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On the Isothermal Compressible Euler Equations with Frictional Damping

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Abstract

This paper aims at initial-boundary value problems(IBVP) for the isothermal compressible Euler equations with damping on bounded domains. We first prove global existence and uniqueness of classical solutions for smooth initial data. Time asymptotically, it is shown that the density converges to its average over the domain and the momentum vanishes as time tends to infinity. Due to diffusion and boundary effects, the convergence rate is shown to be exponential. Second, based on the entropy principle, it is shown that similar results hold for L^∞ entropy weak solutions.

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1 Introduction

In this paper, we consider the compressible Euler equation with frictional damping:

$$\begin{cases} \rho_t + \nabla\cdot(\rho U) = 0, \ \ \mathbf{x}\in\mathbb{R}^d,\ t>0, \\ (\rho U)_t + \nabla\cdot(\rho U\otimes U) + \nabla P(\rho) = -\alpha\rho U. \end{cases} \tag{1.1}$$

Such a system occurs in the mathematical modeling of compressible flow through a porous medium. Here ρ, U and P denotes the density, velocity and pressure respectively, and the constant $\alpha > 0$ models friction. Assuming the flow is a polytropic perfect gas, then $P(\rho) = P_0\rho^\gamma$, with P_0 a positive constant, and $\gamma \geq 1$ the adiabatic gas exponent. The case $\gamma > 1$ is commonly referred as the isentropic case, while $\gamma = 1$ corresponds to the so-called isothermal case which is the main focus of this paper. Without loss of generality, we take $P_0 = \alpha = 1$ throughout this paper.

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In this paper, we consider (1.1) in bounded domains in $\mathbb{R}^d,\ d=1,2,3.$ The system is supplemented by the following initial and boundary conditions:

$$\begin{cases} (\rho,U)(\mathbf{x},0)=(\rho_0,U_0)(\mathbf{x}), & \mathbf{x}\in\Omega,\\ U\cdot\mathbf{n}|_{\partial\Omega}=0, & t\geq 0,\\ \int_\Omega \rho_0(\mathbf{x})d\mathbf{x}=\rho_*>0, \end{cases} \tag{1.2}$$

where $\Omega\subset\mathbb{R}^d$ is a bounded domain with smooth boundary $\partial\Omega$, $\mathbf{n}$ is the unit outward normal vector on the boundary of Ω and the last condition is imposed to avoid the trivial case, $\rho\equiv 0$.

Due to strong physical background and significant mathematical challenge, system (1.1) and its time-asymptotic behavior have received considerable attentions, and investigations have been carried on for decades. Extensive literatures are available for the isentropic case (i.e. $\gamma>1$). When $d=1$, we refer the readers to [7, 8, 12, 13, 14, 15, 16, 17, 18, 21, 23, 32, 35, 37, 38, 39, 40, 41, 42, 43, 47, 51, 53] for both the Cauchy problem and initial-boundary value problems of (1.1) under miscellaneous initial or initial-boundary conditions. When $d>1$, the readers are referred to [44, 48, 49, 50].

For the isothermal case (i.e. $\gamma=1$), when $d=1$, global BV solutions to the Cauchy problem of (1.1) have been constructed by Luskin and Temple in [31] and by Dafermos in [6] where P is allowed to be any smooth, increasing function for the initial data with small oscillation about some fixed equilibrium state, by working on the so-called p-system with damping (i.e. (1.1) in the Lagrangian coordinates). Recently, the qualitative behavior of L^∞ entropy weak solutions to the Cauchy problem of (1.1) with large rough initial data containing vacuum is studied by Huang and Pan in [22]. There are also some results on other related physical models, see e.g. [20, 24, 25].

However, to the best of the author's knowledge, little is known about the initial-boundary value problems for (1.1) with $\gamma=1$ for any $d\geq 1$. The purpose of this paper is to systematically study the qualitative behavior of solutions to (1.1) with $\gamma=1$ in the presence of physical boundaries. The works of current paper are strongly motivated by [43] and [44], where the authors proved global existence and long time behavior of L^∞ entropy weak solutions (small smooth solutions resp.) to (1.1)–(1.2) with $\gamma>1$ when $d=1$ ($d=3$ resp.). However, we observe that, when $\gamma=1$, all the underlying methods used in [43] and [44] fail. Therefore, new approaches have to be developed in order to deal with this extreme case.

We will first study the global existence and long time behavior of classical solutions to (1.1)–(1.2) for $d=1,2,3$. We will show that the classical solution exists globally in time and converges to a constant equilibrium state exponentially as time evolves provided that the initial perturbation around the constant state is small and smooth. On the other hand, when initial data is large or rough or contains vacuum states, local smooth solutions have been demonstrated to be breaking down in finite time [28, 29, 30, 52]. In this situation, one has to switch attention from smooth solutions to entropy weak solutions. In the second part of the present paper, we will prove global existence and long time behavior of L^∞ entropy weak solutions to (1.1)–(1.2) for large and rough initial data containing vacuum for $d=1$.

Concerning the asymptotic behavior of solutions of (1.1)–(1.2), due to the dissipative mechanism in the momentum equation and the boundary effect, the kinetic energy is expected to vanish as time tends to infinity, while the potential energy will converge to a constant. To identify the global attractor of ρ, we integrate the density equation using the boundary condition to get

$$\int_\Omega \rho(\mathbf{x},t)d\mathbf{x} = \int_\Omega \rho_0(\mathbf{x})d\mathbf{x} = \rho_*.$$

This suggests that the asymptotic state of the solution of (1.1)–(1.2) should be $(\rho,U)|_{t\to\infty} = (\rho_*/|\Omega|,0)$. We will verify the conjecture for both small smooth solutions and large rough solutions. We will show that both solutions converge to the constant equilibrium state exponentially as time goes to infinity.

We remark that, for the isentropic case ($\gamma > 1$), the IBVP (1.1)–(1.2) is equivalent to

$$\begin{cases} \tilde{\rho}_t = \Delta P(\tilde{\rho}), \\ \tilde{M} = -\nabla P(\tilde{\rho}); \\ \nabla P(\tilde{\rho})\cdot \mathbf{n}|_{\partial\Omega} = 0, \end{cases} \tag{1.3}$$

for large time, provided that the two systems carry the same initial mass, as has been demonstrated in [43] and [44]. Here, the first equation is the famous porous medium equation and the second one states the classical Darcy's law. Therefore, when $\gamma = 1$, it is natural to conjecture that the density should satisfy the linear heat equation and the momentum obeys the Darcy's law for large time provided that the two systems carry the same initial mass. Indeed, from the profile of the linear heat equation one can see that the solution to (1.3) converges to the constant equilibrium state $(\rho_*/|\Omega|,0)$ too. Hence, the triangle inequality implies immediately the long time equivalence of the two systems. Since the profile of the linear heat equation is clear, we shall not go through the details of this part in the present paper.

