Communications in **M**athematical **A**nalysis
Volume 9, Number 2, pp. 67–76 (2010) www.commun-math-anal.org
ISSN 1938-9787

General Energy Decay Rates for a Weakly Damped Wave Equation

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(Communicated by Irena Lasiecka)

Abstract

In this paper we consider a wave equation with a weak frictional damping. We establish an explicit and general decay rate result, using some properties of convex functions. Our result is obtained without imposing any restrictive growth assumption on the frictional damping term.

AMS Subject Classification: 35B37, 35L55, 74D05, 93D15, 93d20.

Keywords: general decay, weak frictional damping, wave equation, convexity.

1 Introduction

In this paper we are concerned with the following problem

$$\begin{cases} u_{tt} - \Delta u + \alpha(t) g(u_t) = 0, & \text{in } \Omega \times (0,\infty) \\ u = 0, \quad \text{on } \partial\Omega \times (0,\infty) \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ x \in \Omega, \end{cases} \tag{1.1}$$

a weakly damped wave equation associated with homogeneous Dirichlet boundary conditions and initial data in suitable function spaces. Here Ω is a bounded domain of $\mathbb{R}^n$ $(n \geq 1)$ with a smooth boundary $\partial\Omega$ and α, g are specific functions. For $n = 1$ and $\Omega = (0,L)$, for

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instance, u represents the displacement of a vibrating string of length L, which is clamped at its two ends and subjected to the effect of a weak nonlinear dissipation.

Stabilization of the wave equation by internal or boundary feedbacks has been widely studied in the literature. Zuazua [19] proved the exponential stability of the wave equation by a locally distributed internal feedback depending on the velocity in a linear way. Komornik [6] and Nakao [18] extended, with different methods, the result of Zuazua treating the case of a nonlinear damping term with a polynomial growth near the origin. In fact, for system (1.1) with $\alpha \equiv 1$, if g satisfies

$$c_1 \min\{|s|, |s|^q\} \leq |g(s)| \leq c_2 \max\{|s|, |s|^{1/q}\}$$

where c_1, c_2 positive constants and $q > 1$, it was proved that

$$E(t) \leq C(E(0))\, t^{-2/(q-1)}, \qquad \forall t > 0,$$

and for $q = 1$ the decay rate is exponential. Similar results were also obtained for frictional dissipative boundary condition (see [7, 8, 9, 20]). In the presence of a weak frictional damping, Benaissa *et al.* [3] treated system (1.1) for g having a polynomial growth near the origin and established energy decay results depending on α and g.

Decay results for arbitrary growth of the damping term have been given for the first time in the work of Lasiecka and Tataru [10] in which the wave equation is damped by boundary feedback. They showed that the energy decays as fast as the solution of an associated differential equation whose coefficients depend on the damping term. On the same line, Liu and Zuazua [14] studied the case of the internally distributed damping and Lasiecka and Toundykov [11,12] and Cavalcanti *et al.* [4] dealt with the cases where a source is competing with the internal or boundary frictional dissipation considering Dirichlet or Neumann boundary conditions. Martinez [15, 16] treated the damped wave equation and used the piecewise multiplier technique developed by Liu [13] combined with nonlinear integral inequalities to establish explicit decay rate estimates. Though the decay rates obtained in these results are explicit and easy to compute, they are not optimal for some cases including the case of the polynomial growth. In [1], Alabau-Boussouira used a method based on weighted integral inequalities and some convexity arguments to establish a semi-explicit formula for the energy decay rate of the wave equation, damped by a unique internal or boundary feedbacks, for which the optimal exponential and polynomial decay rate estimates are only special cases. It is also worth mentioning the work of Cavalcanti *et al.* [5] concerning the following problem

