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LOCAL EXISTENCE FOR PARTIAL NEUTRAL FUNCTIONAL INTEGRODIFFERENTIAL EQUATIONS WITH INFINITE DELAY

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Abstract

In this paper, we investigate existence for a class of partial neutral functional integrodifferential equations with infinite delay. Using the integrated semigroup theory and the well known Banach fixed point theorem, we establish local existence and uniqueness of integral solutions to these equations. To illustrate our abstract results, we conclude this work by an example.

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1 Introduction

The purpose of this paper is to study local existence and uniqueness of integral solutions for a class of partial neutral functional integrodifferential equations with infinite delay described by the abstract form

$$\begin{cases} \dfrac{\partial}{\partial t}\mathcal{D}x_t = A\mathcal{D}x_t + \int_0^t k(t,s,x(s))ds + F(t,x_t), \quad t \geq 0, \\ x_0 = \varphi \in \mathcal{B}, \end{cases} \tag{1.1}$$

where $A : D(A) \subseteq E \to E$ is a linear operator on a Banach space $(E,|.|)$, the phase space $\mathcal{B}$ is a linear space of functions mapping $(-\infty,0]$ into E satisfying axioms which will be described below; for every $t \geq 0$, the history function $x_t \in \mathcal{B}$ is defined by $x_t(\theta) = x(t+\theta)$, for $\theta \in (-\infty,0]$; $\mathcal{D}$ is a bounded linear operator from $\mathcal{B}$ into E defined by

$$\mathcal{D}\varphi = \varphi(0) - \mathcal{D}_0\varphi, \text{ for any } \varphi \in \mathcal{B},$$

$\mathcal{D}_0$ is a bounded linear operator from $\mathcal{B}$ into E; $k : \Delta(0,T) \times E \to E$ is a continuous function, with $\Delta(0,T) = \{(t,s) : 0 \leq s \leq t \leq T\}$. F is a nonlinear continuous function from $[0,+\infty) \times \mathcal{B}$ into E and $\varphi \in \mathcal{B}$ is given.

In the literature devoted to equations with infinite delay, the phase space, the space of initial data, is the space of continuous functions on $(-\infty,0]$ endowed with the uniform norm topology. When the delay is infinite, the selection of the phase space $\mathcal{B}$ plays an important role in the study of both qualitative and quantitative theories. The study of partial neutral functional differential equations with infinite delay has received much attention in recent years, we refer the reader to [7], [8], [1], [2], [13], [14], [22] and [15], where the basic theory is given. Note that in the case where $k = 0$ we get the following equation

$$\begin{cases} \dfrac{\partial}{\partial t}\mathcal{D}x_t = A\mathcal{D}x_t + F(t,x_t), \quad t \geq 0, \\ x_0 = \varphi \in \mathcal{B}, \end{cases} \tag{1.2}$$

which has been studied by the second author of this paper in [7]. He has investigated local existence and uniqueness of integral solutions using the integrated semigroup theory and the well known Banach fixed point theorem.

In this paper, we use the basic theory developed in [7] for Eq. (1.2) to establish existence of integral solutions to Eq. (1.1). We use the integrated semigroup theory and the strict contraction principle to prove local existence and uniqueness of integral solutions.

As it was stated in [10], we point out that Eq. (1.1) can be transformed into Eq. (1.2) by setting $G(t,x_t) = \int_{-t}^0 k(t,t+s,x_t(s))ds + F(t,x_t)$. However, we have to assume that k is a function from $\Delta(0,T) \times \mathcal{B}$ into E which is different from our assumption that k is defined on $\Delta(0,T) \times E$. In general, the conclusions in [7] cannot be applied to our case.

This work is organized as follows. In section 2, we recall some preliminary results and make some assumptions that will be used throughout. In section 3, we prove local existence and uniqueness of integral solutions to Eq. (1.1). Moreover, some properties of solutions are also studied. To obtain existence and uniqueness of integral solutions, in [6] we have supposed that F and k are globaly Lipschitz continuous. Here, we suppose that they are Lipschitz continuous on the balls of $\mathcal{B}$.

2 Preliminary notes

In this section we give some basic definitions, assumptions and preliminary results. Throughout this paper, we suppose that $(\mathcal{B}, \|.\|_{\mathcal{B}})$ is a (semi)normed abstract linear space of functions mapping $(-\infty,0]$ into E, and satisfies the following fundamental axioms which have been first introduced in [12] and widely discussed in [16].

(**A**) There exist a positive constant H and functions $K(.)$, $M(.) : \mathbb{R}^+ \to \mathbb{R}^+$, with K continuous and M locally bounded, such that for any $\sigma \in \mathbb{R}$ and $a > 0$, if $x : (-\infty, \sigma + a] \to E$, $x_\sigma \in \mathcal{B}$ and $x(.)$ is continuous on $[\sigma, \sigma + a]$, then for every t in $[\sigma, \sigma + a]$ the following conditions hold:
(**i**) $x_t \in \mathcal{B}$,
(**ii**) $|x(t)| \leq H \|x_t\|_{\mathcal{B}}$, which is equivalent to
(**ii**)$'$ $|\varphi(0)| \leq H \|\varphi\|_{\mathcal{B}}$, for every $\varphi \in \mathcal{B}$,
(**iii**) $\|x_t\|_{\mathcal{B}} \leq K(t-\sigma) \sup_{\sigma \leq s \leq t} |x(s)| + M(t-\sigma) \|x_\sigma\|_{\mathcal{B}}$.
(**A**1) For the function $x(.)$ in (**A**), $t \mapsto x_t$ is a $\mathcal{B}$-valued continuous function for t in $[\sigma, \sigma + a]$.
(**B**) The space $\mathcal{B}$ is complete.

Throughout, we also assume that the operator A satisfies the Hille-Yosida condition :

(**H1**) there exist $\overline{M} \geq 0$ and $\overline{\omega} \in \mathbb{R}$ such that $(\overline{\omega}, +\infty) \subset \rho(A)$ and

$$\sup\left\{(\lambda - \overline{\omega})^n \left\|(\lambda I - A)^{-n}\right\| : \ n \in \mathbb{N}, \lambda > \overline{\omega}\right\} \leq \overline{M}. \tag{2.1}$$

Let A_0 be the part of the operator A in $\overline{D(A)}$, which is defined by

$$\begin{cases} D(A_0) = \left\{x \in D(A) : Ax \in \overline{D(A)}\right\}, \\ A_0 x = Ax, \text{ for } x \in D(A_0). \end{cases}$$

It is well known that $\overline{D(A_0)} = \overline{D(A)}$ and the operator A_0 generates a strongly continuous semigroup $(T_0(t))_{t \geq 0}$ on $\overline{D(A)}$.
Recall that ([19]) for all $x \in \overline{D(A)}$ and $t \geq 0$, one has $\int_0^t T_0(s)x \in D(A_0)$ and

$$\left(A \int_0^t T_0(s)xds\right) + x = T_0(t)x. \tag{2.2}$$

We also recall that $(T_0(t))_{t \geq 0}$ coincides on $\overline{D(A_0)}$ with the derivative of the locally Lipschitz integrated semigroup $(S(t))_{t \geq 0}$ generated by A on E. Which is, according to [5] and [17], a family of bounded linear operators on E, that satisfies
(**i**) $S(0) = 0$,
(**ii**) for any $y \in E$, $t \to S(t)y$ is strongly continuous with values in E,

(**iii**) $S(s)S(t) = \int_0^s (S(t+r) - S(r))dr$ for all $t, s \geq 0$,
and for any $\tau > 0$ there exists a constant $l(\tau) > 0$ such that

$$\|S(t) - S(s)\| \leq l(\tau)\,|t-s|\,, \quad \text{for all } t,s \in [0,\tau]\,.$$

This integrated semigroup is exponentially bounded, that is, there exist two constants $\overline{M}$ and $\overline{\omega}$ such that $\|S(t)\| \leq \overline{M}e^{\overline{\omega}t}$ for all $t \geq 0$.

