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SOLUTIONS OF FUZZY DIFFERENTIAL EQUATIONS BASED ON GENERALIZED DIFFERENTIABILITY

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Abstract

The main goal of this paper is to show that the concept of generalized differentiability introduced by the authors in [2] allows to obtain new solutions to the fuzzy differential equations. A very general existence and uniqueness result of two solutions for the fuzzy differential equations with modified argument and based on generalized differentiability is obtained together with a characterization of these solutions by ODEs. Real-world applications consisting in applications to fuzzy pantograph equation that models the cell growth and to fuzzy logistic equation that models the population dynamics under uncertainty are presented. In particular, as collateral consequences we correct some results on generalized differentiability recently obtained by Y. Chalco-Cano and H. Román-Flores in [7] and propagated in a characterization result obtained by J.J. Nieto, A. Khastan and K. Ivaz in [21].

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Introduction

The main goal of the present paper is to obtain improved existence and uniqueness results (uniqueness is understood as uniqueness of two solutions) for fuzzy differential equations

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under the strongly generalized differentiability concept introduced in [2], studied also in [7], [4], [30]. Real-world applications consisting in applications to fuzzy pantograph equation that models the cell growth and to fuzzy logistic equation that models the population dynamics under uncertainty also are presented.

There are, at least, two ways of approaching the phenomenons in the real world : by using models governed by fuzzy differential equations or, by using models governed by stochastic differential equation. Let us observe that the fuzzy differential equations theory rather complete than compete with the stochastic differential equations theory, simply because they deal with complementary phenomenons. Indeed, the probabilistic (stochastic/random) phenomenons and the possibilistic (fuzzy) phenomenons are complementary [34]. A simple example is the famous experience with a coin in probability. When both faces of the coin are clearly distinguishable, the phenomenon is stochastic, while if the both faces of the coin are just partially distinguishable (or even undistinguishable), then the phenomenon is fuzzy. Thus, while the stochastic differential equations model the real-life phenomenons when the conditions/hypothesis under which the phenomenons take place are clear, but the occurrence of the phenomenon is not known, the fuzzy differential equations model the real-life phenomenons when the conditions under which the phenomenons take place are uncertain/fuzzy [35].

Fuzzy differential equations appear to be a natural way to model Epistemic Uncertainties. These are usually due to a lack of knowledge, and they appear naturally in different areas in Science and Engineering [12].

There are several different approaches to fuzzy differential equations. The first approach dates back to [25] and it is based on the Hukuhara derivative. This approach has the drawback that the solution of a fuzzy differential equation needs to have increasing length of its support, so the qualitative theory is in this case very poor compared to ODEs [8]. This shortcoming was overcame in [13], by interpreting a fuzzy differential equation as a system of differential inclusions. The main shortcoming of the method based on differential inclusions is that the concept of the derivative of a fuzzy-number-valued function is missing. Another approach can be found in [5], [6] and it is based on Zadeh's extension principle in order to extend crisp differential equations to the fuzzy case. This approach suffers from the same disadvantage as the approach based on differential inclusions, that is the concept of derivative does not exist.

The strongly generalized differentiability concept introduced in [2] allows us to obtain new solutions of fuzzy differential equations as it was shown in [2] and [7], solutions which may have decreasing length of their support. So, we do not have the well-known drawback of the Hukuhara concept of fuzzy differentiability, i.e., that of ”possibilistic irreversibility” (see [9]). Also, this concept shows to be very promising both in practical applications and in theoretical investigations of qualitative type [1].

The existence and uniqueness theorems for fuzzy differential equations can be traced back to [15], [26], [32], [27], [19], [18] and in all the cases, there was a boundedness-type or local compactness assumption in the statements of these theorems. As it was recently shown in [17], this condition can be eliminated. This helps us to eliminate the boundedness condition in [2] under the strongly generalized differentiability concept too. Also, we show in the present paper a sufficient condition for the existence of some Hukuhara differences for large classes of fuzzy-number-valued functions, bringing improvement over a condition in the corresponding results in [2]. Combining all these ideas, we present a very general new result on the local existence and uniqueness of two solutions for fuzzy differential equations with modified argument, under the generalized differentiability concept. However in the classical setting, Differential Equations with modified argument are well studied (see e.g., [22], [11], [14]) there is not much work done in the fuzzy case. In [17] fuzzy delay differential equations are investigated. Here we consider equations with modified argument under uncertainty.

Recently, characterization of solutions of fuzzy differential equations by ODEs were obtained in [3], see also [24], under the Hukuhara concept of differentiability. In [30], characterization results in the interval-valued setting were studied.

The present paper intends to extend these results to the concept of generalized differentiability.

The plan of the paper goes as follows : Section 1 presents the main concept of generalized differentiability, Section 2 contains some auxiliary results and the main theoretical result on fuzzy differential equations with modified argument, in Section 3 we present characterization results of fuzzy differential equations by classical ODE's equations, Section 4 contains real-world applications consisting in applications to fuzzy pantograph equation that models the cell growth and to fuzzy logistic equation that models the population dynamics under uncertainty, while in Section 5 we present a few corrections to some recent results in [7], propagated also to other papers, as e.g. [21]. The paper ends with Section 6 that presents the conclusions.