$$\begin{cases} u_{tt} - \Delta u + \int_0^t h(t-\tau)\Delta u(\tau)d\tau = 0, & \text{in } \Omega \times (0,\infty) \\ u = 0, \quad \text{on } \Gamma_0 \times (0,\infty) \\ \frac{\partial u}{\partial \nu} - \int_0^t h(t-\tau)\frac{\partial u}{\partial \nu}(\tau)d\tau + g(u_t) = 0, & \text{on } \Gamma_1 \times (0,\infty) \end{cases} \tag{1.2}$$

in which the uniform stability of (1.2) has been proved, with more general assumption on the relaxtion function h and no growth assumption on the boundary frictional damping function g. They established explicit decay rate results, depending on h and g, for some special cases and implicitly for the general case. In [17], Messaoudi and Mustafa improved the results of [5], with weaker conditions on h and g, and obtained an explicit and general formula for the energy decay rate of system (1.2).

Our aim in this work is to investigate (1.1), in which the damping considered is modulated by a time dependent coefficient $\alpha(t)$, and establish an explicit and general decay result, depending on g and α, for which the optimal exponential and polynomial decay rate estimates are only special cases. More precisely, we intend to obtain a general relation between the decay rate for the energy (when t goes to infinity) and the functions g and α without imposing any restrictive growth assumption on the frictional damping term. The result of this paper provides an explicit energy decay formula that allows a larger class of functions g and α, from which the energy decay rates are not necessarily of exponential or polynomial types (see the examples in section 3). The proof is based on the multiplier method and makes use of some properties of convex functions including the use of the general Young's inequality and Jensen's inequality. These convexity arguments were introduced and developed by Lasiecka and co-workers [4, 10-12] and used by Liu and Zuazua [14] and Alabau-Boussouira [1]. The paper is organized as follows. In section 2, we present some notations and material needed for our work. The statement and the proof of our main result will be given in section 3.

2 Preliminaries

We consider the following hypotheses
(H1) $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is a nonincreasing differentiable function.
(H2) $g : \mathbb{R} \to \mathbb{R}$ is a nondecreasing C^0 function such that there exist constants $\varepsilon, c_1, c_2 > 0$ and an increasing function $G \in C^1([0,+\infty))$, with $G(0) = 0$, and G is linear or strictly convex C^2 function on $(0,\varepsilon]$ such that

$$c_1|s| \leq |g(s)| \leq c_2|s|^p \quad \text{if } |s| \geq \varepsilon,$$

$$s^2 + g^2(s) \leq G^{-1}(sg(s)) \quad \text{if } |s| \leq \varepsilon$$

and p satisfies

$$\begin{aligned} 1 &\leq p \leq \frac{n+2}{n-2} && \text{if } n > 2 \\ 1 &\leq p < \infty && \text{if } n \leq 2. \end{aligned}$$

Remarks
1. Hypothesis (H2) implies that $sg(s) > 0$, for all $s \neq 0$.
2. The condition (H2), with $\varepsilon = 1$ and $p = 1$, was introduced and employed by Lasiecka and Tataru [10] in their study of the asymptotic behavior of solutions of nonlinear wave equations with nonlinear boundary damping where they obtained decay estimates that depend on the solution of an explicit nonlinear ordinary differential equation. It was also shown there that the monotonicity and continuity of g guarantee the existence of the function G with the properties stated in (H2). In our present work, we consider the internal damping to be modulated by a time dependent coefficient $\alpha(t)$, and establish an explicit and general energy decay formula, depending on both g and α, without imposing any growth assumption near the origin on g and allowing g to have nonlinear growth at infinity.

