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NONLINEARIZED FOURIER APPROACH, GASDYNAMIC COHERENCE, AND SHOCK-TURBULENCE INTERACTION

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1 Introduction. Main Results

The present paper considers, in a linearized context, the interaction between two gasdynamic objects: a *turbulence model* and, respectively, a *planar shock discontinuity*. The incident turbulence, regarded as a perturbation, is modelled by a nonstatistical/noncorrelative superposition of some finite (or point core) planar vortices. • The *linearized* (with shock) context assumes a *minimal* nonlinearity. It considers a *linearized* problem: a linear problem with a *nonlinear subconscious* (in the sense of P.D. Lax and A. Majda; see §2). The resultant perturbation is regarded as a solution ("interaction solution") of such a *linearized* problem. • In presence of a nonlinear subconscious the interaction solution is essentially constructed as an *admissible* solution.

The turbulence − planar shock interaction is associated with a class of interaction elements. An interaction element formally models the interaction between a planar shock and a *single* incident vortex corresponding to a certain inclination of the vortex axis with respect to the shock. • Modelling the incident turbulence by a superposition of compressible planar vortices appears to correspond to a *first level* of decomposition. In Lighthill's fundamental

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paper [13] the turbulence is acoustically modelled by a distribution of quadrupoles − which is equivalent with a "weighted" distribution of point vortices. • Next, in order to proceed, each incident vortex is Fourier decomposed into planar monochromatic waves − a *second level* of decomposition. • Finally, each incident planar monochromatic wave is Snell passed through the shock discontinuity [Figures 1, 2]. • The composition of the mentioned levels leads to a Fourier−Snell representation of the interaction solution (cf. §§4,5). • The main point of the analysis in §§4,5 is that the result of the passage through the shock can again be presented by two levels of *recombination* so that each incident level of decomposition has a correspondent in the emergent solution.

A Fourier−Snell representation of the linearized interaction between a planar shock discontinuity and a planar compressible finite-core vortex whose axis is *parallel* to the shock has been considered first time by Ribner (1959) in a theoretical attempt consecutive to a pioneering and most suggestive experimental approach of Hollingworth and Richards (1956) concerning the mentioned interaction. An ample and significant series of theoretical and experimental developments has followed the two mentioned works (see Ribner [15] for a thorough review).

The present analysis has essentially two objectives: (a) finding an *explicit, closed, and optimal* form for the interaction solution, and (b) offering an *exhaustively classifying characterization* of this mentioned solution.

Realizing the objective (a) is connected with: (a_1) considering a *singular limit* of the interaction solution, (a_2) considering a *hierarchy of (natural) partitions* of the singular limit, (a_3) inserting some (natural) *gasdynamic factorizations* at a certain level of the mentioned hierarchy and (a_4) noticing a *compatibility* of these factorizations (indicating a gasdynamic *inner coherence*), (a_5) *predicting some exact details* of the interaction solution, (a_6) indicating some parasite singularities [= strictly depending on the method] to be compensated [= pseudosingularities], (a_7) *re-weighting* the singular limit of the interaction solution.

Realizing the objective (b) is connected with finding some *Lorentz arguments of criticity*. The interaction solution appears essentially to (exhaustively) include a *subcritical* and respectively a *supercritical* contribution distinguished by differences of a *"relativistic"* nature. Precisely: in the singular limit of the interaction solution the emergent sound is *singular* in the subcritical contribution and it is *regular* in the supercritical contribution. This "relativistic" discontinuity in the nature of the emergent sound, corresponding to the singular limit of the interaction solution, appears to be dissembled (hidden) in the re-weighted interaction solution.

The present analysis could be set in contrast with a lot of recent studies, devoted to shock-vortex parallel interaction, which allow (analitically or numerically) a *more complete* consideration of the nonlinearity contribution; see for example Grove and Menikoff [7], Han and Yin [8] or Inoue et all [9].

The work of Han and Yin (analytically) allows *more* nonlinearity yet in presence of *a set of (approximating) restrictions* [cf. its page 188]. These authors characterize the context of their work to be "complicated" [page 189]. The present paper identifies, in presence of a *minimal* nonlinearity, some *structuring arguments* (needed to replace a "complicated" context by a *complex* context). • More nonlinearity is (numerically) allowed in the *parallel* interactions considered in the papers by Inoue et all or Grove and Menikoff.

• The structure of the present interaction solution is associated first, from a classifying prospect, to the Lighthill fundamental representation of the shock-turbulence interaction ([13]). • It is noticed that the present interaction solution parallels and extends, from an analytical prospect, the Ribner representation and computational approach corresponding to the interaction between a shock discontinuity and a planar vortex whose axis is parallel to this discontinuity. • The details of the "relativistic" separation between a subcritical character and a supercritical character are essentially and significantly related with the criticity arguments considered in fundamental numerical studies on the shock-turbulence interaction; see S.K. Lele [12].

2 Linearized context. Ingredients of a Fourier−Snell analysis

2.1 Linearized context. Nonlinear subconscious

We begin by presenting the *linearized* context which will be used to describe the turbulence-shock interaction.

We consider, at the zeroth order of a perturbation expansion, a shock (= *admissible* discontinuity). A distinctive feature of the linearized analysis will be therefore that a *triad* is perturbed which includes, in addition to the adjacent (to the shock) constant (left/right) states u_l, u_r, the shock propagation speed D. If the perturbation is two-dimensional a linearized analysis has to begin with the system of equations

$$\frac{\partial u}{\partial t}+\frac{\partial f(u)}{\partial x}+\frac{\partial g(u)}{\partial y}=0 \tag{2.1}$$

together with the jump conditions on the shock

$$[\![u]\!]\frac{\partial \varphi}{\partial t}+[\![f(u)]\!]\frac{\partial \varphi}{\partial x}+[\![g(u)]\!]\partial\varphi\partial y=0 \tag{2.2}$$

where we put $[\![u]\!]=u_r-u_l$ and, similarly, $[\![f(u)]\!]=f(u_r)-f(u_l)$, etc.

We have to develop, with respect to a small parameter ε of the flow

$$0<\varepsilon\ll\min(|u_r|,|u_l|)\quad \text{for } |u_r|\neq 0,\ |u_l|\neq 0,$$

both the dependent and independent variables in (2.1), (2.2). We express x,y,t in terms of X,Y,T (variables which are independent of ε) and ε, cf.

$$x=X+\varphi^{\varepsilon}(Y,T),\ \ t=T,\ \ y=Y;\quad \varphi^{\varepsilon}=DT+\psi^{\varepsilon}(Y,T) \tag{2.3}$$

use the independence of X,Y,T of ε in (2.3), assume that the perturbed data and the perturbed solution

$$u_0^{\varepsilon}(x,y)\equiv U_0^{\varepsilon}(X,Y;\varepsilon),\quad u^{\varepsilon}(x,y,t)\equiv U^{\varepsilon}(X,Y,T;\varepsilon)$$

smoothly depend on ε, and take into account

$$[U_{l,r}^{\varepsilon}]_{\varepsilon=0}=u_{l,r},\ [\psi^{\varepsilon}]_{\varepsilon=0}=0,\ \left[\frac{\mathrm{d}}{\mathrm{d}\varepsilon}U_0^{\varepsilon}\right]_{\varepsilon=0}=\widetilde{U}_0,\ \left[\frac{\mathrm{d}}{\mathrm{d}\varepsilon}U^{\varepsilon}\right]_{\varepsilon=0}=\widetilde{U},\ \left[\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\psi^{\varepsilon}\right]_{\varepsilon=0}=\psi\ ;$$

then, on separating the first order in ε, we are left with the linearized problem

$$\frac{\partial}{\partial T}\widetilde{U}+A\frac{\partial}{\partial X}\widetilde{U}+b\frac{\partial}{\partial Y}\widetilde{U}=0,\quad (X,Y)\in\mathbb{R}^2,\ T>0 \tag{2.4}$$

$$A_r\widetilde{U}_r=A_l\widetilde{U}_l+[\![u]\!]\frac{\partial\psi}{\partial T}+[\![g(u)]\!]\frac{\partial\psi}{\partial Y}\quad \text{for } X=0 \tag{2.5}$$

$$\widetilde{U}(X,Y,0)=\widetilde{U}_0(X,Y),\ \psi(Y,0)=\psi_0(Y),\quad (X,Y)\in\mathbb{R}^2 \tag{2.6}$$

where

$$A_{l,r}=a(u_{l,r})-DI,\quad A=A(X)\equiv A_l[1-H(X)]+A_rH(X);\quad a(u)=f'(u), \tag{2.7}$$

and b results from $(2.7)_2$ when $A_{l,r}$ is replaced by $b(u_{l,r})$; H is the Heaviside function.

We notice that the limit $|u_r-u_l|\to 0$ of the linearized solution fulfils a linear problem; in fact, the limit linear problem *ignores* the contribution of ψ. This contribution could be regarded as a *memory* of an optimal context connected with the linearized problem. • This aspect indicates the importance of a *nonlinear subconscious*. A nonlinear subconscious (in the sense of P.D. Lax and A. Majda; see [11] and [14]) results when the nonlinearity is allowed only at the zeroth order of a perturbation expansion: a *piecewise constant* admissible solution [with shock; zeroth order] is perturbed; one linearize and prove that the zeroth order requirement of admissibility is still active at the first order and essentially structures the linearized description.

In the case of *adiabatic gas dynamics of a perfect inviscid gas* the system (2.4) takes the form

$$\frac{1}{\overline{c}^2}\overline{\mathcal{D}}\widetilde{p}_l+\overline{\rho}\frac{\partial\widetilde{u}_l}{\partial X}+\overline{\rho}\frac{\partial\widetilde{v}_l}{\partial Y}=0,\ \ \overline{\rho}\overline{\mathcal{D}}\widetilde{u}_l+\frac{\partial\widetilde{p}_l}{\partial X}=0,\ \overline{\rho}\overline{\mathcal{D}}\widetilde{v}_l+\frac{\partial\widetilde{p}_l}{\partial Y}=0,\ \ \overline{\mathcal{D}}\widetilde{s}_l=0\ \ \text{for } X<0 \tag{2.8}$$

where

$$\widetilde{p}_l=\overline{c}^2\widetilde{\rho}_l+(p_s)_l\widetilde{s}_l \tag{2.9}$$

and

$$\mathcal{D}\widetilde{p}+\frac{\partial\widetilde{u}}{\partial X}+\frac{\partial\widetilde{v}}{\partial Y}=0,\ \ \mathcal{D}\widetilde{u}+\frac{\partial\widetilde{p}}{\partial X}=0,\ \ \mathcal{D}\widetilde{v}+\frac{\partial\widetilde{p}}{\partial Y}=0,\ \ \mathcal{D}\widetilde{s}=0\quad \text{for } X>0 \tag{2.10}$$

where

$$\widetilde{p}=\widetilde{\rho}+(p_s)_r\widetilde{s} \tag{2.11}$$

and we denote

$$\overline{\mathcal{D}}=\frac{\partial}{\partial T}+\overline{M}\frac{\partial}{\partial X}+M_y\frac{\partial}{\partial Y},\ \ \mathcal{D}=\frac{\partial}{\partial T}+M\frac{\partial}{\partial X}+M_y\frac{\partial}{\partial Y}.$$

Here, in usual notations, we put ρ, p, s, v_x, v_y for the density, pressure, specific entropy and velocity components respectively.