We need to recall the following results.
Consider the following system

$$\begin{cases} \dfrac{\partial}{\partial t}\mathcal{D}u_t = A\mathcal{D}u_t \text{ if } t \geq 0, \\ u(\theta) = \varphi(\theta) \text{ if } \theta \in (-\infty, 0] \text{ with } \varphi \in \mathcal{B}. \end{cases} \tag{2.3}$$

Using equality (2.2), we can see that a necessary condition for $u : (-\infty, b) \to E$, $b > 0$, to be a solution of Eq. (2.3) is that it verifies the following integrated one on $(-\infty, b)$

$$\begin{cases} \mathcal{D}u_t = T_0(t)\mathcal{D}\varphi,\ t \geq 0, \\ u_0 = \varphi, \end{cases} \tag{2.4}$$

where

$$\varphi \in \mathcal{Y} := \left\{ \varphi \in \mathcal{B} : \mathcal{D}\varphi \in \overline{D(A)} \right\}.$$

Proposition 2.1. *[7] Assume that Condition* ($\mathbf{H}1$) *is satisfied and* $\|\mathcal{D}_0\|K(0) < 1$. *Then, for given* $\varphi \in \mathcal{Y}$ *there exists a unique function u which is continuous on* $[0,T)$ *and solves Eq. (2.4) on* $(-\infty, T)$. *Moreover, the family of operators* $(\mathcal{T}(t))_{t\geq 0}$ *defined on* $\mathcal{Y}$ *by* $\mathcal{T}(t)\varphi = u_t(.,\varphi)$ *is a* C_0*-semigroup on* $\mathcal{Y}$.

We now define a fundamental integral solution $Z(t)$ associated to Eq. (1.1). Consider for given $c \in E$ the following equation

$$\begin{cases} \mathcal{D}z_t = S(t)c \ \text{ if } t \geq 0, \\ z(t) = 0 \ \text{ if } t \in (-\infty, 0]\,. \end{cases} \tag{2.5}$$

To our purpose, we make the following condition

($\mathbf{H2}$) There exists a continuous nondecreasing function $\delta : [0, +\infty) \to [0, +\infty)$, $\delta(0) = 0$ and a family of continuous linear operators $W_\varepsilon : \mathcal{B} \to E$, $\varepsilon \in [0, +\infty)$, such that

$$|\mathcal{D}_0\varphi - \mathcal{D}_\varepsilon\varphi| \leq \delta(\varepsilon)\,\|\varphi\|_{\mathcal{B}}\,, \text{ for } \varepsilon \in [0, +\infty) \text{ and } \varphi \in \mathcal{B},$$

where the linear operator $\mathcal{D}_\varepsilon : \mathcal{B} \to E$ is defined, for $\varepsilon \in [0, +\infty)$, by

$$\begin{cases} \mathcal{D}_\varepsilon = W_\varepsilon \circ \tau_\varepsilon, \\ \tau_\varepsilon(\varphi)(\theta) = \varphi(\theta - \varepsilon), \quad \text{for } \varphi \in \mathcal{B} \text{ and } \theta \in (-\infty, 0]\,. \end{cases}$$

Notice that Assumption ($\mathbf{H}2$) means that the operator $\mathcal{D}_0$ does not depend very strongly upon $\varphi(0)$. It is the infinite delay version of the one introduced in [4] and [3].

Proposition 2.2. *[8] Assume that Conditions* $(\mathbf{H}1)$ *and* $(\mathbf{H}2)$ *are satisfied such that* $K(0)\,\|\mathcal{D}_0\| < 1$. *Then, for given* $c \in E$, *Eq. (2.5) has a unique integral solution* $z := z(.)c : (-\infty, +\infty) \to E$. *Moreover, the operator* $Z(t) : E \to \mathcal{B}$ *defined by*

$$Z(t)c = z_t(.)c$$

satisfies, for any continuous function $f : [0, +\infty) \to E$, *the following properties*
(i) *For each* $T > 0$, *there exists a function* $\alpha(.) \in L^\infty([0,T], \mathbb{R}^+)$ *and* $\beta \in \mathbb{R}$ *such that* $\|Z(t)\| \le \alpha(t)e^{t\beta}$ *for all* $t \in [0,T]$;
(ii) $Z(t)(E) \subseteq \mathcal{Y}$, *for all* $t \ge 0$;
(iii) *For all* $\tau > 0$ *there exists a constant* $k(\tau) > 0$ *such that*

$$\|Z(t)c - Z(s)c\|_{\mathcal{B}} \le k(\tau)\,|t-s|\,|c| \quad \textit{for all } t,s \in [0,\tau] \textit{ and } c \in E.$$

(iv) *For any continuous function* $f : [0, +\infty) \to E$, *the functions*

$$t \mapsto \int_0^t Z(t-s)f(s)\,ds \textit{ and } t \mapsto \int_0^t S(t-s)f(s)\,ds$$

are continuously differentiable for all $t \ge 0$ *and satisfy*

$$\frac{d}{dt}\left(\int_0^t Z(t-s)f(s)\,ds\right) = \lim_{h\to 0^+}\frac{1}{h}\int_0^t \mathcal{T}(t-s)Z(h)f(s)ds \ \textit{ for all } t \ge 0.$$

$$\begin{aligned} \mathcal{D}\left(\frac{d}{dt}\int_0^t Z(t-s)f(s)\,ds\right) &= \lim_{h\to 0^+}\frac{1}{h}\int_0^t S'(t-s)S(h)f(s)ds \\ &= \frac{d}{dt}\int_0^t S(t-s)f(s)ds. \end{aligned}$$

We recall the following definition.

Definition 2.3. [6] Let $T > 0$ and $\varphi \in \mathcal{B}$. We say that a function $x := x(.,\varphi) : (-\infty, T) \to E$, $0 < T \le +\infty$, is an integral solution of Eq. (1.1), if
(**i**) x is continuous on $[0,T)$,
(**ii**) $\int_0^t \mathcal{D}x_s\,ds \in D(A)$ for $t \in [0,T)$,
(**iii**) $\mathcal{D}x_t = \mathcal{D}\varphi + A\int_0^t \mathcal{D}x_s\,ds + \int_0^t [\int_0^s k(s,\tau,x(\tau))d\tau]\,ds + \int_0^t F(s,x_s)\,ds$ for $t \in [0,T)$,
(**iv**) $x(t) = \varphi(t)$, for all $t \in (-\infty, 0]$.

We deduce from [1] and [21] that integral solutions of Eq. (1.1) are given for $\varphi \in \mathcal{B}$ such that $\mathcal{D}\varphi \in \overline{D(A)}$ by the following system

$$\begin{cases} \mathcal{D}x_t &= S'(t)\mathcal{D}\varphi + \dfrac{d}{dt}\int_0^t S(t-s)\,[\int_0^s k(s,\tau,x(\tau))d\tau]\,ds \\ & \quad + \dfrac{d}{dt}\int_0^t S(t-s)F(s,x_s)ds, \qquad t \in [0,T), \\ x_0 &= \varphi \in \mathcal{B}. \end{cases} \tag{2.6}$$

We also suppose that $\mathcal{B}$ is normed and satisfies one of the following two extra axioms.

(**C1**) If $(\phi_n)_{n\geq 0}$ is a Cauchy sequence in $\mathcal{B}$ and if $(\phi_n)_{n\geq 0}$ converges compactly to ϕ on $(-\infty,0]$, then ϕ is in $\mathcal{B}$ and $\|\phi_n-\phi\|_{\mathcal{B}}\to 0$, as $n\to\infty$.
(**D1**) For a sequence $(\varphi_n)_{n\geq 0}$ in $\mathcal{B}$, if $\|\varphi_n\|_{\mathcal{B}}\to 0$, as $n\to\infty$, then $|\varphi_n(\theta)|\to 0$, as $n\to\infty$, for each $\theta\in(-\infty,0]$.

Lemma 2.4. *[18] Let $\mathcal{B}$ be a normed space which satisfies Axiom* (**C1**) *and $f:[0,a]\to\mathcal{B}$, $a>0$, be a continuous function such that $f(t)(\theta)$ is continuous for $(t,\theta)\in[0,a]\times(-\infty,0]$. Then*

$$\left[\int_0^a f(t)dt\right](\theta)=\int_0^a f(t)(\theta)dt,\quad \theta\in(-\infty,0].$$

We can obtain a similar result by using Axiom (**D1**).