The plan of the rest part of the paper is organized as follows. In Section 2, we study small smooth solutions to (1.1)–(1.2) for $d = 1,2,3$. We will start with some preliminaries and then show the global existence and long time behavior of small smooth solutions to (1.1)–(1.2) simultaneously. Section 3 is devoted to the study of the one-dimensional model. We will show global existence and long time behavior of L^∞ entropy weak solutions to (1.1)–(1.2). Similar to the layout of Section 2, Section 3 consists of subsections containing preliminary, global existence and large time behavior.

2 Small smooth solutions

In this section, we study global existence and long time behavior of smooth solutions to (1.1)–(1.2). For simplicity, we only present the proof of the case for $d = 3$. The other cases can be treated similarly. When $\gamma = 1$, system (1.1) turns to be

$$\begin{cases} \rho_t + \nabla\cdot(\rho U) = 0, \\ \rho U_t + \rho(U\cdot\nabla)U + \nabla\rho = -\rho U. \end{cases} \tag{2.1}$$

One of the key ingredients of the proof for the isentropic case ($\gamma > 1$) given in [44, 48] is to symmetrize the system of equations by using the nonlinear transformation $\phi = \rho^\theta/\theta$

(commonly referred as the *sound speed*), where $\theta = (\gamma - 1)/2$. Unfortunately, this technique fails when $\gamma = 1$. However, by multiplying the second equation of (2.1) by ρ we get

$$\begin{cases} \rho_t + U \cdot \nabla\rho + \rho\nabla \cdot U = 0, \\ \rho^2 U_t + \rho^2 (U \cdot \nabla) U + \rho \nabla \rho = -\rho^2 U, \end{cases} \tag{2.2}$$

which is a symmetric hyperbolic system of balance laws.

Let (σ, U) be the perturbation of (ρ, U) around the equilibrium state $(\rho_*/|\Omega|, 0)$, i.e., $(\sigma, U) = (\rho - \rho_*/|\Omega|, U)$. After plugging (σ, U) into (2.2), and assuming $\rho_*/|\Omega| = 1$ without loss of generality, we get

$$\begin{cases} \sigma_t + U \cdot \nabla\sigma + \sigma\nabla \cdot U + \nabla \cdot U = 0, \\ (\sigma+1)^2 U_t + (\sigma+1)^2 (U \cdot \nabla) U + \sigma\nabla\sigma + \nabla\sigma = -(\sigma+1)^2 U. \end{cases} \tag{2.3}$$

And the initial and boundary conditions become

$$\begin{cases} (\sigma, U)(\mathbf{x}, 0) = (\rho_0 - 1, U_0)(\mathbf{x}), \quad \mathbf{x} \in \Omega, \\ U \cdot \mathbf{n}|_{\partial\Omega} = 0, \quad t \geq 0. \end{cases} \tag{2.4}$$

Since (2.3) is equivalent to (2.1) for smooth solutions, it suffices to prove global existence and long time behavior for (2.3)–(2.4).

2.1 Preliminaries and main result

The following notations will be used throughout this section.

Notation 2.1. Throughout this section, $\|\cdot\|_{L^p}$, $\|\cdot\|_{L^\infty}$ and $\|\cdot\|_{H^s}$ denotes the norm of the usual Lebesgue measurable function spaces L^p $(1 \leq p < \infty)$, L^∞ and the usual Hilbert space H^s respectively. We denote the norm $\|\cdot\|_{L^2}$ by $\|\cdot\|$ and $\|\cdot\|_{H^s}$ by $\|\cdot\|_s$ respectively. For simplicity, we will use the following notation: For any vector valued function $\vec{f} = (f_1, f_2, \ldots, f_m) : \mathbb{R}^n \to \mathbb{R}^m$, we denote its norm in a function space X by $\|\vec{f}\|_X \equiv \sum_{i=1}^m \|f_i\|_X$. The energy space under consideration is:

$$X_3([0,T], \Omega) \equiv \left\{ F : \Omega \times [0,T] \to \mathbb{R}^3 (\text{or } \mathbb{R}) \;\middle|\; \partial_t^l F \in L^\infty([0,T]; H^{3-l}(\Omega)), l = 0,1,2,3 \right\},$$

equipped with norm

$$\|F\|_{3,T} \equiv \operatorname{ess}\sup_{0 \leq t \leq T} |||F(\cdot,t)||| \equiv \operatorname{ess}\sup_{0 \leq t \leq T} \left[\sum_{l=0}^{3} \|\partial_t^l F(\cdot,t)\|_{3-l}^2 \right]^{1/2},$$

for any $F \in X_3([0,T], \Omega)$. Unless specified, C will denote a generic constant which is independent of time. The value of C may vary line by line according to the context.

The following theorem is the main result of this section, which gives the global existence and long time behavior of smooth solutions to (2.3)–(2.4) for small initial data. For convenience, we denote the total energy of the solution by

$$W(t) \equiv |||\sigma(t)|||^2 + |||U(t)|||^2 = \sum_{l=0}^{3} \left(\|\partial_t^l \sigma(t)\|_{3-l}^2 + \|\partial_t^l U(t)\|_{3-l}^2 \right). \tag{2.5}$$

Then we have

Theorem 2.1. *There exists $\varepsilon > 0$ such that if $W(0) \leq \varepsilon^2$, then there is a unique global classical solution to (2.3)–(2.4) such that there exist positive constants $C > 0, \eta > 0$, which are independent of t, such that*

$$W(t) \leq CW(0)e^{-\eta t}. \tag{2.6}$$

2.2 Proof of Theorem 2.1

The proof of Theorem 2.1 is based on the local existence result and *a priori* energy estimates. Since the local existence result is standard, see for example [37, 46, 48], we only present the *a priori* energy estimates here. First of all, the following lemma (see [1]) plays an important role in the estimation of a vector-valued function $U \in \mathbb{R}^d$, which gives the estimate of ∇U by $\nabla \cdot U$ and $\nabla \times U$.

Lemma 2.2. *Let $U \in H^s(\Omega)$ be a vector-valued function satisfying $U \cdot \mathbf{n}|_{\partial\Omega} = 0$, where $\mathbf{n}$ is the unit outward normal to $\partial\Omega$. Then*

$$\|U\|_s \leq C(\|\nabla \times U\|_{s-1} + \|\nabla \cdot U\|_{s-1} + \|U\|_{s-1}),$$

for $s \geq 1$, and the constant C depends only on s and Ω.