Relations (2.5) take in this case the form

$$(\widetilde{s}_+,\widetilde{p}_+,\widetilde{u}_+,\widetilde{v}_+)^t=\mathfrak{a}(\widetilde{s}_-,\widetilde{p}_-,\widetilde{u}_-,\widetilde{v}_-)^t+\mathfrak{b}\frac{\partial\psi}{\partial T}+\mathfrak{c}\frac{\partial\psi}{\partial Y},\ \ \text{for } X=0 \tag{2.12}$$

where +/− indicate respectively the states behind/ahead of and, in presence of a component M_y in the direction Y for the velocity corresponding to the adjacent constant states, the coefficients $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ have the expressions

$$\begin{cases} \mathfrak{a}_{11} = 1 + \frac{\overline{M}}{2}\mathfrak{b}_1, \ \ \mathfrak{a}_{12} = -\frac{(\gamma^2-1)^2}{2(\gamma+1)} \cdot \frac{(M-\overline{M})^2}{2\gamma M^2 - (\gamma-1)}, \ \ \mathfrak{a}_{13} = -\mathfrak{b}_1, \\ \mathfrak{a}_{21} = -\frac{2}{\gamma+1} M\overline{M}, \ \ \mathfrak{a}_{22} = \frac{(\gamma+1) - 2(\gamma-1)M\overline{M}}{2\gamma M^2 - (\gamma-1)}, \ \ \mathfrak{a}_{23} = -\mathfrak{b}_2, \\ \mathfrak{a}_{31} = M - \frac{\gamma-1}{\gamma+1}\overline{M}, \ \ \mathfrak{a}_{32} = \frac{2}{M} \cdot \frac{\gamma-1}{\gamma+1}, \ \ \mathfrak{a}_{33} = 1 - \mathfrak{b}_3, \\ \mathfrak{a}_{14} = \mathfrak{a}_{24} = \mathfrak{a}_{34} = \mathfrak{a}_{41} = \mathfrak{a}_{42} = \mathfrak{a}_{43} = 0, \ \ \mathfrak{a}_{44} = 1, \\ \mathfrak{b}_1 = -\frac{\gamma-1}{\overline{M}}\mathfrak{c}_4^2, \ \ \mathfrak{b}_2 = -\frac{4M}{\gamma+1}, \ \ \mathfrak{b}_3 = \frac{3-\gamma}{\gamma+1} + \frac{M}{\overline{\overline{M}}}, \ \ \mathfrak{b}_4 = 0, \\ \mathfrak{c}_1 = M_y \mathfrak{b}_1, \ \ \mathfrak{c}_2 = M_y \mathfrak{b}_2, \ \ \mathfrak{c}_3 = M_y \mathfrak{b}_3, \ \ \mathfrak{c}_4 = \overline{M} - M. \end{cases}$$

We notice that the equations (2.8)−(2.12) are presented in a dimensionless form for which the entities of the perturbed flow are divided by the constant unperturbed state behind the shock. We denote by $\widetilde{s}, \widetilde{p}, \widetilde{u}, \widetilde{v}$ the dimensionless perturbation where

$$[x] = L, \ \ [t] = \frac{L}{c_r}, \ \ [\rho] = \rho_r, \ \ [v] = c_r, \ \ [p] = \rho_r c_r^2, \ \ [s] = c_p, \ \ [T] = \frac{c_r^2}{c_p}$$

$$M = \frac{v_{xr} - D}{[v]}, \ \ \overline{M} = \frac{v_{xl} - D}{[v]}, \ \ M_y = \frac{v_y}{[v]}, \ \ \overline{\rho} = \frac{\rho_l}{[\rho]} = \frac{1}{\overline{\tau}}, \ \ \overline{p} = \frac{p_l}{[p]}, \ \ p = \frac{p_r}{[p]}, \ \ \overline{c} = \frac{c_l}{[v]}.$$

2.2 Ingredients of a Fourier−Snell analysis

Two *essential*, *distinct* and *complementary* classes of *admissible (entropy)* solutions of (2.8)−(2.12) are considered in §§3−5: (*a*) solutions evolving from initial data which tend suitably fast to zero at the infinity, and, (*b*) elementary polymodal Fourier−Snell structures of a *real* frequency [an admissible elementary polymodal structure of a strictly complex frequency belongs to the class (*a*)]. • It can be shown that the requirement of admissibility completely structures/determines the [linearized] solutions in each of these classes.

In the *multidimensional* case [in contrast with the one-dimensional case] the stability of these linearized solutions is not unconditionally guaranteed. A distinction between the stable and unstable circumstances is essentially made, in this case, by a *linearization criterion*: see Blokhin and Trakhinin [1] and Dinu [2] for a thorough review. Incidentally, in case of the adiabatic dynamics of a perfect inviscid gas the linearization appears to be active.

This paper aims to present an example of evolution in the class (*a*), constructed as a superposition of elements in the class (*b*).

We complete the present paragraph with a short review of some aspects of a (linearized) Fourier−Snell analysis in presence of a shock (= admissibile discontinuity).

In presence of an admissible discontinuity (shock) the role that a modal monochromatic wave plays in a linear Fourier analysis is taken over, in a *linearized* Fourier type analysis, by an elementary polymodal structure. Such an elementary structure consists in a *finite* (eventually *minimal*) number of *Snell compatible* monochromatic waves.

A monochromatic wave has the form

$$(\widetilde{s}_{l,r}, \widetilde{p}_{l,r}, \widetilde{u}_{l,r}, \widetilde{v}_{l,r})^t = (\widehat{s}_{l,r}, \widehat{p}_{l,r}, \widehat{u}_{l,r}, \widehat{v}_{l,r})^t \exp \mathrm{i}(\alpha_{l,r} X + \beta_{l,r} Y - \omega_{l,r} T), \quad \beta_{l,r} \in \mathbb{R},$$

associated with the propagation vector

$$(\alpha_{l,r}, \beta_{l,r}) = k_{l,r}\,(\cos\kappa_{l,r}, \sin\kappa_{l,r}).$$

As it is well known, there are three gasdynamic distinct modes: a sound mode and a (double) entropy-vorticity mode; therefore, we have at our disposal six modal monochromatic waves (three for each of the two regions adjacent to the shock) to construct an elementary structure and we use, as a *key* element of this construction the following Snell laws of refraction through /reflection at the shock ([10], [2]):

all the monochromatic waves implied in the elementary structure have
(S_1) equal frequencies ω, when measured in the same reference frame, and
(S_2) equal values of β.

Essentially, for the monochromatic waves which contribute in an elementary structure, we use in this construction: the shock relations (2.12) to connect their amplitudes and the mentioned Snell laws (S_1), (S_2) to connect, via the modal dispersion laws, their propagation vectors.

The class (*b*) of elementary structures is presented in our study as an union of two disjoint subclasses (see for example Kontorovich [10], Dinu [2]) : a *pseudohyperbolic* subclass [each element of this subclass includes only monochromatic waves with a real α], and, a *pseudoelliptic* subclass [each element of this subclass has a structure which includes at least a monochromatic wave with a (strictly) complex α]. It can be proven that only four [real frequency] elementary structures are admissible [= have a completely determined/organized linearized evolution] in presence of an admissible discontinuity (see for example Kontorovich [10], Dinu [2]); precisely:

$$\mathcal{V}_{li}\mathcal{S}^{+}_{rd}\mathcal{V}_{rd}\ ,\quad \mathcal{S}^{+}_{li}\mathcal{S}^{+}_{rd}\mathcal{V}_{rd}\ ,\quad \mathcal{S}^{-}_{li}\mathcal{S}^{-}_{rd}\mathcal{V}_{rd}\ ,\quad \mathcal{S}^{-}_{ri}\mathcal{S}^{+}_{rd}\mathcal{V}_{rd}\ , \tag{2.13}$$

where in (2.13) $\mathcal{V}$ and $\mathcal{S}$ indicate, respectively, an entropy-vorticity and a sound contribution [with the subscripts *l/r* for left (ahead of) /right (behind), and *i/d* for incident/divergent (emergent)]. • An interaction solution of the linearized problem is Fourier−Snell constructed by a superposition of certain [admissible, real frequency] elementary polymodal structures (2.13). We have to notice in this respect that the emergent initial data in (2.6) result constructively from the Fourier representation of the incident initial data.

The present paper only considers the details related to the first of the elements (elementary structures) (2.13). If we associate, as a parameter, to this element the inclination of its incident entropy-vorticity propagation vector, cf.

$$\mathfrak{z} = \tan\kappa_{ev,li} = \frac{\beta}{\alpha_{ev,li}} = -\cot\phi_l, \tag{2.14}$$

(see Figure 1) then it can be shown that two cases, a pseudohyperbolic one [for $|\mathfrak{z}| < \mathfrak{z}_c$] and respectively a pseudoelliptic one [for $|\mathfrak{z}| > \mathfrak{z}_c$], are possible for the considered structure, separated by the critical value

$$\mathfrak{z}_c = \frac{\overline{M}}{\sqrt{1-M^2}}. \tag{2.14$_c$}$$

The structures (2.13) replace the four elementary (monochromatic) waves of the gasdynamic Fourier theory of a *linear* problem.

3 The Ribner parallel linearized solution

3.1 Highlights of this work

Paragraphs 3−5 present (thus materializing a suggestion of §1) a *set of arguments needed to structure* the complex construction of the interaction (turbulence-shock) solution. Paragraphs 3,4 consider a *parallel* version (see §1) of the mentioned set of arguments. Then, an *oblique* version of this set of arguments is taken into account in §5.

3.2 Sound contribution in the interaction solution: first constructive details. Gasdynamic partitions (I). Lorentz coordinates

We shall use the Lagrangian reference frames $\underset{\sim}{x}, \underset{\sim}{y}$ (fixed on the undisturbed flow ahead of the shock) and $\widetilde{x}, \widetilde{y}$ (fixed on the undisturbed flow behind the shock) in addition to the frame X, Y fixed on the shock discontinuity. We have

$$\underset{\sim}{x} = X - \overline{M}T,\ \ \widetilde{x} = X - MT = \underset{\sim}{x} + (\overline{M} - M)\,\underset{\sim}{t};\ \ \underset{\sim}{y} = \widetilde{y} = Y;\ \ \underset{\sim}{t} = \tilde{t} = T. \tag{3.1}$$

Now, in the frame $\underset{\sim}{x}, \underset{\sim}{y}$ we consider for the subsystem (2.8), (2.9) the *steady* solution of a vortex with a *finite* core

$$[\widetilde{u}(\underset{\sim}{x}, \underset{\sim}{y}), \widetilde{v}(\underset{\sim}{x}, \underset{\sim}{y})] = \frac{\widetilde{\varepsilon}}{2\pi} \begin{cases} (1/r_*^2)[-\underset{\sim}{y}, \underset{\sim}{x}] \text{ for } r \le r_* \\ (1/r^2)[-\underset{\sim}{y}, \underset{\sim}{x}] \text{ for } r_* \le r \end{cases} \qquad \widetilde{s} \equiv \widetilde{p} \equiv 0. \tag{3.2}$$

where r_* is the radius of the vortex core.

Proposition 3.1. *(Ribner [15]). The solution* (3.2) *is Fourier represented by*

$$\{\tilde{s}, \tilde{p}, \tilde{u}, \tilde{v}\} = -\frac{\widetilde{\varepsilon}}{2\pi^2} \mathrm{Im} \int_0^\infty \frac{1}{k} \cdot \frac{2J_1(kr_*)}{kr_*} \mathrm{d}k \int_{-\pi/2}^{\pi/2} \exp[\mathrm{i}(\alpha_l \underset{\sim}{x} + \beta_l \underset{\sim}{y})]\{0, 0, \beta_l, -\alpha_l\} \mathrm{d}\kappa_l. \tag{3.3}$$

Remark 3.2. (Ribner [15]). (i) The parallel vortex-shock interaction solution results cf. (3.3) and $(2.13)_1$. Precisely: we have to complete, in the region behind the shock, each incident vorticity wave in the sum (3.3) up to an elementary structure $(2.13)_1$. Therefore, a sound contribution and, respectively, an entropy-vorticity contribution are seen to be included by the mentioned interaction solution in the region behind the shock. Only the emergent sound contribution will be *constructed*. The emergent entropy-vorticity contribution will be then *represented* in terms of the emergent sound contribution [see (3.8)−(3.11) here in below]. (ii) We notice (Figure 1) that to *each* elementary structure $(2.13)_1$ which contributes in the representation of the interaction solution an associated frame $\widehat{X}, \widehat{Y}$ coresponds which translates along the shock, cf. $\widehat{X} = X$, $\widehat{Y} = Y + M_y T$, where the velocity M_y is chosen [to annul the frequency of the incident vorticity wave] so as to make steady the elementary structure $(2.13)_1$ associated to it.