1 Preliminaries

Let us recall some known useful concepts. Let $\mathbb{R}_{\mathcal{F}}$ denote the space of fuzzy reals, i.e. the set of normal, fuzzy convex, upper semicontinuous, compactly supported fuzzy sets $x : \mathbb{R} \to [0,1]$. Then, for $x \in \mathbb{R}_{\mathcal{F}}$, $\alpha \in [0,1]$, the $\alpha-$level set $[x]^\alpha = [x_\alpha^-, x_\alpha^+]$ is an interval. The arithmetic operations over the set of fuzzy numbers are induced by Zadeh's extension principle. For $u \in \mathbb{R}_{\mathcal{F}}$, we denote by $len(u) = u_0^+ - u_0^-$, the length of the closure of $[u]^0$-the support of u. The topological structure on $\mathbb{R}_{\mathcal{F}}$ is induced by the Hausdorff distance between fuzzy numbers (see e.g. [9]). In this note we use the Aumann integral ([26]), although the usage of Henstock integral ([33]) would lead to the same outcomes. Let $x, y \in \mathbb{R}_{\mathcal{F}}$. If there exists $z \in \mathbb{R}_{\mathcal{F}}$ such that $x = y + z$, then z is called the H-difference of x and y and it is denoted by $x \ominus y$. We use notation $-u$ for the fuzzy number $(-1) \cdot u$. Let us remark that $x - y$ means in the notations of the present paper $x + (-1)y \neq x \ominus y$. For the H-difference, everywhere we will use the symbol "$\ominus$".

The basic concept used in this paper is the generalized fuzzy differentiability introduced by the authors in [2].

Definition 1.1. ([2]) Let $f : (a,b) \to \mathbb{R}_{\mathcal{F}}$ and $x_0 \in (a,b)$. We say that f is strongly generalized differentiable on x_0, if there exists an element $f'(x_0) \in \mathbb{R}_{\mathcal{F}}$, such that

(i) for all $h > 0$ sufficiently small, $\exists f(x_0 + h) \ominus f(x_0)$, $f(x_0) \ominus f(x_0 - h)$ and the limits (in the metric D)

$$\lim_{h \searrow 0} \frac{f(x_0 + h) \ominus f(x_0)}{h} = \lim_{h \searrow 0} \frac{f(x_0) \ominus f(x_0 - h)}{h} = f'(x_0),$$

or

(ii) for all $h > 0$ sufficiently small, $\exists f(x_0) \ominus f(x_0 + h)$, $f(x_0 - h) \ominus f(x_0)$ and the limits

$$\lim_{h \searrow 0} \frac{f(x_0) \ominus f(x_0 + h)}{(-h)} = \lim_{h \searrow 0} \frac{f(x_0 - h) \ominus f(x_0)}{(-h)} = f'(x_0),$$

or

(iii) for all $h > 0$ sufficiently small, $\exists f(x_0 + h) \ominus f(x_0)$, $f(x_0 - h) \ominus f(x_0)$ and the limits

$$\lim_{h \searrow 0} \frac{f(x_0 + h) \ominus f(x_0)}{h} = \lim_{h \searrow 0} \frac{f(x_0 - h) \ominus f(x_0)}{(-h)} = f'(x_0),$$

or

(iv) for all $h > 0$ sufficiently small, $\exists f(x_0) \ominus f(x_0 + h)$, $f(x_0) \ominus f(x_0 - h)$ and the limits

$$\lim_{h \searrow 0} \frac{f(x_0) \ominus f(x_0 + h)}{(-h)} = \lim_{h \searrow 0} \frac{f(x_0) \ominus f(x_0 - h)}{h} = f'(x_0).$$

Remarks. 1) Note that the concept of generalized differentiability in Definition 1.1 is more general than the concept of one-sided differentiability introduced and used in [7] by Y. Chalco-Cano and H. Román-Flores. Indeed, the one-sided type differentiability in [7] coincides in fact with the cases (i) and (ii) of the above generalized differentiability but does not cover the cases (iii) and (iv). Although in Definition 1.1 the cases (iii) and (iv) are not so important as the cases (i) and (ii) since occur only on a discrete set of points, they can be very useful. As a first argument supporting that, let us consider $c \in \mathbb{R}_{\mathcal{F}} \setminus \mathbb{R}$ be any fuzzy (non real) constant and the very simple fuzzy-number-valued function $f : \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$, $f(t) = c \cdot \cos t$. It is natural to expect that f is differentiable everywhere in his domain. Let us remark that f is differentiable according to Definition 1.1, (ii), on each subinterval $(2k\pi, (2k+1)\pi)$ and differentiable according to Definition 1.1, (i), on each interval of the form $((2k+1)\pi, 2k\pi)$, $k \in \mathbb{Z}$. But, at the points $\{k\pi\}$, $k \in \mathbb{Z}$, none of the cases (i) and (ii) in Definition 1 are fulfilled. Namely, at these points the H-differences $f(k\pi + h) \ominus f(k\pi)$ and $f(k\pi) \ominus f(k\pi - h)$ may not exist simultaneously. Also, the H-differences $f(k\pi) \ominus f(k\pi + h)$ and $f(k\pi - h) \ominus f(k\pi)$ cannot exist simultaneously, so f is not differentiable at $k\pi$ in none of the cases (i) and (ii) of differentiability in Definition 1.1. Instead, it will be differentiable as in the cases (iii) and (iv) in Definition 1.1.

