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Reiterated Homogenization of Nonlinear Pseudo Monotone Degenerate Parabolic Operators

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Abstract

Reiterated deterministic homogenization problem for nonlinear pseudo monotone parabolic type operators is considered beyond the usual periodic setting. We present a new approach based on the generalized Besicovitch type spaces, which allows to consider general assumptions on the coefficients of the operators under consideration. In particular we solve the weakly almost periodic homogenization problem and many new other problems such as the homogenization in the Fourier-Stieltjes algebra. Our approach falls within the scope of multiscale convergence method.

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1 Introduction

Let Ω be a bounded open set in $\mathbb{R}^N$ with Lipschitz boundary and $Q = \Omega \times (0,T)$, where $T > 0$ is a fixed real number. Let $2 \leq p < \infty$ and $p' = p/(p-1)$. We consider the initial-boundary value problem

$$\begin{gathered}
\frac{\partial u_\varepsilon}{\partial t} - \operatorname{div} a\left(x, t, \frac{x}{\varepsilon_1}, \frac{x}{\varepsilon_2}, \frac{t}{\varepsilon}, u_\varepsilon, Du_\varepsilon\right) + a_0\left(x, t, \frac{x}{\varepsilon_1}, \frac{x}{\varepsilon_2}, \frac{t}{\varepsilon}, u_\varepsilon, Du_\varepsilon\right) = f \text{ in } Q \\
u_\varepsilon = 0 \text{ on } \partial\Omega \times (0,T) \\
u_\varepsilon(x,0) = 0 \text{ in } \Omega,
\end{gathered} \tag{1.1}$$

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where $f \in L^{p'}(0,T;W^{-1,p'}(\Omega))$, D and div denote respectively the gradient and divergence operators in Ω, ε_1 and ε_2 are two well-separated functions of ε tending to zero with ε (that is $0 < \varepsilon_1$, ε_2, $\varepsilon_2/\varepsilon_1 \to 0$ as $\varepsilon \to 0$), and the functions $(x,t,y,z,\tau,\mu,\lambda) \mapsto a(x,t,y,z,\tau,\mu,\lambda)$ and $(x,t,y,z,\tau,\mu,\lambda) \mapsto a_0(x,t,y,z,\tau,\mu,\lambda)$ from $\overline{Q} \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N$ to $\mathbb{R}^N$ and $\mathbb{R}$, respectively, satisfy the following assumptions:

For any arbitrary $(\mu,\lambda) \in \mathbb{R} \times \mathbb{R}^N$ and for any $(x,t) \in \overline{Q}$, the functions $a_0(x,t,\cdot,\cdot,\cdot,\mu,\lambda)$ and $a(x,t,\cdot,\cdot,\cdot,\mu,\lambda)$ are measurable; (1.2)

$a(x,t,y,z,\tau,\mu,0) = 0$ a.e. in $(y,z,\tau) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}$, for all $(x,t) \in \overline{Q}$ and all $\mu \in \mathbb{R}$; (1.3)

There are three constants $c_0, c_1, c_2 > 0$ and a continuity modulus ν (i.e., a nondecreasing continuous function on $[0,+\infty)$ such that $\nu(0) = 0, \nu(r) > 0$ if $r > 0$, and $\nu(r) = 1$ if $r > 1$) such that a.e. in $(y,z,\tau) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}$,
(i) $(a(x,t,y,z,\tau,\mu,\lambda) - a(x,t,y,z,\tau,\mu,\lambda')) \cdot (\lambda - \lambda') \geq c_0 |\lambda - \lambda'|^p$
(ii) $a_0(x,t,y,z,\tau,\mu,\lambda)\mu \geq 0$
(iii) $|a_0(x,t,y,z,\tau,\mu,\lambda)| + |a(x,t,y,z,\tau,\mu,\lambda)| \leq c_1(1 + |\mu|^{p-1} + |\lambda|^{p-1})$
(iv) $|a_0(x,t,y,z,\tau,\mu,\lambda) - a_0(x',t',y,z,\tau,\mu',\lambda')| +$
$+ |a(x,t,y,z,\tau,\mu,\lambda) - a(x',t',y,z,\tau,\mu',\lambda')|$
$\leq \nu(|x-x'| + |t-t'| + |\mu-\mu'|)(1 + |\mu|^{p-1} + |\lambda|^{p-1} + |\mu'|^{p-1}$
$+ |\lambda'|^{p-1}) + c_2(1 + |\mu| + |\lambda| + |\mu'| + |\lambda'|)^{p-2} |\lambda - \lambda'|$
for all $(x,t), (x',t') \in \overline{Q}$ and all $(\mu,\lambda), (\mu',\lambda') \in \mathbb{R} \times \mathbb{R}^N$, where the dot denotes the usual Euclidean inner product in $\mathbb{R}^N$ and $|\cdot|$ the associated norm. (1.4)

Remark 1.1. *The positivity constraint (ii) in (1.4) is stated in order to establish the a priori estimates. It plays no role in the process of the existence of solutions to (1.1).*

Assuming the diffusion term in (1.1) is rigorously defined, problem (1.1) admits (at least) a solution $u_\varepsilon \in L^p(0,T;W_0^{1,p}(\Omega))$ for each fixed $\varepsilon > 0$ as it will be seen in Section 2. It is classically known that under some additional conditions on the functions a and a_0, the above problem possesses a unique solution so that we will assume the uniqueness of u_ε for each fixed $\varepsilon > 0$. This therefore furnishes a sequence $(u_\varepsilon)_{\varepsilon>0}$ and our main objective in this paper is the investigation of the asymptotic behavior, as $\varepsilon \to 0$, of u_ε under a suitable realistic assumption on the behavior of the function $(y,z,\tau) \mapsto a_i(x,t,y,z,\tau,\mu,\lambda)$ (for fixed x,t,μ,λ; where a_i for $1 \leq i \leq N$ denotes the ith component of the function a).

Such a problem is referred to as a reiterated homogenization problem. Problems of this type were first introduced by Bruggeman [9]. The real first issue in this area is due to Bensoussan et al. [3] who proved the so-called *iterated* homogenization result for linear monotone elliptic operators. For the case of nonlinear elliptic equations, we refer to [16, 28, 30, 31, 33] for the monotone operators in the periodic setting, to [32] for the monotone operators in the general deterministic framework, and to [10] for the monotone degenerated operators in the periodic setting. Concerning parabolic operators, the reiterated homogenization of nonlinear parabolic operators in the periodic setting has been studied in [25]. In [20], Flodén and Olsson have studied the periodic homogenization of nonlinear parabolic monotone equations. It is worth noting that beyond the periodic setting (particularly in the Besicovitch-almost periodic framework), a thorough analysis of the non-reiterated homogenization problem for the operators considered here is carried out in [21], by means of the G-convergence method. It is also important to know that in the stochastic framework, the homogenization problem for the operators considered in [21] has already been treated by Pankov [40] using once again the G-convergence method for parabolic operators; see also [22]. As seen here above, the problem addressed is the general deterministic (but non stochastic) homogenization problem for variational parabolic equations. However, there exists an enormous bibliography related to the non periodic homogenization of non-variational uniformly elliptic or parabolic equations. We mention here the papers [11, 29, 42, 43] related to the stochastic homogenization of fully nonlinear parabolic equations.

In what considering here, we study the reiterated homogenization problem for (1.1) in a general setting characterized by an assumption on $a_i(x,t,\cdot,\cdot,\cdot,\mu,\lambda)$ (for fixed x,t,μ,λ) covering a great set of concrete behaviors such as the periodicity, the almost periodicity, the weakly almost periodicity, and others. As opposed to what was usually done in the deterministic homogenization theory (see for instance the papers [32, 36, 37, 38, 45]), we present here a new approach based on the generalized Besicovitch type spaces (see Section 3), which widely opens the scope of application of our main homogenization result, Theorem 4.5, which reads as

Theorem 1.2. *Let* $2 \leq p < \infty$. *Suppose* (4.1) *holds and further* $A = A_y \odot A_z \odot A_\tau$ *where the algebras* A_y *and* A_z *are ergodic. For each fixed real number* $\varepsilon > 0$, *let* u_ε *be the (unique) solution of* (1.1). *There exists a subsequence of* ε, *still denoted by* ε, *such that, as* $\varepsilon \to 0$,

$$u_\varepsilon \to u_0 \text{ in } L^p(0,T;W_0^{1,p}(\Omega))\text{-weak}$$

$$\frac{\partial u_\varepsilon}{\partial t} \to \frac{\partial u_0}{\partial t} \text{ in } L^{p'}(0,T;W^{-1,p'}(\Omega))\text{-weak}$$

$$\frac{\partial u_\varepsilon}{\partial x_j} \to \frac{\partial u_0}{\partial x_j} + \frac{\overline{\partial} u_1}{\partial y_j} + \frac{\overline{\partial} u_2}{\partial z_j} \text{ in } L^p(Q)\text{-weak } R\Sigma \ (1 \leq j \leq N),$$

where $\mathbf{u} = (u_0, u_1, u_2) \in \mathbb{F}_0^{1,p}$ *solves the variational problem*

$$\begin{aligned} &\textstyle\int_0^T \langle u_0'(t), v_0(t)\rangle\, dt + \iint_{Q\times\Delta(A)} b(\cdot,\cdot,u_0,\mathbb{D}\mathbf{u}) \cdot \mathbb{D}\mathbf{v} dxdtd\beta + \\ &+ \textstyle\iint_{Q\times\Delta(A)} b_0(\cdot,\cdot,u_0,\mathbb{D}\mathbf{u}) v_0 dxdtd\beta = \int_0^T \langle f(t), v_0(t)\rangle\, dt \\ &\text{for all } \mathbf{v} = (v_0,v_1,v_2) \in \mathbb{F}_0^{1,p}, \end{aligned} \tag{1.5}$$

with $\mathbb{D}\mathbf{w} = Dw_0 + \partial_s \widehat{w}_1 + \partial_r \widehat{w}_2$ *for* $w = (w_0,w_1,w_2) \in \mathbb{F}_0^{1,p}$. *Moreover* u_1 *and* u_2 *are unique and any weak* $R\Sigma$*-limit point in* V^p *of* $(u_\varepsilon)_{\varepsilon>0}$ *is a solution to problem* (1.5).

One may wonder why we chose here functions a and a_0 not depending on several microscopic time variables as is the case for the microscopic spatial variables. The reason is quite simple: as one can see in the paper [46] of the author, multiplying the number of temporal scales does not change the form of the homogenized problem stated here above since it does not influence the number microscopic solutions (represented in (1.5) by u_1

and u_2). In the same register we refer the reader to a more recently work [41] in which the periodic homogenization problem for parabolic operators is considered with m microscopic temporal scales, m being an arbitrarily fixed positive integer. So the problem (1.5) contains all the required information as far as the reiterated homogenization is concerned. Our homogenization approach, the *$R\Sigma$-convergence method*, proceeds with the juxtaposition of the multiscale convergence method [1, 31] and the *algebras with mean value* [48]. This is the so-called deterministic homogenization theory which includes the periodic homogenization theory as a particular case. It is important to note that restricting to the periodic functions, by using the periodic unfolding technique (see [15]), our convergence mode is realized to be equivalent to an ordinary weak convergence for sequences of functions which depend both on the macroscopic and the microscopic variables. sequences of functions which depends both on the global and the local variable.

The paper is organized as follows. Section 2 presents some trace results. Section 3 deals with the concept of algebras with mean value and its connection to the generalized Besicovitch spaces. We also state there some compactness results. Finally in Section 4, we prove the main homogenization result of the paper and we give some applications of the said result.

Unless otherwise specified, the vector spaces throughout are assumed to be real vector spaces, and the scalar functions are assumed to take real values. If X and F denote a locally compact space and a Banach space, respectively, then we write $\mathcal{C}(X;F)$ and $\mathcal{B}(X;F)$ for continuous mappings of X into F and bounded uniformly continuous mappings of X into F, respectively. We shall always assume that $\mathcal{B}(X;F)$ is equipped with the supremum norm $\|u\|_\infty = \sup_{x\in X}\|u(x)\|$ ($\|\cdot\|$ denotes the norm in F). For shortness we will write $\mathcal{C}(X) = \mathcal{C}(X;\mathbb{R})$ and $\mathcal{B}(X) = \mathcal{B}(X;\mathbb{R})$. Likewise the usual space $L^p(X;F)$ and $L^p_{\mathrm{loc}}(X;F)$ (X provided with a positive Radon measure) will be denoted by $L^p(X)$ and $L^p_{\mathrm{loc}}(X)$, respectively, in the case when $F=\mathbb{R}$. Finally, it will always be assumed that the numerical spaces $\mathbb{R}^m$ ($m\geq 1$) and their open sets are each equipped with Lebesgue measure $dy = dy_1...dy_m$.

2 Some trace results

In order that equation (1.1) makes sense we need to precise the meaning of the functions $(x,t)\mapsto a_i(x,t,x/\varepsilon_1,x/\varepsilon_2,t/\varepsilon,v(x,t),Dv(x,t))((x,t)\in\overline{Q})$ for $v\in L^p(0,T;W^{1,p}(\Omega))$. Let $(v_0,\mathbf{v})\in\mathcal{C}(\overline{Q})\times\mathcal{C}(\overline{Q})^N = \mathcal{C}(\overline{Q})^{N+1}$. Then using (1.2) and part (iv) of (1.4), we easily see that the function $((x,t),(x',t'),(y,z,\tau))\mapsto a_i(x,t,y,z,\tau,v_0(x',t'),\mathbf{v}(x',t'))$, from $\overline{Q}\times\overline{Q}\times\mathbb{R}^N_y\times\mathbb{R}^N_z\times\mathbb{R}_\tau$ to $\mathbb{R}$, belongs to $\mathcal{C}(\overline{Q}\times\overline{Q};L^\infty(\mathbb{R}^{N+1}_{y,\tau}\times\mathbb{R}^N_z))$, so that the trace function $(x,t,y,z,\tau)\mapsto a_i(x,t,y,z,\tau,v_0(x,t),\mathbf{v}(x,t))$, of $\overline{Q}\times\mathbb{R}^N_y\times\mathbb{R}^N_z\times\mathbb{R}_\tau$ into $\mathbb{R}$, is naturally defined by

$$a_i(x,t,y,z,\tau,v_0(x,t),\mathbf{v}(x,t)) = a_i(x,t,y,z,\tau,v_0(x',t'),\mathbf{v}(x',t'))\big|_{(x',t')=(x,t)},$$

as element of $\mathcal{C}(\overline{Q};L^\infty(\mathbb{R}^{N+1}_{y,\tau}\times\mathbb{R}^N_z))$. Whence, in view of [46, Proposition 3.1] we can define, for any $\varepsilon>0$, the function $(x,t)\mapsto a_i(x,t,x/\varepsilon_1,x/\varepsilon_2,t/\varepsilon,v_0(x,t),\mathbf{v}(x,t))$ on Q, as an element of $L^\infty(Q)$, denoted by $a_i^\varepsilon(-,v_0,\mathbf{v})$. Let

$$a^\varepsilon(-,v_0,\mathbf{v}) = (a_i^\varepsilon(-,v_0,\mathbf{v}))_{1\leq i\leq N}.$$

We have the following

Proposition 2.1. *Let $2 \leq p < \infty$ and $p' = p/(p-1)$. The transformation $(v_0, \mathbf{v}) \mapsto a_i^\varepsilon(-, v_0, \mathbf{v})$, of $\mathcal{C}(\overline{Q})^{N+1}$ into $L^\infty(Q)$, extends by continuity to a continuous mapping still denoted by $(v_0, \mathbf{v}) \mapsto a_i^\varepsilon(-, v_0, \mathbf{v})$, of $L^p(Q)^{N+1}$ into $L^{p'}(Q)$ verifying*

$$a^\varepsilon(-, v_0, 0) = 0 \text{ a.e. in } Q \tag{2.1}$$

$$(a^\varepsilon(-, v_0, \mathbf{v}) - a^\varepsilon(-, v_0, \mathbf{w})) \cdot (\mathbf{v} - \mathbf{w}) \geq c_0 |\mathbf{v} - \mathbf{w}|^p \text{ a.e. in } Q \tag{2.2}$$

$$a_0^\varepsilon(-, v_0, \mathbf{v}) v_0 \geq 0 \text{ a.e. in } Q \tag{2.3}$$

$$\|a_i^\varepsilon(-, v_0, \mathbf{v})\|_{L^{p'}(Q)} \leq c_1' \left(1 + \|v_0\|_{L^p(Q)}^{p-1} + \|\mathbf{v}\|_{L^p(Q)^N}^{p-1}\right) \tag{2.4}$$

$$\begin{aligned} &\|a_i^\varepsilon(-, v_0, \mathbf{v}) - a_i^\varepsilon(-, v_0, \mathbf{w})\|_{L^{p'}(Q)} \\ &\quad \leq c_2 \|1 + |v_0| + |\mathbf{v}| + |\mathbf{w}|\|_{L^p(Q)}^{p-2} \|\mathbf{v} - \mathbf{w}\|_{L^p(Q)^N} \end{aligned} \tag{2.5}$$

$$\begin{aligned} &|a_i^\varepsilon(-, v_0, \mathbf{v}) - a_i^\varepsilon(-, w_0, \mathbf{v})| \\ &\quad \leq \nu(|v_0 - w_0|) \left(1 + |v_0|^{p-1} + |w_0|^{p-1} + |\mathbf{v}|^{p-1}\right) \text{ a.e. in } Q \end{aligned} \tag{2.6}$$

for all $v_0, w_0 \in L^p(Q)$ and all $\mathbf{v}, \mathbf{w} \in L^p(Q)^N$, where the positive constant c_1' depends only on c_1 and on Q.