We compute:

$$\alpha_l \underset{\sim}{x} + \beta_l \underset{\sim}{y} = \alpha_l \widehat{X} + \beta_l \widehat{Y}.$$

Now, the emergent *sound* monochromatic wave coresponding to the incident vorticity monochromatic wave

$$A_0(0,0,\beta_l,-\alpha_l)\exp[\imath(\alpha_l \underset{\sim}{x} + \beta_l \underset{\sim}{x})]$$

$$A_0 = -\frac{\widetilde{\varepsilon}}{2\pi^2}\cdot\frac{2J_1(kr_*)}{kr_*}\cdot\frac{1}{k}\mathrm{d}k\,\mathrm{d}\kappa_l = -\frac{\widetilde{\varepsilon}}{2\pi^2}\cdot\frac{2J_1(kr_*)}{kr_*}\cdot\frac{1}{k}\cdot\frac{1}{1+\mathfrak{z}^2}\mathrm{d}k\,\mathrm{d}\mathfrak{z}$$

in (3.3) can be presented by:

$$A_1[0,-(M\alpha_s + M_y\beta_l),\alpha_s,\beta_l]\exp[\mathrm{i}(\alpha_s\widehat{X} + \beta_l\widehat{Y})],\ A_1 = a_1 A_0$$

$$a_1 = \mathfrak{z}\frac{(d_{11}\mathfrak{z}^2 + d_{12}) + (d_{13}\mathfrak{z}^2 + d_{14})\check{\varepsilon}\sqrt{|\mathfrak{z}_c^2 - \mathfrak{z}^2|}}{(d_{01}\mathfrak{z}^2 + d_{02}) + (d_{03}\mathfrak{z}^2 + d_{04})\check{\varepsilon}\sqrt{|\mathfrak{z}_c^2 - \mathfrak{z}^2|}}, \qquad \check{\varepsilon} = \begin{cases} 1 \text{ for } |\mathfrak{z}| \le \mathfrak{z}_c \\ \mathrm{i} \text{ for } |\mathfrak{z}| \ge \mathfrak{z}_c \end{cases} \tag{3.4}$$

with

$$d_{01} = \frac{2}{\gamma+1}\frac{\overline{M}}{M}(1-2M^2),\ \ d_{02} = \frac{\overline{M}}{M}d_{01} - \frac{8}{(\gamma+1)^2}\frac{\overline{M}^2}{M^2}(1-M^2)$$

$$d_{03} = -\frac{2}{\gamma+1}\sqrt{1-M^2},\ \ d_{04} = \frac{\overline{M}}{M}d_{03},$$

$$d_{11} = \frac{8}{(\gamma+1)^2}(1-M^2),\ d_{12} = -\frac{\overline{M}}{M}d_{11},\ d_{13} = d_{14} = 0$$

where we use the *Lorentz coordinates*

$$x = \frac{\widetilde{x} + M\widetilde{t}}{\sqrt{1-M^2}} = \frac{X}{\sqrt{1-M^2}},\ \ y = \widetilde{y},\ \ t = \frac{\widetilde{t} + M\widetilde{x}}{\sqrt{1-M^2}} \tag{3.5}$$

to compute, cf. (2.14), (3.1) and Remark 3.2(ii):

$$\frac{\alpha_s}{\beta_l} = \mu = \begin{cases} \dfrac{\sqrt{\mathfrak{z}_c^2 - \mathfrak{z}^2} - M\mathfrak{z}_c}{\mathfrak{z}\sqrt{1-M^2}} & \text{for } |\mathfrak{z}| < \mathfrak{z}_c \\ \dfrac{\mathrm{i}\sqrt{\mathfrak{z}^2 - \mathfrak{z}_c^2} - M\mathfrak{z}_c}{\mathfrak{z}\sqrt{1-M^2}} & \text{for } \mathfrak{z}_c < |\mathfrak{z}| \end{cases}$$

$$\mathrm{i}\left(\alpha_s\widehat{X} + \beta_l\widehat{Y}\right) = \begin{cases} k\dfrac{x\sqrt{\mathfrak{z}_c^2 - \mathfrak{z}^2} - (t\mathfrak{z}_c - y\mathfrak{z})}{\sqrt{1+\mathfrak{z}^2}} & \text{for } |\mathfrak{z}| \le \mathfrak{z}_c \\ -k\dfrac{x\sqrt{\mathfrak{z}^2 - \mathfrak{z}_c^2}}{\sqrt{1+\mathfrak{z}^2}} - \mathrm{i}k\dfrac{t\mathfrak{z}_c - y\mathfrak{z}}{\sqrt{1+\mathfrak{z}^2}} & \text{for } |\mathfrak{z}| \ge \mathfrak{z}_c\ . \end{cases}$$

The *sound* contribution in the constructed solution for $X > 0$ (behind the shock) results from (3.3), (2.14), (3.4) and (3.5) cf. Remark 3.2(i) and consists of a *pseudohyperbolic*

FIGURE 1 Details of the parallel construction part, abbreviated *h*-part, which is a superposition (Figure 1*a,b*) of pseudohyperbolic waves corresponding to $|\mathfrak{z}| \leq \mathfrak{z}_c$,

$$[\widetilde{p}_h(x,y,t), \widetilde{u}_h(x,y,t), \widetilde{v}_h(x,y,t)]$$

$$= \frac{\widetilde{\varepsilon}}{2\pi^2} \int_{-\mathfrak{z}_c}^{\mathfrak{z}_c} I_h(r_*) \left[\frac{M\sqrt{\mathfrak{z}_c^2 - \mathfrak{z}^2} - \mathfrak{z}_c}{\mathfrak{z}\sqrt{1-M^2}}, -\frac{\sqrt{\mathfrak{z}_c^2 - \mathfrak{z}^2} - M\mathfrak{z}_c}{\mathfrak{z}\sqrt{1-M^2}}, -1 \right] \cdot a_1 \cdot \frac{\mathfrak{z}}{\sqrt{1+\mathfrak{z}^2}} \cdot \frac{1}{1+\mathfrak{z}^2} \mathrm{d}\mathfrak{z} \quad (3.6)$$

$$I_h(r_*) = \int_0^\infty \frac{2J_1(kr_*)}{kr_*} \sin \left[k \frac{x\sqrt{\mathfrak{z}_c^2 - \mathfrak{z}^2} - (t\mathfrak{z}_c - y\mathfrak{z})}{\sqrt{1+\mathfrak{z}^2}} \right] \mathrm{d}k \quad (3.6)^*$$

and a <u>*pseudoelliptic*</u> part, abbreviated *e*-part, which is a superposition of pseudoelliptic waves corresponding to $|\mathfrak{z}| \geq \mathfrak{z}_c$,

$$[\widetilde{p}_e(x,y,t), \widetilde{u}_e(x,y,t), \widetilde{v}_e(x,y,t)]$$

$$= \frac{\widetilde{\varepsilon}}{2\pi^2} \mathrm{Im} \left(\int_{-\infty}^{-\mathfrak{z}_c} + \int_{\mathfrak{z}_c}^{\infty} \right) I_e(r_*) \left[\frac{\mathrm{i}M\sqrt{\mathfrak{z}^2 - \mathfrak{z}_c^2} - \mathfrak{z}_c}{\mathfrak{z}\sqrt{1-M^2}}, -\frac{\mathrm{i}\sqrt{\mathfrak{z}^2 - \mathfrak{z}_c^2} - M\mathfrak{z}_c}{\mathfrak{z}\sqrt{1-M^2}}, -1 \right] \cdot a_1 \cdot \frac{\mathfrak{z}}{\sqrt{1+\mathfrak{z}^2}} \cdot \frac{1}{1+\mathfrak{z}^2} \mathrm{d}\mathfrak{z} \quad (3.7)$$

$$I_e(r_*) = \int_0^\infty \frac{2J_1(kr_*)}{kr_*} \exp \left[-k \frac{x\sqrt{\mathfrak{z}^2 - \mathfrak{z}_c^2}}{\sqrt{1+\mathfrak{z}^2}} - \mathrm{i}k \frac{t\mathfrak{z}_c - y\mathfrak{z}}{\sqrt{1+\mathfrak{z}^2}} \right] \mathrm{d}k. \quad (3.7)^*$$

The form (3.6), (3.7) reflects some essential re-arrangements (see Dinu [2]) of the original Ribner's representation.

We could obtain expressions similar to (3.6), (3.7) for the <u>*entropy-vorticity*</u> contribution and the <u>*shock disturbance*</u>. Still, we shall prefer, using the equations (2.8)−(2.11) and the shock relations (2.12), to represent these contributions in terms of the sound contribution cf.:

$$\widetilde{u}_{\mathrm{vorticity}}(\widetilde{x}, \widetilde{y}, \widetilde{t})$$

$$= \widetilde{u}_- \left(\underset{\sim}{y}, \underset{\sim}{t} = T - \frac{X}{M} \right) + \frac{\mathfrak{b}_3}{\mathfrak{b}_2} \widetilde{p}_+ \left(\widetilde{y}, \widetilde{t} = T - \frac{X}{M} \right) - \int_{T-\frac{X}{M}}^{T} \frac{\partial \widetilde{p}}{\partial \widetilde{x}} (\widetilde{x}, \widetilde{y}, \theta) \mathrm{d}\theta - \widetilde{u}_{\mathrm{sound}}(\widetilde{x}, \widetilde{y}, \widetilde{t}) \quad (3.8)$$

$$\widetilde{v}_{\mathrm{vorticity}}(\widetilde{x}, \widetilde{y}, \widetilde{t})$$

$$= \widetilde{v}_- \left(\underset{\sim}{y}, \underset{\sim}{t} = T - \frac{X}{M} \right) + \mathfrak{c}_4 \frac{\partial \psi}{\partial \widetilde{y}} \left(\widetilde{y}, \widetilde{t} = T - \frac{X}{M} \right) - \int_{T-\frac{X}{M}}^{T} \frac{\partial \widetilde{p}}{\partial \widetilde{y}} (\widetilde{x}, \widetilde{y}, \theta) \mathrm{d}\theta - \widetilde{v}_{\mathrm{sound}}(\widetilde{x}, \widetilde{y}, \widetilde{t}) \quad (3.9)$$

$$\widetilde{s}(\widetilde{x}, \widetilde{y}, \widetilde{t}) \equiv \frac{\mathfrak{b}_1}{\mathfrak{b}_2} \widetilde{p}_+ \left(-\frac{\widetilde{x}}{M}, \widetilde{y} \right) \quad (3.10)$$

$$\psi(\widetilde{y}, \widetilde{t}) = \int_{-\infty}^{\widetilde{t}} \left[\frac{1}{\mathfrak{b}_2} \widetilde{p}_+(\widetilde{y}, \theta) + \widetilde{u}_-(\widetilde{y}, \theta) \right] \mathrm{d}\theta \quad (3.11)$$

where we have to insert in (3.8), (3.9), cf. (3.1)

$$T = \tilde{t},\ \ T - \frac{X}{M} = -\frac{\widetilde{x}}{M},\ \ \underset{\sim}{y} = \widetilde{y},$$

and we take into account that $\lim_{T\to-\infty} \psi = 0$ in order to get (3.11).

We motivate by Remark 3.2 to call (3.2), (3.6), (3.7), (3.8)−(3.11) *the Ribner representation of the linearized interaction solution*.

4 Explicit closed form of Ribner's parallel representation

4.1 Two essential elements of the structural analysis

Before presenting the details of the analysis in this paragraph we have to identify, cf. Remark 4.1 here below, two elements essential for structuring this analysis. We denote in $(3.6)^*$

$$\mathcal{E}(\widetilde{x},\widetilde{y},\tilde{t};\mathfrak{z}) \overset{\text{def}}{=} x\sqrt{\mathfrak{z}_c^2-\mathfrak{z}^2} - (t\mathfrak{z}_c - y\mathfrak{z})$$

$$\equiv \frac{\widetilde{x}(\sqrt{\mathfrak{z}_c^2-\mathfrak{z}^2} - M\mathfrak{z}_c) + \widetilde{y}\mathfrak{z}\sqrt{1-M^2} + \widetilde{t}(M\sqrt{\mathfrak{z}_c^2-\mathfrak{z}^2} - \mathfrak{z}_c)}{\sqrt{1-M^2}}, \quad \mathfrak{z} \in (-\mathfrak{z}_c, \mathfrak{z}_c).$$

A straightforward calculation shows that for each $\tilde{t} > 0$ the envelope (corresponding to the *pseudohyperbolic* contribution; depicted, cf. Figure 1, in X,Y with T as a parameter) of the straightlines family $\mathcal{E} = 0$, $\mathfrak{z} \in (-\mathfrak{z}_c, \mathfrak{z}_c)$ has the form of an *arc* of the (dimensionless) *sonic circle*

$$\widetilde{x}^2 + \widetilde{y}^2 - \tilde{t}^2 = x^2 + y^2 - t^2 = 0, \quad X > 0. \tag{4.1}$$

Remark 4.1. (Ribner [15]). *We call the arc* (4.1) *the S-arc; also, the region of the sonic disk belonging to the half-plane* $X > 0$ *is said to be the S-region.*

4.2 The highlights of the parallel analysis

Remark 4.2. We have $\mathcal{E}(\tilde{x},\tilde{y},\tilde{t};\mathfrak{z}) < 0$, $\mathfrak{z} \in (-\mathfrak{z}_c, \mathfrak{z}_c)$ at the interior points of the S-region. Consequently, the phase in $(3.6)^*$ is (strictly) negative at the interior points of the S-region.