The next lemma is an application of Lemma 2.2. The lemma states that the spatial derivatives of the solution to (2.3)–(2.4) are bounded by the temporal derivatives and the vorticity. Let $\omega = \nabla \times U$ and define

$$E(t) \equiv \sum_{l=0}^{3} \left(\|\partial_t^l \sigma\|^2 + \|\partial_t^l U\|^2\right), \text{ and } V(t) \equiv \sum_{l=0}^{2} \|\partial_t^l \omega\|_{2-l}^2. \tag{2.7}$$

Lemma 2.3. *Let (σ, U) be the solution to (2.3)–(2.4). There is a small constant $\bar{\delta}$ such that if $W(t) \leq \bar{\delta}$, then there exists a constant $C_1 > 0$ such that*

$$W(t) \leq C_1\Big(V(t) + E(t)\Big).$$

Proof. From the velocity equation $(2.3)_2$ we have

$$\nabla\sigma = -(\sigma+1)^2 U_t - (\sigma+1)^2 (U \cdot \nabla)U - \sigma\nabla\sigma - (\sigma+1)^2 U, \tag{2.8}$$

which implies, by the smallness of $W(t)$ and the Sobolev embedding $H^2(\Omega) \hookrightarrow C^0(\bar{\Omega})$, that

$$\begin{aligned} \|\nabla\sigma\|^2 &\leq C\big(\|U_t\|^2 + \|U\|^2\big) + C\|U\|_{L^\infty}^2 \|\nabla U\|^2 + \|\sigma\|_{L^\infty}^2 \|\nabla\sigma\|^2 \\ &\leq C\big(\|U\|^2 + \|U_t\|^2\big) + CW(t)^2. \end{aligned} \tag{2.9}$$

The continuity equation $(2.3)_1$ implies

$$\nabla \cdot U = -\sigma_t - U \cdot \nabla\sigma - \sigma\nabla \cdot U. \tag{2.10}$$

Therefore, we obtain

$$\|\nabla \cdot U\|^2 \leq C\Big(\|\sigma_t\|^2 + W(t)^2\Big). \tag{2.11}$$

Using Lemma 2.2 with $s=1$ and (2.11) we have

$$\begin{aligned}\|U\|_1^2 &\le C\big(\|\omega\|^2+\|\nabla\cdot U\|^2+\|U\|^2\big)\\ &\le C\big(\|\omega\|^2+\|\sigma_t\|^2+\|U\|^2+W(t)^2\big).\end{aligned}$$

Next, we take temporal derivatives of (2.8) and (2.10). It is clear that every temporal derivative up to order two of $\nabla\sigma$ and $\nabla\cdot U$ is again bounded by $E(t)$. Furthermore, together with an induction on the number of spatial derivatives, the same is true for any derivative up to order two of $\nabla\sigma$ and $\nabla\cdot U$. By applying Lemma 2.2 with $s=1,2,3$ respectively and using the smallness of $W(t)$ we finally deduce the lemma. This completes the proof of Lemma 2.3. □

Lemma 2.3 reduces the estimate of $W(t)$ to those for $E(t)$ and $V(t)$. Our next goal is to deal with the estimates of $E(t)$ and $V(t)$.

Lemma 2.4. *There is a constant $C>0$ such that*

$$\frac{d}{dt}\sum_{l=0}^{3}\big(\|\partial_t^l\sigma\|^2+\|(\sigma+1)\partial_t^lU\|^2\big)+2\sum_{l=0}^{3}\|(\sigma+1)\partial_t^lU\|^2\le CW(t)^{\frac{3}{2}}. \tag{2.12}$$

Proof. Zero order estimate: Taking L^2 inner product of $(2.3)_1$ with σ and $(2.3)_2$ with U respectively, adding the results and integrating by parts using the boundary condition we get

$$\frac{1}{2}\frac{d}{dt}\big(\|\sigma\|^2+\|(\sigma+1)U\|^2\big)+\|(\sigma+1)U\|^2=-\int_\Omega(\sigma+1)^2\big((U\cdot\nabla)U\big)\cdot U\,d\mathbf{x}+\int_\Omega\sigma_t(\sigma+1)|U|^2d\mathbf{x},$$

from which we derive

$$\frac{1}{2}\frac{d}{dt}\big(\|\sigma\|^2+\|(\sigma+1)U\|^2\big)+\|(\sigma+1)U\|^2\le C\|(\nabla U,\sigma_t)\|_{L^\infty}\|U\|^2.$$

By the Sobolev embedding $H^2(\Omega)\hookrightarrow C^0(\bar{\Omega})$ and (2.5) we get

$$\frac{d}{dt}\big(\|\sigma\|^2+\|(\sigma+1)U\|^2\big)+2\|(\sigma+1)U\|^2\le CW(t)^{\frac{3}{2}}.$$

First order estimate: Differentiating (2.3) with respect to t, and taking L^2 inner products we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\big(\|\sigma_t\|^2+\|(\sigma+1)U_t\|^2\big)+\|(\sigma+1)U_t\|^2\\ &=-\int_\Omega\bigg(\frac{1}{2}\sigma_t^2\nabla\cdot U+\sigma_t(\nabla\sigma\cdot U_t)+2(\sigma+1)\sigma_t\big[(U\cdot\nabla)U\big]\cdot U_t+(\sigma+1)\sigma_t|U_t|^2\\ &\quad(\sigma+1)^2\big[(U_t\cdot\nabla)U\big]\cdot U_t+(\sigma+1)^2\big[(U\cdot\nabla)U_t\big]\cdot U_t+2(\sigma+1)\sigma_tU\cdot U_t\bigg)d\mathbf{x},\end{aligned}$$

where we have used the boundary conditions $U\cdot\mathbf{n}|_{\partial\Omega}=0$ and $U_t\cdot\mathbf{n}|_{\partial\Omega}=0$. Simple estimates then yield

$$\frac{1}{2}\frac{d}{dt}\big(\|\sigma_t\|^2+\|(\sigma+1)U_t\|^2\big)+2\|(\sigma+1)U_t\|^2\le CW(t)^{\frac{3}{2}}.$$

Second order estimate: Repeating the above procedure again for the second order temporal derivatives, and noting that no term with order higher than three will appear, we get the following

$$\frac{d}{dt}\left(\|\sigma_{tt}\|^2+\|(\sigma+1)U_{tt}\|^2\right)+2\|(\sigma+1)U_{tt}\|^2\le CW(t)^{\frac{3}{2}}.$$