For completeness we state, without proof, the following standard existence and regularity result (see [16]).
Proposition 2.1. *Let* $(u_0,u_1)\in H_0^1(\Omega)\times L^2(\Omega)$ *be given. Assume that (H1) and (H2) are satisfied, then problem* (1.1) *has a unique global (weak) solution*

$$u\in C(\mathbb{R}_+;H_0^1(\Omega))\cap C^1(\mathbb{R}_+;L^2(\Omega)).$$

Moreover, if

$$(u_0,u_1)\in (H^2(\Omega)\cap H_0^1(\Omega))\times H_0^1(\Omega)$$

then the solution satisfies

$$u\in L^\infty(\mathbb{R}_+;H^2(\Omega)\cap H_0^1(\Omega))\cap W^{1,\infty}(\mathbb{R}_+;H_0^1(\Omega))\cap W^{2,\infty}(\mathbb{R}_+;L^2(\Omega)).$$

Now, we introduce the energy functional

$$E(t):=\frac{1}{2}\int_\Omega\left(u_t^2+|\nabla u|^2\right)dx. \tag{2.1}$$

We will use c, throughout this paper, to denote a generic positive constant.

3 The main result

In this section we state and prove our main result. For this purpose we establish several lemmas.
Lemma 3.1. *Let u be the solution of* (1.1). *Then the energy functional satisfies*

$$E'(t)=-\alpha(t)\int_\Omega u_t g(u_t)dx\le 0. \tag{3.1}$$

Proof. By multiplying equation (1.1) by u_t and integrating over Ω, using integration by parts, hypotheses (H1)-(H2) and some manipulations, we obtain (3.1) for any regular solution. This equality remains valid for weak solutions by a simple density argument.
Lemma 3.2. *Assume that (H1) and (H2) hold. Then, for some positive constants N, c, and m, the functional F defined by*

$$F(t):=NE(t)+\int_\Omega uu_t dx$$

satisfies, along the solution, the estimate

$$F'(t)\le -mE(t)+c\int_\Omega(u_t^2dx+|ug(u_t)|)dx$$

and

$$F(t)\sim E(t).$$

Proof. Direct computations, using (1.1) and (3.1), yield

$$\begin{aligned} F'(t) &= NE'(t)+\int_\Omega u_t^2 dx+\int_\Omega u\Delta u dx-\alpha(t)\int_\Omega ug(u_t)dx \\ &= NE'(t)+\int_\Omega u_t^2 dx-\int_\Omega |\nabla u|^2 dx-\alpha(t)\int_\Omega ug(u_t)dx \\ &\leq -mE(t)+c\int_\Omega (u_t^2 dx+|ug(u_t)|)dx. \end{aligned}$$

To prove that $F(t)\sim E(t)$, we want to show, for suitable N, that

$$\lambda_1 E(t)\leq F(t)\leq \lambda_2 E(t) \tag{3.2}$$

for some positive constants λ_1 and λ_2. For this, we use Young's and Poincaré's inequalities to get

$$\int_\Omega uu_t dx\leq \frac{1}{2}\int_\Omega (u^2+u_t^2)dx\leq c\int_\Omega (|\nabla u|^2+u_t^2)dx$$

and

$$\int_\Omega uu_t dx\geq -\frac{1}{2}\int_\Omega (u^2+u_t^2)dx\geq -c\int_\Omega (|\nabla u|^2+u_t^2)dx.$$

Then, for N large enough, we clearly obtain (3.2).

Now, let us choose $0<\varepsilon_1\leq\varepsilon$ such that

$$sg(s)\leq \min\{\varepsilon,\ G(\varepsilon)\} \qquad \text{for all } |s|\leq\varepsilon_1. \tag{3.3}$$

Then, it is easy to show that

$$\begin{cases} c_1'|s|\leq |g(s)|\leq c_2'|s|^p & \text{if } |s|\geq\varepsilon_1, \\ s^2+g^2(s)\leq G^{-1}(sg(s)) & \text{if } |s|\leq\varepsilon_1. \end{cases} \tag{3.4}$$