At this point we have to notice that even in presence of the structuring arguments of §3 we may need a bit of "chance" in order to get a successful calculation in the Ribner representation. For example, the attempt to obtain an explicit/closed form for the Ribner *parallel* interaction solution may be fruitless if we are not aware of the presence of a lot of "traps": (*i*) the emergent sound contribution (3.6), (3.7) cannot be computed *directly*; in fact, this contribution can be put in an explicit form directly *only* in the limit $r_* \to 0$ and *only* at the points of the S-region; incidentally it can be predicted (and verified) at the exterior points of the S-region; (*ii*) the emergent entropy-vorticity contribution cannot be computed directly in its Fourier−Snell representation [similar to (3.6), (3.7)] even in the limit $r_* \to 0$; its explicit form results by taking into account its connection (3.8)−(3.11) with the emergent sound contribution; (*iii*) finally, the explicit form of the Ribner nonsingular interaction representation results from a *re-weighting* (a re-set of the weight lost in the limit $r_* \to 0$; cf. Dinu and Dinu [5]).

4.3 Gasdynamic factorizations (I)

Remark 4.3. (Dinu [2]). By rationalizing the denominator of (3.4) we obtain, irrespectively of the circumstances $|\mathfrak{z}| \le \mathfrak{z}_c$ or $|\mathfrak{z}| \ge \mathfrak{z}_c$, the *factorized* expression

$$E(\mathfrak{z}^2) \stackrel{\text{def}}{=} (d_{01}\mathfrak{z}^2 + d_{02})^2 + (d_{03}\mathfrak{z}^2 + d_{04})^2(\mathfrak{z}^2 - \mathfrak{z}_c^2) \equiv d_{03}^2(\mathfrak{z}^2 + a^2)(\mathfrak{z}^2 - b^2)(\mathfrak{z}^2 - c^2) \quad (4.2)$$

with

$$a \stackrel{\text{def}}{=} \frac{\overline{M}}{M}, \;\; \varpi_{\pm}^2 \stackrel{\text{def}}{=} \frac{\overline{M}}{M}\left[(2M\overline{M} - 1) \pm 2M\sqrt{\frac{\gamma-1}{\gamma+1}M\overline{M}}\right], \;\; b^2 \stackrel{\text{def}}{=} \varpi_-^2, \;\; c^2 \stackrel{\text{def}}{=} \varpi_+^2 \quad (4.3)$$

$$a > 1, \quad \begin{cases} b^2 > 0 \quad \text{for } -1 < \gamma < \frac{5}{3}; \quad c^2 > 0 \\ 0 < |b| < |c| < \mathfrak{z}_c \end{cases} \quad (4.4)$$

where a corresponds to the entropy-vorticity contribution while b, c correspond to the sound contribution.

4.4 Singular limit of the sound contribution: (I) Lorentz entities

The computation of the limit $r_* \to 0$ of the sound contribution begins with the following steps.

∎ (†) We explicitly calculate $I_h(r_*)$, $I_e(r_*)$, given by $(3.6)^*$, $(3.7)^*$, and then $\lim_{r_*\to 0} I_h(r_*)$, $\lim_{r_*\to 0} I_e(r_*)$ at the points of the S-region (using the Remark 4.2). We have from $(3.6)^*$ for each interior point of the S-region:

$$I_h(r_*) = -\frac{2\sqrt{1+\mathfrak{z}^2}}{|x\sqrt{\mathfrak{z}_c^2 - \mathfrak{z}^2} - (t\mathfrak{z}_c - y\mathfrak{z})| + \sqrt{|x\sqrt{\mathfrak{z}_c^2 - \mathfrak{z}^2} - (t\mathfrak{z}_c - y\mathfrak{z})|^2 - r_*^2(1+\mathfrak{z}^2)}}$$

and then

$$\lim_{r_*\to 0} I_h(r_*) = \frac{\sqrt{1+\mathfrak{z}^2}}{x\sqrt{\mathfrak{z}_c^2 - \mathfrak{z}^2} - (t\mathfrak{z}_c - y\mathfrak{z})} = -\frac{\sqrt{1+\mathfrak{z}^2}}{x^2+y^2} \cdot \frac{x\sqrt{\mathfrak{z}_c^2 - \mathfrak{z}^2} + (t\mathfrak{z}_c - y\mathfrak{z})}{(\mathfrak{z}-\xi)^2 + \eta^2}.$$

A similar calculation gives for each interior point of the S-region:

$$\lim_{r_*\to 0} I_e(r_*) = \frac{\sqrt{1+\mathfrak{z}^2}}{x^2+y^2} \cdot \frac{x\sqrt{\mathfrak{z}^2 - \mathfrak{z}_c^2} - \mathrm{i}(t\mathfrak{z}_c - y\mathfrak{z})}{(\mathfrak{z}-\xi)^2 + \eta^2}\,.$$

∎ (‡) We use the calculations (†) and the factorization (4.2) to get the limit $r_* \to 0$ of the sound component (3.6), (3.7) at the points of the S-region. We denote

$$\xi = \frac{\mathfrak{z}_c t y}{x^2+y^2} \;\;,\;\; \eta = \frac{\mathfrak{z}_c x\sqrt{t^2 - x^2 - y^2}}{x^2+y^2}$$

and motivate by (3.5) to call ξ, η *Lorentz entities*. Then we set

$$K = \frac{1}{\pi\sqrt{1-M^2}}\overline{K} \quad , \quad \overline{K} = \frac{\widetilde{\varepsilon}}{2\pi} \cdot \frac{1}{d_{03}^2}$$

$$Q_1(\mathfrak{z}^2) \stackrel{\text{def}}{=} d_{11}\mathfrak{z}^2 + d_{12},\ \ Q_2(\mathfrak{z}^2) \stackrel{\text{def}}{=} d_{01}\mathfrak{z}^2 + d_{02},\ \ Q_3(\mathfrak{z}^2) \stackrel{\text{def}}{=} d_{03}\mathfrak{z}^2 + d_{04}$$

to obtain at the points of the S-region:

$$\begin{aligned}
&[\widetilde{p}_h(x,y,t), \widetilde{u}_h(x,y,t), \widetilde{v}_h(x,y,t)] \\
&= -\frac{K}{x^2+y^2} \int_{-\mathfrak{z}_c}^{\mathfrak{z}_c} \frac{x\sqrt{\mathfrak{z}_c^2-\mathfrak{z}^2}+(t\mathfrak{z}_c-y\mathfrak{z})}{(\mathfrak{z}-\xi)^2+\eta^2} \\
&\cdot\left[(M\sqrt{\mathfrak{z}_c^2-\mathfrak{z}^2}-\mathfrak{z}_c), -(\sqrt{\mathfrak{z}_c^2-\mathfrak{z}^2}-M\mathfrak{z}_c), -\mathfrak{z}\sqrt{1-M^2}\right] \\
&\cdot\frac{\mathfrak{z}Q_1(\mathfrak{z}^2)[Q_2(\mathfrak{z}^2)-Q_3(\mathfrak{z}^2)\sqrt{\mathfrak{z}_c^2-\mathfrak{z}^2}]}{(\mathfrak{z}^2+a^2)(\mathfrak{z}^2-b^2)(\mathfrak{z}^2-c^2)(\mathfrak{z}^2+1)}\mathrm{d}\mathfrak{z}
\end{aligned} \tag{4.5}$$

$$\begin{aligned}
&[\widetilde{p}_e(x,y,t), \widetilde{u}_e(x,y,t), \widetilde{v}_e(x,y,t)] \\
&= \frac{K}{x^2+y^2}\mathrm{Im}\left(\int_{-\infty}^{-\mathfrak{z}_c} + \int_{\mathfrak{z}_c}^{\infty}\right) \frac{x\sqrt{\mathfrak{z}^2-\mathfrak{z}_c^2}-\imath(t\mathfrak{z}_c-y\mathfrak{z})}{(\mathfrak{z}-\xi)^2+\eta^2} \\
&\cdot\left[(\mathrm{i}M\sqrt{\mathfrak{z}^2-\mathfrak{z}_c^2}-\mathfrak{z}_c), -(\mathrm{i}\sqrt{\mathfrak{z}^2-\mathfrak{z}_c^2}-M\mathfrak{z}_c), -\mathfrak{z}\sqrt{1-M^2}\right] \\
&\cdot\frac{\mathfrak{z}Q_1(\mathfrak{z}^2)[Q_2(\mathfrak{z}^2)-\mathrm{i}Q_3(\mathfrak{z}^2)\sqrt{\mathfrak{z}^2-\mathfrak{z}_c^2}]}{(\mathfrak{z}^2+a^2)(\mathfrak{z}^2-b^2)(\mathfrak{z}^2-c^2)(\mathfrak{z}^2+1)}\mathrm{d}\mathfrak{z}\ .
\end{aligned} \tag{4.6}$$

4.5 Gasdynamic partitions (II). Gasdynamic factorizations (II). Memory and memory factorization for a second partition

We notice that the representations (4.5), (4.6) have a most suggestive form. They present, for example, through distinct factors, the contribution of the *vortex shape* and the contribution of the *shock-vorticity interaction*; these contributions are connected to the factors $[(\mathfrak{z}-\xi)^2+\eta^2]$ or, respectively, $(\mathfrak{z}^2-\zeta_i)$, $1 \le i \le 4$ where we denote, cf. (4.3), (4.4),

$$\zeta_1 = -a^2,\ \zeta_2 = b^2,\ \zeta_3 = c^2,\ \zeta_4 = -1.$$

We shall add to the partition (3.6), (3.7) a new partition to distinguish between the contribution of the *vortex shape* (label *vs*) and that of the *shock-vorticity interaction* (label *int*); such a partition will take into account the decompositions

$$\begin{aligned}
&\frac{1}{[(\mathfrak{z}-\xi)^2+\eta^2](\mathfrak{z}^2-\zeta_i)} \\
&= \frac{1}{(\xi^2+\eta^2+\zeta_i)^2-4\xi^2\zeta_i}\left\{\frac{(-2\xi)\mathfrak{z}+(3\xi^2-\eta^2-\zeta_i)}{(\mathfrak{z}-\xi)^2+\eta^2}+\frac{(2\xi)\mathfrak{z}+(\xi^2+\eta^2+\zeta_i)}{\mathfrak{z}^2-\zeta_i}\right\}
\end{aligned}$$

$$\begin{aligned}
&\frac{\mathfrak{z}}{[(\mathfrak{z}-\xi)^2+\eta^2](\mathfrak{z}^2-\zeta_i)} \\
&= \frac{1}{(\xi^2+\eta^2+\zeta_i)^2-4\xi^2\zeta_i}\left\{\frac{-(\xi^2+\eta^2+\zeta_i)\mathfrak{z}+2\xi(\xi^2+\eta^2)}{(\mathfrak{z}-\xi)^2+\eta^2}+\frac{(\xi^2+\eta^2+\zeta_i)\mathfrak{z}+(2\xi\zeta_i)}{\mathfrak{z}^2-\zeta_i}\right\}.
\end{aligned}$$

Expression $[(\xi^2+\eta^2+\zeta_i^2)^2-4\xi^2\zeta_i]$ is then revealed as a price paid for separation or, as a *memory* of this separation. It allows a *second gasdynamic factorization* [which uses (3.5)]

$$(\xi^2+\eta^2+\zeta_i)^2-4\xi^2\zeta_i=\frac{1}{(x^2+y^2)^2}\left[\left(\mathfrak{z}_c t-x\sqrt{\mathfrak{z}_c^2-\zeta_i}\right)^2-\zeta_i y^2\right]\left[\left(\mathfrak{z}_c t+x\sqrt{\mathfrak{z}_c^2-\zeta_i}\right)^2-\zeta_i y^2\right]. \tag{4.7}$$