Third order estimate: The third order estimate is tricky. We calculate $\partial_t^3\sigma\partial_t^3(2.1)_1+\partial_t^3U\cdot\partial_t^3(2.1)_2$ and get

$$\frac{d}{dt}\left(\|\sigma_{ttt}\|^2+\|(\sigma+1)U_{ttt}\|^2\right)+2\|(\sigma+1)U_{ttt}\|^2\le CW(t)^{\frac{3}{2}}+CY(t), \tag{2.13}$$

where

$$\begin{aligned}Y(t)=&\int_\Omega\Big(\Big[|U_{tt}||\nabla\sigma_t|+|\sigma_{tt}||\nabla U_t|\Big]|\sigma_{ttt}|+\Big[|\sigma_{tt}||U_{tt}|+|\sigma_{tt}||\nabla U_t|+|U_{tt}||\nabla U_t|+|\sigma_{tt}||\nabla\sigma_t|\Big]|U_{ttt}|\Big)d\mathbf{x}\\&-\int_\Omega\Big(\Big[U\cdot\nabla\sigma_{ttt}+(\sigma+1)\nabla\cdot U_{ttt}\Big]\sigma_{ttt}+\Big[(\sigma+1)^2(U\cdot\nabla)U_{ttt}+(\sigma+1)\nabla\sigma_{ttt}\Big]\cdot U_{ttt}\Big)d\mathbf{x}\\\equiv&I_1+I_2,\end{aligned}$$

which can not be simply estimated by the Sobolev embedding $H^2(\Omega)\hookrightarrow C^0(\bar\Omega)$. However, the term I_1 can be estimated by using the Sobolev inequality $\|f\|_{L^4}\le C\|f\|_{H^1}$. Since all the terms in I_1 have the same structure, to simplify the presentation, we only choose one of them as a representative. For example,

$$\begin{aligned}\int_\Omega|U_{tt}||\nabla\sigma_t||\sigma_{ttt}|d\mathbf{x}\le&\|U_{tt}\|_{L^4}\|\nabla\sigma_t\|_{L^4}\|\sigma_{ttt}\|\\\le&C\|U_{tt}\|_{H^1}\|\nabla\sigma_t\|_{H^1}\|\sigma_{ttt}\|\le CW(t)^{\frac{3}{2}}.\end{aligned}$$

Then we have

$$I_1\le CW(t)^{\frac{3}{2}}. \tag{2.14}$$

For the term I_2, after integration by parts we get

$$I_2=\int_\Omega\Big(\frac{1}{2}\sigma_{ttt}^2\nabla\cdot U+\sigma_{ttt}U_{ttt}\cdot\nabla\sigma+\frac{1}{2}|U_{ttt}|^2\nabla\cdot\big[(\sigma+1)^2U\big]\Big)d\mathbf{x}\le CW(t)^{\frac{3}{2}}. \tag{2.15}$$

Combining (2.13)–(2.15) we have

$$\frac{d}{dt}\left(\|\sigma_{ttt}\|^2+\|(\sigma+1)U_{ttt}\|^2\right)+2\|(\sigma+1)U_{ttt}\|^2\le CW(t)^{\frac{3}{2}}.$$

This completes the proof of Lemma 2.4. □

Lemma 2.4 contains the dissipation in velocity. In the next lemma we are going to explore the dissipation in density.

Lemma 2.5. *There exist constants $c_0, C>0$ such that*

$$\frac{d}{dt}\Big(\sum_{l=1}^3\int_\Omega\big(-\partial_t^{l-1}\sigma\partial_t^l\sigma\big)d\mathbf{x}\Big)+\sum_{l=0}^3\|\partial_t^l\sigma\|^2\le CW(t)^{\frac{3}{2}}+c_0\sum_{l=0}^3\|(\sigma+1)\partial_t^lU\|^2. \tag{2.16}$$

Proof. First of all, by definition and Poincaré's inequality we know that $\|\sigma\|^2 = \|\rho - 1\|^2 \le C\|\nabla\sigma\|^2$. Using (2.9) we obtain

$$\|\sigma\|^2 \le C\big(\|U\|^2 + \|U_t\|^2\big) + W(t)^2.$$

Dividing the second equation of $(2.3)_2$ by $(\sigma+1)^2$ we have

$$U_t + (U\cdot\nabla)U + \frac{\nabla\sigma}{\sigma+1} = -U. \tag{2.17}$$

Calculating $\partial_t(2.3)_1 - \nabla\cdot(2.17)$ we get

$$\sigma_{tt} + \nabla\cdot(U\sigma)_t - \nabla\cdot\Big[U\cdot\nabla U + \frac{\nabla\sigma}{\sigma+1} + U\Big] = 0. \tag{2.18}$$

Taking L^2 inner product of (2.18) with σ we get

$$\frac{d}{dt}\Big(\int_\Omega \sigma\sigma_t d\mathbf{x}\Big) - \|\sigma_t\|^2 - \int_\Omega (U\sigma)_t\cdot\nabla\sigma d\mathbf{x} + \int_\Omega \Big[U\cdot\nabla U + \frac{\nabla\sigma}{\sigma+1} + U\Big]\cdot\nabla\sigma d\mathbf{x} = 0,$$

which gives

$$-\frac{d}{dt}\Big(\int_\Omega \sigma\sigma_t d\mathbf{x}\Big) + \|\sigma_t\|^2 \le C\big(W(t)^{\frac{3}{2}} + \|\nabla\sigma\|^2 + \|U\|^2\big). \tag{2.19}$$

Combining (2.19) and (2.9) we have

$$-\frac{d}{dt}\Big(\int_\Omega \sigma\sigma_t d\mathbf{x}\Big) + \|\sigma_t\|^2 \le C\big(W(t)^{\frac{3}{2}} + \|U\|^2 + \|U_t\|^2\big).$$

Next, we take temporal derivatives of (2.18). Similar derivations show that

$$-\frac{d}{dt}\Big(\int_\Omega \sigma_t\sigma_{tt} d\mathbf{x}\Big) + \|\sigma_{tt}\|^2 \le C\big(W(t)^{\frac{3}{2}} + \|U_t\|^2 + \|U_{tt}\|^2\big),$$
$$-\frac{d}{dt}\Big(\int_\Omega \sigma_{tt}\sigma_{ttt} d\mathbf{x}\Big) + \|\sigma_{ttt}\|^2 \le C\big(W(t)^{\frac{3}{2}} + \|U_{tt}\|^2 + \|U_{ttt}\|^2\big).$$

When $W(t)$ is small enough, we have $|\sigma+1| > 1/2$, which together with the above estimates imply (2.16). This completes the proof of Lemma 2.5. □

Now, we are ready to combine Lemma 2.4 and Lemma 2.5 to characterize the total dissipation. For this purpose, we let $C_2 \equiv \max\{2, c_0\}$, and define

$$G(t) \equiv \sum_{l=0}^{3}\big(\|\partial_t^l\sigma\|^2 + \|(\sigma+1)\partial_t^l U\|^2\big),$$
$$E_1(t) \equiv C_2 G(t) - \sum_{l=1}^{3}\int_\Omega \big(\partial_t^{l-1}\sigma\partial_t^l\sigma\big)d\mathbf{x}$$
$$= C_2\sum_{l=0}^{3}\big(\|\partial_t^l\sigma\|^2 + \|(\sigma+1)\partial_t^l U\|^2\big) - \sum_{l=1}^{3}\int_\Omega\big(\partial_t^{l-1}\sigma\partial_t^l\sigma\big)d\mathbf{x}.$$