2) Another argument for the importance of the cases (iii) and (iv) in Definition 1.1 is pointed out in the Remark after the Theorem 5.1.

2 Improved Existence Result

For the proof of the existence and uniqueness result of two solutions for fuzzy differential equations, we need first to prove an auxiliary lemma. In this sense, we will make use of the well-known L-U representation of a fuzzy number valued function.

Theorem 2.1. *([10], [29]) A fuzzy number is completely determined by any pair $u = (u^-, u^+)$ of functions $u^-, u^+ : [0,1] \to \mathbb{R}$, defining the endpoints of the $\alpha-$level sets, satisfying the following conditions:*

(i) $u^-(\alpha) = u_\alpha^- \in \mathbb{R}$ *is a bounded nondecreasing left-continuous function in* $(0,1]$ *and it is right-continuous at* 0.

(ii) $u^+(\alpha) = u_\alpha^+ \in \mathbb{R}$ *is a bounded nonincreasing left-continuous function in* $(0,1]$ *and it is right-continuous at* 0.

(iii) $u^-(\alpha) \le u^+(\alpha), \forall \alpha \in [0,1]$.

We have :

Lemma 2.2. *Let* $x \in \mathbb{R}_{\mathcal{F}}$ *be such that the functions* $x^\pm$ *defined as in Theorem 2.1 are differentiable, with* x^- *strictly increasing and* x^+ *strictly decreasing on* $[0,1]$*, such that there exist the constants* $c_1 > 0$*,* $c_2 < 0$ *satisfying* $(x_\alpha^-)' \ge c_1$ *and* $(x_\alpha^+)' \le c_2$ *for all* $\alpha \in [0,1]$. *Let* $f : [a,b] \to \mathbb{R}_{\mathcal{F}}$ *be continuous with respect to* t*, having the level sets* $f_\alpha^-(t)$ *and* $f_\alpha^+(t)$ *with bounded partial derivatives* $\frac{\partial f_\alpha^-(t)}{\partial \alpha}$ *and* $\frac{\partial f_\alpha^+(t)}{\partial \alpha}$*, with respect to* $\alpha \in [0,1]$*,* $t \in [a,b]$.

If

a) $x^-(1) < x^+(1)$

or if

b) $x^-(1) = x^+(1)$ *and the core* $[f(s)]^1$ *consists of exactly one element for any* $s \in T = [a,b]$*, then there exists* $h > a$ *such that the H-difference*

$$x \ominus \int_a^t f(s)ds$$

exists for any $t \in [a,h]$.

Proof. We observe that the H-difference of two fuzzy numbers $u \ominus v$ exists if and only if the functions $(u^- - v^-, u^+ - v^+)$ define a fuzzy number. Indeed, let us suppose that $u \ominus v = z$, this is equivalent to $z + v = u$ and taking the α-levels we have equivalently $[z_\alpha^-, z_\alpha^+] + [v_\alpha^-, v_\alpha^+] = [u_\alpha^-, u_\alpha^+]$, i.e., $z_\alpha^- = u_\alpha^- - v_\alpha^-$, $z_\alpha^+ = u_\alpha^+ - v_\alpha^+$, $\alpha \in [0,1]$. By the above Theorem 2.1, the assumption that $u \ominus v$ exists is equivalent with the fact that $(u^- - v^-, u^+ - v^+)$ define a fuzzy number. Moreover, since left and right continuity requirements of Theorem 2.1 obviously hold true whenever u, v are fuzzy numbers, the existence of $u \ominus v$ becomes equivalent to $v^+(1) - v^-(1) \le u^+(1) - u^-(1)$, i.e., $len([v]^1) \le len([u]^1)$, (here len denotes the length of the corresponding interval) and that $u^- - v^-$ is nondecreasing and $u^+ - v^+$ is nonincreasing.

Therefore, in order to prove the existence of $x \ominus \int_a^t f(s)ds$ in the statement, we have to check that

$$\left[\int_a^t f(s)ds\right]^+ (1) - \left[\int_a^t f(s)ds\right]^- (1) \le x^+(1) - x^-(1) = len([x]^1),$$

$$x^-(\alpha) - \left[\int_a^t f(s)ds\right]^- (\alpha) \text{ is nondecreasing w.r.t. } \alpha,$$

$$x^+(\alpha) - \left[\int_a^t f(s)ds\right]^+ (\alpha) \text{ is nonincreasing w.r.t. } \alpha.$$

But

$$\left[\int_a^t f(s)ds\right]^\pm (\alpha) = \int_a^t [f(s)]^\pm(\alpha)ds,$$

which implies that the above conditions are equivalent to

$$\int_a^t len([f(s)]^1)ds \leq len([x]^1),$$

$$(x_\alpha^-)' - \int_a^t \frac{\partial f_\alpha^-(s)}{\partial \alpha} ds \geq 0, \text{ for all } \alpha \in [0,1],$$

$$(x_\alpha^+)' - \int_a^t \frac{\partial f_\alpha^+(s)}{\partial \alpha} ds \leq 0, \text{ for all } \alpha \in [0,1].$$

Since f is continuous, it is bounded and the function $len[f(t)]^1$ is bounded as well. Let M be such that $len[f(t)]^1 \leq M, t \in [a,b]$. Also, note that we always have $\int_a^t len[f(s)]^1 ds \leq M(t-a)$.