We briefly present the succession of the two mentioned partitions by

$$[\widetilde{p},\widetilde{u},\widetilde{v}]=[\widetilde{p}_h,\widetilde{u}_h,\widetilde{v}_h]+[\widetilde{p}_e,\widetilde{u}_e,\widetilde{v}_e]=[\widetilde{p}_{vs},\widetilde{u}_{vs},\widetilde{v}_{vs}]+[\widetilde{p}_{int},\widetilde{u}_{int},\widetilde{v}_{int}]. \tag{4.8}$$

4.6 Some calculation details

The list of integrals corresponding to the *vs*-part consists of

$$\left[I_0(\xi,\eta^2),I_1(\xi,\eta^2)\right]=\int_{-\infty}^{\infty}\frac{[1,\mathfrak{z}]}{(\mathfrak{z}-\xi)^2+\eta^2}\,\mathrm{d}\mathfrak{z}=\frac{\pi}{\eta}[1,\xi] \tag{4.9}$$

$$\begin{aligned}\left[\mathcal{K}_0(\xi,\eta^2),\mathcal{K}_1(\xi,\eta^2)\right]&=\int_{-\mathfrak{z}_c}^{\mathfrak{z}_c}\frac{[1,\mathfrak{z}]}{\sqrt{\mathfrak{z}_c^2-\mathfrak{z}^2}[(\mathfrak{z}-\xi)^2+\eta^2]}\,\mathrm{d}\mathfrak{z}\\&=\frac{1}{\eta\sqrt{2}}\cdot\frac{\sqrt{(\mathfrak{z}_c^2+\eta^2-\xi^2)+\sqrt{(\mathfrak{z}_c^2+\eta^2-\xi^2)^2+4\xi^2\eta^2}}}{\sqrt{(\mathfrak{z}_c^2+\eta^2-\xi^2)^2+4\xi^2\eta^2}}\\&\quad\cdot\left[\frac{2\xi}{(\mathfrak{z}_c^2+\eta^2-\xi^2)+\sqrt{(\mathfrak{z}_c^2+\eta^2-\xi^2)^2+4\xi^2\eta^2}}\ ,\ 1\right]\\&=\frac{1}{\mathfrak{z}_c^3}\cdot\frac{x^2+y^2}{(t^2-y^2)\sqrt{t^2-x^2-y^2}}[y,\mathfrak{z}_c t]\end{aligned} \tag{4.10}$$

$$[\mathcal{J}_0(\xi,\eta^2),\mathcal{J}_1(\xi,\eta^2),\mathcal{J}_2(\xi,\eta^2)]=\int_{-\mathfrak{z}_c}^{\mathfrak{z}_c}\frac{[1,\mathfrak{z},\mathfrak{z}^2]\sqrt{\mathfrak{z}_c^2-\mathfrak{z}^2}}{(\mathfrak{z}-\xi)^2+\eta^2}\mathrm{d}\mathfrak{z}=[\mathcal{J}_0^r,\mathcal{J}_1^r,\mathcal{J}_2^r]+[\mathcal{J}_0^s,\mathcal{J}_1^s,\mathcal{J}_2^s] \tag{4.11}$$

$$\left\{\begin{aligned}&[\mathcal{J}_0^r,\mathcal{J}_1^r,\mathcal{J}_2^r]=\pi\left[-1,\ -2\xi,\ \frac{1}{2}\mathfrak{z}_c^2-3\xi^2+\eta^2\right]\\&[\mathcal{J}_0^s,\mathcal{J}_1^s,\mathcal{J}_2^s]=-\pi\mathfrak{z}_c^2[2\xi,\ 3\xi^2-\eta^2-\mathfrak{z}_c^2,\ 2\xi(2\xi^2-2\eta^2-\mathfrak{z}_c^2)]\mathcal{K}_0\\&\qquad\qquad+\pi[\xi^2+\eta^2+\mathfrak{z}_c^2,\ 2\xi(\xi^2+\eta^2),(\xi^2+\eta^2)(3\xi^2-\eta^2-\mathfrak{z}_c^2)]\mathcal{K}_1\end{aligned}\right. \tag{4.11$_{r,s}$}$$

$$\begin{cases} \mathcal{J}_0^s = \dfrac{t}{\sqrt{t^2-x^2-y^2}}, \ \mathcal{J}_1^s = \dfrac{y}{\sqrt{t^2-x^2-y^2}} \cdot \dfrac{\mathfrak{z}_c}{x^2+y^2}(2t^2-x^2-y^2), \\ \mathcal{J}_2^s = \dfrac{t}{\sqrt{t^2-x^2-y^2}} \left(\dfrac{\mathfrak{z}_c}{x^2+y^2}\right)^2 [t^2(3y^2-x^2)-(x^2+y^2)(2y^2-x^2)]. \end{cases} \tag{4.11$_s$}$$

We complete this list by using the remark that if η^2 is replaced by $(-\overline{\eta}^2)$ in (4.9)$-$(4.11) then we get

$$\begin{cases} I_0(\xi,-\overline{\eta}^2)=0, \ I_1(\xi,-\overline{\eta}^2)=0, \\ \mathcal{K}_0(\xi,-\overline{\eta}^2)=0, \ \mathcal{K}_1(\xi,-\overline{\eta}^2)=0, \end{cases} \tag{4.12}$$

$$\mathcal{J}_0^s(\xi,-\overline{\eta}^2)=0, \ \mathcal{J}_1^s(\xi,-\overline{\eta}^2)=0, \ \mathcal{J}_2^s(\xi,-\overline{\eta}^2)=0. \tag{4.13}$$

A list similar to (4.9)$-$(4.11) can be shown for the *int*-part; the integrals of this list result from (4.9)$-$(4.13) when the details concerning the form of ζ_i, $1 \le i \le 4$, are taken into consideration cf. (4.3), (4.4). We have, for $1 \le i \le 4$,

$$\overline{I}_0(\zeta_i) = I_0(0,-\zeta_i) = (2-i)(3-i)\frac{\pi}{2\sqrt{|\zeta_i|}}, \ \ \overline{I}_1(\zeta_i)=0, \tag{4.14}$$

$$\begin{cases} \overline{\mathcal{J}}_0(\zeta_i) = \mathcal{J}_0(0,-\zeta_i) = \pi\left[(-1)+\dfrac{(2-i)(3-i)}{2}\cdot\sqrt{\dfrac{\mathfrak{z}_c^2-\zeta_i}{|\zeta_i|}}\right] \\ \overline{\mathcal{J}}_1(\zeta_i) = \mathcal{J}_1(0,-\zeta_i) = 0 \\ \overline{\mathcal{J}}_2(\zeta_i) = \mathcal{J}_2(0,-\zeta_i) = \pi\left[\left(\dfrac{1}{2}\mathfrak{z}_c^2-\zeta_i\right)+\dfrac{(2-i)(3-i)}{2}\sqrt{|\zeta_i|(\mathfrak{z}_c^2-\zeta_i)}\right], \end{cases} \tag{4.15}$$

$$[\overline{\mathcal{J}}_0^r(\zeta_i), \overline{\mathcal{J}}_1^r(\zeta_i), \overline{\mathcal{J}}_2^r(\zeta_i) = [\mathcal{J}_0^r(0,-\zeta_i), \mathcal{J}_1^r(0,-\zeta_i), \mathcal{J}_2^r(0,-\zeta_i) = \pi\left[-1, \ ,0, \ \frac{1}{2}\mathfrak{z}_c^2-\zeta_i\right]. \tag{4.15$_r$}$$

4.7 Gasdynamic partitions (III). Gasdynamic factorizations (III). A prefinal form of the sound emergent contribution

Next, the integrals corresponding to the *vs*-contribution appear, cf. section 4.6, to include a part which is *singular*, concurrently with η^{-1}, with respect to the S-arc (4.1). This circumstance naturally completes the sequence (4.8) of partitions with a last element (the labels *r/s* mean regular / singular with respect to the S-arc):

$$[\widetilde{p}_{vs}, \widetilde{u}_{vs}, \widetilde{v}_{vs}] + [\widetilde{p}_{int}, \widetilde{u}_{int}, \widetilde{v}_{int}] = [\widetilde{p}_r, \widetilde{u}_r, \widetilde{v}_r] + [\widetilde{p}_s, \widetilde{u}_s, \widetilde{v}_s]. \tag{4.16}$$

Now, we carry and re-arrange the calculations 4.6 into the last partition of the sequence (4.16). In Dinu [2] it is noticed that, incidentally and remarkably, to the terms of the mentioned last partition in (4.16) a set of *other four gasdynamic factorizations*, compatible with

(4.7), can be naturally associated [via (3.5)]:

$$\begin{aligned}&\mathcal{E}_1^p(\zeta_i)[2(\mathfrak{z}_c^2-\zeta_i)x\xi+2\mathfrak{z}_c\sqrt{\mathfrak{z}_c^2-\zeta_i}\,t\xi-\sqrt{\mathfrak{z}_c^2-\zeta_i}\,y(\xi^2+\eta^2+\zeta_i)]\\&+\mathcal{E}_2^p(\zeta_i)[-2\mathfrak{z}_c t\xi+y(\xi^2+\eta^2+\zeta_i)-2\sqrt{\mathfrak{z}_c^2-\zeta_i}\,x\xi]\\&=[\sqrt{\mathfrak{z}_c^2-\zeta_i}\mathcal{E}_1^p(\zeta_i)-\mathcal{E}_2^p(\zeta_i)][2\sqrt{\mathfrak{z}_c^2-\zeta_i}\,x\xi+2\mathfrak{z}_c t\xi-y(\xi^2+\eta^2+\zeta_i)]\\&=[\sqrt{\mathfrak{z}_c^2-\zeta_i}\mathcal{E}_1^p(\zeta_i)-\mathcal{E}_2^p(\zeta_i)]\left\{[y/(x^2+y^2)][(\mathfrak{z}_c t+x\sqrt{\mathfrak{z}_c^2-\zeta_i})^2-\zeta_i y^2]\right\}\end{aligned}$$

$$\begin{aligned}&\sqrt{\mathfrak{z}_c^2-\zeta_i}\,t[\mathfrak{z}_c^2(t^2-x^2)-\zeta_i(x^2+y^2)]\pm\mathfrak{z}_c x[\mathfrak{z}_c^2(t^2-x^2)-\zeta_i(2t^2-x^2-y^2)]\\&\equiv(t\sqrt{\mathfrak{z}_c^2-\zeta_i}\mp x\mathfrak{z}_c)[(\mathfrak{z}_c t\pm x\sqrt{\mathfrak{z}_c^2-\zeta_i})^2-\zeta_i y^2]\end{aligned}$$

$$\begin{aligned}&-\left\{2\xi\zeta_i\mathcal{T}_1^v(\zeta_i)+(\xi^2+\eta^2+\zeta_i)\mathcal{T}_2^v(\zeta_i)\right\}\\&+\sqrt{\mathfrak{z}_c^2-\zeta_i}\cdot\left\{2\xi\zeta_i[-y\mathcal{E}_1^v(\zeta_i)]+(\xi^2+\eta^2+\zeta_i)[\mathfrak{z}_c t\mathcal{E}_1^v(\zeta_i)-x\mathcal{E}_2^v(\zeta_i)]\right\}\\&=-[\mathcal{E}_2^v(\zeta_i)-\mathcal{E}_1^v(\zeta_i)\sqrt{\mathfrak{z}_c^2-\zeta_i}][(\xi^2+\eta^2+\zeta_i)(\mathfrak{z}_c t+x\sqrt{\mathfrak{z}_c^2-\zeta_i})-2\zeta_i y\xi]\\&=-[\mathcal{E}_2^v(\zeta_i)-\mathcal{E}_1^v(\zeta_i)\sqrt{\mathfrak{z}_c^2-\zeta_i}][1/(x^2+y^2)](\mathfrak{z}_c t-x\sqrt{\mathfrak{z}_c^2-\zeta_i})[(\mathfrak{z}_c t+x\sqrt{\mathfrak{z}_c^2-\zeta_i})^2-\zeta_i y^2]\end{aligned}$$

$$\begin{aligned}&\mathfrak{z}_c[\mathfrak{z}_c^2(t^4-t^2x^2-t^2y^2-x^2y^2)-\zeta_i(t^2y^2-x^2y^2-y^4-t^2x^2)]\\&\pm\sqrt{\mathfrak{z}_c^2-\zeta_i}\,tx[\mathfrak{z}_c^2(t^2-y^2)-(\mathfrak{z}_c^2-\zeta_i)(x^2+y^2)]\\&=[t(\mathfrak{z}_c t\mp x\sqrt{\mathfrak{z}_c^2-\zeta_i})-\mathfrak{z}_c^2y^2][(\mathfrak{z}_c t\pm x\sqrt{\mathfrak{z}_c^2-\zeta_i})^2-\zeta_i y^2]\end{aligned}$$

---

where

$$\mathcal{E}_1^p(\zeta_i)\stackrel{\text{def}}{=}MQ_2(\zeta_i)+\mathfrak{z}_c Q_3(\zeta_i),\quad \mathcal{E}_2^p(\zeta_i)\stackrel{\text{def}}{=}\mathfrak{z}_c Q_2(\zeta_i)+M(\mathfrak{z}_c^2-\zeta_i)Q_3(\zeta_i)$$

$$\mathcal{E}_1^v(\zeta_i)\stackrel{\text{def}}{=}Q_3(\zeta_i),\quad \mathcal{E}_2^v(\zeta_i)\stackrel{\text{def}}{=}Q_2(\zeta_i)$$

$$\mathcal{T}_1^v(\zeta_i)\stackrel{\text{def}}{=}-y\mathcal{E}_2^v(\zeta_i),\quad \mathcal{T}_2^v(\zeta_i)\stackrel{\text{def}}{=}-x(\mathfrak{z}_c^2-\zeta_i)\mathcal{E}_1^v(\zeta_i)+t\mathfrak{z}_c\mathcal{E}_2^v(\zeta_i).$$

We have to notice here that an *analogue* of the first of these factorizations holds true if $\mathcal{E}_1^p,\mathcal{E}_2^p$ are replaced by $\mathcal{E}_1^u,\mathcal{E}_2^u$ corresponding respectively to the u-component of the sound contribution in the interaction solution.