It is easy to see that $E_1(t) \ge 0$ for any $t \ge 0$. Then we have

Lemma 2.6. *There exist constants $C_3, C > 0$ such that*

$$\frac{d}{dt}E_1(t) + C_3 G(t) \le CW(t)^{\frac{3}{2}}. \tag{2.20}$$

Proof. $C_2 \times (2.12) + (2.16)$ yields

$$\frac{d}{dt}E_1(t) + c_0 \sum_{l=0}^{3} \|(\sigma+1)\partial_t^l U\|^2 + \sum_{l=0}^{3} \|\partial_t^l \sigma\|^2 \le CW(t)^{\frac{3}{2}}. \tag{2.21}$$

Let $C_3 = \min\{c_0, 1\}$, then (2.20) follows directly from (2.21). □

The next lemma is contributed to the estimate of $V(t)$ defined in (2.7).

Lemma 2.7. *For $V(t)$ defined in (2.7), there exists a constant $C > 0$ such that*

$$\frac{d}{dt}V(t) + 2V(t) \le CW(t)^{\frac{3}{2}}. \tag{2.22}$$

Proof. Taking the *curl* of (2.17) we have

$$\omega_t + \omega = -U \cdot \nabla\omega + \omega \cdot \nabla U - \omega(\nabla \cdot U).$$

Let ∂ denote any mixed time and spatial derivative of order $0 \le |\partial| \le 2$, then by taking any mixed derivative of the above equation, we get

$$\partial\omega_t + \partial\omega = \partial\{-U \cdot \nabla\omega + \omega \cdot \nabla U - \omega(\nabla \cdot U)\}.$$

Taking L^2 inner product of the above equation with $\partial\omega$ we get

$$\frac{1}{2}\frac{d}{dt}\|\partial\omega(t)\|^2 + \|\partial\omega(t)\|^2 \le CW(t)^{\frac{3}{2}}.$$

Finally, we deduce the lemma by summing up the above inequality for all $0 \le |\partial| \le 2$. This completes the proof of Lemma 2.7. □

Having Lemmas 2.3–Lemma 2.7 in hand, we now prove Theorem 2.1. From the definition of C_2 we can easily see that $G(t)$ and $E_1(t)$ are equivalent, i.e., there exist constants $c_1, c_2 > 0$ such that

$$c_1 E_1(t) \le G(t) \le c_2 E_1(t). \tag{2.23}$$

Then, by (2.20) we have

$$\frac{d}{dt}E_1(t) + c_1 C_3 E_1(t) \le CW(t)^{\frac{3}{2}}. \tag{2.24}$$

Combining (2.22) and (2.24) we get

$$\frac{d}{dt}\Big(V(t) + E_1(t)\Big) + \Big(2V(t) + c_1 C_3 E_1(t)\Big) \le CW(t)^{\frac{3}{2}}.$$

Let $C_4 \equiv \min\{2, c_1 C_3\}$, then we have

$$\frac{d}{dt}\Big(V(t) + E_1(t)\Big) + C_4\Big(V(t) + E_1(t)\Big) \le CW(t)^{\frac{3}{2}}. \tag{2.25}$$

On the other hand, it is easy to see that $G(t)$ and $E(t)$ are equivalent, i.e., there exist constants $c_3, c_4 > 0$ such that

$$c_3 G(t) \le E(t) \le c_4 G(t).$$

From Lemma 2.3 and (2.23) we see that

$$W(t) \le C_1 \Big(V(t) + c_2 c_4 E_1(t) \Big).$$

Let $C_5 \equiv \max\{C_1, c_2 c_4 C_1\}$, then we get

$$W(t) \le C_5 \Big(V(t) + E_1(t) \Big). \tag{2.26}$$

When $W(t)$ is sufficiently small, (2.25) and (2.26) yield

$$\frac{d}{dt}\Big(V(t) + E_1(t) \Big) + C_4 \Big(V(t) + E_1(t) \Big) \le \frac{C_4}{2} \Big(V(t) + E_1(t) \Big).$$

Thus, we get

$$\frac{d}{dt}\Big(V(t) + E_1(t) \Big) + \frac{C_4}{2} \Big(V(t) + E_1(t) \Big) \le 0,$$

which yields the exponential decay of $V(t) + E_1(t)$. Finally, the exponential decay of $W(t)$ follows from (2.26). This completes the proof of Theorem 2.1.

Remark 2.8. Theorem 2.1 applies directly to the case for $d = 2$. When $d = 1$, due to the Sobolev embedding $H^1(\Omega) \hookrightarrow C^0(\bar{\Omega})$ in $\mathbb{R}$, the regularity of the initial data can be lowered to H^2.

3 L^∞ entropy weak solutions

In this section, we will study the qualitative behavior of L^∞ entropy weak solutions to (1.1)–(1.2) in one space dimension for large, rough initial data containing vacuum. Since the velocity u may not be well-defined at vacuum in general, after introducing the momentum $m = \rho u$, we can rewrite the one-dimensional version of (1.1) with $\gamma = 1$ as follows:

$$\begin{cases} \rho_t + m_x = 0, \\ m_t + \Big(\dfrac{m^2}{\rho} + \rho \Big)_x = -m, \end{cases} \tag{3.1}$$

with the initial and boundary conditions

$$\begin{cases} (\rho, m)(x,0) = (\rho_0, m_0)(x), \ \ 0 < x < 1, \\ m|_{x=0} = m|_{x=1} = 0, \ \ t \ge 0, \\ \displaystyle\int_0^1 \rho_0(x)\, dx = \rho_* > 0. \end{cases} \tag{3.2}$$

Since, when the initial data is large or rough, shock will develop in finite time [52], one has to consider weak entropy solutions. One of the main difficulties is that the weak solution may contain the vacuum state, where the system (3.1) experiences resonance since two families of characteristics coincide; see [28], [29] and [30]. In this section, we consider L^∞ entropy admissible weak solutions to (3.1) for physical initial and boundary data. We now give the definition of entropy weak solutions to (3.1)–(3.2).