Suppose we are under assumption a) in statement. Then, since for all $t \in [a, a + len([x]^1)/M]$, we get $M(t-a) \leq len([x]^1)$, by the above inequality it easily follows that $\int_a^t len([f(s)]^1)ds \leq len([x]^1)$.

Let $M_1, M_2 > 0$ be such that $\left|\frac{\partial f_\alpha^-(s)}{\partial \alpha}\right| \leq M_1$ and $\left|\frac{\partial f_\alpha^-(s)}{\partial \alpha}\right| \leq M_2$, for all $s \in [a,b]$ and $\alpha \in [0,1]$. Since $(x_\alpha^-)' \geq c_1$ for all $\alpha \in [0,1]$, we have

$$\int_a^t \frac{\partial f_\alpha^-(s)}{\partial \alpha} ds \leq (t-a)M_1 \leq c_1 \leq \left(x_\alpha^-\right)'$$

for any $t \in \left[a, a + \frac{c_1}{M_1}\right]$ and for all $\alpha \in [0,1]$, which implies that $x_\alpha^- - \int_a^t f_\alpha^-(s)ds$ is nondecreasing with respect to α for $t \in \left[a, a + \frac{c_1}{M_1}\right]$. Similarly, since $(x_\alpha^+)' \leq c_2$ for all $\alpha \in [0,1]$ we have have

$$-\int_a^t \frac{\partial f_\alpha^+(s)}{\partial \alpha} ds \leq (t-a)M_2 \leq |c_2| \leq -\left(x_\alpha^+\right)'$$

and for any $t \in \left[a, a + \frac{|c_2|}{M_2}\right]$ and for all $\alpha \in [0,1]$, i.e. $x_\alpha^+ - \int_a^t f_\alpha^+(s)ds$ is nonincreasing with respect to. α. By the above reasonings it follows that $x \ominus \int_a^t f(s)ds$ exists, for all $t \in [a,h]$, where $h = \min\left\{\frac{c_1}{M_1}, \frac{|c_2|}{M_2}, \frac{len([x]^1)}{M}\right\} > 0$.

If we are under the assumption b), then it follows that $len([f(s)]^1) = 0$, for all $s \in [a,b]$ and $\int_a^t len([f(s)]^1)ds = len([x]^1) = 0$, for all $t \in [a,b]$. The other two required inequalities can be obtained by similar reasonings as above, for all $t \in [a, a+h]$, where $h = \min\left\{\frac{c_1}{M_1}, \frac{|c_2|}{M_2}\right\} > 0$, which proves the lemma. □

The following result is useful in obtaining characterization results for the solutions of fuzzy differential equations by ODEs and it was obtained in [7], Theorem 5, see also, [2], Theorem 8.

Theorem 2.3. *Let $f : (a,b) \to \mathbb{R}_{\mathcal{F}}$ and $x_0 \in (a,b)$.*

(i) If f is strongly generalized differentiable on x_0 as in (i) of Definition 1.1 (i-differentiable) then $[f'(x_0)]_\alpha = \left[(f_\alpha^-)'(x_0), (f_\alpha^+)'(x_0)\right], \forall \alpha \in [0,1]$

(ii) If f is strongly generalized differentiable on x_0 as in (ii) of Definition 1.1 (ii-differentiable) then

$$\left[f'(x_0)\right]_\alpha = \left[\left(f_\alpha^+\right)'(x_0), \left(f_\alpha^-\right)'(x_0)\right], \forall \alpha \in [0,1],$$

where $[f(x_0)]_\alpha = [f_\alpha^-(x_0), f_\alpha^+(x_0)]$ *are the* $\alpha-$*level intervals of* $f(x_0)$, $\alpha \in [0,1]$.

Now, let us denote by $\overline{B}(x_0,q) = \{x \in \mathbb{R}_\mathcal{F} : D(x,x_0) \le q\}$, a closed ball in $\mathbb{R}_\mathcal{F}$, where $x_0 \in \mathbb{R}_\mathcal{F}$ and $D(x,x_0)$ represents the Hausdorff distance between the fuzzy numbers x and x_0 (see e.g. [9]). Let us remark here that it is not compact (since $\mathbb{R}_\mathcal{F}$ is not locally compact) and so, the continuity of a function $f : \overline{B}(x_0,q) \to \mathbb{R}_\mathcal{F}$ does not imply that it is bounded. But as it was recently shown by V. Lupulescu in [17], continuity, together with a local Lipschitz condition are sufficient to ensure boundedness. This also allows us to eliminate some boundedness conditions from the existence results in [27], [32] or locally compactness type conditions in [15] (see [17] for details). The same result will also allow us to eliminate the boundedness conditions in [2].

Lemma 2.4. *([17]) Assume that* $F : [t_0,t_0+p] \times \overline{B}(x_0,q) \times \overline{B}(x_0,q) \to \mathbb{R}_\mathcal{F}$ *is continuous and fulfils the Lipschitz condition*

$$D(F(t,x,u),F(t,y,v)) \le L[D(x,y)+D(u,v)], \forall (t,x,u),(t,y,v) \in R_0.$$

Then F is bounded, and there exists $M > 0$ *such that* $D(F(t,x,u),0) \le M$, *where* 0 *denotes the singleton fuzzy number* $\{0\}$.