Proof. For all $v_0, w_0 \in \mathcal{C}(\overline{Q})$ and all $\mathbf{v} \in \mathcal{C}(\overline{Q})^N$, we have

$$|a_i^\varepsilon(-, v_0, \mathbf{v}) - a_i^\varepsilon(-, w_0, \mathbf{v})| \leq \nu(|v_0 - w_0|)(1 + |v_0|^{p-1} + |w_0|^{p-1} + |\mathbf{v}|^{p-1}) \tag{2.7}$$

a.e. in Q. So, let $\mathbf{v} \in \mathcal{C}(\overline{Q})^N$ be freely fixed. Let $\varepsilon > 0$ be fixed. Define

$$g_\mathbf{v}^\varepsilon : \mathcal{C}(\overline{Q}) \to L^\infty(Q) \text{ by } g_\mathbf{v}^\varepsilon(v_0) = a_i^\varepsilon(-, v_0, \mathbf{v}) \ \ (v_0 \in \mathcal{C}(\overline{Q})).$$

Then $|g_\mathbf{v}^\varepsilon(v_0) - g_\mathbf{v}^\varepsilon(w_0)| \leq \nu(|v_0 - w_0|)\left(1 + |v_0|^{p-1} + |w_0|^{p-1} + |\mathbf{v}|^{p-1}\right)$ a.e. in Q. Let us show that $g_\mathbf{v}^\varepsilon$ is continuous on $\mathcal{C}(\overline{Q})$ endowed with the relative topology on $L^p(Q)$. For this purpose let $v_0 \in \mathcal{C}(\overline{Q})$ and let $v_n \in \mathcal{C}(\overline{Q})$ be such that $v_n \to v_0$ in $L^p(Q)$ as $n \to \infty$. Let's show that $g_\mathbf{v}^\varepsilon(v_n) \to g_\mathbf{v}^\varepsilon(v_0)$ in $L^{p'}(Q)$ as $n \to \infty$. We have by (2.7),

$$\|g_\mathbf{v}^\varepsilon(v_n) - g_\mathbf{v}^\varepsilon(v_0)\|_{L^{p'}(Q)}^{p'} \leq c \int_Q \nu(|v_n - v_0|)^{p'} (1 + |v_n|^p + |v_0|^p + |\mathbf{v}|^p)\, dxdt$$

where $c = 4^{p'-1}$. Set $w_n = v_n - v_0$ and $F_n = 1 + |v_n|^p + |v_0|^p + |\mathbf{v}|^p$. Then $F_n \to 1 + 2|v_0|^p + |\mathbf{v}|^p$ in $L^1(Q)$ as $n \to \infty$ (recall that $v_n \to v_0$ in $L^p(Q)$), and so $F_n \to 1 + 2|v_0|^p + |\mathbf{v}|^p$ in $L^1(Q)$-weak. On the other hand, since $v_n \to v_0$ in $L^p(Q)$, we know by [8, Thm IV-9, p.58] that there exist a subsequence of w_n (still denoted by w_n) and a function $u \in L^p(Q)$ such that

$$\begin{aligned} &w_n \to 0 \text{ a.e. in } Q \text{ (hence } |w_n| \to 0 \text{ a.e. in } Q) \text{ as } n \to \infty \\ &|w_n| \leq u \text{ a.e. in } Q \text{ for all } n. \end{aligned}$$

ν being continuous and in particular at 0 with $\nu(0) = 0$, and moreover being increasing, we deduce that $\nu(|w_n|)$ is measurable for all n (each w_n is measurable and hence $|w_n|$ too) and

(i) $\nu(|w_n|) \to 0$ a.e. in Q as $n \to \infty$

(ii) $\nu(|w_n|)^{p'} \leq \nu(u)^{p'}$ a.e. in Q for all n.

ν being a continuity modulus, the function $\nu(u) \equiv \nu \circ u$ is measurable and essentially bounded on Q, i.e., $\nu(u) \in L^\infty(Q)$. Thus, the sequence $\nu(|w_n|)^{p'}$ is equibounded (see (ii) above) and converges almost pointwise in Q to 0. Therefore, due to Egorov's theorem, one obtains

$$\int_Q \nu(|w_n|)^{p'} F_n dxdt \to 0 \text{ as } n \to \infty.$$

But the above limit being independent of the subsequence w_n, still holds for the whole sequence w_n. We deduce from this that

$$\|g^\varepsilon_{\mathbf{v}}(v_n) - g^\varepsilon_{\mathbf{v}}(v_0)\|_{L^{p'}(Q)} \to 0 \text{ as } n \to \infty.$$

Now, let $v_0 \in L^p(Q)$. We define the function $G^\varepsilon_{\mathbf{v}}$ by setting

$$G^\varepsilon_{\mathbf{v}}(v_0) = \lim_{n\to\infty} g^\varepsilon_{\mathbf{v}}(v_n) \text{ in } L^{p'}(Q)$$

where (v_n) is a sequence in $\mathcal{C}(\overline{Q})$ such that $v_n \to v_0$ in $L^p(Q)$. It is worth noting that, thanks to what has been done above, this limit is independent of the chosen sequence (v_n), and so, $G^\varepsilon_{\mathbf{v}}$ is well defined. Moreover,

$$\begin{aligned} &G^\varepsilon_{\mathbf{v}}(v_0) = g^\varepsilon_{\mathbf{v}}(v_0) \text{ for all } v_0 \in \mathcal{C}(\overline{Q}), \text{ and} \\ &|G^\varepsilon_{\mathbf{v}}(v_0) - G^\varepsilon_{\mathbf{v}}(w_0)| \leq \nu(|v_0 - w_0|)\left(1 + |v_0|^{p-1} + |w_0|^{p-1} + |\mathbf{v}|^{p-1}\right) \\ &\text{a.e. in } Q \text{ and for all } v_0, w_0 \in L^p(Q). \end{aligned} \tag{2.8}$$

In fact, for (2.8), let $v_0, w_0 \in L^p(Q)$, and let $v_n, w_n \in \mathcal{C}(\overline{Q})$ be such that $v_n \to v_0$ in $L^p(Q)$ and $w_n \to w_0$ in $L^p(Q)$ as $n \to \infty$. We easily get, as $n \to \infty$,

$$|G^\varepsilon_{\mathbf{v}}(v_n) - G^\varepsilon_{\mathbf{v}}(w_n)| \to |G^\varepsilon_{\mathbf{v}}(v_0) - G^\varepsilon_{\mathbf{v}}(w_0)| \text{ in } L^{p'}(Q) \text{ (and hence in } L^{p'}(Q)\text{-weak).}$$

Thus, for all $\varphi \in L^p(Q)$,

$$\int_Q |G^\varepsilon_{\mathbf{v}}(v_n) - G^\varepsilon_{\mathbf{v}}(w_n)|\,\varphi dxdt \to \int_Q |G^\varepsilon_{\mathbf{v}}(v_0) - G^\varepsilon_{\mathbf{v}}(w_0)|\,\varphi dxdt \text{ as } n \to \infty.$$

On the other hand, for $\varphi \in L^p(Q)$ with $\varphi \geq 0$ a.e. in Q, we have

$$\int_Q |G^\varepsilon_{\mathbf{v}}(v_n) - G^\varepsilon_{\mathbf{v}}(w_n)|\,\varphi dxdt \leq \int_Q \nu(|v_n - w_n|)(1 + |v_n|^{p-1} + |w_n|^{p-1} + |\mathbf{v}|^{p-1})\varphi dxdt.$$

Following the same lines of proceeding as we have done it to compute $\lim \int_Q \nu(|w_n|)^{p'} F_n dxdt$ and moreover by using the fact that $(1 + |v_n|^{p-1} + |w_n|^{p-1} + |\mathbf{v}|^{p-1})\varphi$ converges weakly to $(1 + |v_0|^{p-1} + |w_0|^{p-1} + |\mathbf{v}|^{p-1})\varphi$ in $L^1(Q)$, we arrive at

$$\begin{aligned} &\textstyle\int_Q \nu(|v_n - w_n|)\left(1 + |v_n|^{p-1} + |w_n|^{p-1} + |\mathbf{v}|^{p-1}\right)\varphi dxdt \to \\ &\quad \to \textstyle\int_Q \nu(|v_0 - w_0|)\left(1 + |v_0|^{p-1} + |w_0|^{p-1} + |\mathbf{v}|^{p-1}\right)\varphi dxdt. \end{aligned}$$

Thus

$$\int_Q |G^\varepsilon_{\mathbf{v}}(v_0) - G^\varepsilon_{\mathbf{v}}(w_0)|\, \varphi dxdt \le \int_Q \nu(|v_0 - w_0|) \left(1 + |v_0|^{p-1} + |w_0|^{p-1} + |\mathbf{v}|^{p-1}\right) \varphi dxdt$$

for all $\varphi \in L^p(Q)$, $\varphi \ge 0$ a.e. in Q. Whence (2.8) follows. We deduce from (2.8) that $G^\varepsilon_{\mathbf{v}}$ is (uniformly) continuous on $L^p(Q)$. Now, fix freely v_0 in $L^p(Q)$ and define

$$F^\varepsilon_{v_0} : \mathcal{C}(\overline{Q})^N \to L^{p'}(Q) \text{ by } F^\varepsilon_{v_0}(\mathbf{v}) = G^\varepsilon_{\mathbf{v}}(v_0)\ (\mathbf{v} \in \mathcal{C}(\overline{Q})^N).$$

For $\mathbf{v}, \mathbf{w} \in \mathcal{C}(\overline{Q})^N$, we have

$$\begin{aligned} \left\| F^\varepsilon_{v_0}(\mathbf{v}) - F^\varepsilon_{v_0}(\mathbf{w}) \right\|_{L^{p'}(Q)} &= \| G^\varepsilon_{\mathbf{v}}(v_0) - G^\varepsilon_{\mathbf{w}}(v_0) \|_{L^{p'}(Q)} \\ &= \lim_{n\to\infty} \| G^\varepsilon_{\mathbf{v}}(v_n) - G^\varepsilon_{\mathbf{w}}(v_n) \|_{L^{p'}(Q)} \\ &= \lim_{n\to\infty} \| g^\varepsilon_{\mathbf{v}}(v_n) - g^\varepsilon_{\mathbf{w}}(v_n) \|_{L^{p'}(Q)}, \end{aligned}$$

where $v_n \in \mathcal{C}(\overline{Q})$ is a sequence such that $v_n \to v_0$ in $L^p(Q)$. Since $v_n \to v_0$ in $L^p(Q)$, one has, as $n \to \infty$,

$$\| 1 + |v_n| + |\mathbf{v}| + |\mathbf{w}| \|^{p-2}_{L^p(Q)} \to \| 1 + |v_0| + |\mathbf{v}| + |\mathbf{w}| \|^{p-2}_{L^p(Q)}.$$

Therefore, using the inequality

$$\| g^\varepsilon_{\mathbf{v}}(v_n) - g^\varepsilon_{\mathbf{w}}(v_n) \|_{L^{p'}(Q)} \le c_2 \| 1 + |v_n| + |\mathbf{v}| + |\mathbf{w}| \|^{p-2}_{L^p(Q)} \| \mathbf{v} - \mathbf{w} \|_{L^p(Q)^N},$$

we arrive at

$$\left\| F^\varepsilon_{v_0}(\mathbf{v}) - F^\varepsilon_{v_0}(\mathbf{w}) \right\|_{L^{p'}(Q)} \le c_2 \| 1 + |v_0| + |\mathbf{v}| + |\mathbf{w}| \|^{p-2}_{L^p(Q)} \| \mathbf{v} - \mathbf{w} \|_{L^p(Q)^N}$$
$$\text{for all } \mathbf{v}, \mathbf{w} \in \mathcal{C}(\overline{Q})^N.$$

Whence, proceeding as in [37, Proposition 2.1], one deduces the existence of a unique mapping $H^\varepsilon_{v_0} : L^p(Q)^N \to L^{p'}(Q)$, that extends by continuity $F^\varepsilon_{v_0}$. Therefore we set

$$a^\varepsilon_i(-, v_0, \mathbf{v}) = H^\varepsilon_{v_0}(\mathbf{v}) \ \ ((v_0, \mathbf{v}) \in L^p(Q)^{N+1}).$$

It is an easy task to see that the transformation $(v_0, \mathbf{v}) \mapsto a^\varepsilon_i(-, v_0, \mathbf{v})$ thus define is continuous on $L^p(Q)^{N+1}$. Furthermore properties (2.1)–(2.6) are satisfied. □

Let $v \in L^p(0,T;W^{1,p}_0(\Omega))$. In Proposition, take $(v_0, \mathbf{v}) = (v, Dv)$; then one sees that the function $a^\varepsilon_i(-, v, Dv)$ is well-defined and so, the elliptic part in equation (1.1) is rigorously justified. Thus, for each fixed $\varepsilon > 0$, equation (1.1) admits (at least) a solution $u_\varepsilon \in L^p(0,T;W^{1,p}_0(\Omega))$ (see e.g., [27, 2]). Moreover u_ε lies in

$$V^p = \{ v \in L^p(0,T;W^{1,p}_0(\Omega)) : v' = \frac{dv}{dt} \in L^{p'}(0,T;W^{-1,p'}(\Omega)) \}.$$

Equipped with the norm $\|v\|_{V^p} = \|v\|_{L^p(0,T;W^{1,p}_0(\Omega))} + \|v'\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))}$, V^p is a Banach space which is continuously embedded in $\mathcal{C}([0,T];L^2(\Omega))$, so that the existence of $u_\varepsilon(0)$ is justified. Therefore u_ε belongs to the space $V_0^p = \{v \in V^p : v(0) = 0\}$, a Banach space with the V^p-norm.

Now, let $(\psi_0, \Psi) \in \mathcal{C}(\overline{Q}; \mathcal{B}(\mathbb{R}_y^N \times \mathbb{R}_z^N \times \mathbb{R}_\tau)^{N+1})$. It follows by mere routine that the function $(\zeta, \theta, \eta) \mapsto a_i(x,t,\cdot,\cdot,\cdot,\psi_0(x',t',\zeta,\theta,\eta), \Psi(x',t',\zeta,\theta,\eta))$ (for $(x,t),(x',t')$ fixed in $\overline{Q}$) sends continuously $\mathbb{R}_\zeta^N \times \mathbb{R}_\theta^N \times \mathbb{R}_\eta$ into $L^\infty(\mathbb{R}_{y,\tau}^{N+1} \times \mathbb{R}_z^N)$ (where $\mathbb{R}_{y,\tau}^{N+1} \times \mathbb{R}_z^N \equiv \mathbb{R}_y^N \times \mathbb{R}_z^N \times \mathbb{R}_\tau$), and so, belongs to $\mathcal{B}(\mathbb{R}_\zeta^N \times \mathbb{R}_\theta^N \times \mathbb{R}_\eta; L^\infty(\mathbb{R}_{y,\tau}^{N+1} \times \mathbb{R}_z^N))$ (the space of bounded continuous real functions of $\mathbb{R}_\zeta^N \times \mathbb{R}_\theta^N \times \mathbb{R}_\eta$ into $L^\infty(\mathbb{R}_{y,\tau}^{N+1} \times \mathbb{R}_z^N)$). Therefore one can define the trace $(y,z,\tau) \mapsto a_i(x,t,y,z,\tau,\psi_0(x',t',y,z,\tau),\Psi(x',t',y,z,\tau))$ in the sense of [36], denoted by $a_i(x,t,\cdot,\cdot,\cdot,\psi_0(x',t',\cdot,\cdot,\cdot),\Psi(x',t',\cdot,\cdot,\cdot))$, as element of $L^\infty(\mathbb{R}_{y,\tau}^{N+1} \times \mathbb{R}_z^N)$. Moreover, thanks to [part (iv) of] (1.4), the function $((x,t),(x',t')) \mapsto a_i(x,t,\cdot,\cdot,\cdot,\psi_0(x',t',\cdot,\cdot,\cdot),\Psi(x',t',\cdot,\cdot,\cdot))$ sends continuously $\overline{Q} \times \overline{Q}$ into $L^\infty(\mathbb{R}_{y,\tau}^{N+1} \times \mathbb{R}_z^N)$. Hence the function $(x,t) \mapsto a_i(x,t,\cdot,\cdot,\cdot,\psi_0(x,t,\cdot,\cdot,\cdot),\Psi(x,t,\cdot,\cdot,\cdot))$ is well-defined and lies in $\mathcal{C}(\overline{Q}; L^\infty(\mathbb{R}_{y,\tau}^{N+1} \times \mathbb{R}_z^N))$. This enables us to define the function $(x,t) \mapsto a_i(x,t,x/\varepsilon_1,x/\varepsilon_2,t/\varepsilon,\psi_0(x,t,x/\varepsilon_1,x/\varepsilon_2,t/\varepsilon),\Psi(x,t,x/\varepsilon_1,x/\varepsilon_2,t/\varepsilon))$ on Q, as element of $L^\infty(Q)$ denoted by $a_i^\varepsilon(-,\psi_0^\varepsilon,\Psi^\varepsilon)$ (for fixed $\varepsilon > 0$).

3 Algebras with mean value

3.1 Algebras with mean value

We begin by stating the concept of algebras with mean value. This concept was first introduced by Zhikov and Krivenko [48] as the main tool useful to tackle nonperiodic deterministic homogenization problems. It is a generalization of the concept of almost periodic functions, so that one can easily introduce the generalized Besicovitch spaces associated to an algebra with mean value as we will see it in the sequel. In the same direction, we mention the work [34] in which the concept of homogenization algebras is introduced, the main difference between those concepts being the separability hypothesis imposed on the latter concept. It is important to note that in [39] the latter concept has just been released from the separability assumption. Thus our approach here follows closely the one in [39]. Before we can state the concept of algebras with mean value, we need to give some preliminaries.