Lemma 2.5. *[1], [9] Let $\mathcal{B}$ satisfy Axiom* (**D1**) *and $f:[0,a]\to\mathcal{B}$ be a continuous function. Then for all $\theta\in(-\infty,0]$, the function $f(.)(\theta)$ is continuous and*

$$\left[\int_0^a f(t)dt\right](\theta)=\int_0^a f(t)(\theta)dt,\quad \theta\in(-\infty,0].$$

Proposition 2.6. *Let $\mathcal{B}$ be a normed space which satisfies Axiom* (**C**1) *or Axiom* (**D1**). *If there exists an integral solution $x:=x(.,\varphi):(-\infty,T)\to E$, $0<T\leq+\infty$, of Eq. (1.1), then for $t\in[0,T)$, the function $t\mapsto x_t\in\mathcal{B}$ satisfies*

$$\begin{aligned} x_t &= \mathcal{T}(t)\varphi+\frac{d}{dt}\int_0^t Z(t-s)\left[\int_0^s k(s,\tau,x(\tau))d\tau\right]ds\\ &\quad +\frac{d}{dt}\int_0^t Z(t-s)F(s,x_s)ds\\ &= \mathcal{T}(t)\varphi+\lim_{h\to 0^+}\frac{1}{h}\int_0^t \mathcal{T}(t-s)Z(h)\left[\int_0^s k(s,\tau,x(\tau))d\tau\right]ds\\ &\quad +\lim_{h\to 0^+}\frac{1}{h}\int_0^t \mathcal{T}(t-s)Z(h)F(s,x_s)ds. \end{aligned}\tag{2.7}$$

Conversely, if there exists a function $y\in\mathcal{C}([0,T),\mathcal{B})$ such that

$$\begin{aligned} y(t) &= \mathcal{T}(t)\varphi+\frac{d}{dt}\int_0^t Z(t-s)\left[\int_0^s k(s,\tau,y(\tau)(0))d\tau\right]ds\\ &\quad +\frac{d}{dt}\int_0^t Z(t-s)F(s,y(s))ds, \qquad t\in[0,T), \end{aligned}\tag{2.8}$$

then $y(t)=x_t$ for all $t\in[0,T)$, where

$$x(t)=\begin{cases} y(t)(0), & t\in[0,T),\\ \varphi(t), & t\in(-\infty,0],\end{cases}$$

and $x(.)$ is an integral solution of Eq. (1.1).

Proof. We use similar arguments as in [8]. First, by Proposition 2.2, it is immediate that for any continuous function $f:[0,T)\to E$,

$$W(t):=\int_0^t Z(t-s)f(s)ds.$$

is continuously differentiable and $W'(0)=0$. Set

$$w(t)=\begin{cases} W(t)(0) \text{ if } t\geq 0,\\ 0 \text{ if } t\in(-\infty,0]. \end{cases}$$

By use of axiom $(\mathbf{A}-(\mathbf{ii})')$, $w(t)$ is continuously differentiable. Lemma 2.4 or Lemma 2.5 implies that for all $t\in[0,T)$

$$\begin{aligned} w(t) &= \left(\int_0^t Z(t-s)f(s)ds\right)(0)\\ &= \int_0^t (Z(t-s)f(s))(0)ds\\ &= \int_0^t z(t-s)f(s)ds. \end{aligned}$$

In general, for all $t\in[0,T)$ and $\theta\in(-\infty,0]$

$$\begin{aligned} (W(t))(\theta) &= \left(\int_0^t Z(t-s)f(s)ds\right)(\theta)\\ &= \int_0^t (Z(t-s)f(s))(\theta)ds\\ &= \int_0^t z(t+\theta-s)f(s)ds. \end{aligned}$$

Besides, since $z(s)=0$ for all $s\in(-\infty,0]$, we get

$$\int_0^t z(t+\theta-s)f(s)ds=\int_0^{t+\theta} z(t+\theta-s)f(s)ds, \tag{2.9}$$

and $(W(t))(\theta)=w(t+\theta)$. Which is equivalent to $W(t)=w_t$. On the other hand, we can see that for all $t\in[0,T)$ and $\theta\in(-\infty,0]$

$$(W'(t))(\theta)=w'(t+\theta).$$

Hence $W'(t)=(w')_t$ for all $t\in[0,T)$.
Now, suppose that $y(.,\varphi)$ is a solution of Eq. (2.8). The function $\mathcal{T}(t)\varphi=u_t$ with $u:(-\infty,T)\to E$ is the integral solution of $\mathcal{D}u_t=S'(t)\mathcal{D}\varphi$ such that $u_0=\varphi$. Set

$$w(t)=\int_0^t z(t-s)\left[\int_0^s k(s,\tau,y(\tau)(0))d\tau\right]ds+\int_0^t z(t-s)F(s,y(s))ds. \tag{2.10}$$

From (2.10) and (2.10), we have for all $t \in [0,T)$ and $\theta \in (-\infty,0]$

$$\begin{aligned}
w(t+\theta) &= \int_0^{t+\theta} z(t+\theta-s)\left[\int_0^s k(s,\tau,y(\tau)(0))d\tau\right]ds \\
&\quad + \int_0^{t+\theta} z(t+\theta-s)F(s,y(s))ds \\
&= \int_0^{t} z(t+\theta-s)\left[\int_0^s k(s,\tau,y(\tau)(0))d\tau\right]ds \\
&\quad + \int_0^{t} z(t+\theta-s)F(s,y(s))ds \\
&= \int_0^{t} z_{t-s}(\theta)\left[\int_0^s k(s,\tau,y(\tau)(0))d\tau\right]ds \\
&\quad + \int_0^{t} z_{t-s}(\theta)F(s,y(s))ds \\
&= \int_0^{t} \left(Z(t-s)\left[\int_0^s k(s,\tau,y(\tau)(0))d\tau\right]\right)(\theta)ds \\
&\quad + \int_0^{t} \left(Z(t-s)F(s,y(s))\right)(\theta)ds \\
&= \begin{pmatrix} \int_0^t Z(t-s)\left[\int_0^s k(s,\tau,y(\tau)(0))d\tau\right]ds \\ +\int_0^t Z(t-s)F(s,y(s))ds \end{pmatrix}(\theta).
\end{aligned}$$

Hence

$$\begin{aligned}
w'(t+\theta) &= \frac{d}{dt}\begin{pmatrix} \int_0^t Z(t-s)\left[\int_0^s k(s,\tau,y(\tau)(0))d\tau\right]ds \\ +\int_0^t Z(t-s)F(s,y(s))ds \end{pmatrix}(\theta) \\
&= \begin{pmatrix} \frac{d}{dt}\int_0^t Z(t-s)\left[\int_0^s k(s,\tau,y(\tau)(0))d\tau\right]ds \\ +\frac{d}{dt}\int_0^t Z(t-s)F(s,y(s))ds \end{pmatrix}(\theta).
\end{aligned}$$

Then by the definition of $y(t)$ in (2.8), we obtain

$$\begin{aligned}
y(t)(\theta) &= (\mathcal{T}(t)\varphi)(\theta) + \left(\frac{d}{dt}\int_0^t Z(t-s)\left[\int_0^s k(s,\tau,y(\tau)(0))d\tau\right]ds\right)(\theta) \\
&\quad + \left(\frac{d}{dt}\int_0^t Z(t-s)F(s,y(s))ds\right)(\theta) \\
&= u_t(\theta) + w'(t+\theta).
\end{aligned}$$

then

$$y(t) = u_t + (w')_t = (u+w')_t\,.$$

If we set $x(t) = u(t) + w'(t)$, we obtain $y(t) = x_t$ and

$$\begin{aligned}
x_t &= \mathcal{T}(t)\varphi + \frac{d}{dt}\int_0^t Z(t-s)\left[\int_0^s k(s,\tau,y(\tau)(0))d\tau\right]ds \\
&\quad + \frac{d}{dt}\int_0^t Z(t-s)F(s,y(s))ds \\
&= \mathcal{T}(t)\varphi + \frac{d}{dt}\int_0^t Z(t-s)\left[\int_0^s k(s,\tau,x(\tau))d\tau\right]ds \\
&\quad + \frac{d}{dt}\int_0^t Z(t-s)F(s,x_s)ds.
\end{aligned}$$

Since $\mathcal{D}(\mathcal{T}(t)\varphi) = S'(t)\mathcal{D}\varphi$ and by Proposition 2.2

$$
\begin{aligned}
&\mathcal{D}\left(\frac{d}{dt}\int_0^t Z(t-s)\,[\int_0^s k(s,\tau,x(\tau))d\tau]\,ds+\frac{d}{dt}\int_0^t Z(t-s)F(s,x_s)ds\right)\\
&=\frac{d}{dt}\int_0^t S(t-s)\,[\int_0^s k(s,\tau,x(\tau))d\tau]+\frac{d}{dt}\int_0^t S(t-s)F(s,x_s)ds,
\end{aligned}
$$