Proof. The proof is given in [17]. We give it for our special case. We observe that

$$\begin{aligned} D(F(t,x,u),0) &\le D(F(t,x,u),F(t,x_0,x_0)) + D(F(t,x_0,x_0),0) \\ &\le L[D(x,x_0)+D(u,x_0)] + D(F(t,x_0,x_0),0) \\ &\le 2qL + \max_{t\in[t_0,t_0+p]} D(F(t,x_0,x_0),0) = M. \end{aligned}$$

□

The main result of our paper is the following.

Theorem 2.5. *Let* $R_0 = [t_0,t_0+p] \times \overline{B}(x_0,q) \times \overline{B}(x_0,q)$, $p,q > 0$, $x_0 \in \mathbb{R}_\mathcal{F}$ *and* $F : R_0 \to \mathbb{R}_\mathcal{F}$ *be continuous such that the following assumptions hold:*

(i) There exist a constant $L > 0$ *such that*

$$D(F(t,x,u),F(t,y,v)) \le L[D(x,y)+D(u,v)], \forall (t,x,u),(t,y,v) \in R_0.$$

(ii) Let $[F(t,x,u)]^\alpha = [F_\alpha^-(t,x,u), F_\alpha^+(t,x,u)]$ *be the representation of F as in Theorem 2.1, then* F_α^-, $F_\alpha^+ : R_0 \to \mathbb{R}$ *have bounded partial derivatives with respect to* $\alpha \in [0,1]$, *the bounds being independent of* $(t,x,u) \in R_0$ *and* $\alpha \in [0,1]$.

(iii) The functions x_0^- *and* x_0^+ *are differentiable, existing* $c_1 > 0$ *with* $x_0^-(\alpha) \ge c_1$, *and* $c_2 < 0$ *with* $x_0^+(\alpha) \le c_2$, *for all* $\alpha \in [0,1]$ *(*$x_0^\pm$ *being defined as in Theorem 2.1) and we have the possibilities*

a) $x_0^-(1) < x_0^+(1)$

or

b) if $x_0^-(1) = x_0^+(1)$ *then the core* $[F(t,x,u)]^1$ *consists in exactly one element for any* $(t,x,u) \in R_0$, *whenever* $[x]^1$ *and* $[u]^1$ *consist in exactly one element.*

If $h : [t_0,t_0+p] \to [t_0,t_0+p]$ *is continuous, then the fuzzy initial value problem*

$$x'(t) = F(t,x(t),x(h(t))), \quad x(t_0) = x_0,$$

has exactly two solutions defined in an interval $[t_0,t_0+k]$ *for some* $k > 0$.

Proof. For the proof one can use the lines in the statements and proofs of Theorems 22 and 25 in [2] (which are generalizations of the existence result in [27], [32]) combined with the previous Lemma 2.4. We will give in the present paper a different proof.

First, let us observe that assumptions (ii) and (iii), by Lemma 2.2 ensure the existence of the H-difference $x_0 \ominus \left(-\int_{t_0}^{t} F(t,x(t),x(h(t)))dt\right)$ for $t \in [t_0,t_0+c]$ for some $0 < c \leq p$. Now we consider $R_1 = [t_0,t_0+c] \times \overline{B}(x_0,q) \times \overline{B}(x_0,q)$, $K_0 = C([t_0,t_0+c],\mathbb{R}_{\mathcal{F}})$ and two operators $P,Q : K_0 \to K_0$ ($C([a,b],\mathbb{R}_{\mathcal{F}})$ being the space of continuous functions $x : [a,b] \to \mathbb{R}_{\mathcal{F}}$) defined as follows:

$$
\begin{aligned}
P(x_0)(t) &= Q(x_0)(t) = x_0 \\
P(x)(t) &= x_0 + \int_{t_0}^{t} F(t,x(t),x(h(t)))dt \\
Q(x)(t) &= x_0 \ominus \left(-\int_{t_0}^{t} F(t,x(t),x(h(t)))dt\right)
\end{aligned}
$$

We observe that P is always well defined, while Q is well defined on $[t_0,t_0+c]$ by the choice of c. By Lemma 2.4, and the Lipschitz condition (i) of the present Theorem, F is bounded and so are both P and Q. Moreover,

$$D(P(x)(t),x_0) \leq \int_{t_0}^{t} D(F(t,x(t),x(h(t)),0)dt \leq M(t-t_0)$$

and

$$D(Q(x)(t),x_0) \leq \int_{t_0}^{t} D(F(t,x(t),x(h(t)),0)dt \leq M(t-t_0),$$

where $M = \sup_{(t,x,u)\in R_1} D(F(t,x,u),0)$ is provided by Lemma 2.4. Let $d = \min\left\{c, \frac{q}{M}\right\}$ and $K_1 = C([t_0,t_0+d],\overline{B}(x_0,q))$. Then for the restrictions $P,Q : K_1 \to C([t_0,t_0+d],\mathbb{R}_{\mathcal{F}})$, we have $D(P(x)(t),x_0) \leq q$ and also $D(Q(x)(t),x_0) \leq q$, i.e., $x \in K_1$ gives $P(x) \in K_1$ and $Q(x) \in K_1$, and K_1 is a complete metric space considered with the uniform distance, as a closed subspace of a complete metric space. Now we will show that P and Q are contractions. Indeed,

$$
\begin{aligned}
D(P(x)(t),P(y)(t)) &\leq \int_{t_0}^{t} D(F(t,x(t),x(h(t)),F(t,y(t),y(h(t)))dt \\
&\leq 2L(t-t_0)D(x,y)
\end{aligned}
$$

and

$$
\begin{aligned}
D(Q(x)(t),Q(y)(t)) &\leq \int_{t_0}^{t} D(F(t,x(t),x(h(t)),F(t,y(t),y(h(t)))dt \\
&\leq 2L(t-t_0)D(x,y).
\end{aligned}
$$

