \in S_\zeta} \chi_{k+T_\zeta}$ (a locally finite sum) or more suitably

$$\chi_{\Theta_\zeta} = \sum_{k \in \mathbb{Z}^N} \theta_\zeta(k) \chi_{k+T_\zeta},$$

where χ_{k+T_ζ} denotes the characteristic function of $k+T_\zeta$ in $\mathbb{R}^N_\zeta$ and θ_ζ is that of S_ζ in $\mathbb{Z}$. We shall refer to θ_ζ as the distribution function of the holes[38].

We have the following result without which the multiscale perforation set up earlier would be useless.

Proposition 4.1. *Let A be an algebra wmv on $\mathbb{R}^N$ (for $\mathcal{H}$ or $\mathcal{H}'$). Assume that the distribution function of the holes θ belongs to the space of essential function on $\mathbb{Z}^N$, $ES(\mathbb{Z}^N)$ (see [36]). On the other hand, assume that for every $\varphi \in \mathcal{K}(\mathbb{T})$ (the space of all continuous complex functions on $\mathbb{R}^N$ with compact supports contained in $\mathbb{T} = \left(-\frac{1}{2}, \frac{1}{2}\right)^N$), the function $\sum_{k\in\mathbb{Z}^N} \theta(k)\tau_k\varphi$ (where $\tau_k\varphi(a) = \varphi(a-k)$ for $a \in \mathbb{R}^N$) lies in A. Then $\chi_\Theta \in B^2_A$ and further*

$$M(\chi_\Theta) = \mathfrak{M}(\theta)\lambda(T)$$

where λ is the Lebesgue measure on $\mathbb{R}^N$ and $\mathfrak{M}(\theta)$ the essential mean of θ [36].

Proof. The proof is an adaptation of that of [38, Proposition 3.1] where we replace there $\mathfrak{X}^p_\Sigma(\mathbb{R}^N)$ by $B^2_A(\mathbb{R}^N)$. □

Corollary 4.2. *Let $A = A_y \odot A_z$ be an algebra wmv where A_y (resp. A_z) is an algebra wmv on $\mathbb{R}^N_y$ (resp. $\mathbb{R}^N_z$) for $\mathcal{H}$ (resp. $\mathcal{H}'$) and suppose A_y (resp. A_z) verify each with its action the hypothesis of Proposition 4.1. Then (3.3) and (3.4) hold true.*

Proof. Let $\zeta = y, z$. By Proposition 4.1, we have on the one hand $M_\zeta(\chi_{\Theta_\zeta}) = \mathfrak{M}(\theta_\zeta)\lambda(T_\zeta)$. On the other hand we have (3.3) as a direct consequence of the equality $\chi_{G_\zeta} = 1 - \chi_{\Theta_\zeta}$ which combined with $\mathfrak{M}(\theta) \leq 1$ and $\lambda(T_\zeta) < \lambda(\mathbb{T}) = 1$ leads to: $\beta_\zeta(\widehat{G}_\zeta) = M_\zeta(\chi_{G_\zeta}) > 0$. But, $M(\chi_G) = \beta(\widehat{G}) = \beta_y \otimes \beta_z(\widehat{G}_y \times \widehat{G}_z) = \beta_y(\widehat{G}_y)\beta_z(\widehat{G}_z) = M_y(\chi_{G_y})M_z(\chi_{G_z}) > 0$. □

4.2 Double equidistribution of the holes

Throughout this section we assume that $\theta_y(k) = \theta_z(k) = 1$ for all $k \in \mathbb{Z}^N$ which is equivalent to $S_y = S_z = \mathbb{Z}^N$. This precisely means that each cell $k+Y$ (resp. $k+Z$) contains a hole $k+T_y$ (resp. $k+T_z$), $k \in \mathbb{Z}^N$. This is usually called double periodicity[20, 21, 28] but we find it more convenient to be refereed to as double equidistribution of the holes. $L^2_{per}(Y)$ denoting the space of Y-periodic functions in $L^2_{loc}(\mathbb{R}^N_y)$ and $\mathcal{C}_{per}(Y)$ its subspace made up with continuous functions, it is classic that $L^2_{per}(Y)$ is the closure of $\mathcal{C}_{per}(Y)$ in $L^2_{loc}(\mathbb{R}^N_y)$ with respect to the norm $\|\cdot\|_2$ here defined by $\|u\|_2 = \left(\int_Y |u(y)|^2 dy\right)^{\frac{1}{2}}$. It is also easily seen that $L^2_{per}(Y) = B^2_{\mathcal{C}_{per}(Y)}$. Under The previous perforation hypothesis, we have [38, Section 3.2] that

$$\chi_{G_y} \in L^2_{per}(Y), \qquad M_y(\chi_{G_y}) > 0 \tag{4.1}$$

$$\chi_{G_z} \in L^2_{per}(Z), \qquad M_z(\chi_{G_z}) > 0. \tag{4.2}$$

Hence (3.3) and (3.4) follow.