$x(t)$ is an integral solution of Eq. (1.1).
Conversely, let $x(.,\varphi)$ be an integral solution of Eq. (1.1) on $(-\infty,T)$, then

$$
\begin{aligned}
\mathcal{D}x_t &= S'(t)\mathcal{D}\varphi+\frac{d}{dt}\int_0^t S(t-s)\,[\int_0^s k(s,\tau,x(\tau))d\tau]\,ds\\
&\quad+\frac{d}{dt}\int_0^t S(t-s)F(s,x_s)ds.
\end{aligned}
$$

By definition of $\mathcal{T}(t)$, we get

$$
\begin{aligned}
\mathcal{D}x_t &= \mathcal{D}(\mathcal{T}(t)\varphi)+\mathcal{D}(\frac{d}{dt}\int_0^t Z(t-s)\,[\int_0^s k(s,\tau,x(\tau))d\tau]\,ds\\
&\quad+\frac{d}{dt}\int_0^t Z(t-s)F(s,x_s)ds)\\
&= \mathcal{D}(T(t)\varphi)+\frac{d}{dt}\int_0^t Z(t-s)\,[\int_0^s k(s,\tau,x(\tau))d\tau]\,ds\\
&\quad+\frac{d}{dt}\int_0^t Z(t-s)F(s,x_s)ds)\\
&= \mathcal{D}(u_t+(w')_t),
\end{aligned}
$$

where $u:(-\infty,T)\to E$ is the integral solution of $\mathcal{D}u_t=S'(t)\mathcal{D}\varphi$, and $w(t)$ is defined by

$$
w(t)=\int_0^t z(t-s)\left[\int_0^s k(s,\tau,y(\tau)(0))d\tau\right]ds+\int_0^t z(t-s)F(s,y(s))ds.
$$

We deduce that, $\mathcal{D}[(x-(u+w'))_t]=0$, and hence $x-(u+w')=0$. Consequently,

$$
\begin{aligned}
x_t &= u_t+(w')_t\\
&= \mathcal{T}(t)\varphi+\frac{d}{dt}\int_0^t Z(t-s)\,[\int_0^s k(s,\tau,x(\tau))d\tau]\,ds\\
&\quad+\frac{d}{dt}\int_0^t Z(t-s)F(s,x_s)ds.
\end{aligned}
$$

Which ends the proof of the proposition. □

Remark 2.7. The proof of the above proposition needs computing integrals in $\mathcal{B}$ from integrals in E. That is why we suppose that $\mathcal{B}$ is normed and satisfies one of the two extra axioms (**C**1) or (**D1**). See [18], [1] or [8] for more details.

3 Main Results

To obtain our main results on local existence and uniqueness of integral solutions to Eq. (1.1), we make the following assumptions.
(**H3**) $F:[0,+\infty[\times\mathcal{B}$ is Lipschitz continuous with respect to φ on the balls of $\mathcal{B}$, i.e., for each $r_0>0$, there exists a constant $L_0(r_0)>0$ such that if $t\geq 0$, $\varphi_1,\varphi_2\in\mathcal{B}$ and $\|\varphi_1\|_{\mathcal{B}},\|\varphi_2\|_{\mathcal{B}}\leq r_0$ then

$$
|F(t,\varphi_1)-F(t,\varphi_2)|\leq L_0(r_0)\,\|\varphi_1-\varphi_2\|_{\mathcal{B}}.
$$

$(\mathbf{H}4)$ $k:\Delta(0,T)\times E\to E$, with $\Delta(0,T)=\{(t,s):0\le s\le t\le T\}$, is continuous and satisfies the local Lipschitz condition:

For each $r_1>0$, there exists $L_1(r_1)>0$ such that

$$\begin{array}{rcl} |k(t,s,x_1)-k(t,s,x_2)| & \le & L_1(r_1)\,|x_1-x_2|\,, \\ \text{for } (t,s) & \in & \Delta(0,T) \text{ and } |x_1|,\ |x_2|\le r_1. \end{array}$$

Theorem 3.1. *Let $\mathcal{B}$ be a normed space which satisfies Axiom $(\mathbf{C}1)$ or Axiom $(\mathbf{D1})$. Assume that $(\mathbf{H}1)$, $(\mathbf{H}2)$, $(\mathbf{H}3)$ and $(\mathbf{H}4)$ hold. Let $\varphi\in\mathcal{B}$ such that $\mathcal{D}\varphi\in\overline{D(A)}$. Then, there exists a maximal interval of existence $\left(-\infty,b_\varphi\right)$, $b_\varphi>0$, and a unique integral solution $u(.,\varphi)$ of Eq. (1.1), defined on $\left(-\infty,b_\varphi\right)$ and either $b_\varphi=+\infty$ or*

$$\limsup_{t\to b_\varphi^-}|u(t,\varphi)|=+\infty.$$

Proof. Let $b\in\,]0,T]$. Note that $(\mathbf{H}4)$ implies that for each $r_1>0$ there exists $L_1(r_1)>0$ such that

$$\begin{array}{rcl} |k(t,s,x)| & \le & |k(t,s,x)-k(t,s,0)|+|k(t,s,0)|\,, \\ & \le & L_1(r_1)\,|x|+|k(t,s,0)|\,, \\ & \le & r_1L_1(r_1)+\sup\limits_{0\le s\le t\le b}|k(t,s,0)|\,, \qquad \text{for } |x|\le r_1. \end{array}$$

From $(\mathbf{H}3)$ it follows that for each $r_0>0$ there exists $L_0(r_0)>0$ such that

$$|F(t,\varphi)|\le r_0L_0(r_0)+\sup_{t\in[0,b]}|F(t,0)|\,,\ \text{for } t\in[0,b]\,,\ \varphi\in\mathcal{B}\ \text{ and }\ \|\varphi\|_{\mathcal{B}}\le r_0.$$

We set

$$c_0=r_0L_0(r_0)+\sup_{t\in[0,b]}|F(t,0)|\,, \tag{3.1}$$

and

$$c_1=r_1L_1(r_1)+\sup_{0\le s\le t\le b}|k(t,s,0)|\,. \tag{3.2}$$

Let $\varphi\in\mathcal{B}$, with $\mathcal{D}\varphi\in\overline{D(A)}$, $r_0=\|\varphi\|_{\mathcal{B}}+1$ and $r_1=H(\|\varphi\|_{\mathcal{B}}+1)$.

We define the set Ω_φ by

$$\Omega_\varphi:=\left\{y\in\mathcal{C}\left([0,b]\,,\mathcal{B}\right):\ \sup_{0\le s\le b}\|y(s)-\varphi\|_{\mathcal{B}}\le 1\right\},$$

where $\mathcal{C}\left([0,b]\,,\mathcal{B}\right)$ is endowed with the uniform convergence topology. We can easily see that Ω_φ is a closed set of $\mathcal{C}\left([0,b]\,,\mathcal{B}\right)$. Consider the mapping

$$J:\Omega_\varphi\to\mathcal{C}\left([0,b]\,,\mathcal{B}\right),$$

defined for $y\in\Omega_\varphi$ and $t\in[0,b]$ by

$$\begin{aligned} J(y)(t) \quad &:= \mathcal{T}(t)\varphi + \frac{d}{dt}\int_0^t Z(t-s)\left[\int_0^s k(s,\tau,y(\tau)(0))d\tau\right]ds \\ &+ \frac{d}{dt}\int_0^t Z(t-s)F(s,y(s))ds, \\ &= \mathcal{T}(t)\varphi + \lim_{h\to 0^+}\frac{1}{h}\int_0^t \mathcal{T}(t-s)Z(h)\left[\int_0^s k(s,\tau,y(\tau)(0))d\tau\right]ds \\ &+ \lim_{h\to 0^+}\frac{1}{h}\int_0^t \mathcal{T}(t-s)Z(h)F(s,y(s))ds. \end{aligned}$$