Finally, we naturally get for the sound emergent contribution a more suggestive (prefinal) form (see Dinu [2] for the calculation details):

$$\begin{aligned}\widetilde{p}_r(x,y,t)=\ &\frac{\pi y}{2(x^2+y^2)^2}\sum_{i=1}^{4}\frac{(2-i)(3-i)\sqrt{|\zeta_i|}\,\widetilde{k}_i(\zeta)}{(\xi^2+\eta^2+\zeta_i)^2-4\xi^2\zeta_i}\left[\sqrt{\mathfrak{z}_c^2-\zeta_i}\mathcal{E}_1^p(\zeta_i)-\mathcal{E}_2^p(\zeta_i)\right]\\&\cdot\left[\left(\mathfrak{z}_c t+x\sqrt{\mathfrak{z}_c^2-\zeta_i}\right)^2-\zeta_i y^2\right]\qquad(4.17)_p\end{aligned}$$

$$\widetilde{u}_r(x,y,t) = -\frac{\pi y}{2(x^2+y^2)^2}\sum_{i=1}^{4}\frac{(2-i)(3-i)\sqrt{|\zeta_i|}\,\widetilde{k}_i(\zeta)}{(\xi^2+\eta^2+\zeta_i)^2-4\xi^2\zeta_i}\left[\sqrt{\mathfrak{z}_c^2-\zeta_i}\,\mathcal{E}_1^u(\zeta_i)-\mathcal{E}_2^u(\zeta_i)\right] \cdot\left[\left(\mathfrak{z}_c t + x\sqrt{\mathfrak{z}_c^2-\zeta_i}\right)^2-\zeta_i y^2\right] \quad (4.17)_u$$

$$\widetilde{v}_r(x,y,t) = \frac{\pi\sqrt{1-M^2}}{2(x^2+y^2)^2}\sum_{i=1}^{4}\frac{(2-i)(3-i)\sqrt{|\zeta_i|}\,\widetilde{k}_i(\zeta)}{(\xi^2+\eta^2+\zeta_i)^2-4\xi^2\zeta_i}\left[\sqrt{\mathfrak{z}_c^2-\zeta_i}\,\mathcal{E}_1^v(\zeta_i)-\mathcal{E}_2^v(\zeta_i)\right] \cdot\left(\mathfrak{z}_c t - x\sqrt{\mathfrak{z}_c^2-\zeta_i}\right)\left[\left(\mathfrak{z}_c t + x\sqrt{\mathfrak{z}_c^2-\zeta_i}\right)^2-\zeta_i y^2\right] \quad (4.17)_v$$

$$\widetilde{p}_s(x,y,t) = -\frac{y}{\sqrt{t^2-x^2-y^2}}\cdot\frac{\pi}{(x^2+y^2)^2}\sum_{i=1}^{4}\frac{\widetilde{k}_i(\zeta)}{(\xi^2+\eta^2+\zeta_i)^2-4\xi^2\zeta_i} \cdot\left\{\left[\sqrt{\mathfrak{z}_c^2-\zeta_i}\,\mathcal{E}_1^p(\zeta_i)-\mathcal{E}_2^p(\zeta_i)\right]\left(t\sqrt{\mathfrak{z}_c^2-\zeta_i}-\mathfrak{z}_c x\right)\left[\left(\mathfrak{z}_c t+x\sqrt{\mathfrak{z}_c^2-\zeta_i}\right)^2-\zeta_i y^2\right] + \left[\sqrt{\mathfrak{z}_c^2-\zeta_i}\,\mathcal{E}_1^p(\zeta_i)+\mathcal{E}_2^p(\zeta_i)\right]\left(t\sqrt{\mathfrak{z}_c^2-\zeta_i}+\mathfrak{z}_c x\right)\left[\left(\mathfrak{z}_c t-x\sqrt{\mathfrak{z}_c^2-\zeta_i}\right)^2-\zeta_i y^2\right]\right\} \quad (4.18)_p$$

$$\widetilde{u}_s(x,y,t) = \frac{y}{\sqrt{t^2-x^2-y^2}}\cdot\frac{\pi}{(x^2+y^2)^2}\sum_{i=1}^{4}\frac{\widetilde{k}_i(\zeta)}{(\xi^2+\eta^2+\zeta_i)^2-4\xi^2\zeta_i} \cdot\left\{\left[\sqrt{\mathfrak{z}_c^2-\zeta_i}\,\mathcal{E}_1^u(\zeta_i)-\mathcal{E}_2^u(\zeta_i)\right]\left(t\sqrt{\mathfrak{z}_c^2-\zeta_i}-\mathfrak{z}_c x\right)\left[\left(\mathfrak{z}_c t+x\sqrt{\mathfrak{z}_c^2-\zeta_i}\right)^2-\zeta_i y^2\right] + \left[\sqrt{\mathfrak{z}_c^2-\zeta_i}\,\mathcal{E}_1^u(\zeta_i)+\mathcal{E}_2^u(\zeta_i)\right]\left(t\sqrt{\mathfrak{z}_c^2-\zeta_i}+\mathfrak{z}_c x\right)\left[\left(\mathfrak{z}_c t-x\sqrt{\mathfrak{z}_c^2-\zeta_i}\right)^2-\zeta_i y^2\right]\right\} \quad (4.18)_u$$

$$\widetilde{v}_s(x,y,t) = -\frac{1}{\sqrt{t^2-x^2-y^2}}\cdot\frac{\pi\sqrt{1-M^2}}{(x^2+y^2)^2}\sum_{i=1}^{4}\frac{\widetilde{k}_i(\zeta)}{(\xi^2+\eta^2+\zeta_i)^2-4\xi^2\zeta_i}\cdot\frac{\zeta_i}{\sqrt{\mathfrak{z}_c^2-\zeta_i}} \cdot\left\{\left[\sqrt{\mathfrak{z}_c^2-\zeta_i}\,\mathcal{E}_1^v(\zeta_i)-\mathcal{E}_2^v(\zeta_i)\right]\left[t\left(\mathfrak{z}_c t-x\sqrt{\mathfrak{z}_c^2-\zeta_i}\right)-\mathfrak{z}_c y^2\right]\left[\left(\mathfrak{z}_c t+x\sqrt{\mathfrak{z}_c^2-\zeta_i}\right)^2-\zeta_i y^2\right] - \left[\sqrt{\mathfrak{z}_c^2-\zeta_i}\,\mathcal{E}_1^v(\zeta_i)+\mathcal{E}_2^v(\zeta_i)\right]\left[t\left(\mathfrak{z}_c t+x\sqrt{\mathfrak{z}_c^2-\zeta_i}\right)-\mathfrak{z}_c y^2\right]\left[\left(\mathfrak{z}_c t-x\sqrt{\mathfrak{z}_c^2-\zeta_i}\right)^2-\zeta_i y^2\right]\right\} \quad (4.18)_v$$

where

$$\widetilde{k}_i(\zeta) = \frac{\widetilde{\varepsilon}}{2\pi^2}\cdot\frac{1}{d_{03}^2\sqrt{1-M^2}}\cdot\frac{Q_1(\zeta_i)}{\prod_{j\neq i}(\zeta_i-\zeta_j)}.$$

4.8 A special nature of the gasdynamic context. Inner coherence

It is interesting to remark that to the factorizations mentioned here above [in sections 4.3, 4.5 and 4.7] we have to add the coefficients factorizations and other particular relations included in 4.10. A *special nature* is shown therefore for the gasdynamic context. This special nature is even more extensive; in fact, we have to notice a *factoring compatibility* ("inner coherence") of the factorizations mentioned here above [see comparatively (4.7) and (4.17), (4.18)].

We have to notice, on the other hand, that the mentioned factorizations may become *immaterial* if the gasdynamic context is extended /lost (see Dinu and Dinu [6]).

4.9 The singular limit of the sound contribution: (II) an optimal closed form

Next, we take into account the mentioned compatibility (gasdynamic "inner coherence") − precisely: we use (4.7) into (4.17) and (4.18) − to finally get the following *optimal* form of the limit $r_* \to 0$ of the sound emergent contribution (H is the Heaviside function)

$$[\widetilde{p}_r(\widetilde{x},\widetilde{y},\widetilde{t}),\widetilde{u}_r(\widetilde{x},\widetilde{y},\widetilde{t}),\widetilde{v}_r(\widetilde{x},\widetilde{y},\widetilde{t})] = -\overline{K}\sum_{i=1}^{4}\frac{k_i^r(\zeta)Q^-(\zeta_i)}{[\tilde{t}\hat{k}^-(\zeta_i)+\widetilde{x}\check{k}^-(\zeta_i)]^2-\zeta_i\widetilde{y}^2}[\hat{k}^-(\zeta_i)\widetilde{y},-\check{k}^-(\zeta_i)\widetilde{y},\ \tilde{t}\hat{k}^-(\zeta_i)+\widetilde{x}\check{k}^-(\zeta_i)] \tag{4.19}$$

$$\widetilde{p}_s(\widetilde{x},\widetilde{y},\widetilde{t}) = -\frac{\overline{K}}{\sqrt{\tilde{t}^2-\widetilde{x}^2-\widetilde{y}^2}}\cdot H(\tilde{t}-\sqrt{\widetilde{x}^2+\widetilde{y}^2}) \cdot\left\{\sum_{i=1}^{4}\overline{k}_i(\zeta)Q^-(\zeta_i)\hat{k}^-(\zeta_i)\frac{\widetilde{y}[\tilde{t}\check{k}^-(\zeta_i)+\widetilde{x}\hat{k}^-(\zeta_i)]}{[\tilde{t}\hat{k}^-(\zeta_i)+\widetilde{x}\check{k}^-(\zeta_i)]^2-\zeta_i\widetilde{y}^2} + \sum_{i=1}^{4}\overline{k}_i(\zeta)Q^+(\zeta_i)\hat{k}^+(\zeta_i)\frac{\widetilde{y}[\tilde{t}\check{k}^+(\zeta_i)+\widetilde{x}\hat{k}^-(\zeta_i)]}{[\tilde{t}\hat{k}^+(\zeta_i)+\widetilde{x}\check{k}^+(\zeta_i)]^2-\zeta_i\widetilde{y}^2}\right\} \tag{4.20$_p$}$$