4.2.1 Problem I: Periodic homogenization.

We assume here that for each fixed $x \in \overline{\Omega}$ and for any $1 \leq i, j \leq N$, the function $(y,z) \to a_{ij}(x,y,z)$ satisfies the following periodicity hypothesis:

$$\begin{cases} \text{For each } k \in \mathbb{Z}^N \text{ and each } l \in \mathbb{Z}^N, \text{ we have} \\ a_{ij}(x,y+k,z+l) = a_{ij}(x,y,z) \end{cases} \tag{4.3}$$

which is expressed by saying that $a_{ij}(x,y,z)$ is $Y \times Z$-periodic in (y,z). Hypothesis (4.3) leads at once to

$$a_{ij}(x,\cdot,\cdot) \in L^\infty_{per}(Y \times Z) \text{ for any } x \in \overline{\Omega} \quad (1 \leq i, j \leq N).$$

The suitable algebras wmv for this problem are $A_y = \mathcal{C}_{per}(Y)$, $A_z = \mathcal{C}_{per}(Z)$ and $A = \mathcal{C}_{per}(Y) \odot \mathcal{C}_{per}(Z) = \mathcal{C}_{per}(Y \times Z)$. Hence, $B^2_{A_y} = \mathcal{B}^2_{A_y} = L^2_{per}(Y)$, $B^2_{A_z} = \mathcal{B}^2_{A_z} = L^2_{per}(Z)$, $B^2_A = \mathcal{B}^2_A = L^2_{per}(Y \times Z)$ and

$$a_{ij}(x,\cdot,\cdot) \in L^2_{per}(Y \times Z) \text{ for all } x \in \overline{\Omega} \quad (1 \leq i, j \leq N).$$

Thus the conclusion of Theorem 3.10 is achieved under hypothesis (4.3).

For the sake of clarity we state the outlines of the homogenization theorem in this setting. Before we can do that we need a few details. We have $\mathcal{B}^{1,2}_{\#A_y} = H^1_\#(Y)$, $\mathcal{B}^{1,2}_{\#A_z} = H^1_\#(Z)$ and the Haar measure $\beta = \beta_y \otimes \beta_z$ is just the Lebesgue measure $dydz$ on $\mathbb{R}^N_y \times \mathbb{R}^N_z$. Put $Y^* = Y \setminus T_y$, $Z^* = Z \setminus T_z$ and bear in mind that the mean value of a function $u \in L^2_{per}(Y)$ is merely expressed by $M(u) = \int_Y u(y)dy$. It follows from (4.1) that $M(\chi_{G_y}) = \int_Y \chi_{G_y}(y)dy = |Y^*| > 0$ (similar remark for $|Z^*|$). Hence $|Y^* \times Z^*| > 0$. Fix $x \in \overline{\Omega}$ and let $\chi^j(x) = (\chi^j_1(x), \chi^j_2(x))$ $(1 \leq j \leq N)$ be a solution to the following periodic meso-local problem

$$\begin{cases} \chi^{\mathbf{j}}(x) = (\chi^j_1(x), \chi^j_2(x)) \in \mathcal{H} = H^1_\#(Y) \times L^2_{per}(Y; H^1_\#(Z)) \\ \displaystyle\sum_{k,l=1}^N \iint_{Y^* \times Z^*} a_{kl} \left(\frac{\partial \chi^j_1}{\partial y_l} + \frac{\partial \chi^j_2}{\partial z_l} \right) \left(\overline{\frac{\partial v_1}{\partial y_l}} + \overline{\frac{\partial v_2}{\partial z_l}} \right) dydz = \sum_{m=1}^N \iint_{Y^* \times Z^*} a_{mj} \left(\overline{\frac{\partial v_1}{\partial y_m}} + \overline{\frac{\partial v_2}{\partial z_m}} \right) dydz \\ \forall \mathbf{v} = (v_1, v_2) \in \mathcal{H}. \end{cases}$$