Next we show that

$$J\left(\Omega_\varphi\right) \subseteq \Omega_\varphi.$$

One can remark, as in the proof of Proposition 2.2 of [17], that

$$\limsup_{h\to 0^+}\frac{1}{h}\|Z(h)\| < +\infty.$$

Then we can set

$$c := \limsup_{h\to 0^+}\frac{1}{h}\|Z(h)\|. \tag{3.3}$$

We have for suitable constants $\overline{M}$ and ω

$$\begin{aligned} \|J(y)(t)-\varphi\|_{\mathcal{B}} \quad &\leq \|\mathcal{T}(t)\varphi-\varphi\|_{\mathcal{B}} \\ &+\overline{M}e^{\omega t}\int_0^t e^{-\omega s}\tfrac{1}{h}\|Z(h)\|\int_0^s |k(s,\tau,y(\tau)(0))|d\tau ds \\ &+\overline{M}e^{\omega t}\int_0^t e^{-\omega s}\tfrac{1}{h}\|Z(h)\|\,|F(s,y(s))|ds. \end{aligned}$$

We can assume here without loss of generality that $\omega > 0$. Thus we obtain the estimate

$$\begin{aligned} \|J(y)(t)-\varphi\|_{\mathcal{B}} \quad &\leq \quad \|\mathcal{T}(t)\varphi-\varphi\|_{\mathcal{B}} + \overline{M}ce^{\omega t}\int_0^t\int_0^s |k(s,\tau,y(\tau)(0))|d\tau ds \\ &+\overline{M}ce^{\omega t}\int_0^t |F(s,y(s))|ds. \end{aligned}$$

Since $\|y(s)-\varphi\|_{\mathcal{B}} \leq 1$, for $s\in[0,b]$ and $r_0 = \|\varphi\|_{\mathcal{B}}+1$, we have that $\|y(s)\|_{\mathcal{B}} \leq \|y(s)-\varphi\|_{\mathcal{B}} + \|\varphi\|_{\mathcal{B}} \leq r_0$, for $s\in[0,b]$, then

$$|F(s,y(s))| \leq L_0(r_0)\|y(s)\|_{\mathcal{B}} + |F(s,0)| \leq c_0.$$

From Axiom $(\mathbf{A}-\mathbf{ii})$, we have

$$\begin{aligned} |y(\tau)(0)| \quad &\leq H\|y(\tau)\|_{\mathcal{B}} \\ &\leq H\left(\|y(s)-\varphi\|_{\mathcal{B}} + \|\varphi\|_{\mathcal{B}}\right) \\ &\leq H\left(1+\|\varphi\|_{\mathcal{B}}\right), \qquad\qquad \text{for } 0\leq\tau\leq s\leq t\leq b. \end{aligned}$$

Since $r_1 = H(\|\varphi\|_{\mathcal{B}}+1)$ then

$$\begin{aligned}
|k(s,\tau,y(\tau)(0))| &\leq L_1(r_1)\,|y(\tau)(0)| + \sup_{0\leq\tau\leq s\leq b}|k(s,\tau,0)| \\
&\leq L_1(r_1)H\,\|y(\tau)\|_{\mathcal{B}} + \sup_{0\leq\tau\leq s\leq b}|k(s,\tau,0)| \\
&\leq L_1(r_1)H\,(1+\|\varphi\|_{\mathcal{B}}) + \sup_{0\leq\tau\leq s\leq b}|k(s,\tau,0)| \\
&\leq c_1.
\end{aligned}$$

Consider $b > 0$ sufficiently small such that

$$\sup_{0\leq s\leq b}\left\{\|\mathcal{T}(s)\varphi-\varphi\|_{\mathcal{B}} + \overline{M}ce^{\omega s}c_1s^2 + \overline{M}ce^{\omega s}c_0s\right\} < 1.$$

Note that $\lim\limits_{s\to 0}\|\mathcal{T}(s)\varphi-\varphi\|_{\mathcal{B}} = 0$ since $\mathcal{B}$ is normed.
Then, we deduce that for $t \in [0,b]$

$$\|J(v)(t)-\varphi\|_{\mathcal{B}} \leq \|\mathcal{T}(t)\varphi-\varphi\|_{\mathcal{B}} + \overline{M}ce^{\omega s}c_1t^2 + \overline{M}ce^{\omega t}c_0t < 1.$$

Hence

$$J\left(\Omega_\varphi\right) \subseteq \Omega_\varphi.$$

On the other hand, let $u, v \in \Omega_\varphi$ and $t \in [0,b]$. Then we have

$$\begin{aligned}
\|J(u)(t)-J(v)(t)\|_{\mathcal{B}} \quad &\leq \overline{M}e^{\omega t}\int_0^t e^{-\omega s}\tfrac{1}{h}\|Z(h)\|\int_0^s \left|\begin{array}{c} k(s,\tau,u(\tau)(0)) \\ -k(s,\tau,v(\tau)(0)) \end{array}\right| d\tau ds \\
&\quad + \overline{M}e^{\omega t}\int_0^t e^{-\omega s}\tfrac{1}{h}\|Z(h)\|\,|F(s,u(s))-F(s,v(s))|\,ds.
\end{aligned}$$

We can assume here without loss of generality that $\omega > 0$. Thus we obtain the estimate

$$\begin{aligned}
\|J(u)(t)-J(v)(t)\|_{\mathcal{B}} \quad &\leq \overline{M}ce^{\omega t}\int_0^t\int_0^s |k(s,\tau,u(\tau)(0))-k(s,\tau,v(\tau)(0))|\,d\tau ds \\
&\quad + \overline{M}ce^{\omega t}\int_0^t |F(s,u(s))-F(s,v(s))|\,ds \\
&\leq \overline{M}ce^{\omega t}L_1(r_1)\int_0^t\int_0^s |u(\tau)(0)-v(\tau)(0)|\,d\tau ds \\
&\quad + \overline{M}ce^{\omega t}L_0(r_0)\int_0^t \|u(s)-v(s)\|_{\mathcal{B}}\,ds \\
&\leq \overline{M}ce^{\omega t}L_1(r_1)H\int_0^t\int_0^s \|u(\tau)-v(\tau)\|_{\mathcal{B}}\,d\tau ds \\
&\quad + \overline{M}ce^{\omega t}L_0(r_0)\int_0^t \|u(s)-v(s)\|_{\mathcal{B}}\,ds \\
&\leq (\overline{M}ce^{\omega b}L_1(r_1)Hb^2 \\
&\quad + \overline{M}ce^{\omega b}L_0(r_0)b)\,\|u-v\|_{C([0,b],\mathcal{B})}\,.
\end{aligned}$$

Since $r_0 = \|\varphi\|_{\mathcal{B}} + 1 > 0$, by definition of c_0 in (3.1), we have $L_0(r_0) \leq c_0$. Similarly, since $c_1 = r_1L_1(r_1) + \sup\limits_{0\leq s\leq t\leq b}|k(t,s,0)|$ with $r_1 = H(\|\varphi\|_{\mathcal{B}}+1)$, then we have $HL_1(r_1) \leq c_1$. Consequently

$$\begin{aligned}
\overline{M}ce^{\omega b}L_1(r_1)\,b^2H + \overline{M}ce^{\omega b}L_0(r_0)\,b \quad &\leq \sup_{0\leq s\leq b}\{\|\mathcal{T}(s)\varphi-\varphi\|_{\mathcal{B}} + \overline{M}ce^{\omega s}c_1s^2 \\
&\quad + \overline{M}ce^{\omega s}c_0s\} \\
&< 1,
\end{aligned}$$

then

$$\|J(u)(t)-J(v)(t)\|_{\mathcal{B}}<\|u-v\|_{\mathcal{C}([0,b],\mathcal{B})}.$$

This means that J is a strict contraction in $\left(\Omega_{\varphi},\ \|.\|_{\mathcal{B}}\right)$.We conclude that there exists a unique function $y\in\Omega_{\varphi}$ such that $J(y)=y$. Then, Eq. (1.1) has one and only one integral solution $x:(-\infty,b)\to E$ defined by

$$x(t)=\begin{cases} y(t)(0), & t\in[0,b),\\ \varphi(t), & t\in(-\infty,0].\end{cases}$$