Now choosing $k < \min\{d, \frac{1}{2L}\}$ both P and Q become contractions. Banach's fixed point theorem implies the existence of x^i fixed point of P and x^{ii} fixed point of Q. To conclude, we observe that x^i is strongly generalized differentiable as in case (i) of Definition 1.1, while x^{ii} is strongly generalized differentiable as in case (ii) of Definition 1.1 and both are solutions of the fuzzy initial value problem

$$x'(t) = F(t,x(t),x(h(t))), \quad x(t_0) = x_0,$$

for any $t \in [t_0, t_0 + k]$. The uniqueness of the two solutions locally on $[t_0, t_0 + k]$ follows by the uniqueness of the fixed points for P and Q. □

3 Characterization Results

Recently, in [3] a characterization result was presented for the solution of a fuzzy differential equation under Hukuhara differentiability. Also, this result was used in [24] to solve Hybrid fuzzy differential equations. For the strongly generalized differentiability in the interval setting, characterization results were obtained in [30]. Let us observe that these characterization results are highly dependent on existence and uniqueness theorems of Peano type. So, the better the existence theorem, the better the characterization result. This shows us, that it is possible to extend the result in [3] to the strongly generalized differentiability setting and simultaneously to improve the results. Also, the result in [3] had a boundedness condition, which was eliminated in a Corollary presented in [24] and it can be eliminated here too, based on the results in [17].

Theorem 3.1. *Let $R_0 = [t_0, t_0 + p] \times \overline{B}(x_0, q) \times \overline{B}(x_0, q)$, $p, q > 0$, $x_0 \in \mathbb{R}_{\mathcal{F}}$ and $F : R_0 \to \mathbb{R}_{\mathcal{F}}$ be such that*

$$[F(t,x,u)]^{\alpha} = [F_{\alpha}^{-}(t, x_{\alpha}^{-}, x_{\alpha}^{+}, u_{\alpha}^{-}, u_{\alpha}^{+}), F_{\alpha}^{+}(t, x_{\alpha}^{-}, x_{\alpha}^{+}, u_{\alpha}^{-}, u_{\alpha}^{+})], \forall \alpha \in [0,1]$$

and the following assumptions hold:

(i) $F_{\alpha}^{\pm}(t, x_{\alpha}^{-}, x_{\alpha}^{+}, u_{\alpha}^{-}, u_{\alpha}^{+})$ are equicontinuous, uniformly Lipschitz in their second through fifth arguments (i.e., there exist a constant $L > 0$ such that

$$\begin{aligned}&|F_{\alpha}^{\pm}(t, x_{\alpha}^{-}, x_{\alpha}^{+}, u_{\alpha}^{-}, u_{\alpha}^{+}) - F_{\alpha}^{\pm}(t, y_{\alpha}^{-}, y_{\alpha}^{+}, v_{\alpha}^{-}, v_{\alpha}^{+})| \\ &\leq L(|x_{\alpha}^{-} - y_{\alpha}^{-}| + |x_{\alpha}^{+} - y_{\alpha}^{+}| + |u_{\alpha}^{-} - v_{\alpha}^{-}| + |u_{\alpha}^{+} - v_{\alpha}^{+}|),\end{aligned}$$

$\forall (t,x,u), (t,y,v) \in R_0, \alpha \in [0,1]$).

(ii) F_{α}^{-}, $F_{\alpha}^{+} : R_0 \to \mathbb{R}$ have bounded partial derivatives with respect to $\alpha \in [0,1]$, the bounds being independent of $(t,x,u) \in R_0$ and $\alpha \in [0,1]$.

(iii) The functions x_0^{-} and x_0^{+} are differentiable, existing $c_1 > 0$ with $x_0^{-}(\alpha) \geq c_1$, and $c_2 < 0$ with $x_0^{+}(\alpha) \leq c_2$, for all $\alpha \in [0,1]$ ($x_0^{\pm}$ being defined as in Theorem 2.1) and we have the possibilities

a) $x_0^{-}(1) < x_0^{+}(1)$

or

b) if $x_0^{-}(1) = x_0^{+}(1)$ then the core $[F(t,x,u)]^1$ consists in exactly one element for any $(t,x,u) \in R_0$, whenever $[x]^1$ and $[u]^1$ consist in exactly one element.