Let m be a positive integer. Let $\mathcal{H} = (H_\varepsilon)_{\varepsilon>0}$ be the following action of $\mathbb{R}_+^*$ (the multiplicative group of positive real numbers) on the numerical space $\mathbb{R}^m$ defined by

$$H_\varepsilon(x) = \frac{x}{\varepsilon_1} \quad (x \in \mathbb{R}^m) \tag{3.1}$$

where ε_1 is a positive function of ε tending to 0 with ε. For a given $\varepsilon > 0$, let

$$u^\varepsilon(x) = u(H_\varepsilon(x)) \quad (x \in \mathbb{R}^m)$$

for $u \in L^1_{\mathrm{loc}}(\mathbb{R}_y^m)$ (as usual, $\mathbb{R}_y^m$ denotes the numerical space $\mathbb{R}^m$ of variables $y = (y_1, \ldots, y_m)$). In view of $(\mathrm{H})_3$, u^ε lies in $L^1_{\mathrm{loc}}(\mathbb{R}_x^m)$. More generally, if u lies in $L^p_{\mathrm{loc}}(\mathbb{R}^m)$ (resp. $L^p(\mathbb{R}^m)$), $1 \leq p < +\infty$, then so also is u^ε.

A function $u \in \mathcal{B}(\mathbb{R}_y^m)$ (the $\mathcal{C}^*$-algebra of bounded uniformly continuous functions on $\mathbb{R}_y^m$) is said to have a mean value for $\mathcal{H}$, if there exists a real number $M(u)$ such that $u^\varepsilon \to M(u)$ in $L^\infty(\mathbb{R}_x^m)$-weak $*$ as $\varepsilon \to 0$. The real number $M(u)$ is called the mean value of u (for

$\mathcal{H}$). It is evident that this defines a mapping M which is a positive linear form (on the space of functions $u \in \mathcal{B}(\mathbb{R}^m_y)$ with mean value) attaining the value 1 on the constant function 1 and verifying the inequality $|M(u)| \leq \|u\|_\infty \equiv \sup_{y\in\mathbb{R}^m} |u(y)|$ for all such u's. The mapping M is called the mean value on $\mathbb{R}^m$ for $\mathcal{H}$; see [35] for further details. It is also a fact, as the characteristic function of all relatively compact set in $\mathbb{R}^m$ lies in $L^1(\mathbb{R}^m)$, that

$$M(u) = \lim_{R\to+\infty} \frac{1}{|B_R|} \int_{B_R} u(y)dy \tag{3.2}$$

where B_R stands for the bounded open ball in $\mathbb{R}^m$ with radius R, and $|B_R|$ denotes its Lebesgue measure. Expression (3.2) also holds for $u \in L^1_{\text{loc}}(\mathbb{R}^m)$ provided that the above limit makes sense.

This being so, by an *algebra with mean value on* $\mathbb{R}^m$ *for* $\mathcal{H}$ (algebra wmv, in short) is meant any Banach subalgebra A of $\mathcal{B}(\mathbb{R}^m_y)$ which is translation invariant ($u(\cdot + a) \in A$ for any $u \in A$ and each $a \in \mathbb{R}^m$), contains the constants and whose each element possesses a mean value for $\mathcal{H}$.

Let A be an algebra wmv on $\mathbb{R}^m_y$. It is known that A (endowed with the sup norm topology) is a commutative $\mathcal{C}$*-algebra with identity. We denote by $\Delta(A)$ the spectrum of A and by $\mathcal{G}$ the Gelfand transformation on A. We recall that $\Delta(A)$ (a subset of the topological dual A' of A) is the set of all nonzero multiplicative linear functionals on A, and $\mathcal{G}$ is the mapping of A into $\mathcal{C}(\Delta(A))$ such that $\mathcal{G}(u)(s) = \langle s, u\rangle$ $(s \in \Delta(A))$, where $\langle , \rangle$ denotes the duality pairing between A' and A. We endow $\Delta(A)$ with the relative weak$*$ topology on A'. Then using the well-known theorem of Stone (see e.g., [26]) one can easily show that the spectrum $\Delta(A)$ is a compact topological space, and the Gelfand transformation $\mathcal{G}$ is an isometric isomorphism identifying A with $\mathcal{C}(\Delta(A))$ (the continuous functions on $\Delta(A)$) as $\mathcal{C}$*-algebras. Next, since each element of A possesses a mean value, this yields an application $u \mapsto M(u)$ (denoted by M and called the mean value) which is a nonnegative continuous linear functional on A with $M(1) = 1$, and so provides us with a linear nonnegative functional $\psi \mapsto M_1(\psi) = M(\mathcal{G}^{-1}(\psi))$ defined on $\mathcal{C}(\Delta(A)) = \mathcal{G}(A)$, which is clearly bounded. Therefore, by the Riesz-Markov theorem, $M_1(\psi)$ is representable by integration with respect to some Radon measure β (of total mass 1) in $\Delta(A)$, called the *M-measure* for A [34]. It is evident that we have

$$M(u) = \int_{\Delta(A)} \mathcal{G}(u)d\beta \text{ for } u \in A.$$

To enhance the comprehension of the notion of the spectrum of an algebra wmv, let us give one well-known example: if A is the periodic algebra wmv $\mathcal{C}_{\text{per}}(Y)$ $(Y = (0,1)^m)$ of Y-periodic continuous real-valued functions on $\mathbb{R}^m$, then $\Delta(A)$ can be identified with the m-dimensional torus $\mathbb{T}^m = (\mathbb{R}/\mathbb{Z})^m$.

Next, the partial derivative of index i $(1 \leq i \leq m)$ on $\Delta(A)$ is defined to be the mapping $\partial_i = \mathcal{G} \circ \partial/\partial y_i \circ \mathcal{G}^{-1}$ (usual composition) of $\mathcal{D}^1(\Delta(A)) = \{\varphi \in \mathcal{C}(\Delta(A)) : \mathcal{G}^{-1}(\varphi) \in A^1\}$ into $\mathcal{C}(\Delta(A))$, where $A^1 = \{\psi \in \mathcal{C}^1(\mathbb{R}^m) : \psi, D_{y_i}\psi \in A \ (1 \leq i \leq m)\}$ with $D_{y_i}\psi = \partial\psi/\partial y_i$. Higher order derivatives are defined analogously. At the present time, let A^∞ be the space of $\psi \in \mathcal{C}^\infty(\mathbb{R}^m_y)$ such that $D^\alpha_y\psi = \frac{\partial^{|\alpha|}\psi}{\partial y_1^{\alpha_1}\cdots\partial y_m^{\alpha_m}} \in A$ for every $\alpha = (\alpha_1, ..., \alpha_m) \in \mathbb{N}^m$, and let $\mathcal{D}(\Delta(A)) = \{\varphi \in \mathcal{C}(\Delta(A)) : \mathcal{G}^{-1}(\varphi) \in A^\infty\}$. Endowed with a suitable locally convex topology (see [34]), A^∞ (resp. $\mathcal{D}(\Delta(A))$) is a Fréchet space and further, $\mathcal{G}$ viewed as defined on A^∞ is a topological isomorphism of A^∞ onto $\mathcal{D}(\Delta(A))$.

Analogously to the space $\mathcal{D}'(\mathbb{R}^m)$, we now define the space of distributions on $\Delta(A)$ to be the space of all continuous linear form on $\mathcal{D}(\Delta(A))$. We denote it by $\mathcal{D}'(\Delta(A))$ and we endow it with the strong dual topology. It is an easy exercise to see that since A^∞ is dense in A (see [47, Proposition 2.3]), the space $L^p(\Delta(A))$ $(1 \leq p \leq \infty)$ is a subspace of $\mathcal{D}'(\Delta(A))$ (with continuous embedding), so that one may define the Sobolev spaces on $\Delta(A)$ as follows.

$$W^{1,p}(\Delta(A)) = \{u \in L^p(\Delta(A)) : \partial_i u \in L^p(\Delta(A)) \ (1 \leq i \leq m)\} \ (1 \leq p < \infty)$$

where the derivative $\partial_i u$ is taken in the distribution sense on $\Delta(A)$. We equip $W^{1,p}(\Delta(A))$ with the norm

$$||u||_{W^{1,p}(\Delta(A))} = \left[||u||^p_{L^p(\Delta(A))} + \sum_{i=1}^m ||\partial_i u||^p_{L^p(\Delta(A))}\right]^{\frac{1}{p}} \quad \left(u \in W^{1,p}(\Delta(A))\right),$$
$$1 \leq p < \infty,$$

which makes it a Banach space. To that space are attached some other spaces such as $W^{1,p}(\Delta(A))/\mathbb{C} = \{u \in W^{1,p}(\Delta(A)) : \int_{\Delta(A)} ud\beta = 0\}$ and its separated completion $W^{1,p}_\#(\Delta(A))$; we refer to [36] for a documented presentation of these spaces.

However, the notion of a product algebra wmv needs a few further details. Suppose $m = m_1 + ... + m_n$ $(n \geq 2)$, where m_i $(1 \leq i \leq n)$ are positive integers. Thus

$$\mathbb{R}^m = \mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_n}.$$

For each integer i $(1 \leq i \leq n)$, let $\mathcal{H}_i = (H^i_\varepsilon)_{\varepsilon>0}$ be an action of $\mathbb{R}^*_+$ on $\mathbb{R}^{m_i}$ defined in a same way as the action $\mathcal{H}$ in (3.1). For fixed $\varepsilon > 0$, let $H_\varepsilon = H^1_\varepsilon \times \cdots \times H^n_\varepsilon$ (direct product), i.e.,

$$H_\varepsilon(x) = (H^1_\varepsilon(pr_1(x)), ..., H^n_\varepsilon(pr_n(x))) \quad (x \in \mathbb{R}^m)$$

where pr_i denotes the natural projection of $\mathbb{R}^m$ onto $\mathbb{R}^{m_i}$. There is no difficulty in checking that the family $\mathcal{H} = (H_\varepsilon)_{\varepsilon>0}$ thus defined is an action of $\mathbb{R}^*_+$ on $\mathbb{R}^m$. This is referred to as the product of the actions $\mathcal{H}_i$ $(1 \leq i \leq n)$, and is denoted by $\mathcal{H}' = \prod_{i=1}^n \mathcal{H}_i = \mathcal{H}_1 \times \cdots \times \mathcal{H}_n$.

Now, if A_i is an algebra wmv on $\mathbb{R}^{m_i}$ for $\mathcal{H}_i$, then we define the product algebra wmv $A_1 \odot ... \odot A_n$ as the closure in $\mathcal{B}(\mathbb{R}^m)$ of the tensor product $A_1 \otimes ... \otimes A_n = \{\sum_{\text{finite}} u_{i_1} \otimes ... \otimes u_{i_n} : u_{i_j} \in A_j\}$. This defines an algebra wmv on $\mathbb{R}^m$ for $\mathcal{H}'$. However we need to characterize such products. To this end, let A_y and A_z be two algebras wmv on $\mathbb{R}^N_y$ and $\mathbb{R}^m_z$ respectively, N and m being two positive integers. Let $A = A_y \odot A_z$ be their product, which is an algebra wmv on $\mathbb{R}^{N+m}_{y,z}$. We have the following result whose proof can be found in [39].

Theorem 3.1. *Let A_y, A_z and A be as above. For $f \in \mathcal{B}(\mathbb{R}^{N+m}_{y,z})$, we define $f_y \in \mathcal{B}(\mathbb{R}^m_z)$ and $f^z \in \mathcal{B}(\mathbb{R}^N_y)$ by*

$$f_y(z) = f^z(y) = f(y,z) \text{ for } (y,z) \in \mathbb{R}^N_y \times \mathbb{R}^m_z$$

and put

$$B_f = \{f^z : z \in \mathbb{R}^m\},\ C_f = \{f_y : y \in \mathbb{R}^N\}.$$

Then $B_f \subset A_y$ and $C_f \subset A_z$ for every $f \in A$. Also for $f \in A$ both B_f and C_f are relatively compact in A_y and in A_z respectively (in the sup norm topology).

Let $AP(\mathbb{R}^m)$ denote the space of all Bohr almost periodic functions on $\mathbb{R}^m$ [5, 6], that is the algebra of functions in $\mathcal{B}(\mathbb{R}^m)$ that are uniformly approximated by finite linear combinations of functions in the set $\{\cos(k\cdot z), \sin(k\cdot z) : k \in \mathbb{R}^m\}$. It is well-known that $AP(\mathbb{R}^m)$ is an algebra wmv on $\mathbb{R}^m$. As a first consequence of the preceding proposition we have the following

Proposition 3.2. *Let $A_y = AP(\mathbb{R}^N_y)$ and $A_z = AP(\mathbb{R}^m_z)$ be two almost periodic algebras wmv. Then $A \equiv A_y \odot A_z = AP(\mathbb{R}^N_y \times \mathbb{R}^m_z)$.*

Proof. The result is a consequence of the following fact: A function $f \in AP(\mathbb{R}^N_y \times \mathbb{R}^m_z)$ is in A if and only if either C_f or B_f is relatively compact (in the sup norm topology). $\square$

One can also easily show that $\mathcal{C}_{\mathrm{per}}(Y) \odot \mathcal{C}_{\mathrm{per}}(Z) = \mathcal{C}_{\mathrm{per}}(Y \times Z)$ where $Y = (0,1)^N$ and $Z = (0,1)^m$. This follows from the identification $\mathcal{C}_{\mathrm{per}}(Y) = \mathcal{C}(\mathbb{T}^N)$ where $\mathbb{T}^N$ is the N-torus in $\mathbb{R}^N$. Similarly we have $\mathcal{C}_{\mathrm{per}}(Y) \odot AP(\mathbb{R}^m_z) \odot \mathcal{C}_{\mathrm{per}}(\mathcal{T}) = \mathcal{C}_{\mathrm{per}}(Y \times \mathcal{T}; AP(\mathbb{R}^m_z))$ where $\mathcal{T} = (0,1)$. Other examples of product algebras wmv can be given.

3.2 The generalized Besicovitch spaces

We can define the generalized Besicovitch spaces associated to an algebra wmv. The notations are those of the preceding subsection. Let A be an algebra wmv on $\mathbb{R}^m$ (integer $m \geq 1$). Let $1 \leq p < \infty$. If $u \in A$, then $|u|^p \in A$ with $\mathcal{G}(|u|^p) = |\mathcal{G}(u)|^p$. Hence the limit $\lim_{R\to+\infty} \frac{1}{|B_R|}\int_{B_R} |u(y)|^p\, dy$ exists and we have

$$\lim_{R\to+\infty} \frac{1}{|B_R|}\int_{B_R} |u(y)|^p\, dy = M(|u|^p) = \int_{\Delta(A)} |\mathcal{G}(u)|^p\, d\beta.$$

Hence, for $u \in A$, put

$$\|u\|_p = (M(|u|^p))^{1/p}.$$

This defines a seminorm on A with which A is not complete. We denote by B^p_A the completion of A with respect to $\|\cdot\|_p$. B^p_A is a Fréchet space, and an argument due to Besicovitch [5] states that B^p_A is a complete subspace of $L^p_{\mathrm{loc}}(\mathbb{R}^m_y)$. We have the following properties that can be achieved using the theory of the completion; see e.g. [7, Chap. II].

Proposition 3.3. *The following hold true:*

(i) A is dense in B^p_A;

(ii) If F is a Banach space then any continuous linear mapping l from A to F extends by continuity to a unique continuous linear mapping L, of B^p_A into F.

Now, let $1 \leq p \leq q < \infty$. Obviously we have $B^q_A \subset B^p_A$, so that one may naturally define the space B^∞_A as follows:

$$B^\infty_A = \left\{ f \in \bigcap_{1\leq p<\infty} B^p_A : \sup_{1\leq p<\infty} \|f\|_p < \infty \right\}.$$

We endow B^∞_A with the seminorm $[f]_\infty = \sup_{1\leq p<\infty} \|f\|_p$, which makes it a Fréchet space.

Next, thanks to the preceding proposition, the following properties are worth noting:

(1) The Gelfand transformation $\mathcal{G}: A \to \mathcal{C}(\Delta(A))$ extends by continuity to a unique continuous linear mapping, still denoted by $\mathcal{G}$, of B_A^p into $L^p(\Delta(A))$. Furthermore if $u \in B_A^p \cap L^\infty(\mathbb{R}_y^m)$ then $\mathcal{G}(u) \in L^\infty(\Delta(A))$ and $\|\mathcal{G}(u)\|_{L^\infty(\Delta(A))} \leq \|u\|_{L^\infty(\mathbb{R}_y^m)}$.

(2) The mean value M viewed as defined on A, extends by continuity to a positive continuous linear form (still denoted by M) on B_A^p satisfying $M(u) = \int_{\Delta(A)} \mathcal{G}(u)d\beta$ $(u \in B_A^p)$. Furthermore, $M(\tau_a u) = M(u)$ for each $u \in B_A^p$ and all $a \in \mathbb{R}^m$, where $\tau_a u(y) = u(y-a)$ for almost all $y \in \mathbb{R}^m$.

(3) Let $1 \leq p,q,r < \infty$ be such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} \leq 1$. The usual multiplication $A \times A \to A$; $(u,v) \mapsto uv$, extends by continuity to a bilinear form $B_A^p \times B_A^q \to B_A^r$ with

$$\|uv\|_r \leq \|u\|_p \|v\|_q \text{ for } (u,v) \in B_A^p \times B_A^q.$$

The following result will be of great interest in the work.