$$\widetilde{u}_s(\widetilde{x},\widetilde{y},\widetilde{t}) = \frac{\overline{K}}{\sqrt{\tilde{t}^2-\widetilde{x}^2-\widetilde{y}^2}}\cdot H(\tilde{t}-\sqrt{\widetilde{x}^2+\widetilde{y}^2}) \cdot\left\{\sum_{i=1}^{4}\overline{k}_i(\zeta)Q^-(\zeta_i)\check{k}^-(\zeta_i)\frac{\widetilde{y}[\tilde{t}\check{k}^-(\zeta_i)+\widetilde{x}\hat{k}^-(\zeta_i)]}{[\tilde{t}\hat{k}^-(\zeta_i)+\widetilde{x}\check{k}^-(\zeta_i)]^2-\zeta_i\widetilde{y}^2} + \sum_{i=1}^{4}\overline{k}_i(\zeta)Q^+(\zeta_i)\check{k}^+(\zeta_i)\frac{\widetilde{y}[\tilde{t}\check{k}^+(\zeta_i)+\widetilde{x}\hat{k}^+(\zeta_i)]}{[\tilde{t}\hat{k}^+(\zeta_i)+\widetilde{x}\check{k}^+(\zeta_i)]^2-\zeta_i\widetilde{y}^2}\right\} \tag{4.20$_u$}$$

$$\widetilde{v}_s(\widetilde{x},\widetilde{y},\widetilde{t}) = -\frac{\overline{K}}{\sqrt{\tilde{t}^2-\widetilde{x}^2-\widetilde{y}^2}}\cdot H(\tilde{t}-\sqrt{\widetilde{x}^2+\widetilde{y}^2}) \cdot\left\{\sum_{i=1}^{4}\overline{k}_i(\zeta)Q^-(\zeta_i)\,\mathring{k}(\zeta_i)\frac{(\tilde{t}+M\widetilde{x})[\tilde{t}\hat{k}^-(\zeta_i)+\widetilde{x}\check{k}^-(\zeta_i)]-\overline{M}\widetilde{y}^2}{[\tilde{t}\hat{k}^-(\zeta_i)+\widetilde{x}\check{k}^-(\zeta_i)]^2-\zeta_i\widetilde{y}^2} + \sum_{i=1}^{4}\overline{k}_i(\zeta)Q^+(\zeta_i)\,\mathring{k}(\zeta_i)\frac{(\tilde{t}+M\widetilde{x})[\tilde{t}\hat{k}^+(\zeta_i)+\widetilde{x}\check{k}^+(\zeta_i)]-\overline{M}\widetilde{y}^2}{[\tilde{t}\hat{k}^+(\zeta_i)+\widetilde{x}\check{k}^+(\zeta_i)]^2-\zeta_i\widetilde{y}^2}\right\} \tag{4.20$_v$}$$

where we denote

$$k_i^r(\zeta) = \frac{(2-i)(3-i)}{2}\bar{k}_i(\zeta)\sqrt{|\zeta_i|}, \quad \bar{k}_i(\zeta) = \left[\prod_{j\neq i}(\zeta_i - \zeta_j)\right]^{-1}$$

$$\hat{k}^{\pm}(\zeta_i) = \frac{\mathfrak{z}_c \pm M\sqrt{\mathfrak{z}_c^2 - \zeta_i}}{\sqrt{1-M^2}},\ \check{k}^{\pm}(\zeta_i) = \frac{M\mathfrak{z}_c \pm \sqrt{\mathfrak{z}_c^2 - \zeta_i}}{\sqrt{1-M^2}},\ \mathring{k}(\zeta_i) = \frac{\zeta_i}{\sqrt{1-M^2}\sqrt{\mathfrak{z}_c^2 - \zeta_i}}$$

$$Q^{\pm}(\zeta_i) = Q_1(\zeta_i)\left[Q_2(\zeta_i) \pm Q_3(\zeta_i)\sqrt{\mathfrak{z}_c^2 - \zeta_i}\right].$$

4.10 A few useful gasdynamic relations

We notice at this point a few useful gasdynamic relations:

$$\begin{aligned}
\mathfrak{z}_c - M\sqrt{\mathfrak{z}_c^2 + a^2} &= 0 \\
Q_2(-a^2) + Q_3(-a^2)\sqrt{\mathfrak{z}_c^2 + a^2} &= 0 \\
Q_2(b^2) - Q_3(b^2)\sqrt{\mathfrak{z}_c^2 - b^2} &= 0, \quad b^2 < \mathfrak{z}_c^2 \\
Q_2(c^2) - Q_3(c^2)\sqrt{\mathfrak{z}_c^2 - c^2} &= 0, \quad c^2 < \mathfrak{z}_c^2
\end{aligned}$$

$$\begin{aligned}
\sqrt{\mathfrak{z}_c^2 - \zeta_i}\mathcal{E}_1^p(\zeta_i) \pm \mathcal{E}_2^p(\zeta_i) &= \left(M\sqrt{\mathfrak{z}_c^2 - \zeta_i} \pm \mathfrak{z}_c\right)\left[Q_2(\zeta_i) \pm \sqrt{\mathfrak{z}_c^2 - \zeta_i}Q_3(\zeta_i)\right] \\
\sqrt{\mathfrak{z}_c^2 - \zeta_i}\mathcal{E}_1^u(\zeta_i) \pm \mathcal{E}_2^u(\zeta_i) &= \left(\sqrt{\mathfrak{z}_c^2 - \zeta_i} \pm M\mathfrak{z}_c\right)\left[Q_2(\zeta_i) \pm \sqrt{\mathfrak{z}_c^2 - \zeta_i}Q_3(\zeta_i)\right] \\
\sqrt{\mathfrak{z}_c^2 - \zeta_i}\mathcal{E}_1^v(\zeta_i) \pm \mathcal{E}_2^v(\zeta_i) &= \pm\left[Q_2(\zeta_i) \pm \sqrt{\mathfrak{z}_c^2 - \zeta_i}Q_3(\zeta_i)\right]
\end{aligned}$$

$$\begin{aligned}
\sqrt{\mathfrak{z}_c^2 - \zeta_i}\mathcal{E}_1^p(\zeta_i) - \mathcal{E}_2^p(\zeta_i) &= 0, \quad 1 \le i \le 3 \\
\sqrt{\mathfrak{z}_c^2 - \zeta_i}\mathcal{E}_1^u(\zeta_i) - \mathcal{E}_2^u(\zeta_i) &= 0, \quad i = 2,3 \\
\sqrt{\mathfrak{z}_c^2 - \zeta_i}\mathcal{E}_1^v(\zeta_i) - \mathcal{E}_2^v(\zeta_i) &= 0, \quad i = 2,3
\end{aligned}$$

$$\mathcal{E}_1^p(\zeta_1) = 0, \quad \mathcal{E}_2^p(\zeta_1) = 0$$

to get that

$$k_2^r(\zeta) = 0,\ k_3^r(\zeta) = 0;\ \hat{k}^-(\zeta_1) = 0;\ Q^+(\zeta_1) = 0,\ Q^-(\zeta_2) = 0,\ Q^-(\zeta_3) = 0$$

in the coefficients of (4.19), (4.20) − thus annulling some of these coefficients.

4.11 Final notes

This sound contribution corresponds to the incidence [obtained from (3.2) in the limit $r_*\to 0$]

$$[\widetilde{u}(\underset{\sim}{x},\underset{\sim}{y}),\widetilde{v}(\underset{\sim}{x},\underset{\sim}{x})] = \frac{\widetilde{\varepsilon}}{2\pi}\cdot\frac{[-\underset{\sim}{y},\underset{\sim}{x}]}{\underset{\sim}{x}^2+\underset{\sim}{x}^2}, \quad \widetilde{s}\equiv\widetilde{p}\equiv 0; \quad (\widetilde{x},\widetilde{y})\neq(0,0). \tag{4.21}$$

We have to complete these results with the explicit form of the rest of the limit $r_*\to 0$ of the Ribner solution by carrying (4.19), (4.20) and (4.21) into (3.8)−(3.11).

The singular structure of the cumulative contribution of (4.19) and (4.20) consists in the sound singularities continuously distributed along the *S*-arc and is completed with a vorticity singularity laid at the point ($\widetilde{x}=0,\widetilde{y}=0$). The other singularities of (4.19), (4.20) are proven to be pseudosingularities: they appear to be *compensated* in the sums $\widetilde{p}_r+\widetilde{p}_s$, $\widetilde{u}_r+\widetilde{u}_s$, $\widetilde{v}_r+\widetilde{v}_s$. In fact this result is suggested by Figure 1.

The presence, at $\tilde{t}>0$, of the *S*-arc − which supports a continuous distribution of sound singularities − could be regarded as a *widening* of an incident vorticity singularity, in presence of a nonlinear subconscious. We notice the *irreversible* character of this solution.

5 An oblique extension of Ribner's parallel solution

5.1 Details of the oblique extension

The explicit closed form of the sound contribution in the parallel interaction solution could be tentatively presented, in its singular limit, by

$$\begin{cases} \widetilde{p}_r+\widetilde{p}_s & \equiv & \widetilde{p}_{\|}(x,y,t;\ \zeta_1,\zeta_2,\zeta_3,\zeta_4;\ \mathfrak{z}_c;\ Q_1,Q_2,Q_3) \\ \widetilde{u}_r+\widetilde{u}_s & \equiv & \widetilde{u}_{\|}(x,y,t;\ \zeta_1,\zeta_2,\zeta_3,\zeta_4;\ \mathfrak{z}_c;\ Q_1,Q_2,Q_3) \\ \widetilde{v}_r+\widetilde{v}_s & \equiv & \widetilde{v}_{\|}(x,y,t;\ \zeta_1,\zeta_2,\zeta_3,\zeta_4;\ \mathfrak{z}_c;\ Q_1,Q_2,Q_3) \end{cases}. \tag{5.1}$$

The turbulence − planar shock interaction is associated with a class of interaction elements. An interaction element formally models the interaction between a planar shock and a *single* incident vortex corresponding to a certain inclination of the vortex axis with respect to the shock. • Modelling the incident turbulence by a superposition of compressible planar vortices appears to correspond to a *first level* of decomposition. In Lighthill's fundamental paper [13] the turbulence is acoustically modelled by a distribution of quadrupoles − which is equivalent with a "weighted" distribution of point vortices. • Next, in order to proceed, each incident vortex is Fourier decomposed into planar monochromatic waves − a *second level* of decomposition. • Finally, each incident planar monochromatic wave is Snell passed through the shock discontinuity (Figure 2). • The result of the passage through the shock can again be presented by two levels of *recombination* so that each incident level of decomposition has a correspondent in the emergent solution.

The description in sections 5.2−5.4 considers, formally, an interaction element − taken from the superposition associated to a vorticity incident turbulence. Let δ be the axis of the (oblique) incident vortex, let Π be the plane of the shock, and we consider the point

$O \in \delta \cap \Pi$. We consider the frame X, Y, Z with the origin at O and whose axes Y, Z are included in Π; direction OZ is placed along the projection of δ on Π. In Figure 2 we particularly depict the passage through the shock of a plane of zero phase corresponding to a certain incident monochromatic vorticity wave in the Fourier representation [analogous to (3.3)] of the incident vortex; let d be the intersection of this plane with the shock plane. Let θ be an angle (Figure 2) expressing the inclination of δ with respect to Π. Let ϖ be the angle between the line d and the axis OZ. We adapt the details of construction in Figure 1 to the constructive analysis around d [particularly, we use a bar over the notations around d which are analogous to those of Figure 1; so, ϕ_l, ϕ, ϕ' of the Figure 1 become $\bar{\phi}_l, \bar{\phi}, \bar{\phi}'$ around the line d]. Let $\pi(d_1, d_2)$ the plane spanned by two concurrent lines d_1, d_2. We use the facts of Figure 1 in order to characterize the refraction of the plane $\pi(d, \delta)$. To complete the Fourier$-$Snell representation of the considered passage through the shock we need the expression of the dihedral angle $\bar{\phi}_l$ of the planes $\pi(d, \delta)$ and X, Y in terms of the angles θ and ϖ. We compute

$$\cot \bar{\phi}_l = -\frac{\sqrt{1+\tan^2 \varpi}}{\tan\theta \tan\varpi} \stackrel{\text{def}}{=} -\bar{\mathfrak{z}}; \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \tag{5.2}$$

which parallels (2.14).

• The same as in the parallel case, it is easy to be seen that the envelope of the refracted zero-phase vorticity planes, which result from the passage of the mentioned incident vortex,

FIGURE 2 Details of the oblique construction is a straightline δ' – the axis of the refracted vorticity (Figure 2). The planes $\pi(\delta,\delta')$ and X,Z are coincident. A straightforward geometrical analysis shows that $\tan\theta' = \frac{\overline{M}}{M}\tan\theta$ – where θ' is an angle (Figure 2) expressing the inclination of δ' with respect to Π. • Finally, in order to complete the comparison between an oblique and a parallel refraction, in sections 5.2–5.4 it is considered, in its dependence on the inclination of the incident vortex, the nature of the sound contribution which results from interaction. These sections distinguish between two types of inclinations (subcritical, supercritical) and make evidence of a "relativistic" (critical) separation between the sound contributions respectively associated.