If $h : [t_0, t_0 + p] \to [t_0, t_0 + p]$ is continuous, then the fuzzy initial value problem

$$x'(t) = F(t, x(t), x(h(t))), \quad x(t_0) = x_0, \tag{3.1}$$

is equivalent on some interval $[t_0, t_0 + k]$ with the union of the following two ODEs:

$$\begin{cases} (x_{\alpha}^{-})'(t) = F_{\alpha}^{-}(t, x_{\alpha}^{-}(t), x_{\alpha}^{+}(t), x_{\alpha}^{-}(h(t)), x_{\alpha}^{+}(h(t))) \\ (x_{\alpha}^{+})'(t) = F_{\alpha}^{+}(t, x_{\alpha}^{-}(t), x_{\alpha}^{+}(t), x_{\alpha}^{-}(h(t)), x_{\alpha}^{+}(h(t))) \quad , \alpha \in [0,1] \\ x_{\alpha}^{-}(t_0) = (x_0)_{\alpha}^{-}, x_{\alpha}^{+}(t_0) = (x_0)_{\alpha}^{+} \end{cases} \tag{3.2}$$

$$\begin{cases} (x_\alpha^-)'(t) = F_\alpha^+(t, x_\alpha^-(t), x_\alpha^+(t), x_\alpha^-(h(t)), x_\alpha^+(h(t))) \\ (x_\alpha^+)'(t) = F_\alpha^-(t, x_\alpha^-(t), x_\alpha^+(t), x_\alpha^-(h(t)), x_\alpha^+(h(t))) \\ \quad x_\alpha^-(t_0) = (x_0)_\alpha^-, x_\alpha^+(t_0) = (x_0)_\alpha^+ \end{cases}, \alpha \in [0,1] \tag{3.3}$$

Proof. Condition (i) of the theorem ensures the existence of a unique solution for each of the equations (3.2) and (3.3) for any fixed $\alpha \in [0,1]$. Let us denote these solutions by $(x_\alpha^-)^i, (x_\alpha^+)^i$ and $(x_\alpha^-)^{ii}, (x_\alpha^+)^{ii}$ respectively. The equicontinuity of $F_\alpha^\pm$ ensures the continuity of F, while the Lipshitz condition in (i) is sufficient for the corresponding Lipschitz property of F. All these conditions together ensure the existence and uniqueness of two solutions x^i and x^{ii} for the fuzzy initial value problem (3.1). Let $[x^i]_\alpha = [(x^i)_\alpha^-, (x^i)_\alpha^+]$ and $[x^{ii}]_\alpha = [(x^{ii})_\alpha^-, (x^{ii})_\alpha^+]$, $\alpha \in [0,1]$. Then, by Theorem 2.3, since x^i Hukuhara differentiable (or i-differentiable) then

$$\left[(x^i)'\right]_\alpha = [\left((x^i)_\alpha^-\right)', \left((x^i)_\alpha^+\right)']$$

and $(x^i)_\alpha^-, (x^i)_\alpha^+$ is a solution of (3.2). Since $(x_\alpha^-)^i, (x_\alpha^+)^i$ is the unique solution of (3.2) we obtain

$$\left((x^i)_\alpha^-, (x^i)_\alpha^+\right) = \left((x_\alpha^-)^i, (x_\alpha^+)^i\right), \forall \alpha \in [0,1].$$

Similar reasoning for the ii-differentiable solution implies

$$\left[(x^{ii})'\right]_\alpha = [\left((x^{ii})_\alpha^+\right)', \left((x^{ii})_\alpha^-\right)'],$$

so $(x^{ii})_\alpha^+, (x^{ii})_\alpha^-$ is a solution of (3.3), so

$$\left((x^{ii})_\alpha^-, (x^{ii})_\alpha^+\right) = \left((x_\alpha^-)^{ii}, (x_\alpha^+)^{ii}\right), \forall \alpha \in [0,1].$$

As a conclusion the solutions of (3.1) are exactly those of (3.2) and (3.3). □

Remark. The conditions in Theorems 2.5 and 3.1 are fulfilled by e.g. any triangular initial value x_0 with strictly increasing (strictly decreasing) slopes for the sides, together with any function F having triangular values and satisfying the Lipschitz property in Theorem 2.5, (i). Moreover, the conditions are fulfilled for any trapezoidal initial condition, having strictly increasing (strictly decreasing) slopes for the sides, together with any F having trapezoidal values with the Lipschitz property in Theorem 2.5, (i). So, the conditions of Theorem 2.5, are very relaxed, and we can find large classes of fuzzy differential equations for which locally, the two solutions exist and are unique. Also, notice that a weaker existence result for triangular numbers as initial vale data (and for $h(t) = t$) was presented in [2]. Therefore our theorem generalizes Theorem 25 in [2] from two points of view : firstly, the form of differential equation is more general (that is, with deviated argument) and secondly, the initial values and the values taken by the function F can be a general fuzzy number.

In the followings we formulate a particularization of the existence, uniqueness and characterization Theorem 3.1 for fuzzy initial value problems with triangular data. We denote by $\mathbb{R}_\mathcal{T}$ the space of triangular fuzzy numbers. Any triangular fuzzy number can be characterized by the endpoints of the 0-level set and the core (1-level set) which is a singleton. As a consequence we can denote $x = (x^-, x^1, x^+) \in \mathbb{R}_\mathcal{T}$ any triangular fuzzy number. Also, in the next Theorem we consider $h(t) = t$ for simplicity. Similar result holds as in the previous Theorem 3.1 with general continuous h, but we formulate a simpler result in view of possible applications.