Proposition 3.4. *Let A be a algebra wmv on $\mathbb{R}^m$. Then A^∞ is dense in B_A^p.*

Proof. Indeed by [47, Proposition 2.3] A^∞ is dense in A. The result follows therefore by [part (i) of] Proposition 3.3. □

Now, let $u \in B_A^p$ $(1 \leq p < \infty)$; then $|u|^p \in B_A^1$ (this is easily seen) and so, by part **(2)** above one has $M(|u|^p) = \int_{\Delta(A)} |\mathcal{G}(u)|^p d\beta = \|\mathcal{G}(u)\|^p_{L^p(\Delta(A))}$. Thus for $u \in B_A^p$ we have $\|u\|_p = (M(|u|^p))^{1/p}$, and $\|u\|_p = 0$ if and only if $\mathcal{G}(u) = 0$. Unfortunately, the mapping $\mathcal{G}$ (defined on B_A^p) is not in general injective. So let $\mathcal{N} = Ker\mathcal{G}$ (the kernel of $\mathcal{G}$) and let

$$\mathcal{B}_A^p = B_A^p/\mathcal{N}.$$

Endowed with the norm

$$\|u + \mathcal{N}\|_{\mathcal{B}_A^p} = \|u\|_p \quad (u \in B_A^p),$$

$\mathcal{B}_A^p$ is a Banach space with the following property.

Theorem 3.5. *The mapping $\mathcal{G} : B_A^p \to L^p(\Delta(A))$ induces an isometric isomorphism $\mathcal{G}_1$ of $\mathcal{B}_A^p$ onto $L^p(\Delta(A))$.*

Proof. This result has already been proved in [39]. However for the sake of convenience, we give a sketch of the proof here. Since $\mathcal{G}$ is an isometry (indeed $\|\mathcal{G}(u)\|_{L^p(\Delta(A))} = \|u\|_p$ for $u \in \mathcal{B}_A^p$) it suffices to show that $\mathcal{G}$ is surjective. Firstly we have that $\mathcal{G}(B_A^p)$ is dense in $L^p(\Delta(A))$. Now, let $v \in L^p(\Delta(A))$; then there exists a sequence $(u_n)_n \subset B_A^p$ such that $\mathcal{G}(u_n) \to v$ in $L^p(\Delta(A))$ as $n \to \infty$. Since $\|u_n - u_{n'}\|_p = \|\mathcal{G}(u_n) - \mathcal{G}(u_{n'})\|_{L^p(\Delta(A))} \to 0$ as $n, n' \to \infty$, the sequence $(u_n)_n$ is a Cauchy sequence in B_A^p which therefore converge to some function $u \in B_A^p$, hence $\mathcal{G}(u_n) \to \mathcal{G}(u)$ in $L^p(\Delta(A))$ as $n \to \infty$. We deduce that $v = \mathcal{G}(u)$. This shows that $\mathcal{G}$ is surjective. Therefore, by the first isomorphism theorem, the mapping $\mathcal{G}_1 : \mathcal{B}_A^p = B_A^p/\mathcal{N} \to L^p(\Delta(A))$ defined by

$$\mathcal{G}_1(u + \mathcal{N}) = \mathcal{G}(u) \text{ for } u \in B_A^p$$

is an algebraic isomorphism. But $\mathcal{G}_1$ is a topological isometric isomorphism since

$$\left\|\mathcal{G}_1(u+\mathcal{N})\right\|_{L^p(\Delta(A))} = \|u\|_p = \left\|u+\mathcal{N}\right\|_{\mathcal{B}_A^p} \text{ for } u \in B_A^p.$$

This completes the proof. □

As a first consequence of the preceding theorem one can also define the mean value of $u+\mathcal{N}$ (for each $u \in B_A^p$) as follows:

$$M_1(u+\mathcal{N}) = M(u), \text{ so that } M_1(u+\mathcal{N}) = \lim_{R\to+\infty} \frac{1}{|B_R|}\int_{B_R} u(y)dy.$$

One crucial result that can be derived from the preceding theorem is the following

Corollary 3.6. *The following hold true:*

(i) The spaces $\mathcal{B}_A^p$ are reflexive for $1 < p < \infty$;

(ii) The topological dual of the space $\mathcal{B}_A^p$ $(1 \le p < \infty)$ is the space $\mathcal{B}_A^{p'}$ $(p' = p/(p-1))$, the duality being given by

$$\left\langle u+\mathcal{N}, v+\mathcal{N}\right\rangle_{\mathcal{B}_A^{p'},\mathcal{B}_A^p} = M(uv) = \int_{\Delta(A)} \mathcal{G}_1(u+\mathcal{N})\mathcal{G}_1(v+\mathcal{N})d\beta$$

for $u \in B_A^{p'}$ and $v \in B_A^p$.

This result is easily proven by using the properties of L^p-spaces and the above isometric isomorphism.

Remark 3.7. *The space $\mathcal{B}_A^p$ is the separated completion of B_A^p and the canonical mapping of B_A^p into $\mathcal{B}_A^p$ is just the canonical surjection of B_A^p onto $\mathcal{B}_A^p$; see once more [7] for the theory of completion.*

Another definition which will be of great interest in the sequel is

Definition 3.8. *An algebra wmv A on $\mathbb{R}^m$ is* ergodic *if for every $u \in B_A^1$ such that $\|u - u(\cdot + a)\|_1 = 0$ for every $a \in \mathbb{R}^m$ we have $\|u - M(u)\|_1 = 0$.*

An equivalent property stated by Casado and Gayte [12] is given in the following proposition.

Proposition 3.9 ([12])**.** *An algebra wmv A on $\mathbb{R}^m$ is ergodic if and only if*

$$\lim_{R\to+\infty}\left\|\frac{1}{|B_R|}\int_{B_R} u(\cdot+y)dy - M(u)\right\|_p = 0 \text{ for all } u \in B_A^p,\ 1 \le p < \infty. \tag{3.3}$$

The following result provides us with a few examples of ergodic algebras (see next section for its application).

Lemma 3.10. *Let A be an algebra wmv on $\mathbb{R}^m$ with the following property:*

$$\lim_{R\to+\infty}\frac{1}{|B_R|}\int_{B_R} u(x+y)dx = M(u) \text{ uniformly with respect to } y. \tag{3.4}$$

Then A is ergodic.

Proof. As A is dense in B_A^p, it suffices to check (3.3) for $u \in A$. Let $\eta > 0$ be freely fixed. For such u's, according to (3.4) there exists some $R_0 > 0$ such that

$$\left| \frac{1}{|B_R|} \int_{B_R} u(x+y)dx - M(u) \right| \leq \eta \text{ for } R > R_0$$

and all $y \in \mathbb{R}^m$. This leads at once at

$$M_y \left(\left| \frac{1}{|B_R|} \int_{B_R} u(x+y)dx - M(u) \right|^p \right) \leq \eta^p \text{ for } R > R_0.$$

The ergodicity of A follows thereby. □

In order to simplify the presentation of the paper we will from now on, use the same letter u (if there is no danger of confusion) to denote the equivalence class of an element $u \in B_A^p$. The symbol ρ will denote the canonical mapping of B_A^p onto $\mathcal{B}_A^p = B_A^p/\mathcal{N}$. Our goal here is to define another space attached to $\mathcal{B}_A^p$. Let $u \in L^p(\Delta(A))$, and let $1 \leq i \leq m$. We know that $\partial_i u \in \mathcal{D}'(\Delta(A))$ exists and is defined by

$$\langle \partial_i u, \varphi \rangle = - \langle u, \partial_i \varphi \rangle \text{ for any } \varphi \in \mathcal{D}(\Delta(A)).$$

If we assume further that $\partial_i u \in L^p(\Delta(A))$, then there exists a unique $u_i \in \mathcal{B}_A^p$ such that $\partial_i u = \mathcal{G}_1(u_i)$. This leads to the following definition.

Definition 3.11. *By a* formal derivative of index $1 \leq i \leq m$, *of a function* $u \in \mathcal{B}_A^p$ *is meant the unique element* $\overline{\partial} u/\partial y_i$ *of* $\mathcal{B}_A^p$ *(if there exists) such that*

$$\mathcal{G}_1 \left(\overline{\partial} u/\partial y_i \right) = \partial_i \mathcal{G}_1(u). \tag{3.5}$$

Remark 3.12. *For* $u \in B_A^{1,p}$ *(that is the space of* $u \in B_A^p$ *such that* $D_y u \in (B_A^p)^m$*) we have*

$$\mathcal{G}_1 \left(\rho \left(\frac{\partial u}{\partial y_i} \right) \right) = \mathcal{G} \left(\frac{\partial u}{\partial y_i} \right) = \partial_i \mathcal{G}(u) = \partial_i \mathcal{G}_1 (\rho(u)) = (\textit{by definition})\ \mathcal{G}_1 \left(\frac{\overline{\partial}}{\partial y_i} (\rho(u)) \right),$$

hence

$$\rho \left(\frac{\partial u}{\partial y_i} \right) = \frac{\overline{\partial}}{\partial y_i} (\rho(u)),$$

or equivalently,

$$\rho \circ \frac{\partial}{\partial y_i} = \frac{\overline{\partial}}{\partial y_i} \circ \rho \textit{ on } B_A^{1,p}. \tag{3.6}$$

Now, set (for $1 \leq p < \infty$)

$$\mathcal{B}_A^{1,p} = \left\{ u \in \mathcal{B}_A^p : \frac{\overline{\partial} u}{\partial y_i} \in \mathcal{B}_A^p, \text{ for } 1 \leq i \leq m \right\}.$$

We endow $\mathcal{B}_A^{1,p}$ with the norm

$$\|u\|_{\mathcal{B}_A^{1,p}} = \left[\|u\|_p^p + \sum_{i=1}^m \left\| \frac{\overline{\partial} u}{\partial y_i} \right\|_p^p \right]^{1/p} \quad (u \in \mathcal{B}_A^{1,p})$$

which makes it a Banach space with the property that the restriction of $\mathcal{G}_1$ to $\mathcal{B}_A^{1,p}$ is an isometric isomorphism of $\mathcal{B}_A^{1,p}$ onto $W^{1,p}(\Delta(A))$. However we will be mostly concerned with the subspace $\mathcal{B}_A^{1,p}/\mathbb{C}$ of $\mathcal{B}_A^{1,p}$ consisting of functions $u \in \mathcal{B}_A^{1,p}$ with $M_1(u) \equiv M(u) = 0$. Equipped with the seminorm

$$\|u\|_{\mathcal{B}_A^{1,p}/\mathbb{C}} = \left\| \overline{D}_y u \right\|_p := \left[\sum_{i=1}^m \left\| \frac{\overline{\partial} u}{\partial y_i} \right\|_p^p \right]^{1/p} \quad (u \in \mathcal{B}_A^{1,p}/\mathbb{C})$$

where $\overline{D}_y = (\overline{\partial}/\partial y_i)_{1\leq i\leq m}$, $\mathcal{B}_A^{1,p}/\mathbb{R}$ is a locally convex topological space which is in general not separate and not complete. We denote by $\mathcal{B}_{\#A}^{1,p}$ the separated completion of $\mathcal{B}_A^{1,p}/\mathbb{C}$ with respect to $\|\cdot\|_{\mathcal{B}_A^{1,p}/\mathbb{C}}$, and by J_1 the canonical mapping of $\mathcal{B}_A^{1,p}/\mathbb{R}$ into $\mathcal{B}_{\#A}^{1,p}$. By the theory of completion of the uniform spaces [7] it is a fact that the mapping $\overline{\partial}/\partial y_i : \mathcal{B}_A^{1,p}/\mathbb{R} \to \mathcal{B}_A^p$ extends by continuity to a unique continuous linear mapping still denoted by $\overline{\partial}/\partial y_i : \mathcal{B}_{\#A}^{1,p} \to \mathcal{B}_A^p$ and satisfying

$$\frac{\overline{\partial}}{\partial y_i} \circ J_1 = \frac{\overline{\partial}}{\partial y_i} \text{ and } \|u\|_{\mathcal{B}_{\#A}^{1,p}} = \left\| \overline{D}_y u \right\|_p \quad (u \in \mathcal{B}_{\#A}^{1,p}) \tag{3.7}$$

where $\overline{D}_y = (\overline{\partial}/\partial y_i)_{1\leq i\leq m}$. Since $\mathcal{G}_1$ is an isometric isomorphism of $\mathcal{B}_A^{1,p}$ onto $W^{1,p}(\Delta(A))$ we have by (3.5) that the restriction of $\mathcal{G}_1$ to $\mathcal{B}_A^{1,p}/\mathbb{R}$ sends isometrically and isomorphically $\mathcal{B}_A^{1,p}/\mathbb{R}$ onto $W^{1,p}(\Delta(A))/\mathbb{R}$. So by [7, Chap. II] there exists a unique isometric isomorphism $\overline{\mathcal{G}}_1 : \mathcal{B}_{\#A}^{1,p} \to W_{\#}^{1,p}(\Delta(A))$ such that

$$\overline{\mathcal{G}}_1 \circ J_1 = J \circ \mathcal{G}_1 \tag{3.8}$$

and

$$\partial_i \circ \overline{\mathcal{G}}_1 = \mathcal{G}_1 \circ \frac{\overline{\partial}}{\partial y_i} \quad (1 \leq i \leq m). \tag{3.9}$$

We recall that J is the canonical mapping of $W^{1,p}(\Delta(A))/\mathbb{R}$ into its separated completion $W_{\#}^{1,p}(\Delta(A))$ while J_1 is the canonical mapping of $\mathcal{B}_A^{1,p}/\mathbb{R}$ into $\mathcal{B}_{\#A}^{1,p}$. Furthermore, as $J_1(\mathcal{B}_A^{1,p}/\mathbb{R})$ is dense in $\mathcal{B}_{\#A}^{1,p}$ (this is classical) and since A^∞ is dense in A, it follows that $(J_1 \circ \rho)(A^\infty/\mathbb{R})$ is dense in $\mathcal{B}_{\#A}^{1,p}$, where $A^\infty/\mathbb{R} = \{u \in A^\infty : M(u) = 0\}$.

3.3 The $R\Sigma$-convergence

We rewrite here the definition of the multiscale convergence method. Thanks to the equality $\mathcal{G}_1(\mathcal{B}_A^p) = L^p(\Delta(A))$ we will merely work on the Besicovitch spaces $\mathcal{B}_A^p$. To this end, let ε_1 and ε_2 be two well-separated scales as at the beginning of Section 1, and let A_y (resp. A_z, A_τ)

be an algebra wmv on $\mathbb{R}^N$ (resp. $\mathbb{R}^N$, $\mathbb{R}$) for the action $\mathcal{H} = (H_\varepsilon)_{\varepsilon>0}$ (resp. $\mathcal{H}' = (H'_\varepsilon)_{\varepsilon>0}$, $\mathcal{H}_0 = (H^0_\varepsilon)_{\varepsilon>0}$) of $\mathbb{R}^*_+$ on $\mathbb{R}^N$ (resp. $\mathbb{R}^N$, $\mathbb{R}$) given by $H_\varepsilon(x) = x/\varepsilon_1$ (resp. $H'_\varepsilon(x) = x/\varepsilon_2$, $H^0_\varepsilon(t) = t/\varepsilon$), and finally let $A = A_y \odot A_z \odot A_\tau$ be their product which is an algebra wmv on $\mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}$ for the product action $\mathcal{H}^* = \mathcal{H} \times \mathcal{H}' \times \mathcal{H}_0$ of $\mathbb{R}^*$ on $\mathbb{R}^{2N+1} = \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}$ given by $\mathcal{H}^* = (H^*_\varepsilon)_{\varepsilon>0}$ with

$$H^*(x,x',t) = \left(\frac{x}{\varepsilon_1}, \frac{x'}{\varepsilon_2}, \frac{t}{\varepsilon}\right) \quad \text{for } x,x' \in \mathbb{R}^N, t \in \mathbb{R}\ (\varepsilon > 0).$$

We will denote by the same letter M, the mean value on $\mathbb{R}^N$ for $\mathcal{H}$ and for $\mathcal{H}'$, on $\mathbb{R}$ for $\mathcal{H}^0$, and on $\mathbb{R}^{2N+1}$ for $\mathcal{H}^*$ as well. The same letter $\mathcal{G}$ will denote the Gelfand transformation on A_y, A_z, A_τ and A, as well. Points in $\Delta(A_y)$ (resp. $\Delta(A_z)$, $\Delta(A_\tau)$) are denoted by s (resp. r, s_0). The compact space $\Delta(A_y)$ (resp. $\Delta(A_z)$, $\Delta(A_\tau)$) is equipped with the M-measure β_y (resp. β_z, β_τ), for A_y (resp. A_z, A_τ). We recall that $\Delta(A) = \Delta(A_y) \times \Delta(A_z) \times \Delta(A_\tau)$ (Cartesian product) and hence the M-measure for A, with which $\Delta(A)$ is equipped, is just the product measure $\beta = \beta_y \otimes \beta_z \otimes \beta_\tau$.

The letter E will throughout denote exclusively a family of positive real numbers admitting 0 as an accumulation point. In the particular case where $E = (\varepsilon_n)_{n\in\mathbb{N}}$ with $0 < \varepsilon_n \leq 1$ and $\varepsilon_n \to 0$ as $n \to \infty$, we will refer to E as a *fundamental sequence*.

Definition 3.13. *A sequence $(u_\varepsilon)_{\varepsilon>0} \subset L^p(Q)$ $(1 \leq p < \infty)$ is said to* weakly $R\Sigma$-converge *in $L^p(Q)$ to some $u_0 \in u_0 \in L^p(Q; \mathcal{B}^p_A)$ if as $\varepsilon \to 0$, we have*

$$\int_Q u_\varepsilon(x,t) v\left(x,t,\frac{x}{\varepsilon_1},\frac{x}{\varepsilon_2},\frac{t}{\varepsilon}\right) dxdt \to \iint_{Q\times\Delta(A)} \widehat{u}_0(x,t,s,r,s_0)\widehat{v}(x,t,s,r,s_0)dxdtd\beta \qquad (3.10)$$

for every $v \in L^{p'}(Q;A)$ $(1/p' = 1 - 1/p)$, where $\widehat{u}_0 = \mathcal{G}_1 \circ u_0$ and $\widehat{v} = \mathcal{G}_1 \circ (\rho \circ v) = \mathcal{G} \circ v$. We express this by $u_\varepsilon \to u_0$ in $L^p(Q)$-weak $R\Sigma$.