5.2 Subcritical and supercritical inclinations

Remark 5.1. We put $\Theta = \frac{\pi}{2} - \theta\mathrm{sign}\theta$ and notice that the requirement

$$|\bar{\mathfrak{z}}| \leq \mathfrak{z}_c \tag{5.3}$$

consists, cf. (5.2), in

$$1 \leq \frac{\sqrt{1+\tan^2\varpi}}{|\tan\varpi|} \leq \frac{\mathfrak{z}_c}{\tan\Theta}$$

or, *equivalently*, in

$$\tan\Theta \leq \mathfrak{z}_c \tag{5.4}$$

together with

$$|\tan\varpi| \geq \frac{\tan\Theta}{\sqrt{\mathfrak{z}_c^2 - \tan^2\Theta}}. \tag{5.5}$$

This suggests that we ought to distinguish between the <u>*supercritical*</u> and <u>*subcritical*</u> inclinations of the incident vortex axis respectively characterized by $\tan\Theta \geq \mathfrak{z}_c$ and $\tan\Theta \leq \mathfrak{z}_c$. In fact, for a supercritical inclination of the mentioned axis the possibility (5.3) is excluded cf. (5.4) and we must require $|\bar{\mathfrak{z}}| \geq \mathfrak{z}_c$, so that the sound component of the refracted solution is entirely *pseudoelliptic*. On the other hand, for a subcritical inclination of the mentioned axis a pseudohyperbolic part, isolated by the requirement (5.5), is allowed in a *mixed type* sound component of the refracted solution.

5.3 Extended Lorentz coordinates. The subcritical case

Let us consider next the case of *subcritical* incident vortices (see Dinu and Dinu [4] for the details of the supercritical case). This case is largely similar, to the (subcritical) case considered (for $\Theta = 0$) in §4. It is easy to show that the zero-phase planes corresponding to the sound component of the emergent solution envelope a circular *sonic cone* [cf. the representation in section 5.4 below, and via (5.1), (4.19) and (4.20) above, and (5.7) below] with the axis δ' and the vertex angle 2χ where

$$\sin\chi = \frac{1}{M}\cos\theta' = \frac{\cos\theta}{\sqrt{\overline{M}^2 + (M^2 - \overline{M}^2)\cos^2\theta}}$$

and we notice that for a real χ we must require $\tan\Theta \leq \mathfrak{z}_c$, i.e. subcriticity.

FIGURE 3 The simplest deterministic model of turbulence refraction ($\tilde{t} > 0$).

In the sequel we parallel (3.5) by introducing the *extended Lorentz coordinates*

$$\begin{cases} x = \dfrac{\mathfrak{z}_c\cos\Theta}{\sqrt{\overline{M}^2+(M^2-\overline{M}^2)\sin^2\Theta}}\widetilde{x}+\dfrac{M\mathfrak{z}_c}{\overline{M}}\widetilde{t}+\dfrac{M\mathfrak{z}_c}{\overline{M}}\cdot\dfrac{(\mathrm{sign}\theta)\sin\Theta}{\sqrt{\overline{M}^2+(M^2-\overline{M}^2)\sin^2\Theta}}\widetilde{z}=\dfrac{X}{\sqrt{1-M^2}}; \\ y = \widetilde{y};\;\; z=\widetilde{z} \\ t = \dfrac{M\mathfrak{z}_c^*(\Theta)\cos\Theta}{\sqrt{\overline{M}^2+(M^2-\overline{M}^2)\sin^2\Theta}}\widetilde{x}+\dfrac{\mathfrak{z}_c^2}{\overline{M}\mathfrak{z}_c^*(\Theta)}\widetilde{t}+\dfrac{\mathfrak{z}_c^2}{\overline{M}\mathfrak{z}_c^*(\Theta)}\cdot\dfrac{(\mathrm{sign}\theta)\sin\Theta}{\sqrt{\overline{M}^2+(M^2-\overline{M}^2)\sin^2\Theta}}\widetilde{z} \end{cases} \tag{5.6}$$

where

$$\mathfrak{z}_c^*(\Theta)\overset{\mathrm{def}}{=}\sqrt{\mathfrak{z}_c^2-\tan^2\Theta},\;\; \mathfrak{z}^*=\frac{\mathfrak{z}}{\cos\Theta},$$

and notice that

$$t^2-x^2-y^2=\left[\widetilde{z}+(\mathrm{sign}\theta)\frac{\widetilde{t}}{\sin\chi}\right]^2\tan^2\chi-(\widetilde{x}^2+\widetilde{y}^2). \tag{5.7}$$

5.4 The simplest nonstatistical model of turbulence refraction and its relation with Lighthill's model

We remark that the (5.1) has a *"Lorentz type"* arguments structure [which could be regarded as being a code ("cipher") which filters out the passage to an oblique approach]. Incidentally, this form appears to be *extensible* to the case of oblique interactions. The nature of the extension is suggested in Figure 3. We use (5.1) to put the limit $r_* \to 0$ of the sound emergent contribution corresponding to the mentioned *subcritical* interaction into the form

$$\begin{aligned} \widetilde{p}(x,y,t) &= \{1+[\mathfrak{z}_c^*(\Theta)-\mathfrak{z}_c]\} \\ &\qquad \cdot\widetilde{p}_{\|}[x,y,t;a^{*2},\varepsilon_b b^{*2},\varepsilon_c c^{*2},\upsilon^{*2};\mathfrak{z}_c^*(\Theta);Q_1^*,Q_2^*,Q_3^*] \\ &\quad +M[\mathfrak{z}_c^*(\Theta)-\mathfrak{z}_c] \\ &\qquad \cdot\widetilde{u}_{\|}[x,y,t;a^{*2},\varepsilon_b b^{*2},\varepsilon_c c^{*2},\upsilon^{*2};\mathfrak{z}_c^*(\Theta);Q_1^*,Q_2^*,Q_3^*] \end{aligned}$$

$$\begin{aligned} \widetilde{u}(x,y,t) &= \overline{M}\left\{1+\frac{M^2}{\overline{M}^2}\mathfrak{z}_c\left[\mathfrak{z}_c^*(\Theta)-\mathfrak{z}_c-\frac{1}{\mathfrak{z}_c}\tan^2\Theta\right]\right\}\cdot\frac{\cos\Theta}{\sqrt{\overline{M}^2+(M^2-\overline{M}^2)\sin^2\Theta}} \\ &\qquad \cdot\widetilde{u}_{\|}[x,y,t;a^{*2},\varepsilon_b b^{*2},\varepsilon_c c^{*2},\upsilon^{*2};\mathfrak{z}_c^*(\Theta);Q_1^*,Q_2^*,Q_3^*] \\ &\quad +\frac{M}{\overline{M}}\mathfrak{z}_c\left[\mathfrak{z}_c^*(\Theta)-\mathfrak{z}_c-\frac{1}{\mathfrak{z}_c}\tan^2\Theta\right]\cdot\frac{\cos\Theta}{\sqrt{\overline{M}^2+(M^2-\overline{M}^2)\sin^2\Theta}} \\ &\qquad \cdot\widetilde{p}_{\|}[x,y,t;a^{*2},\varepsilon_b b^{*2},\varepsilon_c c^{*2},\upsilon^{*2};\mathfrak{z}_c^*(\Theta);Q_1^*,Q_2^*,Q_3^*] \end{aligned}$$

$$\widetilde{v}(x,y,t) = \widetilde{v}_{\|}[x,y,t;a^{*2},\varepsilon_b b^{*2},\varepsilon_c c^{*2},\upsilon^{*2};\mathfrak{z}_c^*(\Theta);Q_1^*,Q_2^*,Q_3^*]$$

$$\widetilde{w}(x,y,t) = M\left\{2+\frac{M^2}{\overline{M}^2}\mathfrak{z}_c[\mathfrak{z}_c^*(\Theta)-\mathfrak{z}_c]\right\}\cdot\frac{(\mathrm{sign}\theta)\sin\Theta}{\sqrt{\overline{M}^2+(M^2-\overline{M}^2)\sin^2\Theta}}$$

$$
\cdot \widetilde{u}_{\|}[x,y,t;a^{*2},\varepsilon_b b^{*2},\varepsilon_c c^{*2},\upsilon^{*2};\mathfrak{z}_c^*(\Theta);Q_1^*,Q_2^*,Q_3^*]
$$

$$
+\left\{1+\frac{M^2}{\overline{M}^2}\mathfrak{z}_c[\mathfrak{z}_c^*(\Theta)-\mathfrak{z}_c]\right\}\cdot\frac{(\operatorname{sign}\theta)\sin\Theta}{\sqrt{\overline{M}^2+(M^2-\overline{M}^2)\sin^2\Theta}}
$$

$$
\cdot \widetilde{p}_{\|}[x,y,t;a^{*2},\varepsilon_b b^{*2},\varepsilon_c c^{*2},\upsilon^{*2};\mathfrak{z}_c^*(\Theta);Q_1^*,Q_2^*,Q_3^*]
$$

where x,y,t depend on $\widetilde{x},\widetilde{y},\widetilde{z},\widetilde{t}$ cf. (5.6) and we denote

$$
a^{*2}=a^2+\tan^2\Theta,\ b^{*2}=|b^2-\tan^2\Theta|,\ c^{*2}=|c^2-\tan^2\Theta|,\ \upsilon^{*2}=1+\tan^2\Theta
$$

$$
\varepsilon_b=\operatorname{sign}(\tan^2\Theta-b^2),\ \varepsilon_c=\operatorname{sign}(\tan^2\Theta-c^2)
$$

$$
\begin{cases}
Q_1^*(\mathfrak{z}^{*2})\overset{\text{def}}{=}d_{11}\mathfrak{z}^{*2}+(d_{11}\tan^2\Theta+d_{12})\equiv Q_1(\bar{\mathfrak{z}}^2)\\
Q_2^*(\mathfrak{z}^{*2})\overset{\text{def}}{=}d_{01}\mathfrak{z}^{*2}+(d_{01}\tan^2\Theta+d_{02})\equiv Q_2(\bar{\mathfrak{z}}^2)\\
Q_3^*(\mathfrak{z}^{*2})\overset{\text{def}}{=}d_{03}\mathfrak{z}^{*2}+(d_{03}\tan^2\Theta+d_{04})\equiv Q_3(\bar{\mathfrak{z}}^2).
\end{cases}
$$

A suggestive description concerning the refraction of a turbulence model through a shock discontinuity is considered in Figure 3. This description brings together and compares the passage through the discontinuity of an incident point vortex whose axis is parallel to the shock and the passage through the same shock of a point vortex whose axis is oblique − subcritical or supercritical.

In the singular limit of the interaction solution the *subcritical* contribution and the *supercritical* one are distinguished by differences of a *"relativistic"* nature. Precisely: in the singular limit of the interaction solution the emergent sound is *singular* in the subcritical contribution and it is *regular* in the supercritical contribution ([4]; see for example the refraction of an incident vortex whose axis is orthogonal to the plane of the shock).

In Dinu and Dinu [5] it is shown that the "relativistic" discontinuity in the nature of the emergent sound, corresponding to the singular limit of the interaction solution, appears to be dissembled (hidden) in the re-weighted interaction solution.

In Lighthill's fundamental paper [13] the turbulence is acoustically modelled by a distribution of quadrupoles − which is equivalent with a "weighted" distribution of point vortices. • We notice that the *explicit* character of Figure 3 induces an *exhaustive* nonstatistical classification into Lighthill's *implicit* description.

The details of the "relativistic" separation between a subcritical character and a supercritical character are essentially and significantly related with the criticity arguments considered in some fundamental numerical studies on the shock-turbulence interaction; see S.K. Lele [12].

Some amplifications of the approach in the present work are included in papers [3]−[6].

The paper Dinu [3] replaces the vorticity incident turbulence with a sound turbulence.

In Dinu and Dinu [4] some supercritical details, omitted in the present analysis, are taken into account.

In Dinu and Dinu [6] some remarks are included concerning the case in which the discontinuity implied in the interaction has an ionizing nature.

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