Considering the following partition of Ω

$$\Omega_1=\{x\in\Omega : |u_t|\leq\varepsilon_1\ \}, \qquad \Omega_2=\{x\in\Omega : |u_t|>\varepsilon_1\}$$

and using the embedding $H_0^1(\Omega)\hookrightarrow L^{p+1}(\Omega)$ and Hölder's inequality give

$$\begin{aligned} \int_{\Omega_2}|ug(u_t)|\,dx &\leq \left(\int_{\Omega_2}|u|^{p+1}\,dx\right)^{\frac{1}{p+1}}\left(\int_{\Omega_2}|g(u_t)|^{1+\frac{1}{p}}\,dx\right)^{\frac{p}{p+1}} \\ &\leq c\,\|u\|_{H_0^1(\Omega)}\left(\int_{\Omega_2}|g(u_t)|^{1+\frac{1}{p}}\,dx\right)^{\frac{p}{p+1}}. \end{aligned}$$

Then, Poincaré's inequality and (3.4) yield

$$\begin{aligned} \int_{\Omega_2}(u_t^2 dx+|ug(u_t)|)dx &\leq c\int_{\Omega_2}u_t g(u_t)dx+c\left(\int_\Omega |\nabla u|^2\,dx\right)^{\frac{1}{2}}\left(\int_{\Omega_2}u_t g(u_t)dx\right)^{\frac{p}{p+1}} \\ &\leq -cE'(t)+cE(t)^{\frac{1}{2}}\left(-E'(t)\right)^{\frac{p}{p+1}}, \end{aligned}$$

which, using Young's inequality and the boundedness of E, leads to

$$\int_{\Omega_2} (u_t^2 dx + |ug(u_t)|)dx \le c\varepsilon E(t)^{\frac{p+1}{2}} - C_\varepsilon E'(t) \le c\varepsilon E(t) - C_\varepsilon E'(t). \tag{3.5}$$

Also,

$$\begin{aligned}\int_{\Omega_1} (u_t^2 dx + |ug(u_t)|)dx &\le \int_{\Omega_1} u_t^2 dx + \varepsilon \int_{\Omega_1} u^2 dx + C_\varepsilon \int_{\Omega_1} g(u_t)^2 dx \\ &\le \int_{\Omega_1} u_t^2 dx + c\varepsilon E(t) + C_\varepsilon \int_{\Omega_1} g(u_t)^2 dx. \end{aligned} \tag{3.6}$$

Hence, Lemma 3.2, (3.5), and (3.6) imply, for ε small enough, that the functional $\mathcal{L} = F + C_\varepsilon E$ satisfies

$$\mathcal{L}'(t) \le -dE(t) + c \int_{\Omega_1} (u_t^2 dx + g(u_t)^2) dx \tag{3.7}$$

and

$$\mathcal{L}(t) \sim E(t) \tag{3.8}$$

We are now ready to state and prove our main result.

Theorem 3.3. *Assume that (H1) and (H2) hold. Then there exist positive constants* k_1, k_2, k_3 *and* ε_0 *such that the solution of* (1.1) *satisfies*

$$E(t) \le k_3 G_1^{-1}(k_1 \int_0^t \alpha(s) ds + k_2) \qquad \forall t \ge 0, \tag{3.9}$$

where

$$G_1(t) = \int_t^1 \frac{1}{G_2(s)} ds \qquad \text{and} \qquad G_2(t) = tG'(\varepsilon_0 t).$$

Here, G_1 *is strictly decreasing and convex on* $(0,1]$, *with* $\lim_{t\to 0} G_1(t) = +\infty$.

Proof. We multiply (3.7) by $\alpha(t)$ to get

$$\alpha(t)\mathcal{L}'(t) \le -d\alpha(t)E(t) + c\alpha(t) \int_{\Omega_1} (u_t^2 dx + g(u_t)^2) dx. \tag{3.9}$$