Let $\left(-\infty,b_{\varphi}\right)$ be the maximal interval of existence of x. Assume that $b_{\varphi}<+\infty$ and

$$\limsup_{t\to b_{\varphi}^{-}}|x(t,\varphi)|<+\infty.$$

Consider $t,t+h\in\left[0,b_{\varphi}\right)$ and $h>0$,

$$\begin{aligned} x_{t+h} &= \mathcal{T}(t+h)\varphi+\lim_{d\to0^+}\frac{1}{d}\int_0^{t+h}\mathcal{T}(t+h-s)Z(d)\left[\int_0^s k(s,\tau,x(\tau))d\tau\right]ds\\ &\quad+\lim_{d\to0^+}\frac{1}{d}\int_0^{t+h}\mathcal{T}(t+h-s)Z(d)F(s,x_s)ds\\ &=\mathcal{T}(t+h)\varphi\\ &\quad+\lim_{d\to0^+}\frac{1}{d}\begin{pmatrix}\int_0^h\mathcal{T}(t)\mathcal{T}(h-s)Z(d)\left[\int_0^s k(s,\tau,x(\tau))d\tau\right]ds\\ +\int_h^{t+h}\mathcal{T}(t+h-s)Z(d)\left[\int_0^s k(s,\tau,x(\tau))d\tau\right]ds\end{pmatrix}\\ &\quad+\lim_{d\to0^+}\frac{1}{d}\begin{pmatrix}\int_0^h\mathcal{T}(t)\mathcal{T}(h-s)Z(d)F(s,x_s)ds\\ +\int_h^{t+h}\mathcal{T}(t+h-s)Z(d)F(s,x_s)ds\end{pmatrix},\end{aligned}$$

and

$$\begin{aligned} x_t &= \mathcal{T}(t)\varphi+\lim_{d\to0^+}\frac{1}{d}\int_0^t\mathcal{T}(t-s)Z(d)\left[\int_0^s k(s,\tau,x(\tau))d\tau\right]ds\\ &\quad+\lim_{d\to0^+}\frac{1}{d}\int_0^t\mathcal{T}(t-s)Z(d)F(s,x_s)ds.\end{aligned}$$

Since

$$\int_h^{t+h}\mathcal{T}(t+h-s)Z(d)F(s,x_s)ds=\int_0^t\mathcal{T}(t-s)Z(d)F(s+h,x_{s+h})ds,$$

and

$$\begin{aligned}&\int_h^{t+h}\mathcal{T}(t+h-s)Z(d)\left[\int_0^s k(s,\tau,x(\tau))d\tau\right]ds\\ &=\int_0^t\mathcal{T}(t-s)Z(d)\left[\int_0^{s+h}k(s+h,\tau,x(\tau))d\tau\right]ds,\end{aligned}$$

we have

$$\begin{aligned} x_{t+h}-x_t &= \mathcal{T}(t+h)\varphi-\mathcal{T}(t)\varphi\\ &\quad+\lim_{d\to0^+}\frac{1}{d}\begin{pmatrix}\mathcal{T}(t)\int_0^h\mathcal{T}(h-s)Z(d)\left[\int_0^s k(s,\tau,x(\tau))d\tau\right]ds+\\ \int_0^t\mathcal{T}(t-s)Z(d)\begin{bmatrix}\int_0^{s+h}k(s+h,\tau,x(\tau))d\tau\\ -\int_0^s k(s,\tau,x(\tau))d\tau\end{bmatrix}ds\end{pmatrix}\\ &\quad+\lim_{d\to0^+}\frac{1}{d}\begin{pmatrix}\mathcal{T}(t)\int_0^h\mathcal{T}(h-s)Z(d)F(s,x_s)ds\\ +\int_0^t\mathcal{T}(t-s)Z(d)\left(F(s+h,x_{s+h})-F(s,x_s)\right)ds\end{pmatrix}.\end{aligned}\tag{3.4}$$

From **(iv)** in Proposition 2.2, we know that

$$\lim_{d\to 0^+}\frac{1}{d}\int_0^h \mathcal{T}(h-s)Z(d)F(s,x_s)ds=\frac{d}{dh}\int_0^h Z(h-s)F(s,x_s)ds,$$

and

$$\begin{aligned}&\lim_{d\to 0^+}\frac{1}{d}\int_0^h \mathcal{T}(h-s)Z(d)\left[\int_0^s k(s,\tau,x(\tau))d\tau\right]ds\\ &=\frac{d}{dh}\int_0^h Z(h-s)\left[\int_0^s k(s,\tau,x(\tau))d\tau\right]ds.\end{aligned}$$

Then (3.4) becomes

$$\begin{aligned}x_{t+h}-x_t\ &=\ \mathcal{T}(t+h)\varphi-\mathcal{T}(t)\varphi\\ &+\lim_{d\to 0^+}\frac{1}{d}\mathcal{T}(t)\int_0^h \mathcal{T}(h-s)Z(d)\left[\int_0^s k(s,\tau,x(\tau))d\tau\right]ds\\ &+\lim_{d\to 0^+}\frac{1}{d}\int_0^t \mathcal{T}(t-s)Z(d)\begin{bmatrix}\int_0^{s+h}k(s+h,\tau,x(\tau))d\tau\\ -\int_0^s k(s,\tau,x(\tau))d\tau\end{bmatrix}ds\\ &+\lim_{d\to 0^+}\frac{1}{d}\mathcal{T}(t)\int_0^h \mathcal{T}(h-s)Z(d)F(s,x_s)ds\\ &+\lim_{d\to 0^+}\frac{1}{d}\int_0^t \mathcal{T}(t-s)Z(d)\left(F(s+h,x_{s+h})-F(s,x_s)\right)ds.\end{aligned}\tag{3.5}$$

Now we estimate the above four limits

$$\begin{aligned}&\left\|\mathcal{T}(t)\int_0^h \mathcal{T}(h-s)Z(d)\left[\int_0^s k(s,\tau,x(\tau))d\tau\right]ds\right\|_{\mathcal{B}}\\ &\le \overline{M}e^{\omega(t+h)}\int_0^h\|Z(d)\|\int_0^s|k(s,\tau,x(\tau))|\,d\tau ds\\ &\le \overline{M}e^{\omega(t+h)}\|Z(d)\|\,h^2\left(r_1L_1(r_1)+\sup_{0\le\tau\le s\le b_\varphi}|k(s,\tau,0)|\right).\end{aligned}$$

Notice that $\sup_{0\le\tau\le s\le b_\varphi}|k(s,\tau,0)|<\infty$ since $b_\varphi<\infty$. Hence the first limit is estimated as

$$\left\|\lim_{d\to 0^+}\frac{1}{d}\mathcal{T}(t)\int_0^h \mathcal{T}(h-s)Z(d)\left[\int_0^s k(s,\tau,x(\tau))d\tau\right]ds\right\|_{\mathcal{B}}\le \overline{M}e^{\omega h}e^{\omega b_\varphi}ch^2c_3,$$

where

$$c_3:=r_1L_1(r_1)+\sup_{0\le\tau\le s\le b_\varphi}|k(s,\tau,0)|\,.$$

Let us make the following decomposition

$$\begin{aligned}&\int_0^t \mathcal{T}(t-s)Z(d)\left[\int_0^{s+h}k(s+h,\tau,x(\tau))d\tau-\int_0^s k(s,\tau,x(\tau))d\tau\right]ds\\ &=\int_0^t \mathcal{T}(t-s)Z(d)\begin{bmatrix}\int_0^h k(s+h,\tau,x(\tau))d\tau\\ +\int_0^s k(s+h,\tau+h,x(\tau+h))d\tau-\int_0^s k(s,\tau,x(\tau))d\tau\end{bmatrix}ds\\ &=\int_0^t \mathcal{T}(t-s)Z(d)\left[\int_0^h k(s+h,\tau,x(\tau))d\tau\right]ds\\ &+\int_0^t \mathcal{T}(t-s)Z(d)\left[\int_0^s\left(k(s+h,\tau+h,x(\tau))-k(s,\tau,x(\tau))\right)d\tau\right]ds\\ &+\int_0^t \mathcal{T}(t-s)Z(d)\left[\int_0^s\left(k(s+h,\tau+h,x(\tau+h))-k(s+h,\tau+h,x(\tau))\right)d\tau\right]ds\\ &:=\ I_1+I_2+I_3.\end{aligned}\tag{3.6}$$

Then, the first integral I_1 is estimated as

$$\begin{aligned} \| I_1\|_{\mathcal{B}} &\le \int_0^t \overline{M}e^{\omega(t-s)} \|Z(d)\| \left[\int_0^h |k(s+h,\tau,x(\tau))|\, d\tau\right] ds \\ &\le \overline{M}e^{\omega b_\varphi} \|Z(d)\| b_\varphi h c_3. \end{aligned} \tag{3.7}$$