Definition 3.1. For every $T>0$, we define a L^∞ entropy admissible weak solution to (3.1)–(3.2) to be a pair of bounded measurable functions $v(x,t)=\big(\rho(x,t),m(x,t)\big)$ satisfying the following pair of integral identities:

$$\int_0^T\int_0^1(\rho\psi_t+m\psi_x)\,dx\,dt+\int_{t=0}\rho_0\psi\,dx=0,$$

$$\int_0^T\int_0^1\left(m\psi_t+\Big(\frac{m^2}{\rho}+\rho\Big)\psi_x-m\psi\right)dx\,dt+\int_{t=0}m_0\psi\,dx=0,$$

for all $\psi\in C^\infty(I_T)$ satisfying $\psi(x,T)=0$ for $0\le x\le 1$ and $\psi(0,t)=\psi(1,t)=0$ for $t\ge 0$, where $I_T=(0,1)\times(0,T)$, and the initial and boundary conditions in (3.2) are satisfied in the sense of trace and section (see Section 5 of [11]). Moreover, the entropy inequality

$$\partial_t\eta_e+\partial_x q_e+m\partial_m\eta_e\le 0$$

is satisfied in the sense of distribution, where the entropy-entropy flux pair is given by

$$\eta_e=\frac{m^2}{2\rho}+\int_0^\rho\log s\,ds,\qquad q_e=\frac{m^3}{2\rho^2}+m\log\rho.$$

Remark 3.2. In hyperbolic conservation/balance laws, when weak solutions are concerned, uniqueness is usually lost. The entropy condition provides a selection criterion for physically relevant solutions to (3.1)–(3.2).

The following two theorems are the main results of this section.

Theorem 3.3. *Suppose that the initial data (ρ_0,m_0) satisfy the conditions*

$$0\le\rho_0(x)\le M_1,\ \ \rho_0\not\equiv 0,\ \ |m_0(x)|\le M_2\rho_0(x),$$

for some positive constants $M_i(i=1,2)$. Then, the initial-boundary value problem (3.1)–(3.2) has a global weak solution $(\rho(x,t),m(x,t))$, as defined in Definition 3.1, satisfying the following estimates:

$$0\le\rho(x,t)\le M_3,\ \ |m(x,t)|\le\rho(x,t)\big(M_4+|\log\rho(x,t)|\big)\ \text{ a.e.}$$

for some positive constants M_3 and M_4.

Concerning the long time behavior of the physical solution obtained in Theorem 3.3, we have

Theorem 3.4. *Suppose $\int_0^1\rho_0(x)dx=\rho_*>0$. Let (ρ,m) be any L^∞ entropy weak solution to (3.1)–(3.2) defined in Definition 3.1, satisfying the estimates*

$$0\le\rho(x,t)\le M_5<\infty,\quad |m(x,t)|\le\rho(x,t)\big(M_6+|\log\rho(x,t)|\big),$$

where M_5, M_6 are positive constants. Then, there exist constants $C, \beta > 0$ depending on ρ_*, M_5 and initial data such that

$$\left\|(\rho-\rho_*, m)(\cdot,t)\right\|^2_{L^2([0,1])} \le Ce^{-\beta t} \quad \text{as } t \to \infty.$$

The proof of Theorem 3.3 is in the spirit of [4, 10, 11, 22, 24]. We construct the approximate solutions by the method of vanishing viscosity. The ε, T-dependent lower bound and the ε-independent upper bound of the approximate solutions are established by using the arguments in [4, 10] and the invariant region theory [5, 33, 34] respectively. The compensated compactness framework [22, 24] for the case $\gamma = 1$ is then applied to the sequence of approximate solutions to obtain a global weak entropy solution. The initial and boundary conditions are satisfied in the sense of trace and section which are clearly stated in Section 5 of [11], see also [19, 25, 43, 45], and we will omit the details.

The proof of Theorem 3.4 relies heavily on an elementary lemma which plays an important role in the control of singularities near vacuum. Due to the roughness of the solution, standard energy estimates can not be performed in this situation. Instead, we will start the proof with defining an anti-derivative through the mass conservation law in order to gain differentiability. The first step of our energy estimate will be carried out on the equation satisfied by the anti-derivative, which is a nonlinear wave equation with source terms. Then the entropy inequality will be implemented in order to deal with the nonlinearities in the resulting energy estimate obtained in the first step. Although the initial and boundary conditions are satisfied in the weak sense, the theory of divergence measure fields [3] guarantees the eligibility of the calculations. Finally, Poincaré's inequality on bounded domains will be utilized to yield the exponential decay of the solution.

3.1 Global existence

There are several approaches to construct entropy weak solutions to (3.1)–(3.2). Here we sketch a proof based on the approach of [11, 22, 24] by viscosity approximation. To construct global L^∞ entropy weak solutions to (3.1), the following program is to be carried out:

- To construct smooth approximate solutions via viscous perturbation of the hyperbolic system and obtain an uniform ε-independent L^∞ upper bound and a ε, T-dependent lower bound of the sequence of approximate solutions in order to extend any local smooth solution to a global one.
- To show that the entropy dissipation pair $\eta_t(v^\varepsilon) + q_x(v^\varepsilon)$ is compact in H^{-1}_{loc} and apply the *div-curl* lemma [36] to reduce the Young measure associated with the flux function to the Dirac measure in order to conclude that the sequence converges strongly in the L^∞ topology.

Usually, the first bullet can be accomplished by applying standard results on parabolic equations together with the invariant region theory. When $\gamma > 1$, the compensated compactness frameworks established in [9, 10, 26, 27] are sufficient to conclude the second bullet. However, when $\gamma = 1$, the story changes drastically. In this case, since the entropy equation is completely different from the case for $\gamma > 1$, all the frameworks established in

[9, 10, 26, 27] fail here. Fortunately, Huang and Pan [22] and Huang and Wang [24] gave an important approach to recover this case. Based on their results, the second bullet can still be achieved for the case $\gamma = 1$.

Step 1. *Construction of approximate solutions and uniform L^∞ bound.* Following the general procedure in [4, 10, 11, 22, 24], let us consider the artificial viscous approximation to the original system:

$$\begin{cases} \rho^\varepsilon_t + m^\varepsilon_x = \varepsilon \rho^\varepsilon_{xx}, \\ m^\varepsilon_t + \Big(\dfrac{(m^\varepsilon)^2}{\rho^\varepsilon}\Big)_x + \rho^\varepsilon_x = -m^\varepsilon + \varepsilon m^\varepsilon_{xx}, \end{cases} \tag{3.3}$$

with the initial and boundary conditions:

$$\begin{cases} m^\varepsilon|_{x=0} = m^\varepsilon|_{x=1} = 0, \\ \rho^\varepsilon_x|_{x=0} = \rho^\varepsilon_x|_{x=1} = 0; \\ (\rho^\varepsilon, m^\varepsilon)(x,0) = (\rho^\varepsilon_0, m^\varepsilon_0)(x), \end{cases} \tag{3.4}$$

where

$$\begin{aligned} &\rho^\varepsilon_0 = B^\varepsilon_0 + \varepsilon, \quad B^\varepsilon_0 \in C^\infty_0([0,1]), \quad 0 \le B^\varepsilon_0(x) \le \|\rho_0\|_{L^\infty}, \\ &m^\varepsilon_0 = \rho^\varepsilon_0 u^\varepsilon_0, \quad u^\varepsilon_0 \in C^\infty_0([0,1]), \quad |u^\varepsilon_0(x)| \le \|u_0\|_{L^\infty} \end{aligned}$$

and B^ε_0 converges to ρ_0 in the weak* topology of $L^\infty([0,1])$ and u^ε_0 converges to u_0 in the strong topology of $L^2([0,1])$. We remark that, the parabolic boundary conditions in (3.4) are compatible with the hyperbolic boundary condition (3.2) according to [11], and (3.2) will be recovered in the limiting process. Furthermore, under this setting, it holds that m^ε_0 converges to m_0 in the weak* topology of $L^\infty([0,1])$.