• Case 1. G is linear on $[0, \varepsilon]$: Then, we deduce that

$$\alpha(t)\mathcal{L}'(t) \le -d\alpha(t)E(t) + c\alpha(t) \int_{\Omega_1} u_t g(u_t) dx = -d\alpha(t)E(t) - cE'(t),$$

which gives, using (H1),

$$(\alpha\mathcal{L} + cE)'(t) \le -d\alpha(t)E(t).$$

Hence, using the fact that $\alpha\mathcal{L} + cE \sim E$, we easily obtain

$$E(t) \le c' e^{-c'' \int_0^t \alpha(s) ds} = c' G_1^{-1}(c'' \int_0^t \alpha(s) ds).$$

• Case 2. G is nonlinear on $[0, \varepsilon]$: To estimate the last integral in (3.10), we use, for $I(t)$ defined by

$$I(t) := \frac{1}{|\Omega_1|} \int_{\Omega_1} u_t g(u_t) dx,$$

Jensen's inequality to get

$$G^{-1}(I(t)) \geq c\int_{\Omega_1} G^{-1}(u_t g(u_t))dx. \tag{3.10}$$

Thus, using (3.4) and (3.11), we get

$$\alpha(t)\int_{\Omega_1}({u_t}^2+g^2(u_t))dx \leq \alpha(t)\int_{\Omega_1} G^{-1}(u_t g(u_t))dx \leq c\alpha(t)G^{-1}(I(t)).$$

Therefore, (3.10) becomes

$$R_0'(t) \leq -d\alpha(t)E(t)+c\alpha(t)G^{-1}(I(t)), \tag{3.11}$$

where $R_0 = \alpha\mathcal{L}+E$, and $R_0 \sim E$ because of (3.8).
Now, for $\varepsilon_0 < \varepsilon$ and $c_0 > 0$, using (3.12) and the fact that $E' \leq 0$, $G' > 0, G'' > 0$ on $(0,\varepsilon]$, we find that the functional R_1, defined by

$$R_1(t) := G'(\varepsilon_0\frac{E(t)}{E(0)})R_0(t)+c_0E(t),$$

satisfies, for some $a_1,a_2 > 0$,

$$a_1R_1(t) \leq E(t) \leq a_2R_1(t) \tag{3.12}$$

and

$$\begin{aligned} R_1'(t) &= \varepsilon_0\frac{E'(t)}{E(0)}G''(\varepsilon_0\frac{E(t)}{E(0)})R_0(t)+G'(\varepsilon_0\frac{E(t)}{E(0)})R_0'(t)+c_0E'(t) \\ &\leq -d\alpha(t)E(t)G'(\varepsilon_0\frac{E(t)}{E(0)})+c\alpha(t)G'(\varepsilon_0\frac{E(t)}{E(0)})G^{-1}(I(t))+c_0E'(t). \end{aligned} \tag{3.13}$$

Let G^* be the convex conjugate of G in the sense of Young (see [2] p. 61-64), then

$$G^*(s) = s(G')^{-1}(s)-G[(G')^{-1}(s)], \qquad \text{if } s \in (0,G'(\varepsilon)] \tag{3.14}$$

and G^* satisfies the following Young's inequality

$$AB \leq G^*(A)+G(B), \qquad \text{if } A \in (0,G'(\varepsilon)], B \in (0,\varepsilon]. \tag{3.15}$$

With $A = G'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)$ and $B = G^{-1}(I(t))$, using (3.1), (3.3) and (3.14)-(3.16), we arrive at

$$\begin{aligned} R_1'(t) &\leq -d\alpha(t)E(t)G'(\varepsilon_0\frac{E(t)}{E(0)})+c\alpha(t)G^*\left(G'(\varepsilon_0\frac{E(t)}{E(0)})\right)+c\alpha(t)I(t)+c_0E'(t) \\ &\leq -d\alpha(t)E(t)G'(\varepsilon_0\frac{E(t)}{E(0)})+c\varepsilon_0\alpha(t)\frac{E(t)}{E(0)}G'(\varepsilon_0\frac{E(t)}{E(0)})-cE'(t)+c_0E'(t). \end{aligned}$$