The homogenized coefficients are given in this setting by

$$q_{ij}(x) = \iint_{Y^* \times Z^*} a_{ij}(x,y,z)dydz - \sum_{l=1}^N \iint_{Y^* \times Z^*} a_{il}(x,y,z) \left(\frac{\partial \chi^j_1}{\partial y_l}(x,y) + \frac{\partial \chi^j_2}{\partial z_l}(x,y,z) \right) dydz$$

and the homogenization result states as

Theorem 4.3. *For each $k \geq 1$ and each $\varepsilon \in E$, let $(\lambda^k_\varepsilon, u^k_\varepsilon)$ be the k'th eigencouple to (1.1). Then, there exists a subsequence E' of E such that*

$$\begin{array}{rcl} \lambda^k_\varepsilon & \to & \lambda^k_0 \quad \text{in } \mathbb{C} \text{ as } E \ni \varepsilon \to 0 \\ P_\varepsilon u^k_\varepsilon & \to & u^k_0 \quad \text{in } H^1_0(\Omega)\text{-weak as } E' \ni \varepsilon \to 0 \\ P_\varepsilon u^k_\varepsilon & \to & u^k_0 \quad \text{in } L^2(\Omega) \text{ as } E' \ni \varepsilon \to 0 \\ \dfrac{\partial P_\varepsilon u^k_\varepsilon}{\partial x_j} & \to & \dfrac{\partial u^k_0}{\partial x_j} + \dfrac{\partial u^k_1}{\partial y_j} + \dfrac{\partial u^k_2}{\partial z_j} \text{ in } L^2(\Omega)\text{-weak } R\Sigma \text{ as } E' \ni \varepsilon \to 0 (1 \leq j \leq N) \end{array}$$

where $(\lambda_0^k, u_0^k) \in \mathbb{C} \times H_0^1(\Omega)$ *is the k'th eigencouple to the spectral problem*

$$\begin{cases} -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(\frac{q_{ij}(x)}{|Y^* \times Z^*|}\frac{\partial u_0}{\partial x_j}\right) = \lambda_0 u_0 \quad in\ \Omega \\ u_0 = 0 \quad on\ \partial\Omega \\ \int_\Omega |u_0|^2 dx = \frac{1}{|Y^* \times Z^*|}, \end{cases}$$

and where $(u_1^k, u_2^k) \in L^2(\Omega; H_\#^1(Y) \times L^2_{per}(Y; H_\#^1(Z)))$.

4.2.2 Problem II

Let $\mathcal{B}_{\infty,\mathbb{Z}^N}(\mathbb{R}^N_z)$ denotes the space of all finite sum

$$\sum_{finite} \varphi_i u_i \quad \text{with } \varphi_i \in \mathcal{B}_\infty(\mathbb{R}^N_z),\ \ u_i \in \mathcal{C}_{per}(Z)$$

where $\mathcal{B}_\infty(\mathbb{R}^N_z)$ is the space of those continuous complex function converging finitely at infinity. This is obviously an algebra wmv. Under the hypothesis that

$$a_{ij}(x,\cdot,\cdot) \in L^2_{per}(Y; \mathcal{B}_{\infty,\mathbb{Z}^N}(\mathbb{R}^N_z)) \text{ for all } x \in \overline{\Omega} \quad (1 \le i,j \le N), \tag{4.4}$$

the conclusion of Theorem 3.10 holds with $A = \mathcal{C}_{per}(Y) \odot \mathcal{B}_{\infty,\mathbb{Z}^N}(\mathbb{R}^N_z)$. It is worth noticing that hypothesis (4.4) generalizes the case when

$$a_{ij}(x,\cdot,\cdot) \in L^2_{per}(Y; \mathcal{B}_\infty(\mathbb{R}^N_z)) \text{ for any } x \in \overline{\Omega} \quad (1 \le i,j \le N).$$

4.2.3 Problem III

We study here the homogenization problem for (1.1) under the following assumption

$$a_{ij}(x,\cdot,\cdot) \in B^2_{AP}(\mathbb{R}^N_y \times \mathbb{R}^N_z) \text{ for all } x \in \overline{\Omega} \quad (1 \le i,j \le N),$$

where $B^2_{AP}(\mathbb{R}^N_y \times \mathbb{R}^N_z)$ denotes the space of functions in $L^2_{loc}(\mathbb{R}^N_y \times \mathbb{R}^N_z)$ that are almost periodic in the Besicovitch sence[10]. We get at once the conclusion of theorem 3.10 with $A = AP(\mathbb{R}^N_y) \odot AP(\mathbb{R}^N_z)$.