The corresponding Riemann invariants of the hyperbolic system associated with (3.3) are

$$w^\varepsilon = \rho^\varepsilon e^{u^\varepsilon}, \ \ z^\varepsilon = \rho^\varepsilon e^{-u^\varepsilon}. \tag{3.5}$$

The set

$$\Sigma = \{(\rho^\varepsilon, m^\varepsilon) \mid 0 \le z^\varepsilon \le M_7, \ 0 \le w^\varepsilon \le M_7\}$$

is an invariant region for (3.3) due to [5, 34] (see also [22, 24]). Here, $M_7 > 0$ is a suitably large constant independent of ε such that $(\rho^\varepsilon_0, m^\varepsilon_0)$ lies in Σ. This, together with the fact that $\rho^\varepsilon > \delta(\varepsilon, T) > 0$ due to [4, 10] (In fact, according to Theorem 2.1 in [4], the solution to the first equation of (3.3) can be expressed in terms of the fundamental solution of the heat equation and the even periodic extensions(with period 2) of the initial data and the function $u^\varepsilon = m^\varepsilon/\rho^\varepsilon$. Then by using the arguments in Section 4 of [10] and a corrected version of Lemma 4.1 of [10] given by [2], one can show the lower bound of ρ^ε), gives

$$0 < \rho^\varepsilon(x,t) \le M_8, \ \ |m^\varepsilon(x,t)| \le \rho^\varepsilon(x,t)\big(M_9 + |\log \rho^\varepsilon(x,t)|\big) \tag{3.6}$$

for some constants $M_8, M_9 > 0$ independent of ε. We remark that the velocity u_ε may tend to infinity as ρ^ε tends to zero (which is different from the isentropic case $\gamma > 1$). However, one infers from (3.6) that $|u_\varepsilon| \le M_9 + |\log \rho^\varepsilon|$. It is easy to see that the global existence of smooth solutions to (3.3)–(3.4) follows from local existence result and the *a priori* estimate (3.6). This completes the first step.

Step 2. H^{-1}_{loc} *compactness and strong convergence.* In order to obtain global solutions to (3.1), it suffices to show the strong convergence of the sequence of approximate solutions $(\rho^\varepsilon, m^\varepsilon)$, extracting to a subsequence if necessary. However, with the uniform L^∞ estimate of the approximate solutions in hand, one can only infer the convergence in the weak* topology, which is insufficient to handle the nonlinear term in (3.3). One thus has to use the theory of compensated compactness. We then apply the frameworks developed in [22] and [24] to conclude that there exist functions $(\rho, m)(x,t) \in L^\infty((0,1)\times(0,\infty))$ such that

$$(\rho^\varepsilon, m^\varepsilon) \to (\rho, m) \text{ a.e. in } (0,1)\times(0,\infty) \text{ as } \varepsilon \to 0,$$

and satisfy, by virtue of (3.6):

$$0 \le \rho(x,t) \le M_8, \ \ |m(x,t)| \le \rho(x,t)\big(M_9 + |\log\rho(x,t)|\big) \text{ a.e. in } (0,1)\times(0,\infty).$$

Then it is straightforward to verify that (ρ, m) is an entropy weak solution to (3.1). Furthermore, the solution satisfies the initial and boundary conditions in the sense of trace and section, see [11, 19, 25, 43, 45]. This completes the proof of Theorem 3.3.

3.2 Long time behavior

We now turn to the proof of Theorem 3.4. First of all, we give a simply lemma which will play an important role in controlling singularities near vacuum.

Lemma 3.5. *Let $0 \le \rho \le \Lambda < \infty$ and $0 < a \le \rho_* \le \Lambda$ for some constants a and Λ. Then there exist constants $d_1, d_2 > 0$ depending on ρ_* and Λ such that*

$$d_1(\rho-\rho_*)^2 \le \int_0^\rho \log s\, ds - \int_0^{\rho_*} \log s\, ds - \log\rho_*(\rho-\rho_*) \le d_2(\rho-\rho_*)^2. \tag{3.7}$$

Proof. **Step 1.** Let us consider the function

$$f(\rho) = \frac{2}{\rho_*}(\rho-\rho_*)^2 - \int_0^\rho \log s\, ds + \int_0^{\rho_*} \log s\, ds + \log\rho_*(\rho-\rho_*).$$

It is obvious that f is a continuous function of $\rho \ge 0$. Since $f(0) = \rho_* > 0$ and $f(\rho_*) = 0$, there exists a $\bar\rho$ satisfying $0 < \bar\rho < \rho_*$ such that

$$f(\rho) \ge \frac{\rho_*}{2} > 0, \quad \forall\, 0 \le \rho \le \bar\rho.$$

Step 2. We fix $\bar\rho$ and consider the function

$$g(\rho) = \frac{1}{2\bar\rho}(\rho-\rho_*)^2 - \int_0^\rho \log s\, ds + \int_0^{\rho_*} \log s\, ds + \log\rho_*(\rho-\rho_*), \ \ for\ \ \bar\rho < \rho \le \Lambda.$$

We note that $g(\rho_*) = 0$ and $g'(\rho_*) = 0$. Moreover, since

$$g''(\rho) = \frac{1}{\bar\rho} - \frac{1}{\rho} > 0, \ \ for\ \ \bar\rho < \rho \le \Lambda,$$

we conclude that

$$g(\rho) \geq 0, \;\; for \;\; \bar{\rho} < \rho \leq \Lambda.$$

Step 3. Let $d_1 = \max\{2/\rho_*, 1/2\bar{\rho}\}$. Then it is easy to see, from the first two steps, that

$$d_1(\rho-\rho_*)^2 - \int_0^{\rho} \log s \, ds - \int_0^{\rho_*} \log s \, ds - \log \rho_*(\rho-\rho_*) \geq 0, \;\; \forall \, 0 \leq \rho \leq \Lambda.$$

This proves the second inequality of (3.7).