Consequently, with a suitable choice of ε_0 and c_0, we obtain

$$R_1'(t) \leq -k\alpha(t)\left(\frac{E(t)}{E(0)}\right)G'\left(\varepsilon_0\frac{E(t)}{E(0)}\right) = -k\alpha(t)G_2(\frac{E(t)}{E(0)}), \tag{3.16}$$

where $G_2(t) = tG'(\varepsilon_0 t)$.
Since

$$G_2'(t) = G'(\varepsilon_0 t) + \varepsilon_0 t G''(\varepsilon_0 t)$$

then, using the strict convexity of G on $(0,\varepsilon]$, we find that $G_2'(t), G_2(t) > 0$ on $(0,1]$. Thus, with $R(t) = \frac{a_1 R_1(t)}{E(0)}$ and using (3.13) and (3.17), we have

$$R(t) \sim E(t) \tag{3.17}$$

and, for some $k_1 > 0$,

$$R'(t) \le -k_1 \alpha(t) G_2(R(t)).$$

Then, a simple integration gives, for some $k_2 > 0$,

$$R(t) \le G_1^{-1}(k_1 \int_0^t \alpha(s)ds + k_2), \tag{3.18}$$

where $G_1(t) = \int_t^1 \frac{1}{G_2(s)} ds$. Here, we used, based on the properties of G_2, the fact that G_1 is strictly decreasing on $(0,1]$. Using (3.18)-(3.19), we obtain (3.9).

Remark. If g satisfies

$$g_0(|s|) \le |g(s)| \le g_0^{-1}(|s|) \qquad \text{for all } |s| \le \varepsilon \tag{3.19}$$

and

$$c_1 |s| \le |g(s)| \le c_2 |s| \qquad \text{for all } |s| \ge \varepsilon$$

for some strictly increasing function $g_0 \in C^1([0,+\infty))$, with $g_0(0) = 0$, and positive constants c_1, c_2, ε and the function G, defined by $G(s) = \sqrt{\frac{s}{2}} g_0(\sqrt{\frac{s}{2}})$, is strictly convex C^2 function on $(0,\varepsilon]$ when g_0 is nonlinear, then (H2) is satisfied. This kind of hypotheses, where (H2) is weaker, was considered by Liu and Zuazua [14], and Alabau-Boussouira [1].

Examples. We give some examples to illustrate the energy decay rates given by Theorem 3.3. Here, we assume that g satisfies (3.20) near the origin with the following various examples for g_0 :

(**1**) If $g_0(s) = cs^q$ and $q \ge 1$, then $G(s) = cs^{\frac{q+1}{2}}$ satisfies (H2). Then, using Theorem 3.3, we easily obtain

$$E(t) \le ce^{-c' \int_0^t \alpha(s)ds} \qquad \text{if } q = 1,$$

$$E(t) \le c\left(c' \int_0^t \alpha(s)ds + c''\right)^{-\frac{2}{q-1}} \qquad \text{if } q > 1.$$

(**2**) If $g_0(s) = e^{-1/s}$, then (H2) is satisfied for $G(s) = \sqrt{\frac{s}{2}} e^{-\sqrt{2}/\sqrt{s}}$ near zero. Therefore, we get

$$E(t) \le c\left(\ln(c' \int_0^t \alpha(s)ds + c'')\right)^{-2}.$$

(**3**) If $g_0(s) = \frac{1}{s}e^{-1/s^2}$, then (H2) is satisfied for $G(s) = e^{-2/s}$ near zero. Then, we obtain

$$E(t) \leq c\left(\ln(c'\int_0^t \alpha(s)ds + c'')\right)^{-1}.$$

(**4**) If $g_0(s) = \frac{1}{s}e^{-(\ln s)^2}$, then (H2) is satisfied for $G(s) = e^{-\frac{1}{4}(\ln\frac{s}{2})^2}$ near zero. Thus, we have the following energy decay rate

$$E(t) \leq ce^{-2\left(\ln(c'\int_0^t \alpha(s)ds + c'')\right)^{\frac{1}{2}}}.$$

Acknowledgments

The authors thank PSU for its support. This work has been funded by PSU under Project code: IBRP-MATH-2008-12-10.