For I_2, we have

$$\begin{aligned} \|I_2\|_{\mathcal{B}} &\le \int_0^t \overline{M}e^{\omega(t-s)} \|Z(d)\| \int_0^s |k(s+h,\tau+h,x(\tau)) - k(s,\tau,x(\tau))|\, d\tau ds \\ &\le \overline{M}e^{\omega b_\varphi} \|Z(d)\| \int_0^t \int_0^s |k(s+h,\tau+h,x(\tau)) - k(s,\tau,x(\tau))|\, d\tau ds. \end{aligned}$$

Set

$$g(t,h) := \int_0^t \int_0^s |k(s+h,\tau+h,x(\tau)) - k(s,\tau,x(\tau))|\, d\tau ds.$$

For I_3, from $(\mathbf{A}-(\mathbf{ii})')$, we have that $|x_{\tau+h}(0) - x_\tau(0)| \le H \|x_{\tau+h} - x_\tau\|_{\mathcal{B}}$, then

$$\begin{aligned} \|I_3\|_{\mathcal{B}} &\le \int_0^t \overline{M}e^{\omega(t-s)} \|Z(d)\| \int_0^s \left| \begin{array}{c} k(s+h,\tau+h,x(\tau+h)) \\ -k(s+h,\tau+h,x(\tau)) \end{array} \right| d\tau ds \\ &\le \overline{M}e^{\omega b_\varphi} \|Z(d)\| b_\varphi L_1(r_1) \int_0^t \sup_{0\le\tau\le s} |x(\tau+h) - x(\tau)|\, ds \\ &\le \overline{M}e^{\omega b_\varphi} \|Z(d)\| b_\varphi L_1(r_1) H \int_0^t \sup_{0\le\tau\le s} \|x_{\tau+h} - x_\tau\|_{\mathcal{B}}\, ds. \end{aligned}$$

Then we have the estimate for the second limit

$$\begin{aligned} &\left\| \lim_{d\to 0^+} \frac{1}{d} \int_0^t \mathcal{T}(t-s)Z(d) \left[\int_0^{s+h} k(s+h,\tau,x(\tau))d\tau - \int_0^s k(s,\tau,x(\tau))d\tau\right] ds \right\|_{\mathcal{B}} \\ &\le \overline{M}e^{\omega b_\varphi} c \left(b_\varphi h c_3 + g(t,h) + b_\varphi L_1(r_1) H \int_0^t \sup_{0\le\tau\le s} \|x_{\tau+h} - x_\tau\|_{\mathcal{B}}\, ds \right). \end{aligned}$$

Similarly, for the third and fourth limits, we can see that

$$\left\| \lim_{d\to 0^+} \frac{1}{d} \mathcal{T}(t) \int_0^h \mathcal{T}(h-s)Z(d)F(s,u_s)ds \right\|_{\mathcal{B}} \le \overline{M}e^{\omega h}e^{\omega b_\varphi} chc_2,$$

where

$$c_2 := r_0 L_0(r_0) + \sup_{s\in[0,b_\varphi)} |F(s,0)|,$$

and

$$\begin{aligned} &\left\| \lim_{d\to 0^+} \frac{1}{d} \int_0^t \mathcal{T}(t-s)Z(d) \left(F(s+h,u_{s+h}) - F(s,u_s)\right) ds \right\|_{\mathcal{B}} \\ &\le \overline{M}e^{\omega b_\varphi} c \left(L_0(r_0) \int_0^t \|u_{s+h} - u_s\|\, ds + f(t,h) \right), \end{aligned}$$

where

$$f(t,h) := \int_0^t |F(s+h,u_s) - F(s,u_s)|\, ds.$$

Thus we infer

$$\begin{aligned}\|x_{t+h}-x_t\|_{\mathcal{B}} &\le \sup_{0\le\sigma\le t}\|x_{\sigma+h}-x_\sigma\|_{\mathcal{B}}\\ &\le \overline{M}e^{\omega b_\varphi}\|\mathcal{T}(h)\varphi-\varphi\|_{\mathcal{B}}+\overline{M}e^{\omega h}e^{\omega b_\varphi}ch^2c_3+\overline{M}e^{\omega b_\varphi}cf(t,h)\\ &+\overline{M}e^{\omega b_\varphi}cb_\varphi hc_3+\overline{M}e^{\omega b_\varphi}cg(t,h)+\overline{M}e^{\omega h}e^{\omega b_\varphi}chc_2\\ &+\overline{M}e^{\omega b_\varphi}cb_\varphi L_1(r_1)H\sup_{0\le\sigma\le t}\int_0^\sigma\sup_{0\le\tau\le s}\|x_{\tau+h}-x_\tau\|_{\mathcal{B}}\,ds\\ &+\overline{M}e^{\omega b_\varphi}cL_0(r_0)\sup_{0\le\sigma\le t}\int_0^\sigma\|x_{s+h}-x_s\|_{\mathcal{B}}\,ds.\end{aligned}$$

We can see that

$$\sup_{0\le\sigma\le t}\int_0^\sigma\sup_{0\le\tau\le s}\|x_{\tau+h}-x_\tau\|_{\mathcal{B}}\,ds=\int_0^t\sup_{0\le\tau\le s}\|x_{\tau+h}-x_\tau\|_{\mathcal{B}}\,ds,$$

and

$$\begin{aligned}\sup_{0\le\sigma\le t}\int_0^\sigma\|x_{s+h}-x_s\|_{\mathcal{B}}\,ds &= \int_0^t\|x_{s+h}-x_s\|_{\mathcal{B}}\,ds\\ &\le \int_0^t\sup_{0\le\tau\le s}\|x_{\tau+h}-x_\tau\|_{\mathcal{B}}\,ds.\end{aligned}$$

We set

$$\begin{aligned}J_1 \quad &:= \overline{M}e^{\omega b_\varphi}\|\mathcal{T}(h)\varphi-\varphi\|_{\mathcal{B}}+\overline{M}e^{\omega h}e^{\omega b_\varphi}ch^2c_3+\overline{M}e^{\omega b_\varphi}cf(t,h)\\ &+\overline{M}e^{\omega b_\varphi}cb_\varphi hc_3+\overline{M}e^{\omega b_\varphi}cg(t,h)+\overline{M}e^{\omega h}e^{\omega b_\varphi}chc_2,\end{aligned}$$

and

$$J_2 \quad := \overline{M}e^{\omega b_\varphi}cb_\varphi L_1(r_1)H+\overline{M}e^{\omega b_\varphi}cL_0(r_0).$$

Then

$$\begin{aligned}\|x_{t+h}-x_t\|_{\mathcal{B}} &\le \sup_{0\le\sigma\le t}\|x_{\sigma+h}-x_\sigma\|_{\mathcal{B}}\\ &\le J_1+J_2\int_0^t\sup_{0\le\tau\le s}\|x_{\tau+h}-x_\tau\|_{\mathcal{B}}\,ds.\end{aligned}$$

By Gronwall's lemma, it follows that

$$\|x_{t+h}-x_t\|_{\mathcal{B}}\le J_1\exp(J_2).$$

The bounded convergence theorem by Lebesgue implies that

$$\lim_{h\to0}f(b_\varphi,h)=0 \text{ and } \lim_{h\to0}g(b_\varphi,h)=0.$$

Therefore,

$$\lim_{h\to0}\|x_{t+h}(.,\varphi)-x_t(.,\varphi)\|_{\mathcal{B}}=0$$

uniformly for $t\in\left[0,b_\varphi\right)$. Consequently, Axiom $(\mathbf{A}-\mathbf{ii})$ implies that

$$\lim_{h\to0}|x(t+h,\varphi)-x(t,\varphi)|=0.$$

Using the same reasoning, one can show a similar result for $h<0$. We deduce that x is uniformly continuous on $\left[0,b_\varphi\right)$, and $\lim_{t\to b_\varphi}|x(t,\varphi)|$ exists. This implies that the solution $x(.,\varphi)$ can be extended to the right in b_φ, which contradicts the maximality of $\left(-\infty,b_\varphi\right)$. This completes the proof of Theorem 3.1. □

4 Application

Let $E := C([0,\pi];\mathbb{R})$ be the space of continuous functions from $[0,\pi]$ to $\mathbb{R}$ endowed with the uniform norm topology and define the operator $A : D(A) \subset E \to E$ by

$$\begin{array}{l} D(A) = \left\{ y \in C^2([0,\pi],\mathbb{R}) : y(0) = y(\pi) = 0 \right\}, \\ Ay = y''. \end{array}$$

Lemma 4.1. *[11].* $(0,+\infty) \subset \rho(A)$, $\left\| (\lambda I - A)^{-1} \right\| \le \dfrac{1}{\lambda}$ *for* $\lambda > 0$, *and*

$$\overline{D(A)} = \{ y \in E : y(0) = y(\pi) = 0 \} \neq E.$$

Let $\gamma > 0$. We choose the phase space

$$\mathcal{B} := \left\{ \phi \in C((-\infty,0],\, E) \text{ such that} \lim_{\theta \to -\infty} e^{\gamma\theta}\phi(\theta) \text{ exists in } E \right\}.$$

Lemma 4.2. *([16] and [20]).* $\mathcal{B}$ *with the norm* $\|\phi\|_{\mathcal{B}} = \sup_{\theta \le 0} e^{\gamma\theta} |\phi(\theta)|$ *for* $\phi \in \mathcal{B}$, *satisfies the axioms* $(\mathbf{A})$, $(\mathbf{A1})$, $(\mathbf{B})$, $(\mathbf{C1})$ *and* $(\mathbf{D1})$ *with* $K(0) = 1$.