Step 4. To show the first part of (3.7), we notice, by Taylor's Theorem, that

$$\int_0^{\rho} \log s \, ds - \int_0^{\rho_*} \log s \, ds - \log \rho_*(\rho-\rho_*) = \frac{1}{2\xi}(\rho-\rho_*)^2$$

for some ξ satisfying $0 \leq \xi \leq \Lambda$. Thus, we have

$$\int_0^{\rho} \log s \, ds - \int_0^{\rho_*} \log s \, ds - \log \rho_*(\rho-\rho_*) \geq \frac{1}{2\Lambda}(\rho-\rho_*)^2.$$

This completes the proof of Lemma 3.5. □

We then set

$$w = \rho - \rho_*, \quad z = m,$$

which satisfy

$$\begin{cases} w_t + z_x = 0, \\ z_t + \left(\dfrac{m^2}{\rho}\right)_x + (\rho-\rho_*)_x + z = 0, \end{cases} \tag{3.8}$$

and

$$\int_0^1 w(x,t)dx = 0.$$

Define

$$y(x,t) = -\int_0^x w(\xi,t)d\xi,$$

which implies that

$$y_x = -w = \rho_* - \rho, \quad y_t = z.$$

Since

$$\int_0^1 \rho(x,t)dx = \int_0^1 \rho_0(x)dx = \rho_*,$$

we have

$$y(0,t) = y(1,t) = 0.$$

Therefore the second equation of (3.8) turns into

$$y_{tt} + \left(\frac{m^2}{\rho}\right)_x + (\rho-\rho_*)_x + y_t = 0. \tag{3.9}$$

Taking L^2 inner product of (3.9) with y we have

$$\frac{d}{dt}\int_0^1 \left(y_t y + \frac{1}{2}y^2\right)dx - \int_0^1 y_t^2 dx + \int_0^1 (\rho-\rho_*)^2 dx = \int_0^1 \frac{m^2}{\rho} y_x dx.$$

By definition of y, we get

$$\frac{d}{dt}\int_0^1 \Big(y_t y + \frac{1}{2}y^2\Big)dx + \int_0^1 y_x^2 dx = \int_0^1 y_t^2 \frac{\rho_*}{\rho} dx. \tag{3.10}$$

In order to deal with the nonlinearity, we now use the entropy inequality, rather than the usual energy method. Let

$$\eta_e = \frac{m^2}{2\rho} + \int_0^\rho \log s\, ds, \qquad q_e = \frac{m^3}{2\rho^2} + m\log\rho$$

be the mechanical energy and related flux. We define

$$\eta_* = \eta_e - \int_0^{\rho_*} \log s\, ds - \log\rho_*(\rho - \rho_*).$$

Thus, by the definition of entropy weak solutions, the following entropy inequality holds in the sense of distribution:

$$\eta_{*t} + \log\rho_*(\rho-\rho_*)_t + q_{ex} + \frac{m^2}{\rho} \le 0.$$

By the conservation of mass and theory of divergence-measure fields [3], we have

$$\frac{d}{dt}\int_0^1 \eta_* dx + \int_0^1 \frac{y_t^2}{\rho} dx \le 0. \tag{3.11}$$

Choosing $K = \max\{2,\, 2M_5 + \rho_*\}$, where M_5 is the same constant appearing in Theorem 3.4, we add (3.10) to $(3.11) \times K$ and use the expression of η_* to get

$$\frac{d}{dt}\int_0^1 \Big(\frac{K}{2\rho}y_t^2 + yy_t + \frac{1}{2}y^2 + KG(\rho)\Big)dx + \int_0^1 y_x^2 dx + \int_0^1 \frac{K-\rho_*}{\rho} y_t^2 dx \le 0, \tag{3.12}$$

where

$$G(\rho) = \int_0^\rho \log s\, ds - \int_0^{\rho_*} \log s\, ds - \log\rho_*(\rho-\rho_*).$$

Our next goal is to compare the terms inside the temporal derivative with the diffusion terms of (3.12). Clearly, Lemma 3.5 implies

$$\int_0^1 KG(\rho)dx \le d_2 K \int_0^1 y_x^2 dx.$$

On the other hand, since $0 \le \rho \le M_5$ and $y|_{x=0,1} = 0$, Poincaré's inequality implies that there is a positive constant d_3 such that

$$\begin{aligned}\int_0^1 \Big(\frac{K}{2\rho}y_t^2 + yy_t + \frac{1}{2}y^2\Big)dx &\le \int_0^1 \Big(\frac{K}{2\rho}y_t^2 + \frac{1}{2}y_t^2 + y^2\Big)dx \\ &\le d_3\bigg(\int_0^1 \frac{K-\rho_*}{\rho} y_t^2 dx + \int_0^1 y_x^2 dx\bigg).\end{aligned}$$

Therefore, for $d_4 = 2\max\{d_2K, d_3\}$, it holds that

$$\int_0^1 \Big(\frac{K}{2\rho}y_t^2 + yy_t + \frac{1}{2}y^2 + KG(\rho)\Big)dx \le d_4\Big(\int_0^1 \frac{K-\rho_*}{\rho}y_t^2dx + \int_0^1 y_x^2dx\Big). \tag{3.13}$$

From (3.12)–(3.13), we conclude that

$$\frac{d}{dt}\int_0^1 \Big(\frac{K}{2\rho}y_t^2 + yy_t + \frac{1}{2}y^2 + KG(\rho)\Big)dx + d_5\int_0^1 \Big(\frac{K}{2\rho}y_t^2 + yy_t + \frac{1}{2}y^2 + KG(\rho)\Big)dx \le 0,$$

where $d_5 = d_4^{-1}$. This implies that

$$\int_0^1 \Big(\frac{K}{2\rho}y_t^2 + yy_t + \frac{1}{2}y^2 + KG(\rho)\Big)dx \le d_6\exp\{-d_5t\}. \tag{3.14}$$

Furthermore, since $K > 2M_5 \ge 2\rho$, we know that

$$\begin{aligned}\Big(\frac{K}{2\rho}y_t^2 + yy_t + \frac{1}{2}y^2 + KG(\rho)\Big) &\ge y_t^2 + yy_t + \frac{1}{2}y^2 + d_1K(\rho-\rho_*)^2\\ &\ge \frac{1}{2}y_t^2 + d_1K(\rho-\rho_*)^2,\end{aligned}$$

where we have used the first inequality of (3.7). Therefore, (3.14) yields

$$\int_0^1 m^2 + (\rho-\rho_*)^2dx \le d_7\exp\{-d_5t\}.$$

This completes the proof of Theorem 3.4.

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