The above convergence result (3.10) strongly rely on the following convergence property (see [47, Proposition 2.4] for the proof): For each ψ in A we have, as $\varepsilon \to 0$,

$$\psi^\varepsilon \to M(\psi) \text{ in } L^\infty(\mathbb{R}^N_x)\text{-weak } *,$$

where ψ^ε is defined in an obvious way by $\psi^\varepsilon(x) = \psi(x/\varepsilon_1, x/\varepsilon_2, t/\varepsilon)$ for $x \in \mathbb{R}^N$ and $t \in \mathbb{R}$, and M is the mean value on $\mathbb{R}^{2N+1} = \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}$ for the product action $\mathcal{H}^*$. The letter "R" stands for reiteratively. The above definition is more accurate than the previous one in [32]; this is due to the equality $\mathcal{G}_1(\mathcal{B}^p_A) = L^p(\Delta(A))$. Indeed one immediately sees that the right-hand side of (3.10) is equal to $\int_Q M(u_0(x,t,\cdot,\cdot,\cdot)v(x,t,\cdot,\cdot,\cdot))dxdt$. In particular when $A = \mathcal{C}_{\text{per}}(Y) \odot \mathcal{C}_{\text{per}}(Z) \odot \mathcal{C}_{\text{per}}(\mathcal{T})$ one is led at once to the convergence result

$$\int_Q u_\varepsilon(x,t) v\left(x,t,\frac{x}{\varepsilon_1},\frac{x}{\varepsilon_2},\frac{t}{\varepsilon}\right) dxdt \to \int_Q \int_Y \int_Z \int_{\mathcal{T}} u_0(x,t,y,z,\tau)v(x,t,y,z,\tau)d\tau dzdydxdt$$

where $u_0 \in L^p(Q \times Y \times Z \times \mathcal{T})$, which is the original definition of the multiscale convergence [1]. Moreover the uniqueness of the limit u_0 is ensured, and it is also a fact that the weak $R\Sigma$-convergence in L^p implies the weak convergence in L^p.

The following result is the point of departure of all compactness results involved in the framework of Σ-convergence, see [39] for the proof.

Proposition 3.14. *Let $(u_\varepsilon)_{\varepsilon\in E}$ be a bounded sequence in $L^p(Q)$ $(1<p<\infty)$, E being a fundamental sequence. Then there exist a subsequence E' from E and a function u in $L^p(Q;\mathcal{B}^p_A)$ such that the sequence $(u_\varepsilon)_{\varepsilon\in E'}$ weakly $R\Sigma$-converges in $L^p(Q)$ to u.*

Next, for $1<p<\infty$, set

$$V^p=\{v\in L^p(0,T;W_0^{1,p}(\Omega)) : v'=\partial v/\partial t\in L^{p'}(0,T;W^{-1,p'}(\Omega))\},$$

a Banach space with the norm $\|v\|_{V^p}=\|v\|_{L^p(0,T;W_0^{1,p}(\Omega))}+\|v'\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))}$ $(v\in V^p)$. We have that V^p (for $p\geq 2$) is, continuously embedded in the space $\mathcal{C}([0,T];L^2(\Omega))$ and compactly embedded in the space $L^2(Q)$. With all that in mind, we have the following important compactness result whose proof can be found in [39].

Theorem 3.15. *Let $1<p<\infty$. Let Ω be an open subset in $\mathbb{R}^N$. Let $A=A_y\odot A_z\odot A_\tau$ where A_y and A_z are ergodic algebras on $\mathbb{R}^N$. Finally, let $(u_\varepsilon)_{\varepsilon\in E}$ be a bounded sequence in V^p. There exist a subsequence E' from E and a triple $\mathbf{u}=(u_0,u_1,u_2)\in V^p\times L^p(Q;\mathcal{B}^p_{A_\tau}(\mathbb{R}_\tau;\mathcal{B}^{1,p}_{\#A_y}))\times L^p(Q;\mathcal{B}^p_{A_y\odot A_\tau}(\mathbb{R}^{N+1}_{y,\tau};\mathcal{B}^{1,p}_{\#A_z}))$ such that, as $E'\ni\varepsilon\to 0$,*

$$u_\varepsilon\to u_0 \text{ in } V^p\text{-weak}$$

and

$$\frac{\partial u_\varepsilon}{\partial x_j}\to\frac{\partial u_0}{\partial x_j}+\frac{\overline{\partial}u_1}{\partial y_j}+\frac{\overline{\partial}u_2}{\partial z_j} \text{ in } L^p(Q)\text{-weak } R\Sigma\ (1\leq j\leq N).$$

We give below a few examples of algebras wmv which satisfy hypotheses of Theorem 3.15.

3.3.1 Example 3.1. Periodic setting

Let $A_y=\mathcal{C}_{\mathrm{per}}(Y)$ $(Y=(0,1)^N)$ be the algebra of Y-periodic continuous functions on $\mathbb{R}^N_y$. It is classically known that A_y is an ergodic algebra so that Theorem 3.15 applies with $A=\mathcal{C}_{\mathrm{per}}(Y)\odot\mathcal{C}_{\mathrm{per}}(Y)\odot A_\tau$ for any algebra wmv A_τ on $\mathbb{R}_\tau$.

3.3.2 Example 3.2. Almost periodic setting

Let $AP(\mathbb{R}^N)$ be the algebra of Bohr continuous almost periodic functions $\mathbb{R}^N$. We recall that a function $u\in\mathcal{B}(\mathbb{R}^N)$ is in $AP(\mathbb{R}^N)$ if the set of translates $\{\tau_a u : a\in\mathbb{R}^N\}$ is relatively compact in $\mathcal{B}(\mathbb{R}^N)$. An argument due to Bohr [6] specifies that $u\in AP(\mathbb{R}^N)$ if and only if u may be uniformly approximated by finite linear combinations of functions in the set $\{\cos(k\cdot y),\sin(k\cdot y) : k\in\mathbb{R}^N\}$.

It is also a classical result that A_y is an ergodic algebra; see e.g. [18]. Therefore Theorem 3.15 applies with $A_y=AP(\mathbb{R}^N_y)$, $A_z=AP(\mathbb{R}^N_z)$ and A_τ any algebra wmv on $\mathbb{R}_\tau$.

3.3.3 Example 3.3. The convergence at infinity setting

Let $\mathcal{B}_\infty(\mathbb{R}^N)$ denote the space of all continuous functions on $\mathbb{R}^N$ that converge at infinity, that is of all $u\in\mathcal{B}(\mathbb{R}^N)$ such that $\lim_{|y|\to\infty}u(y)\in\mathbb{R}$. It is an easy exercise to see that $\mathcal{B}_\infty(\mathbb{R}^N)$ is an ergodic algebra [18]. Moreover the mean value of a function $u\in\mathcal{B}_\infty(\mathbb{R}^N)$ is given by $M(u)=\lim_{|y|\to\infty}u(y)$. Therefore we have the conclusion of Theorem 3.15 with $A_y=\mathcal{B}_\infty(\mathbb{R}^N)=A_z$ and for any A_τ as in the preceding subsection.

3.3.4 Example 3.4. The weakly almost periodic setting

We begin with the notion of weakly almost periodicity due to Eberlein [18].

Definition 3.16. *A bounded continuous function u on $\mathbb{R}^N$ is weakly almost periodic if the set of translates $\{\tau_a u : a \in \mathbb{R}^N\}$ is relatively weakly compact in for the sup norm topology.*

We denote by $WAP(\mathbb{R}^N_y)$ the set of all weakly almost periodic functions on $\mathbb{R}^N_y$; $WAP(\mathbb{R}^N_y)$ is a vector space over $\mathbb{R}$. Endowed with the norm sup topology, $WAP(\mathbb{R}^N_y)$ is a Banach algebra with the usual multiplication.

The space $WAP(\mathbb{R}^N_y)$ is sometimes called the space of Eberlein's functions. As examples of Eberlein's functions we have the continuous Bohr almost periodic functions, the continuous functions vanishing at infinity, the positive definite functions (hence Fourier-Stieltjes transforms); see [18] for more details.

The following properties are worth mentioning (see [18] for details):

(P1) $WAP(\mathbb{R}^N_y)$ is a translation invariant $\mathcal{C}^*$-subalgebra of $\mathcal{B}(\mathbb{R}^N_y)$.

(P2) A weakly almost periodic function is uniformly continuous and bounded.

(P3) A weakly almost periodic function possesses a mean value with

$$M(u) = \lim_{R\to+\infty} \frac{1}{|B_R|} \int_{B_R} u(y+a)dy,$$

the convergence being uniform in $a \in \mathbb{R}^N$.

(P4) If $u, v \in WAP(\mathbb{R}^N_y)$ the convolution defined by the mean value $w(y) = (u\widehat{*}v)(y) = M_z(u(y-z)v(z)) = M_z(u(z)v(y-z))$ is an usual Bohr almost periodic function, where M_z stands for the mean value with respect to z.

(P5) ([19, Theorem 1]) Every $u \in WAP(\mathbb{R}^N_y)$ admits the unique decomposition $u = v + w$, v being a Bohr almost periodic function and w a continuous function of quadratic mean value zero: $M(|w|^2) = 0$.

Property (P5) above, knowing as a *decomposition theorem*, is crucial in the definition of the weak almost periodic algebra. Indeed let $W_0(\mathbb{R}^N_y)$ denote the subset of $WAP(\mathbb{R}^N_y)$ consisting of elements with quadratic mean value zero. One easily observes that the set of bounded continuous functions on $\mathbb{R}^N$ of quadratic mean value zero is a complete vector subspace of the algebra of bounded continuous functions on $\mathbb{R}^N$. Hence $W_0(\mathbb{R}^N_y)$ is a complete vector subspace of $WAP(\mathbb{R}^N_y)$. Property (P5) states as follows: $WAP(\mathbb{R}^N_y)$ is a direct sum of the two spaces $AP(\mathbb{R}^N_y)$ and $W_0(\mathbb{R}^N_y)$:

$$WAP(\mathbb{R}^N_y) = AP(\mathbb{R}^N_y) \oplus W_0(\mathbb{R}^N_y).$$

Another representation of the space $W_0(\mathbb{R}^N_y)$ is given by de Leeuw and Glicksberg [17]:

$$W_0(\mathbb{R}^N_y) = \{u \in WAP(\mathbb{R}^N_y) : M(|u|) = 0\}.$$

Proposition 3.17. *The vector space $A = AP(\mathbb{R}^N_y) \oplus W_0(\mathbb{R}^N_y)$ is an ergodic algebra.*

Proof. The ergodicity of A is a consequence of Lemma 3.10 and of property (P3). The remainder is a mere consequence of properties (P1) and (P2) above. $\square$

We have the same conclusion as in the preceding examples with $A_y = WAP(\mathbb{R}^N) = A_z = A_{y_i}$. Also, since any algebra wmv of all the preceding examples is a subalgebra of $WAP(\mathbb{R}^N)$, the conclusion of Theorem 3.15 follows if we take instead of $WAP(\mathbb{R}^N)$, any of the algebras of the above examples. In particular, Theorem 3.15 holds with $A = WAP_{\mathcal{R}}(\mathbb{R}^N_y) \odot AP(\mathbb{R}^N_z) \odot A_\tau$, A_τ any algebra wmv on $\mathbb{R}$.

The following result will allow us to see that weakly almost periodic algebra wmv is not an homogenization algebra.

Theorem 3.18. *The algebra $W_0(\mathbb{R}^m)$ is not separable.*

For the proof of this theorem we need the following lemma.

Lemma 3.19. *Let G and H be two Banach spaces, and let φ be a surjective continuous linear mapping of G onto H. If G is separable then so also is H.*

Proof. Let $(a_n)_{n\geq 1}$ be a countable dense set in G, and let c be a positive constant such that $\|\varphi(g)\|_H \leq c\|g\|_G$ for all $g \in G$, where $\|\cdot\|_G$ and $\|\cdot\|_H$ denote respectively the norms in G and in H. Set $B = (\varphi(a_n))_{n\geq 1} \subset H$, and let us show that B is dense in H. For that, let $h \in H$ and let $\eta > 0$ be freely fixed; let finally $g \in G$ be such that $h = \varphi(g)$. There exists some $n_0 \geq 1$ such that $\|g - a_{n_0}\|_G < \eta/c$, hence $\|h - \varphi(a_{n_0})\|_H \leq c\|g - a_{n_0}\|_G < \eta$. Thus B is a countable dense subset of H, and the proof is completed. $\square$

One can now prove Theorem 3.18.

Proof of Theorem 3.18. Set $G = W_0(\mathbb{R}^m)$, $H = W_0(\mathbb{R}^m)/\mathcal{C}_0(\mathbb{R}^m)$ ($\mathcal{C}_0(\mathbb{R}^m)$ denotes the algebra of all continuous functions on $\mathbb{R}^m$ that vanish at infinity) and φ the natural (canonical) homomorphism of G onto H. Assume G is separable (with the sup norm topology), then according to the above lemma, H is separable. But we know by [13, Theorem 4.6] that the quotient space H contains a linear isometric copy of ℓ^∞ and hence is not separable. This contradicts our assumption, and hence G is not separable. $\square$

Since $W_0(\mathbb{R}^m) \subset WAP(\mathbb{R}^m)$ we conclude by Theorem 3.18 that the ergodic algebra $WAP(\mathbb{R}^m)$ is not separable (with respect to the sup norm topology). It is well-known that the algebra $AP(\mathbb{R}^m)$ induces homogenization algebras in the sense of [34]; it suffices to consider those functions in $AP(\mathbb{R}^m)$ with spectrum contained in a countable subgroup $\mathcal{R}$ of $\mathbb{R}^m$ (that we denoted below by $AP_{\mathcal{R}}(\mathbb{R}^m)$), that is the space of those functions in $AP(\mathbb{R}^N_y)$ that can be uniformly approximated by finite linear combinations in the set $\{\cos(k\cdot y), \sin(k\cdot y) : k \in \mathcal{R}\}$. However this is not the case for $WAP(\mathbb{R}^m)$ since for every countable subgroup $\mathcal{R}$ of $\mathbb{R}^m$ the algebra $AP_{\mathcal{R}}(\mathbb{R}^m) + W_0(\mathbb{R}^m)$ is never separable. So we have in hands an example of algebra wmv which contrary to the algebra $AP(\mathbb{R}^m)$, induces no homogenization algebra, says $WAP(\mathbb{R}^m)$. This is therefore sufficient to show that the concept of homogenization algebra cannot handle weakly almost periodic homogenization problems. This is a significant contribution to the theory.

Now, let $\mathcal{C}_0(\mathbb{R}^N_y)$ denote the space of all continuous functions on $\mathbb{R}^N_y$ that vanish at infinity. It is well known that $A_y = AP(\mathbb{R}^N_y) + \mathcal{C}_0(\mathbb{R}^N_y)$ is a proper subalgebra of the algebra of weakly almost periodic functions on $\mathbb{R}_y^N$. A_y is an algebra wmv called the algebra of *perturbed* almost periodic functions. It generates the homogenization algebras as follows. Considering some countable subgroup $\mathcal{R}$ of $\mathbb{R}_y^N$ we define $AP_{\mathcal{R}}(\mathbb{R}_y^N)$ as above; it is known that $AP_{\mathcal{R}}(\mathbb{R}_y^N)$ is an homogenization algebra. If we set $A_{y,\mathcal{R}} = AP_{\mathcal{R}}(\mathbb{R}_y^N) + \mathcal{C}_0(\mathbb{R}_y^N)$, then, as $\mathcal{C}_0(\mathbb{R}_y^N)$ is a separable subalgebra of $\mathcal{B}(\mathbb{R}_y^N)$, we define by this a further homogenization algebra [34]. So A_y generates homogenization algebras. Besides, as seen by Theorem 3.18, even though $\mathcal{R}$ is a countable subgroup of $\mathbb{R}_y^N$, $AP_{\mathcal{R}}(\mathbb{R}_y^N) + W_0(\mathbb{R}_y^N)$ is never separable, so that A_y above is probably the bigger subalgebra of $WAP(\mathbb{R}_y^N)$ that generates homogenization algebras.

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4 Homogenization problem for (1.1)

4.1 Setting of the abstract problem and preliminary results

The notations are those of the preceding sections. Let $A = A_y \odot A_z \odot A_\tau$ be an algebra wmv and let $p' = p/(p-1)$ with $2 \leq p < \infty$. It is easy to see that Property (3.10) (in Definition 3.13) still holds for $v \in \mathcal{C}(\overline{Q}; B_A^{p',\infty})$ instead of $v \in L^{p'}(Q;A)$ mutatis mutandis, where $B_A^{p',\infty} = B_A^{p'} \cap L^\infty(\mathbb{R}_y^N \times \mathbb{R}_z^N \times \mathbb{R}_\tau)$ and $p' = p/(p-1)$. Furthermore, if we provide the space $B_A^{p',\infty}$ with the $L^\infty(\mathbb{R}_y^N \times \mathbb{R}_z^N \times \mathbb{R}_\tau)$-norm, it can be shown that, for $u \in B_A^{p',\infty}$, we have $\mathcal{G}(u) \in L^\infty(\Delta(A))$ and $\|\mathcal{G}(u)\|_{L^\infty(\Delta(A))} \leq \|u\|_{L^\infty(\mathbb{R}_{y,z,\tau}^{2N+1})}$, $\mathcal{G}$ being the canonical mapping of $B_A^{p'}$ into $L^{p'}(\Delta(A))$.