Consider the following equation

$$\left\{ \begin{array}{l} \dfrac{\partial}{\partial t}\mathcal{D}x_t = A\mathcal{D}x_t + \int_0^t k(t,s,x(s))ds + F(t,x_t), \quad t \ge 0, \\ x_0 = \varphi, \end{array} \right. \tag{4.1}$$

with

$$\left\{ \begin{array}{l} x(t)(\xi) = u(t,\xi), \quad t \ge 0,\ \xi \in [0,\pi], \\ \varphi(\theta)(\xi) = u_0(\theta,\xi), \quad \theta \le 0,\ \xi \in [0,\pi], \end{array} \right.$$

and for all $\xi \in [0,\pi]$ and $\phi \in \mathcal{B}$

$$\left\{ \begin{array}{l} k(t,s,x(s))(\xi) = g_2(t-s, x(s)(\xi)),\ t \ge s \ge 0, \\ F(t,\phi)(\xi) = c\int_{-\infty}^0 g_1(\theta)\phi(\theta)(\xi)d\theta + a(t)h_1(\phi(-r))(\xi) + h_2(t,\xi),\ t \ge 0, \end{array} \right.$$

where r and c are positive constants, g_1 is a positive integrable function, $g_2 : [0,\infty) \times \mathbb{R} \to \mathbb{R}$, $a : [0,\infty) \to \mathbb{R}$, $h_1 : \mathbb{R} \to \mathbb{R}$, $h_2 : [0,\infty) \times [0,\pi] \to \mathbb{R}$ and $u_0 : (-\infty,0] \times [0,\pi] \to \mathbb{R}$ are continuous functions and $\mathcal{D} : \mathcal{B} \to E$ is a bounded linear operator defined by $\mathcal{D}\varphi = \varphi(0) - \mathcal{D}_0\varphi$ for any $\varphi \in \mathcal{B}$, $\mathcal{D}_0$ is a bounded linear operator from $\mathcal{B}$ into E.
The part A_0 of A in $\overline{D(A)}$ is defined by

$$\begin{array}{l} D(A_0) = \left\{ y \in C^2([0,\pi],\mathbb{R}) : y(0) = y(\pi) = y''(0) = y''(\pi) = 0 \right\}, \\ A_0 y = y'', \end{array}$$

with $\overline{D(A_0)} = \overline{D(A)}$.
Eq. (4.1) is the abstract form of the following model

$$\begin{cases} \dfrac{\partial}{\partial t}\left[u(t,\xi) - (\mathcal{D}_0 u_t)(\xi)\right] = \dfrac{\partial^2}{\partial \xi^2}\left[u(t,\xi) - (\mathcal{D}_0 u_t)(\xi)\right] + h_2(t,\xi) \\ \qquad\qquad +c\int_{-\infty}^{0} g_1(\theta) u(t+\theta,\xi) d\theta + \int_0^t g_2(t-s, u(s,\xi))\, ds \\ \qquad\qquad +a(t) h_1(u(t-r,\xi)), \quad t \geq 0,\ 0 \leq \xi \leq \pi, \\ u(t,0) - (\mathcal{D}_0 u_t)(0) = u(t,\pi) - (\mathcal{D}_0 u_t)(\pi) = 0,\ t \geq 0, \\ u(\theta,\xi) = u_0(\theta,\xi), \quad -\infty < \theta \leq 0, \quad 0 \leq \xi \leq \pi. \end{cases} \tag{4.2}$$

Lemma (4.1) implies that assumption (**H1**) is satisfied.
We assume that
(**i**) h_1 is locally Lipschitz continuous,
(**ii**) $g_1(.)e^{-\gamma \cdot}$ is integrable on $(-\infty,0]$,
(**iii**) $\lim_{\theta \to -\infty} \left(e^{\gamma\theta} \sup_{0\leq\xi\leq\pi} |u_0(\theta,\xi)| \right)$ exists.
(**iv**) a is bounded on $[0,+\infty)$,
(**v**) $g_2 : [0,\infty) \times I\!R \to I\!R$ is locally Lipschitz continuous with respect to the second argument.

We have, for every $\phi_1, \phi_2 \in \mathcal{B}$,

$$\begin{aligned} &\sup_{0\leq\xi\leq\pi} \int_{-\infty}^{0} g_1(\theta) \left|\phi_1(\theta)(\xi) - \phi_2(\theta)(\xi)\right| d\theta \\ &= \sup_{0\leq\xi\leq\pi} \int_{-\infty}^{0} e^{-\gamma\theta} g_1(\theta) \left(e^{\gamma\theta} \left|\phi_1(\theta)(\xi) - \phi_2(\theta)(\xi)\right|\right) d\theta, \\ &\leq \left(\int_{-\infty}^{0} e^{-\gamma\theta} g_1(\theta) d\theta\right) \|\phi_1 - \phi_2\|_{\mathcal{B}}. \end{aligned}$$

Let $\alpha_0 > 0$ and $\phi_1, \phi_2 \in \mathcal{B}$ such that $\|\phi_1\|_{\mathcal{B}}, \|\phi_2\|_{\mathcal{B}} \leq \alpha_0$.
Then, $|\phi_1(-r)(\xi)|, |\phi_2(-r)(\xi)| \leq \alpha_0 e^{\gamma r}$ for every $\xi \in [0,\pi]$. Thus, from Assumption (**i**), there exists a positive constant $b_0(\alpha_0)$ which depends only on α_0 such that

$$\begin{aligned} &\sup_{s\geq 0}|a(s)| \sup_{0\leq\xi\leq\pi} \left|h_1(\phi_1(-r)(\xi)) - h_1(\phi_2(-r)(\xi))\right| \\ &\quad \leq |a|_\infty b_0(\alpha_0) \sup_{0\leq\xi\leq\pi} \left|\phi_1(-r)(\xi) - \phi_2(-r)(\xi)\right|, \\ &\quad \leq |a|_\infty b_0(\alpha_0) e^{\gamma r} \|\phi_1 - \phi_2\|_{\mathcal{B}}. \end{aligned}$$

We conclude that assumptions (**i**), (**ii**) and (**iv**) imply that F satisfies the hypothesis (**H3**). Moreover, assumption (**v**) implies that for $\alpha_1 > 0$, there exists a positive constant $b_1(\alpha_1)$ such that

$$\begin{aligned} |k(t,s,x_1) - k(t,s,x_2)| &= \sup_{0\leq\xi\leq\pi} \left|g_2(t-s, x_1(\xi)) - g_2(t-s, x_2(\xi))\right| \\ &\leq \sup_{0\leq\xi\leq\pi} b_1(\alpha_1) |x_1(\xi) - x_2(\xi)| \\ &\leq b_1(\alpha_1) |x_1 - x_2|, \end{aligned}$$

for $t \geq 0$, $t \geq s \geq 0$ and $x_1, x_2 \in E$, with $|x_1|, |x_2| \leq \alpha_1$. It follows that condition (**H4**) is satisfied. From assumption (**iii**) we have $\varphi \in \mathcal{B}$.
Finally, if operator $\mathcal{D}$ satisfies assumption (**H2**), with $D\varphi \in \overline{D(A)}$, then Theorem 3.1 ensures the existence of a maximal interval of existence $(-\infty, b_{u_0})$ and a unique integral solution $u(t,\xi)$ of Eq. (4.1) on $(-\infty, b_{u_0}) \times [0,\pi]$.

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