4.2.4 Problem IV

Under the following hypothesis

$$a_{ij}(x,\cdot,\cdot) \in B^2_{WAP}(\mathbb{R}^N_y; B^2_{WAP}(\mathbb{R}^N_z)) \text{ for any } x \in \overline{\Omega} \quad (1 \le i,j \le N),$$

the conclusion of Theorem 3.10 holds true with $A = WAP(\mathbb{R}^N_y) \odot WAP(\mathbb{R}^N_z)$. We recall that periodic functions are weakly almost periodic. We also emphasize that in contrast to what happens in Problem III, we have $WAP(\mathbb{R}^N_y \times \mathbb{R}^N_z) \neq WAP(\mathbb{R}^N_y) \odot WAP(\mathbb{R}^N_z)$ (see [43, Corollary 4.13]).

4.2.5 Problem V

Homogenization in Fourier-Stieltjes algebras. The Fourier-Stieljes algebra on $\mathbb{R}^N$, $FS(\mathbb{R}^N)$, is defined as the closure in $\mathcal{B}(\mathbb{R}^N)$ of the space

$$FS_*(\mathbb{R}^N) = \left\{ f : \mathbb{R}^N \to \mathbb{C},\ f(x) = \int_{\mathbb{R}^N} exp(ix \cdot y) d\nu(y) \text{ for some } \nu \in \mathcal{M}_*(\mathbb{R}^N) \right\}$$

where $\mathcal{M}_*(\mathbb{R}^N)$ denotes the space of complex valued measures ν with finite total variation: $|\nu|(\mathbb{R}^N) < \infty$. This a proper ergodic subalgebra of $WAP(\mathbb{R}^N)$ (see e.g., [43]) that contains the periodic functions. Under the following hypothesis

$$a_{ij}(x,\cdot,\cdot) \in B^2_{FS}(\mathbb{R}^N_y; B^2_{FS}(\mathbb{R}^N_z)) \text{ for any } x \in \overline{\Omega} \quad (1 \le i,j \le N),$$

we reach the conclusion of Theorem 3.10 with $A = FS(\mathbb{R}^N_y) \odot FS(\mathbb{R}^N_z)$.

4.3 Double periodicity: The holes are periodically distributed on each scale

We assume that for $\zeta = y,z$ the function θ_ζ is periodic, that is, there exist a network R_ζ in $\mathbb{R}^N_\zeta$ with $R_\zeta \subset \mathbb{Z}^N$ such that

$$\theta_\zeta(k+r) = \theta_\zeta(k) \text{ for all } k \in \mathbb{Z}^N \text{ and all } r \in R_\zeta.$$

Let $P_{R_\zeta}(\mathbb{R}^N_\zeta)$ be the periodic algebra wmv on $\mathbb{R}^N_\zeta$ represented by the group of period R_ζ, that is, the algebra of continuous functions u on $\mathbb{R}^N_\zeta$ satisfying

$$u(\xi + r) = u(\xi) \text{ for all } \xi \in \mathbb{R}^N \text{ and all } r \in R_\zeta.$$

Arguing exactly as in Section 4.1 (see also [38, Section 3.3] we get

$$\chi_{G_\zeta} \in B^2_{P_{R_\zeta}(\mathbb{R}^N_\zeta)}(\mathbb{R}^N_\zeta),\ \ M_\zeta(\chi_{G_\zeta}) > 0$$

and leave to the reader to check that Problems I-V of the previous subsection carry over without slightest change to the present setting. The reader may also consider a double almost periodicity perforation and solve Problems III-V of the previous subsection without any slightest meditation.