This being so, the main purpose of this section is to investigate the asymptotic analysis, as $\varepsilon \to 0$, of u_ε (the solution of (1.1)) under the hypothesis

$$\begin{aligned}&a_i(x,t,\cdot,\cdot,\cdot,\mu,\lambda) \in B_A^{p'} \text{ for any } (x,t) \in \overline{Q} \text{ and all } (\mu,\lambda) \in \mathbb{R} \times \mathbb{R}^N,\\ &0 \leq i \leq N\end{aligned} \tag{4.1}$$

where $p' = p/(p-1)$ with $2 \leq p < \infty$.

The following result is the cornerstone of the homogenization process. It allows us to go from a concrete hypothesis to the abstract one which is fundamental in the proof of the main result of the paper.

Proposition 4.1. *Assume* (4.1) *holds true. Then, for every* $(\psi_0, \Psi) \in A \times (A)^N = (A)^{N+1}$ *and every* $(x,t) \in \overline{Q}$, *the function* $(y,z,\tau) \mapsto a_i(x,t,y,z,\tau,\psi_0(y,z,\tau),\Psi(y,z,\tau))$ *denoted below by* $a_i(x,t,\cdot,\cdot,\cdot,\psi_0,\Psi)$, *lies in* $B_A^{p'}$.

Proof. Let $K \subset \mathbb{R} \times \mathbb{R}^N$ be a compact set such that $(\psi_0(y,z,\tau),\Psi(y,z,\tau)) \in K$ for all $(y,z,\tau) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}$. By viewing a_i as a function $(x,t,\mu,\lambda) \mapsto a_i(x,t,\cdot,\cdot,\cdot,\mu,\lambda)$ of $\overline{Q} \times \mathbb{R} \times \mathbb{R}^N$ into $B_A^{p'}$, we have that a_i belongs to $\mathcal{C}(\overline{Q} \times \mathbb{R} \times \mathbb{R}^N; B_A^{p'})$ (combine (4.1) with [part (iv) of] (1.4)). Still denoting by a_i the restriction of this function to $\overline{Q} \times K$, it immediately follows that $a_i \in \mathcal{C}(\overline{Q} \times K; B_A^{p'})$. Hence using the density of $\mathcal{C}(\overline{Q} \times K) \otimes B_A^{p'}$ in $\mathcal{C}(\overline{Q} \times K; B_A^{p'})$, one may consider a sequence $(q_n)_{n\geq 1}$ in $\mathcal{C}(\overline{Q}\times K)\otimes B_A^{p'}$ such that

$$\sup_{(x,t)\in\overline{Q}}\sup_{(\mu,\lambda)\in K}\left\|q_n(x,t,\cdot,\cdot,\cdot,\mu,\lambda)-a_i(x,t,\cdot,\cdot,\cdot,\mu,\lambda)\right\|_{p'}\to 0$$
$$\text{as } n\to\infty.$$

As

$$\begin{aligned}&\left\|q_n(x,t,\cdot,\cdot,\cdot,\psi_0,\Psi)-a_i(x,t,\cdot,\cdot,\cdot,\psi_0,\Psi)\right\|_{p',\infty}\leq\\&\sup_{(x,t)\in\overline{Q}}\sup_{(\mu,\lambda)\in K}\left\|q_n(x,t,\cdot,\cdot,\cdot,\mu,\lambda)-a_i(x,t,\cdot,\cdot,\cdot,\mu,\lambda)\right\|_{p'}\end{aligned}$$

we have $q_n(x,t,\cdot,\cdot,\cdot,\psi_0,\Psi)\to a_i(x,t,\cdot,\cdot,\cdot,\psi_0,\Psi)$ in $B_A^{p'}$ as $n\to\infty$. Thus, the proposition is shown if we can verify that each $q_n(x,t,\cdot,\cdot,\cdot,\psi_0,\Psi)$ lies in $B_A^{p'}$. However this will follow in an obvious way once we have checked that for any function $q:\overline{Q}\times\mathbb{R}_y^N\times\mathbb{R}_z^N\times\mathbb{R}_\tau\times K\to\mathbb{R}$ of the form

$$\begin{aligned}&q(x,t,y,z,\tau,\mu,\lambda)=\chi(x,t,\mu,\lambda)\Phi(y,z,\tau)\ (y,z,\lambda\in\mathbb{R}^N,\ \mu,\tau\in\mathbb{R},\ (x,t)\in\overline{Q})\\&\text{with } \chi\in\mathcal{C}(\overline{Q}\times K)\text{ and }\Phi\in B_A^{p'},\end{aligned}$$

we have $q(x,t,\cdot,\cdot,\cdot,\psi_0,\Psi)\in B_A^{p'}$. But given q as above, we know by the Stone-Weierstrass theorem that there is a sequence $(f_n)_{n\geq 1}$ of polynomials in $(x,t,\mu,\lambda)\in\overline{Q}\times K$ such that $f_n\to\chi$ in $\mathcal{C}(\overline{Q}\times K)$ as $n\to\infty$, hence $f_n(x,t,\psi_0,\Psi)\to\chi(x,t,\psi_0,\Psi)$ in $\mathcal{B}(\mathbb{R}_y^N\times\mathbb{R}_z^N\times\mathbb{R}_\tau)$ as $n\to\infty$. Therefore, it follows that $\chi(x,t,\psi_0,\Psi)$ lies in A, since the same is true for each $f_n(x,t,\psi_0,\Psi)$ (recall that A is an algebra). We conclude that

$$q(x,t,\cdot,\cdot,\cdot,\psi_0,\Psi)=\chi(x,t,\psi_0,\Psi)\Phi\in B_A^{p'}$$

as the product of an element of A by an element of $B_A^{p'}$. This concludes the proof. □

Now, assume (4.1) holds. Let $(\varphi,\Psi)\in(A)^{N+1}$; then by Proposition 4.1 we have $a_i(x,t,\cdot,\cdot,\cdot,\varphi,\Psi)\in B_A^{p'}\cap L^\infty(\mathbb{R}_y^N\times\mathbb{R}_z^N\times\mathbb{R}_\tau)=B_A^{p',\infty}$.

Proposition 4.2. *Let $2\leq p<\infty$ and let $0\leq i\leq N$. Suppose* (4.1) *holds. For any $(\psi_0,\Psi)\in\mathcal{C}(\overline{Q};(A)^{N+1})$ we have*

$$a_i^\varepsilon(-,\psi_0^\varepsilon,\Psi^\varepsilon)\to a_i(-,\psi_0,\Psi)\text{ in } L^{p'}(Q)\text{-weak } R\Sigma \text{ as } \varepsilon\to 0. \tag{4.2}$$

Let $a(-,\psi_0,\Psi)=(a_i(-,\psi_0,\Psi))_{1\leq i\leq N}$. The mapping $(\psi_0,\Psi)\mapsto(a_0(-,\psi_0,\Psi),a(-,\psi_0,\Psi))$ of $\mathcal{C}(\overline{Q};(A)^{N+1})$ into $L^{p'}(Q;B_A^{p'})^{N+1}$ extends by continuity to a unique mapping still denoted by (a_0,a), of $L^p(Q;(B_A^p)^{N+1})$ into $L^{p'}(Q;B_A^{p'})^{N+1}$ such that

$$\begin{aligned}&(a(-,u,\mathbf{v})-a(-,u,\mathbf{w}))\cdot(\mathbf{v}-\mathbf{w})\geq c_1\left|\mathbf{v}-\mathbf{w}\right|^p \text{ a.e. in } Q\times\mathbb{R}_y^N\times\mathbb{R}_z^N\times\mathbb{R}_\tau\\&\left\|a_i(-,u,\mathbf{v})\right\|_{L^{p'}(Q;B_A^{p'})}\leq c_2''\left(1+\left\|u\right\|_{L^p(Q;B_A^p)}^{p-1}+\left\|\mathbf{v}\right\|_{L^p(Q;(B_A^p)^N)}^{p-1}\right)\\&\left\|a_i(-,u,\mathbf{v})-a_i(-,u,\mathbf{w})\right\|_{L^{p'}(Q;B_A^{p'})}\\&\quad\leq c_0\left\|1+|u|+|\mathbf{v}|+|\mathbf{w}|\right\|_{L^p(Q;B_A^p)}^{p-2}\left\|\mathbf{v}-\mathbf{w}\right\|_{L^p(Q;(B_A^p)^N)}\end{aligned}\tag{4.3}$$

$$\begin{aligned}&|a_i(x,t,y,z,\tau,u,\mathbf{w})-a_i(x',t',y,z,\tau,v,\mathbf{w})|\leq\\&\leq \nu(|x-x'|+|t-t'|+|u-v|)\left(1+|u|^{p-1}+|v|^{p-1}+|\mathbf{w}|^{p-1}\right)\\&\textit{a.e. in } Q\times\mathbb{R}_y^N\times\mathbb{R}_z^N\times\mathbb{R}_\tau\end{aligned}$$

for all $u,v\in L^p(Q;B_A^p)$, $\mathbf{v},\mathbf{w}\in L^p(Q;(B_A^p)^N)$ *and all* $(x,t),(x',t')\in Q$, *where the constant* c_2'' *depends only on* c_2 *and on* Q.

Proof. We see that the function $a_i(-,\psi_0,\Psi)$ lies in $\mathcal{C}(\overline{Q};B_A^{p',\infty})$. Since Property (3.10) (in Definition 3.13) still holds for $f\in\mathcal{C}(\overline{Q};B_A^{p',\infty})$ the convergence result (4.2) follows at once. On the other hand, by the definition of the function $a_i(-,\psi_0,\Psi)$ (for $(\psi_0,\Psi)\in\mathcal{C}(\overline{Q};(A)^{N+1})$) it is immediate that this function verifies properties of the same type as in Proposition 2.1 (see for instance properties (2.2)-(2.6) therein). Therefore arguing as in the proof of the latter proposition we get the remainder of Proposition 4.2. □

The preceding proposition has several corollaries as seen below. To see this, for $(\psi_0,\Psi)\in L^p(Q;(B_A^p)^{N+1})$ we set $b_i(\cdot,\cdot,\widehat{\psi}_0,\widehat{\Psi})=\mathcal{G}(a_i(-,\psi_0,\Psi))$ $(0\leq i\leq N)$, which defines a mapping from $L^p(Q;(B_A^p)^{N+1})$ to $L^{p'}(Q\times\Delta(A))^{N+1}$, where $\widehat{\psi}_0=\mathcal{G}\circ\psi_0$ (a similar definition for $\widehat{\Psi}$), $\mathcal{G}$ being the canonical mapping of $B_A^{p'}$ into $L^{p'}(\Delta(A))$.

Corollary 4.3. *Let* $(u_\varepsilon)_{\varepsilon\in E}$ *be a sequence in* $L^p(Q)$ *such that* $u_\varepsilon\to u_0$ *in* $L^p(Q)$ *(strong) as* $E\ni\varepsilon\to 0$, *where* $u_0\in L^p(Q)$. *Let* $\Psi\in\mathcal{C}(\overline{Q};(A)^N)$, *and finally let* $0\leq i\leq N$. *Then, as* $E\ni\varepsilon\to 0$,

$$a_i^\varepsilon(-,u_\varepsilon,\Psi^\varepsilon)\to a_i(-,u_0,\Psi).\ \textit{in } L^{p'}(Q)\textit{-weak } R\Sigma.$$

Proof. Let $f\in L^p(Q;A)$, and let $(\psi_j)_j$ be a sequence in $\mathcal{C}_0^\infty(Q)$ such that $\psi_j\to u_0$ in $L^p(Q)$ as $j\to\infty$. We have

$$\begin{aligned}&\textstyle\int_Q a_i^\varepsilon(-,u_\varepsilon,\Psi^\varepsilon)f^\varepsilon dxdt-\iint_{Q\times\Delta(A)}b_i(\cdot,\cdot,u_0,\widehat{\Psi})\widehat{f}dxdtd\beta\\&\textstyle=\int_Q[a_i^\varepsilon(-,u_\varepsilon,\Psi^\varepsilon)-a_i^\varepsilon(-,u_0,\Psi^\varepsilon)]f^\varepsilon dxdt+\\&\textstyle\quad+\int_Q[a_i^\varepsilon(-,u_0,\Psi^\varepsilon)-a_i^\varepsilon(-,\psi_j,\Psi^\varepsilon)]f^\varepsilon dxdt+\\&\textstyle\quad+\int_Q a_i^\varepsilon(-,\psi_j,\Psi^\varepsilon)f^\varepsilon dxdt-\iint_{Q\times\Delta(A)}b_i(\cdot,\cdot,u_0,\widehat{\Psi})\widehat{f}dxdtd\beta\\&=A_\varepsilon+B_{\varepsilon,j}+C_{\varepsilon,j}\end{aligned}$$

where:

$$\begin{aligned}&\textstyle A_\varepsilon=\int_Q[a_i^\varepsilon(-,u_\varepsilon,\Psi^\varepsilon)-a_i^\varepsilon(-,u_0,\Psi^\varepsilon)]f^\varepsilon dxdt,\\&\textstyle B_{\varepsilon,j}=\int_Q[a_i^\varepsilon(-,u_0,\Psi^\varepsilon)-a_i^\varepsilon(-,\psi_j,\Psi^\varepsilon)]f^\varepsilon dxdt,\\&\textstyle C_{\varepsilon,j}=\int_Q a_i^\varepsilon(-,\psi_j,\Psi^\varepsilon)f^\varepsilon dxdt-\iint_{Q\times\Delta(A)}b_i(\cdot,\cdot,u_0,\widehat{\Psi})\widehat{f}dxdtd\beta\end{aligned}$$

We proceed in three steps.
Step 1. We first evaluate $\lim_{E\ni\varepsilon\to 0}A_\varepsilon$. We have

$$|A_\varepsilon|\leq\int_Q\nu(|u_\varepsilon-u_0|)\left(1+|u_\varepsilon|^{p-1}+|u_0|^{p-1}+|\Psi^\varepsilon|^{p-1}\right)|f^\varepsilon|\,dxdt.$$

Let $F_\varepsilon=\left(1+|u_\varepsilon|^{p-1}+|u_0|^{p-1}+|\Psi^\varepsilon|^{p-1}\right)|f^\varepsilon|$. We have, on one hand, $F_\varepsilon\in L^1(Q)$ and $(F_\varepsilon)_{\varepsilon\in E}$ weakly converges in $L^1(Q)$ as $E\ni\varepsilon\to 0$ (this is easily seen). On the other hand, since $v_\varepsilon \equiv u_\varepsilon - u_0 \to 0$ in $L^p(Q)$ as $E \ni \varepsilon \to 0$, we know by [8, Thm IV-9, p.58] that there exist a subsequence E' from E and a function $g \in L^p(Q)$ such that

$$\begin{aligned}&v_\varepsilon \to 0 \text{ (hence } |v_\varepsilon| \to 0) \text{ a.e. in } Q \text{ as } E' \ni \varepsilon \to 0\\ &|v_\varepsilon| \leq g \text{ a.e. in } Q \text{ for all } \varepsilon \in E'.\end{aligned}$$

We conclude as in [the first part of] the proof of Proposition 2.1 that $A_\varepsilon \to 0$ as $E \ni \varepsilon \to 0$.
Step 2. As $C_{\varepsilon,j}$ is concerned. The function $(x,t) \mapsto a_i(x,t,\cdot,\cdot,\cdot,\psi_j(x,t),\Psi(x,t,\cdot,\cdot,\cdot))$ belongs to $\mathcal{C}(\overline{Q};B_A^{p',\infty})$. Thus, using Proposition 4.2, we are led at once at

$$C_{\varepsilon,j} \to \iint_{Q\times\Delta(A)} \left(b_i(\cdot,\cdot,\psi_j,\widehat{\Psi}) - b_i(\cdot,\cdot,u_0,\widehat{\Psi})\right)\widehat{f}dxdtd\beta \equiv \widehat{C}_j \text{ as } E \ni \varepsilon \to 0.$$

But

$$\left|\widehat{C}_j\right| \leq \iint_{Q\times\Delta(A)} \nu\left(\left|\psi_j - u_0\right|\right)\left(1+\left|\psi_j\right|^{p-1}+|u_0|^{p-1}+\left|\widehat{\Psi}\right|^{p-1}\right)\left|\widehat{f}\right|dxdtd\beta.$$

Therefore, proceeding as we have done it in Step 1 above, we obtain $\widehat{C}_j \to 0$ as $j \to \infty$.
Step 3. For the term $B_{\varepsilon,j}$, the same analysis conducted in Steps 1) and 2) yields

$$\lim_{E\ni\varepsilon\to 0}\lim_{j\to\infty} B_{\varepsilon,j} = 0.$$

Finally, since

$$\begin{aligned}&\lim_{E\ni\varepsilon\to0}\left(\int_Q a_i^\varepsilon(-,u_\varepsilon,\Psi^\varepsilon)f^\varepsilon dxdt - \iint_{Q\times\Delta(A)} b_i(\cdot,\cdot,u_0,\widehat{\Psi})\widehat{f}dxdtd\beta\right)\\ &= \lim_{E\ni\varepsilon\to0}A_\varepsilon + \lim_{E\ni\varepsilon\to0}\lim_{j\to\infty}B_{\varepsilon,j} + \lim_{E\ni\varepsilon\to0}\lim_{j\to\infty}C_{\varepsilon,j} = 0,\end{aligned}$$

the result follows. □

Thanks to Corollary 4.3 and to Proposition 4.2, we have the following important corollary whose proof follows the same outlines of [36, Corollary 4.1] (see also [46]), and is therefore left to the reader.