References

[1] F. Alabau-Boussouira, Convexity and weighted integral inequalities for energy decay rates of nonlinear dissipative hyperbolic systems, *Appl. Math. Optim.* **51** (2005), pp 61-105.

[2] V.I. Arnold, Mathematical methods of classical mechanics *Springer-Verlag*, New York (1989).

[3] A. Benaissa and S. Mimouni, Energy decay of solutions of a wave equation of p-Laplacian type with a weakly nonlinear dissipation. *JIPM.J. Inequal. Pure Appl. Math.* **7(1)** (2006), Article 15, pp 8.

[4] M.M. Cavalcanti, V.N. Domingos Cavalcanti and I. Lasiecka, Well-posedness and optimal decay rates for the wave equation with nonlinear boundary damping-source interaction.*J. Differential Equations.***236** (2007), pp 407-459.

[5] M.M. Cavalcanti, V.N. Domingos Cavalcanti and P. Martinez , General decay rate estimates for viscoelastic dissipative systems. *Nonlinear Anal.,* **68(1)** (2008), pp 177-193.

[6] V. Komornik, Decay estimates for the wave equation with internal damping, *International Series of Numerical Mathematics.* **118** (1994), pp 253-266.

[7] V. Komornik and E. Zuazua, A direct method for the boundary stabilization of the wave equation. *J. Math. Pures Appl.* **69**(1990), pp 33-54.

[8] I. Lasiecka, Global uniform decay rates for the solution to the wave equation with nonlinear boundary conditions .*Appl. Anal.,* **47** (1992), pp 191-212.

[9] I. Lasiecka, Stabilization of wave and plate-like equation with nonlinear dissipation on the boundary .*J. Differential Equations*, **79** (1989), pp 340-381.

[10] I. Lasiecka and D. Tataru, Uniform boundary stabilization of semilinear wave equation with nonlinear boundary damping .*Differential Integral Equations*, **8** (1993), pp 507-533.

[11] I. Lasiecka and D. Toundykov, Energy decay rates for the semilinear wave equation with nonlinear localized damping and source terms.*Nonlinear Anal.,* **64** (2006), pp 1757-1797.

[12] I. Lasiecka and D. Toundykov, Regularity of higher energies of wave equation with nonlinear localized damping and a nonlinear source .*Nonlinear Anal.,* **69** (2008), pp 898-910.

[13] K. Liu, Locally distributed control and damping for the conservative systems.*SIAM J. Control Optim.* **35** (1997), pp 1574-1590.

[14] W.J. Liu and E. Zuazua, Decay rates for dissipative wave equations.*Ricerche Mat.* **48** (1999), pp 61-75.

[15] P. Martinez, A new method to decay rate estimates for dissipative systems.*ESIAM Control Optim. Cal. Var.* **4** (1999), pp 419-444.

[16] P. Martinez, A new method to obtain decay rate estimates for dissipative systems with localized damping.*Rev. Mat. Complut.* **12(1)** (1999), pp 251-283.

[17] S. A. Messaoudi and M.I. Mustafa, On the control of solutions of viscoelastic equations with boundary feedback .*Nonlinear Anal.,* **RWA 10** (2009), pp 3132-31340.

[18] M. Nakao, Decay of solutions of the wave equation with a local nonlinear dissipation.*Math. Ann.,* **305** (1996), pp 403-417.

[19] E. Zuazua, Exponential decay for the semilinear wave equation with locally distributed damping. *Comm. Partial Differential Equations.* **15** (1990), pp 205-235.

[20] E. Zuazua, Uniform Stabilization of the wave equation by nonlinear boundary feedback.*SIAM J. Control Optim.,* **28(2)** (1990), pp 466-477.