4.4 Mixed distribution of the holes

We present here, by way of illustration, just the case when the tiny holes are concentrated in a neighborhood of the origin in $\mathbb{R}^N$ whereas the big ones are almost periodically distributed. Thus, we assume that Ω contains the origin of $\mathbb{R}^N$. Assuming that θ_y is almost periodic in the sense that the translates $\tau_h\theta$ ($h \in \mathbb{Z}^N$) form a relatively compact set in $l^\infty(\mathbb{Z}^N)$, then (see [38]) there exists a countable subgroup R_y of $\mathbb{R}^N_y$ such that

$$\chi_{G_y} \in B^2_{AP_{R_y}(\mathbb{R}^N_y)}(\mathbb{R}^N_y) \text{ with } M_y(\chi_{G_y}) > 0.$$

We denote by $\mathcal{B}_\infty(\mathbb{Z}^N)$ the space of all mapping $u:\mathbb{Z}^N\to\mathbb{C}$ that converges finitely at infinity and assume that $\theta_z\in\mathcal{B}_\infty(\mathbb{Z}^N)$. Following the same line of reasoning as in [38] we can prove that

$$\chi_{G_z}\in B^2_{\mathcal{B}^0_\infty(\mathbb{R}^N_z)}(\mathbb{R}^N_z) \text{ with } M_z(\chi_{G_z})>0,$$

where we recall that on letting F stands for the set of all complex continuous functions f on $\mathbb{R}^N_z$ of the form $f=\sum_{k\in\mathbb{Z}^N} d(k)\tau_k(\varphi)$ with $d\in\mathcal{B}_\infty(\mathbb{Z}^N)$ and $\varphi\in\mathcal{K}(\mathbb{T})$ ($\mathbb{T}$ and $\mathcal{K}(\mathbb{T})$ as in Proposition 4.1), $\mathcal{B}^0_\infty(\mathbb{R}^N_z)$ is the closure in $\mathcal{B}(\mathbb{R}^N_z)$ of the space of all complex functions of the form $\psi=c+\sum_{finite} f_i$ with $c\in\mathbb{C}$ and $f_i\in F$. $\mathcal{B}^0_\infty(\mathbb{R}^N_z)$ is an ergodic algebra wmv.

4.4.1 Problem VI

Let $\mathcal{B}_{\infty,AP}(\mathbb{R}^N)$ denote the space defined as $\mathcal{B}_{\infty,\mathbb{Z}^N}(\mathbb{R}^N_z)$ by replacing $\mathcal{C}_{\mathrm{per}}(Z)$ by $AP(\mathbb{R}^N)$. Then it can be shown that $\mathcal{B}_{\infty,AP}(\mathbb{R}^N)=AP(\mathbb{R}^N)\oplus\mathcal{C}_0(\mathbb{R}^N)$ where $\mathcal{C}_0(\mathbb{R}^N)$ stands for the space of those u in $\mathcal{B}(\mathbb{R}^N)$ that vanish at infinity. The space $\mathcal{B}_{\infty,AP}(\mathbb{R}^N)$ is the space of *perturbed* almost periodic functions. We know that $\mathcal{B}_{\infty,AP}(\mathbb{R}^N)$ is a closed subalgebra of the algebra of weakly almost periodic continuous functions on $\mathbb{R}^N$ [22], and so that each element of $\mathcal{B}_{\infty,AP}(\mathbb{R}^N)$ possesses a mean value. With this in mind and under the following assumption

$$a_{ij}(x,\cdot,\cdot)\in B^2_{\infty,AP}(\mathbb{R}^N_y;\mathcal{B}^0_\infty(\mathbb{R}^N_z)) \text{ for all } x\in\overline{\Omega}\quad(1\le i,j\le N),$$

(where $B^2_{\infty,AP}(\mathbb{R}^N_y)$ denotes the completion of $\mathcal{B}_{\infty,AP}(\mathbb{R}^N_y)$ with respect to the Besicovitch seminorm $\|\cdot\|_2$) the homogenization problem for (1.1) can be solved. More precisely, the conclusions of Theorem 3.10 holds true with $A=\mathcal{B}_{\infty,AP}(\mathbb{R}^N_y)\odot\mathcal{B}^0_\infty(\mathbb{R}^N_z)$.

Remark 4.4. The few problems listed here are just for illustration. In this setting and many more we may solve the homogenization problem for (1.1) under a large class of structure hypothesis on the coefficients a_{ij} the trick being to take $A=WAP(\mathbb{R}^N_y)\odot WAP(\mathbb{R}^N_z)$ (the product of the biggest[43] ergodic algebras wmv available so far in the literature), though in some cases it might not be the appropriate algebra wmv for the problem under consideration.

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