Corollary 4.4. *Let* $0 \leq i \leq N$.

$$\Phi_\varepsilon = \psi_0 + \varepsilon_1\psi_1^\varepsilon + \varepsilon_2\psi_2^\varepsilon, \tag{4.4}$$

i.e., $\Phi_\varepsilon(x,t) = \psi_0(x,t) + \varepsilon_1\psi_1(x,t,x/\varepsilon_1,t/\varepsilon) + \varepsilon_2\psi_2(x,t,x/\varepsilon_1,x/\varepsilon_2,t/\varepsilon)$ *for* $(x,t) \in Q$, *where* $\psi_0 \in \mathcal{D}(Q) = \mathcal{C}_0^\infty(Q)$, $\psi_1 \in \mathcal{D}(Q)\otimes A_y^\infty \otimes A_\tau^\infty$ *and* $\psi_2 \in \mathcal{D}(Q)\otimes A^\infty$. *Then as* $\varepsilon \to 0$,

$$(i)\ a_i^\varepsilon(-,\Phi_\varepsilon,D\Phi_\varepsilon) \to a_i(-,\psi_0,D\psi_0 + D_y\psi_1 + D_z\psi_2) \textit{ in } L^{p'}(Q)\textit{-weak } R\Sigma.$$

Assume in addition that $(u_\varepsilon)_{\varepsilon\in E}$ *is a sequence in* $L^p(Q)$ *such that* $u_\varepsilon \to u_0$ *in* $L^p(Q)$ *as* $E \ni \varepsilon \to 0$ *where* $u_0 \in L^p(Q)$. *Then, when* $E \ni \varepsilon \to 0$, *one has*

$$(ii)\ a_i^\varepsilon(-,u_\varepsilon,D\Phi_\varepsilon) \to a_i(-,u_0,D\psi_0 + \partial_s\widehat{\psi}_1 + \partial_r\widehat{\psi}_2) \textit{ in } L^{p'}(Q)\textit{-weak } R\Sigma.$$

Moreover, if $(v_\varepsilon)_{\varepsilon\in E}$ *is a sequence in* $L^p(Q)$ *such that* $v_\varepsilon \to v_0$ *in* $L^p(Q)$*-weak* $R\Sigma$ *as* $E \ni \varepsilon \to 0$ *(where* $v_0 \in L^p(Q;\mathcal{B}_A^p)$*), then, as* $E \ni \varepsilon \to 0$,

$$(iii)\ \int_Q a_i^\varepsilon(-,u_\varepsilon,D\Phi_\varepsilon)v_\varepsilon dxdt \to \iint_{Q\times\Delta(A)} b_i(\cdot,\cdot,u_0,D\psi_0 + \partial_s\widehat{\psi}_1 + \partial_r\widehat{\psi}_2)\widehat{v}_0 dxdtd\beta.$$

4.2 Homogenization result

The basic notation and hypotheses are as in the preceding sections. Let $A = A_y \odot A_z \odot A_\tau$ be an algebra wmv, where A_y and A_z are ergodic algebras. For $2 \leq p < \infty$ we put $\mathbb{F}_0^{1,p} = V_0^p \times L^p(Q; \mathcal{B}^p_{A_\tau}(\mathbb{R}_\tau; \mathcal{B}^{1,p}_{\#A_y})) \times L^p(Q; \mathcal{B}^p_{A_y \odot A_\tau}(\mathbb{R}^{N+1}_{y,\tau}; \mathcal{B}^{1,p}_{\#A_z}))$. We endow $\mathbb{F}_0^{1,p}$ with the norm

$$\begin{aligned} \|\mathbf{u}\|_{\mathbb{F}_0^{1,p}} &= \sum_{i=1}^{N} \left[\|D_{x_i} u_0\|_{L^p(\Omega)} + \left\|\overline{D}_{y_i} u_1\right\|_{L^p(\Omega; \mathcal{B}^p_{A_y \odot A_\tau})} + \left\|\overline{D}_{z_i} u_2\right\|_{L^p(\Omega; \mathcal{B}^p_A)} \right] \\ \mathbf{u} &= (u_0, u_1, u_2) \in \mathbb{F}_0^{1,p}. \end{aligned}$$

$\mathbb{F}_0^{1,p}$ is a Banach space which admitting $F_0^\infty = \mathcal{D}(Q) \times (\mathcal{D}(Q) \otimes [\rho_\tau(A_\tau^\infty) \otimes (J_1^y \circ \rho_y)(A_y^\infty/\mathbb{R})]) \times (\mathcal{D}(Q) \otimes [\rho_y(A_y^\infty) \otimes \rho_\tau(A_\tau^\infty) \otimes (J_1^z \circ \rho_z)(A_z^\infty/\mathbb{R})])$ as a dense subspace. For $\mathbf{u} = (u_0, u_1, u_2) \in \mathcal{V}$ we put $\mathbb{D}_i \mathbf{u} = \partial u_0/\partial x_i + \partial_i \widehat{u}_1 + \partial_i \widehat{u}_2$ $(1 \leq i \leq N)$ and $\mathbb{D}\mathbf{u} = Du_0 + \partial_s \widehat{u}_1 + \partial_r \widehat{u}_2 = (\mathbb{D}_i \mathbf{u})_{1 \leq i \leq N}$, where $\partial_i \widehat{u}_1 = \mathcal{G}_1(\overline{\partial} u_1/\partial y_i)$ and $\partial_i \widehat{u}_2 = \mathcal{G}_1(\overline{\partial} u_2/\partial z_i)$.

We are now in a position to state and prove the main result of the paper.

Theorem 4.5. *Let $2 \leq p < \infty$. Suppose* (4.1) *holds and further $A = A_y \odot A_z \odot A_\tau$ where the algebras A_y and A_z are ergodic algebras. For each fixed real number $\varepsilon > 0$, let u_ε be the (unique) solution of* (1.1). *There exists a subsequence of* ε, *still denoted by* ε, *such that, as* $\varepsilon \to 0$,

$$u_\varepsilon \to u_0 \text{ in } L^p(0,T; W_0^{1,p}(\Omega))\text{-weak} \tag{4.5}$$

$$\frac{\partial u_\varepsilon}{\partial t} \to \frac{\partial u_0}{\partial t} \text{ in } L^{p'}(0,T; W^{-1,p'}(\Omega))\text{-weak} \tag{4.6}$$

$$\frac{\partial u_\varepsilon}{\partial x_j} \to \frac{\partial u_0}{\partial x_j} + \frac{\overline{\partial} u_1}{\partial y_j} + \frac{\overline{\partial} u_2}{\partial z_j} \text{ in } L^p(Q)\text{-weak } R\Sigma \ (1 \leq j \leq N), \tag{4.7}$$

where $\mathbf{u} = (u_0, u_1, u_2) \in \mathbb{F}_0^{1,p}$ solves the variational problem

$$\begin{aligned} &\textstyle\int_0^T \langle u_0'(t), v_0(t)\rangle\, dt + \iint_{Q \times \Delta(A)} b(\cdot,\cdot,u_0,\mathbb{D}\mathbf{u}) \cdot \mathbb{D}\mathbf{v} dxdtd\beta + \\ &\quad + \textstyle\iint_{Q \times \Delta(A)} b_0(\cdot,\cdot,u_0,\mathbb{D}\mathbf{u}) v_0 dxdtd\beta = \int_0^T \langle f(t), v_0(t)\rangle\, dt \\ &\text{for all } \mathbf{v} = (v_0, v_1, v_2) \in \mathbb{F}_0^{1,p}. \end{aligned} \tag{4.8}$$

Moreover u_1 and u_2 are unique and any weak Σ-limit point in V^p of $(u_\varepsilon)_{\varepsilon>0}$ is a solution to problem (4.8).

Proof. We first show that $(u_\varepsilon)_{\varepsilon>0}$ is bounded in V_0^p. Let $v \in V^p$; we have

$$\begin{aligned} &\textstyle\int_0^T \langle u_\varepsilon'(t), v(t)\rangle\, dt + \int_Q a^\varepsilon(-,u_\varepsilon,Du_\varepsilon) \cdot Dvdxdt + \int_Q a_0^\varepsilon(-,u_\varepsilon,Du_\varepsilon) vdxdt \\ &\quad = \textstyle\int_0^T \langle f(t), v(t)\rangle\, dt. \end{aligned} \tag{4.9}$$

Taking in particular $v = u_\varepsilon$ in (4.9) and using inequality $\int_Q a_0^\varepsilon(-,u_\varepsilon,Du_\varepsilon) u_\varepsilon dxdt \geq 0$ (this comes from (2.3)) and the fact that $\int_0^T \langle u_\varepsilon'(t), u_\varepsilon(t)\rangle\, dt = \frac{1}{2} \|u_\varepsilon(T)\|^2_{L^2(\Omega)} \geq 0$, and finally thanks to property (2.1), we get

$$\sup_{\varepsilon>0} \|u_\varepsilon\|_{L^p(0,T;W_0^{1,p}(\Omega))} < \infty.$$

and hence

$$\sup_{\varepsilon>0}\left\{\|a_0^{\varepsilon}(-,u_\varepsilon,Du_\varepsilon)\|_{L^{p'}(Q)}+\|a^{\varepsilon}(-,u_\varepsilon,Du_\varepsilon)\|_{L^{p'}(Q)^N}\right\}<\infty. \tag{4.10}$$

It follows that

$$\sup_{\varepsilon>0}\left\|\frac{\partial u_\varepsilon}{\partial t}\right\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))}<\infty$$

and hence $(u_\varepsilon)_{\varepsilon>0}$ is bounded in V^p, and so, in V_0^p.

Let E be a fundamental sequence. Because of Theorem 3.15, there exist a subsequence E' from E and a triplet $\mathbf{u}=(u_0,u_1,u_2)\in\mathbb{F}_0^{1,p}$ such that (4.5)-(4.7) hold when $E'\ni\varepsilon\to 0$. Next, we show that $\mathbf{u}$ is solution to the variational equation (4.8). To this end $\Phi=(\psi_0,(J_1^y\circ\rho_y)(\psi_1),(J_1^z\circ\rho_z)(\psi_2))\in F_0^\infty$ with $\psi_0\in\mathcal{D}(Q)$, $\psi_1\in\mathcal{D}(Q)\otimes[(A_y^\infty/\mathbb{R})\otimes A_\tau^\infty]$, $\psi_2\in\mathcal{D}(Q)\otimes[A_y^\infty\otimes(A_z^\infty/\mathbb{R})\otimes A_\tau^\infty]$, where $A_y^\infty/\mathbb{R}=\{\psi\in A_y^\infty : M(\psi)=0\}$ and similar definition for $A_z^\infty/\mathbb{R}$. Let $\Phi_\varepsilon=\psi_0+\varepsilon_1\psi_1^\varepsilon+\varepsilon_2\psi_2^\varepsilon$ be defined as in (4.4). We have $\Phi_\varepsilon\in\mathcal{D}(Q)$ and

$$\begin{aligned}0\le &\int_0^T\langle f(t)-u_\varepsilon'(t),u_\varepsilon(t)-\Phi_\varepsilon(\cdot,t)\rangle\,dt-\int_Q a^\varepsilon(-,u_\varepsilon,D\Phi_\varepsilon)\cdot D(u_\varepsilon-\Phi_\varepsilon)dxdt\\ &-\int_Q a_0^\varepsilon(-,u_\varepsilon,Du_\varepsilon)(u_\varepsilon-\Phi_\varepsilon)dxdt\end{aligned}$$

or equivalently, owing to equality $\int_0^T\langle u_\varepsilon'(t),u_\varepsilon(t)\rangle\,dt=\frac{1}{2}\|u_\varepsilon(T)\|_{L^2(\Omega)}^2$,

$$\begin{aligned}\tfrac{1}{2}\|u_\varepsilon(T)\|_{L^2(\Omega)}^2\le &\int_0^T\langle f(t),u_\varepsilon(t)-\Phi_\varepsilon(\cdot,t)\rangle\,dt+\int_0^T\langle u_\varepsilon'(t),\Phi_\varepsilon(\cdot,t)\rangle\,dt-\\ &-\int_Q a^\varepsilon(\text{-},u_\varepsilon,D\Phi_\varepsilon)\cdot D(u_\varepsilon-\Phi_\varepsilon)dxdt-\int_Q a_0^\varepsilon(\text{-},u_\varepsilon,Du_\varepsilon)(u_\varepsilon-\Phi_\varepsilon)dxdt.\end{aligned} \tag{4.11}$$

Since the sequence $(a_0^\varepsilon(-,u_\varepsilon,Du_\varepsilon))_{\varepsilon>0}$ is bounded in $L^{p'}(Q)$ (see (4.10)), there exist (see Proposition 3.14) a subsequence from E', still denoted by E', and a function $\chi\in L^{p'}(Q;\mathcal{B}_A^{p'})$ such that , as $E'\ni\varepsilon\to 0$,

$$a_0^\varepsilon(-,u_\varepsilon,Du_\varepsilon)\to\chi \text{ in } L^{p'}(Q)\text{-weak } R\Sigma. \tag{4.12}$$

Therefore proceeding as in the proof of [37, Theorem 3.1] we obtain by mere routine

$$\begin{aligned}0\le &\int_0^T\langle f(t)-u_0'(t),u_0(t)-\psi_0(\cdot,t)\rangle\,dt-\iint_{Q\times\Delta(A)}b(\cdot,\cdot,u_0,\mathbb{D}\Phi)\cdot\mathbb{D}(\mathbf{u}-\Phi)dxdtd\beta\\ &-\iint_{Q\times\Delta(A)}\chi(u_0-\psi_0)dxdtd\beta.\end{aligned} \tag{4.13}$$

By a density argument, (4.13) still holds for $\Phi\in\mathbb{F}_0^{1,p}$. Hence taking in (4.13) the particular functions $\Phi=\mathbf{u}-\lambda\mathbf{v}$ with $\lambda>0$ and $\mathbf{v}\in\mathbb{F}_0^{1,p}$ we also obtain by mere routine, the following equation:

$$\begin{aligned}&\int_0^T\langle u_0'(t),v_0(t)\rangle\,dt+\iint_{Q\times\Delta(A)}b(\cdot,\cdot,u_0,\mathbb{D}\mathbf{u})\cdot\mathbb{D}\mathbf{v}dxdtd\beta\\ &\quad+\iint_{Q\times\Delta(A)}\chi v_0dxdtd\beta=\int_0^T\langle f(t),v_0(t)\rangle\,dt\\ &\text{for all } \mathbf{v}=(v_0,v_1,v_2)\in\mathbb{F}_0^{1,p}.\end{aligned} \tag{4.14}$$

To obtain (4.8), it therefore remains to show that $\chi=a_0(-,u_0,\overline{\mathbb{D}}\mathbf{u})$ where $\overline{\mathbb{D}}\mathbf{u}=Du_0+\overline{D}_yu_1+\overline{D}_zu_2$. For this purpose, let $\eta>0$ be arbitrarily fixed, and let $B_\#'(u_2,\eta)$ (resp. $B_\#'(u_1,\eta)$, $B_0'(u_0,\eta)$) denote the closed ball of $L^p(Q;\mathcal{B}^p_{A_y\odot A_\tau}(\mathbb{R}^{N+1}_{y,\tau};\mathcal{B}^{1,p}_{\#A_z}))$ (resp. $L^p(Q;\mathcal{B}^p_{A_\tau}(\mathbb{R}_\tau;\mathcal{B}^{1,p}_{\#A_y}))$, $L^p(0,T;W_0^{1,p}(\Omega))$) centered at u_2 (resp. u_1, u_0) and of radius η (resp. η, η). The spaces $L^p(0,T;W_0^{1,p}(\Omega))$, $L^p(Q;\mathcal{B}^p_{A_y\odot A_\tau}(\mathbb{R}^{N+1}_{y,\tau};$

$\mathcal{B}^{1,p}_{\#A_z})$) and $L^p(Q;\mathcal{B}^p_{A_\tau}(\mathbb{R}_\tau;\mathcal{B}^{1,p}_{\#A_y}))$ are reflexive since the same is true for $W^{1,p}_0(\Omega)$, $\mathcal{B}^{1,p}_{\#A_y}$ (and hence for $\mathcal{B}^p_{A_\tau}(\mathbb{R}_\tau;\mathcal{B}^{1,p}_{\#A_y}) = \mathcal{H}_{y,\tau}$) and $\mathcal{B}^{1,p}_{\#A_z}$ (and hence for $\mathcal{B}^p_{A_y\odot A_\tau}(\mathbb{R}^{N+1}_{y,\tau};\mathcal{B}^{1,p}_{\#A_z}) = \mathcal{H}_{y,z,\tau}$). Thus the above balls are weakly compact. So, let

$$h(v_0,v_1,v_2) = \| 1+|u_0|+|Du_0+\partial_s v_1+\partial_r v_2|+|Dv_0+\partial_s v_1+\partial_r v_2|\|^{p-2}_{L^p(Q;\mathcal{B}^p_A)}$$

for $(v_0,v_1,v_2)\in\mathbb{F}^{1,p}_0$, and

$$d = \sup_{v_2\in B'_\#(u_2,\eta)}\ \sup_{v_1\in B'_\#(u_1,\eta)}\ \sup_{v_0\in B'_0(u_0,\eta)} h(v_0,v_1,v_2)$$

and finally set

$$k = c_2 d + 1. \tag{4.15}$$

(4.15) defines a positive real number depending on $\mathbf{u}$ and on η, and verifying $k>1$. Using a density argument, we choose $\psi_0\in\mathcal{D}(Q)$, $\psi_1\in\mathcal{D}(Q)\otimes[(A^\infty_y/\mathbb{R})\otimes A^\infty_\tau]$ and $\psi_2\in\mathcal{D}(Q)\otimes[A^\infty_y\otimes(A^\infty_z/\mathbb{R})\otimes A^\infty_\tau]$ such that

$$\begin{aligned} &\|u_0-\psi_0\|_{L^p(0,T;W^{1,p}_0(\Omega))} < \tfrac{\eta}{4k}, \\ &\left\|u_1-(J^y_1\circ\rho_y)(\psi_1)\right\|_{L^p(Q;\mathcal{H}_{y,\tau})} < \tfrac{\eta}{4k}, \\ &\|u_2-(J^z_1\circ\rho_z)(\psi_2)\|_{L^p(Q;\mathcal{H}_{y,z,\tau})} < \tfrac{\eta}{4k}. \end{aligned} \tag{4.16}$$

Clearly $\psi_0\in B'_0(u_0,\eta)$, $(J^y_1\circ\rho_y)(\psi_1)\in B'_\#(u_1,\eta)$ and $(J^z_1\circ\rho_z)(\psi_2)\in B'_\#(u_2,\eta)$. Set $\Phi = (\psi_0,(J^y_1\circ\rho_y)(\psi_1),(J^z_1\circ\rho_z)(\psi_2))\in F^\infty_0$ and define Φ_ε as in (4.4). Then we have

$$\begin{aligned} &\|b_0(\cdot,\cdot,u_0,\mathbb{D}\mathbf{u})-\widehat{\chi}\|_{L^{p'}(Q\times\Delta(A))} \le \\ &\le \|b_0(\cdot,\cdot,u_0,\mathbb{D}\mathbf{u})-b_0(\cdot,\cdot,u_0,Du_0+\partial_s u_1+\partial_r\widehat{\psi}_2)\|_{L^{p'}(Q\times\Delta(A))} \\ &\quad + \|b_0(\cdot,\cdot,u_0,Du_0+\partial_s u_1+\partial_r\widehat{\psi}_2)-b_0(\cdot,\cdot,u_0,Du_0+\partial_s\widehat{\psi}_1+\partial_r\widehat{\psi}_2)\|_{L^{p'}(Q\times\Delta(A))} \\ &\quad + \|b_0(\cdot,\cdot,u_0,Du_0+\partial_s\widehat{\psi}_1+\partial_r\widehat{\psi}_2)-b_0(\cdot,\cdot,u_0,\mathbb{D}\Phi)\|_{L^{p'}(Q\times\Delta(A))} \\ &\quad + \|b_0(\cdot,\cdot,u_0,\mathbb{D}\Phi)-\widehat{\chi}\|_{L^{p'}(Q\times\Delta(A))}, \end{aligned}$$

and, thanks to (4.15)-(4.16),

$$\|b_0(\cdot,\cdot,u_0,\mathbb{D}\mathbf{u})-\widehat{\chi}\|_{L^{p'}(Q\times\Delta(A))} \le \frac{3\eta}{4} + \|b_0(\cdot,\cdot,u_0,\mathbb{D}\Phi)-\widehat{\chi}\|_{L^{p'}(Q\times\Delta(A))}.$$

On the other hand, using (4.12) and [part (ii) of] Corollary 4.3 (note that $u_\varepsilon\to u_0$ in $L^p(Q)$ as $E'\ni\varepsilon\to 0$ since V^p is compactly embedded in $L^p(Q)$; this is a classical result), we get

$$\|b_0(\cdot,\cdot,u_0,\mathbb{D}\Phi)-\widehat{\chi}\|_{L^{p'}(Q\times\Delta(A))} \le \liminf_{E'\ni\varepsilon\to 0}\|a^\varepsilon_0(-,u_\varepsilon,D\Phi_\varepsilon)-a^\varepsilon_0(-,u_\varepsilon,Du_\varepsilon)\|_{L^{p'}(Q)}.$$

But

$$\begin{aligned} &\left\|a^\varepsilon_0(-,u_\varepsilon,D\Phi_\varepsilon)-a^\varepsilon_0(-,u_\varepsilon,Du_\varepsilon)\right\|_{L^{p'}(Q)} \\ &\le c_2\,\|1+|u_\varepsilon|+|Du_\varepsilon|+|D\Phi_\varepsilon|\|^{p-2}_{L^p(Q)}\,\|Du_\varepsilon-D\Phi_\varepsilon\|_{L^p(Q)^N}. \end{aligned}$$

By inequality $|D\Phi_\varepsilon|\le|Du_\varepsilon-D\Phi_\varepsilon|+|Du_\varepsilon|$, we are led at once at

$$\begin{aligned} &\|1+|u_\varepsilon|+|Du_\varepsilon|+|D\Phi_\varepsilon|\|^{p-2}_{L^p(Q)} \\ &\le \left(\|1+|u_\varepsilon|+2\,|Du_\varepsilon|\|_{L^p(Q)}+\|Du_\varepsilon-D\Phi_\varepsilon\|_{L^p(Q)^N}\right)^{p-2} \\ &\le \left(c+\|Du_\varepsilon-D\Phi_\varepsilon\|_{L^p(Q)^N}\right)^{p-2} \end{aligned}$$

where $c>0$ is a constant independent of ε, the last inequality been obtained thanks to the fact that $(u_\varepsilon)_{\varepsilon>0}$ is bounded in $L^p(0,T;W_0^{1,p}(\Omega))$. Thus,

$$\begin{aligned}&\|b_0(\cdot,\cdot,u_0,\mathbb{D}\Phi)-\widehat{\chi}\|_{L^{p'}(Q\times\Delta(A))}\\&\quad\leq\liminf_{E'\ni\varepsilon\to 0}\left(c+\|Du_\varepsilon-D\Phi_\varepsilon\|_{L^p(Q)^N}\right)^{p-2}\|Du_\varepsilon-D\Phi_\varepsilon\|_{L^p(Q)^N}.\end{aligned}$$

But due to (2.2),

$$\begin{aligned}&c_0\,\|Du_\varepsilon-D\Phi_\varepsilon\|^p_{L^p(Q)^N}\\&\quad\leq\int_Q(a^\varepsilon(-,u_\varepsilon,Du_\varepsilon)-a^\varepsilon(-,u_\varepsilon,D\Phi_\varepsilon))\cdot(Du_\varepsilon-D\Phi_\varepsilon)dxdt.\end{aligned}$$

Following the same line of reasoning as we have done it to obtain (4.14), we are led to

$$\begin{aligned}B_\varepsilon&\equiv\int_Q(a^\varepsilon(-,u_\varepsilon,Du_\varepsilon)-a^\varepsilon(-,u_\varepsilon,D\Phi_\varepsilon))\cdot(Du_\varepsilon-D\Phi_\varepsilon)dxdt\\&\to\iint_{Q\times\Delta(A)}(b(\cdot,\cdot,u_0,\mathbb{D}\mathbf{u})-b(\cdot,\cdot,u_0,\mathbb{D}\Phi))\cdot\mathbb{D}(\mathbf{u}-\Phi)dxdtd\beta\equiv B\end{aligned}$$

when $E'\ni\varepsilon\to 0$. Thus we deduce the existence of $\varepsilon_0>0$ such that we have $B_\varepsilon\leq B+\eta/4$ whenever $E'\ni\varepsilon\leq\varepsilon_0$. Mere computations lead us to

$$B\leq k\left(\frac{3}{k}\right)^2\left(\frac{\eta}{4}\right)^2$$

since $B\leq k\,\|\mathbb{D}(\mathbf{u}-\Phi)\|^2_{L^p(Q\times\Delta(A))}$ and $\|\mathbb{D}(\mathbf{u}-\Phi)\|_{L^p(Q\times\Delta(A))}\leq\frac{3\eta}{4k}$. We therefore obtain $B_\varepsilon\leq k\left(\frac{3}{k}\right)^2\left(\frac{\eta}{4}\right)^2+\frac{\eta}{4}$ provided $E'\ni\varepsilon\leq\varepsilon_0$. Whence

$$\|Du_\varepsilon-D\Phi_\varepsilon\|_{L^p(Q)^N}\leq\left[\frac{1}{c_0}\left(k\left(\frac{3}{k}\right)^2\left(\frac{\eta}{4}\right)^2+\frac{\eta}{4}\right)\right]^{\frac{1}{p}}\ \text{for } E'\ni\varepsilon\leq\varepsilon_0.$$

Set $K(\eta)=\left[\frac{1}{c_0}\left(9\left(\frac{\eta}{4}\right)^2+\frac{\eta}{4}\right)\right]^{\frac{1}{p}}$; then we have $\left[\frac{1}{c_0}\left(k\left(\frac{3}{k}\right)^2\left(\frac{\eta}{4}\right)^2+\frac{\eta}{4}\right)\right]^{\frac{1}{p}}\leq K(\eta)$ since $k>1$, and further

$$\|b_0(\cdot,\cdot,u_0,\mathbb{D}\mathbf{u})-\widehat{\chi}\|_{L^{p'}(Q\times\Delta(A))}\leq\frac{3\eta}{4}+K(\eta)\,[c+K(\eta)]^{p-2}.$$

Note that $K(\eta)$ is independent of k. The above inequality holds for all $\eta>0$. Since $K(\eta)\to 0$ when $\eta\to 0$, letting $\eta\to 0$ yields $\widehat{\chi}=b_0(\cdot,\cdot,u_0,\mathbb{D}\mathbf{u})$, that is, $\chi=a_0(-,u_0,\overline{\mathbb{D}}\mathbf{u})$.

The uniqueness of u_1 and u_2 is obtained by a mere routine (see the proof of [45, Theorem 3.5]), from which the proof is complete. □

4.3 Some applications of Theorem 4.5

One can work out some homogenization problems related to problem (1.1)-(4.1). In particular one can solve:

(P)$_1$ **The periodic homogenization problem** stated as follows: For each fixed $0\leq i\leq N$ and any $(x,t,\mu,\lambda)\in Q\times\mathbb{R}\times\mathbb{R}^N$, the function $(y,z,\tau)\mapsto a_i(x,t,y,z,\tau,\mu,\lambda)$ is Y-periodic in $y\in\mathbb{R}^N$, Z-periodic in $z\in\mathbb{R}^N$ and $\mathcal{T}$-periodic in $\tau\in\mathbb{R}$, where $Y=Z=(0,1)^N$ and $\mathcal{T}=(0,1)$. Here we get the homogenization of (1.1) with $A=\mathcal{C}_{\mathrm{per}}(Y)\odot\mathcal{C}_{\mathrm{per}}(Z)\odot\mathcal{C}_{\mathrm{per}}(\mathcal{T})$. Theorem 4.5 in this case reads as

Theorem 4.6. *Let $2 \leq p < \infty$. For each fixed $\varepsilon > 0$, let u_ε be the unique solution to* (1.1)*. Then, there exists a subsequence of ε not relabeled such that, as $\varepsilon \to 0$,*

$$u_\varepsilon \to u_0 \text{ in } L^p(0,T;W_0^{1,p}(\Omega))\text{-weak}$$

$$\frac{\partial u_\varepsilon}{\partial t} \to \frac{\partial u_0}{\partial t} \text{ in } L^{p'}(0,T;W^{-1,p'}(\Omega))\text{-weak}$$

$$\frac{\partial u_\varepsilon}{\partial x_i} \to \frac{\partial u_0}{\partial x_i} + \frac{\partial u_1}{\partial y_i} + \frac{\partial u_2}{\partial z_i} \text{ in } L^p(Q)\text{-weak } R\Sigma \ (1 \leq i \leq N)$$

where the vector function $\mathbf{u} = (u_0, u_1, u_2)$ solves the variational equation

$$\begin{cases} \mathbf{u} = (u_0,u_1,u_2) \in \mathbb{F}_0^{1,p} = V_0^p \times L^p(Q \times \mathcal{T}; W_\#^{1,p}(Y)) \times L^p(Q \times Y \times \mathcal{T}; W_\#^{1,p}(Z)) : \\ \int_0^T \langle u_0'(t), v_0(t)\rangle\, dt + \iint_{Q\times Y\times Z\times \mathcal{T}} a(x,t,y,z,\tau,u_0,\mathbb{D}\mathbf{u}) \cdot \mathbb{D}\mathbf{v} dxdtdydzd\tau \\ + \iint_{Q\times Y\times Z\times \mathcal{T}} a_0(x,t,y,z,\tau,u_0,\mathbb{D}\mathbf{u}) v_0 dxdtdydzd\tau = \int_0^T \langle f(t), v_0(t)\rangle\, dt \\ \text{for all } \mathbf{v} = (v_0,v_1,v_2) \in \mathbb{F}_0^{1,p} \end{cases} \tag{4.17}$$

with $\mathbb{D}\mathbf{v} = Dv_0 + D_y v_1 + D_z v_2$ for $\mathbf{v} = (v_0,v_1,v_2) \in \mathbb{F}_0^{1,p}$, $W_\#^{1,p}(Y) = \{u \in W_{\text{per}}^{1,p}(Y;\mathbb{R}) : \int_Y u(y)dy = 0\}$ and a similar definition for $W_\#^{1,p}(Z)$, $W_{\text{per}}^{1,p}(Y;\mathbb{R})$ being the space of Y-periodic functions in $W_{\text{loc}}^{1,p}(\mathbb{R}^N;\mathbb{R})$. Moreover u_1 and u_2 are unique and any weak Σ-limit point in V^p of $(u_\varepsilon)_{\varepsilon>0}$ is a solution to problem (4.17)*.*

One can also solve the following homogenization problems for (1.1):

(P)$_2$ The almost periodic homogenization problem stated as follows:

$a_i(x,t,\cdot,\cdot,\cdot,\mu,\lambda) \in B_{AP}^{p'}(\mathbb{R}_y^N \times \mathbb{R}_z^N \times \mathbb{R}_\tau)$ for any $(x,t,\mu,\lambda) \in \overline{Q} \times \mathbb{R}^{N+1}$, $0 \leq i \leq N$

which yields the homogenization of (1.1) with $A = AP(\mathbb{R}_y^N) \odot AP(\mathbb{R}_z^N) \odot AP(\mathbb{R}_\tau) = AP(\mathbb{R}_y^N \times \mathbb{R}_z^N \times \mathbb{R}_\tau)$.

(P)$_3$ The weakly almost periodic homogenization problem I:

$a_i(x,t,\cdot,\cdot,\cdot,\mu,\lambda) \in B_{WAP}^{p'}(\mathbb{R}_z^N; B_{AP}^{p'}(\mathbb{R}_{y,\tau}^{N+1}))$ for any $(x,t,\mu,\lambda) \in \overline{Q} \times \mathbb{R}^{N+1}$, $0 \leq i \leq N$

which leads to the homogenization of (1.1) with $A = AP(\mathbb{R}_y^N) \odot WAP(\mathbb{R}_z^N) \odot AP(\mathbb{R}_\tau)$.

(P)$_4$ The weakly almost periodic homogenization problem II:

$a_i(x,t,\cdot,\cdot,\cdot,\mu,\lambda) \in B_{AP}^{p'}(\mathbb{R}_y^N; B_{WAP}^{p'}(\mathbb{R}_z^N; B_{WAP}^{p'}(\mathbb{R}_\tau)))$ for any $(x,t,\mu,\lambda) \in \overline{Q} \times \mathbb{R}^{N+1}$, $0 \leq i \leq N$

which yields the homogenization of (1.1) with $A = AP(\mathbb{R}_y^N) \odot WAP(\mathbb{R}_z^N) \odot WAP(\mathbb{R}_\tau)$.

(P)$_5$ The fully weakly almost periodic homogenization problem III:

$a_i(x,t,\cdot,\cdot,\cdot,\mu,\lambda) \in B_{WAP}^{p'}(\mathbb{R}_y^N; B_{WAP}^{p'}(\mathbb{R}_z^N; B_{WAP}^{p'}(\mathbb{R}_\tau)))$ for any $(x,t,\mu,\lambda) \in \overline{Q} \times \mathbb{R}^{N+1}$, $0 \leq i \leq N$.

Here the suitable algebra wmv is $A = WAP(\mathbb{R}_y^N) \odot WAP(\mathbb{R}_z^N) \odot WAP(\mathbb{R}_\tau)$.

(P)$_6$ The homogenization problem in the Fourier-Stieltjes algebra. We first need to define the Fourier-Stieltjes algebra $FS(\mathbb{R}^N)$ on $\mathbb{R}^N$.

Definition 4.7. *The Fourier-Stieltjes algebra on $\mathbb{R}^N$ is defined as the closure in $\mathcal{B}(\mathbb{R}^N)$ of the space*

$$FS_*(\mathbb{R}^N) = \left\{ f : \mathbb{R}^N \to \mathbb{R},\ f(x) = \int_{\mathbb{R}^N} \exp(ix\cdot y)d\nu(y) \textit{ for some } \nu \in \mathcal{M}_*(\mathbb{R}^N) \right\}$$

where $\mathcal{M}_*(\mathbb{R}^N)$ denotes the space of complex valued measures ν with finite total variation: $|\nu|(\mathbb{R}^N) < \infty$. We denote it by $FS(\mathbb{R}^N)$.

Since by [18] any function in $FS_*(\mathbb{R}^N)$ is a weakly almost periodic continuous function, we have that $FS(\mathbb{R}^N) \subset WAP(\mathbb{R}^N)$. Moreover thanks to [14, Theorem 4.5] $FS(\mathbb{R}^N)$ is a proper subalgebra of $WAP(\mathbb{R}^N)$.

As $FS(\mathbb{R}^N)$ is an ergodic algebra which is translation invariant (this is easily seen: indeed $FS_*(\mathbb{R}^N)$ is translation invariant) we see that hypotheses of Theorem 3.15 are satisfied with any algebra $A = FS(\mathbb{R}^N_y) \odot FS(\mathbb{R}^N_z) \odot A_\tau$, A_τ being any algebra wmv on $\mathbb{R}_\tau$.

With all this in mind, one can solve the homogenization problem for (1.1) under the assumption that

$$a_i(x,t,\cdot,\cdot,\cdot,\mu,\lambda) \in B^{p'}_{FS}(\mathbb{R}_\tau; B^{p'}_{FS}(\mathbb{R}^N_y; B^{p'}_{FS}(\mathbb{R}^N_z))) \text{ for any } (x,t,\mu,\lambda) \in \overline{Q} \times \mathbb{R}^{N+1},$$
$$0 \le i \le N.$$

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