Theorem 3.2. *Let* $R_0 = [t_0, t_0 + p] \times \left(\overline{B}(x_0, q) \cap \mathbb{R}_{\mathcal{T}}\right)$, $p, q > 0$, $x_0 \in \mathbb{R}_{\mathcal{T}}$ *and* $F : R_0 \to \mathbb{R}_{\mathcal{T}}$ *be such that*

$$F(t,x) = (F^-(t,x^-,x^1,x^+), F^1(t,x^-,x^1,x^+), F^+(t,x^-,x^1,x^+))$$

and the following assumptions hold:

(i) F^-, F^1, F^+ *are continuous, Lipschitz in their second through last arguments.*

(ii) $x_0 = (x_0^-, x_0^1, x_0^+)$ *is a nontrivial triangular number such that* $x_0^- < x_0^1 < x_0^+$. *Then*

(a) The fuzzy initial value problem

$$x'(t) = F(t, x(t)), \quad x(t_0) = x_0, \tag{3.4}$$

has exactly two triangular valued solutions on some interval $[t_0, t_0 + k]$.

(b) Problem (3.4) is equivalent to the union of the following two ODEs:

$$\begin{cases} (x^-)' = F^-(t,x^-,x^1,x^+) \\ \left(x^1\right)' = F^1(t,x^-,x^1,x^+) \\ (x^+)' = F^+(t,x^-,x^1,x^+) \\ x^-(t_0) = x_0^-, x^1(t_0) = x_0^1, x^+(t_0) = x_0^+ \end{cases}, \tag{3.5}$$

$$\begin{cases} (x^-)' = F^+(t,x^-,x^1,x^+) \\ \left(x^1\right)' = F^1(t,x^-,x^1,x^+) \\ (x^+)' = F^-(t,x^-,x^1,x^+) \\ x^-(t_0) = x_0^-, x^1(t_0) = x_0^1, x^+(t_0) = x_0^+ \end{cases}, \tag{3.6}$$

Proof. It is easy to check that since F is triangular valued, the solutions of (3.4) are triangular valued. Further the conditions (i) and (ii) ensure that the problem (3.4) has two unique solutions locally. It is easy to check that the conditions in Theorem 3.1 are fulfilled with $h(t) = t$ and that the problems (3.2) and (3.3), in the case of triangular-valued functions are equivalent with (3.5) and (3.6) respectively, and the proof is complete. □

4 Examples

In the following fist we present on example to show the practical applicability of the previous result for FIVP with triangular data.

Example 1 We consider the Fuzzy Initial Value Problem

$$x'(t) = -tx(t) + (1,2,3)t^2, x(0) = (2,4,7) \tag{4.1}$$

We observe that $F(t,x) = -tx + (1,2,3)t^2$ is a triangular fuzzy number valued function (whenever x is triangular) and

$$F^-(t,x^-,x^1,x^+) = -tx^+ + t^2$$
$$F^1(t,x^-,x^1,x^+) = -tx^1 + 2t^2$$
$$F^+(t,x^-,x^1,x^+) = -tx^- + 3t^2.$$

It is easy to see that conditions (i) and (ii) are obviously satisfied and as a conclusion, on some interval $[t_0, t_0 + k]$ (where the choice of k depends on the existence of the Hukuhara difference in the definition of the operator Q in Theorem 2.5). The second result shows that the FIVP of this example is equivalent to the union of two ODEs:

$$\begin{cases} (x^-)' = -tx^+ + t^2 \\ \left(x^1\right)' = -tx^1 + 2t^2 \\ (x^+)' = -tx^- + 3t^2 \\ (x^-, x^1, x^+) = (2,4,7) \end{cases}$$

and

$$\begin{cases} (x^-)' = -tx^- + 3t^2 \\ \left(x^1\right)' = -tx^1 + 2t^2 \\ (x^+)' = -tx^+ + t^2 \\ (x^-, x^1, x^+) = (2,4,7) \end{cases}.$$

The two solutions of the FIVP are shown in the following Figure 1 (by showing the end-points of the 0-level set and the core (1-level set).

Fig. 1. Solutions x^i (left) and x^{ii} (right) of the FIVP 4.1.

The choice of $k = 1.4$ arises from the fact that the operator Q and the Hukuhara differences in Theorem 2.5 are well defined on the $[0, 1.4]$ interval.

The next example shows the applicability of the strongly generalized differentiability in real-world situations and also it illustrates the existence result in Theorem 2.5.

Example 2 In [20] a logistic fuzzy model is analized in detail and the findings are that it is an adequate model for population dynamics under uncertainty. We use the strongly generalized differentiability and the results of the present paper to analyze a similar fuzzy logistic equation

$$x' = ax(K \ominus_g x),\ x(0) = x_0$$

where aKx is the growth rate, ax^2 is the inhibition term and K is the environment capacity. The generalized Hukuhara difference

$$K \ominus_g x = \begin{cases} K \ominus x, \text{ if } K \ominus g \text{ exists} \\ -(x \ominus K) \text{ if } x \ominus K \text{ exists} \end{cases}$$

proposed in [28] is used here. The advantage of this difference over the Hukuhara difference is that the generalized Hukuhara difference exists in situations when the usual Hukuhara difference fails to exist. We consider $a,K \in \mathbb{R}_{\mathcal{T}}$ symmetric triangular numbers and $x_0 \in \mathbb{R}$. We use in our simulations $x_0 = 500$, $a = (0.98,1,1.02)\cdot 10^{-3}$, $K = (9,10,11)\cdot 10^3$. Let us remark that the function $f(t,x) = ax(K \ominus_g x)$ is not a triangular-number-valued function even if the inputs are triangular numbers, since the product of two triangular fuzzy numbers is not generally triangular. In this case for any $\alpha \in [0,1]$,

$$\begin{aligned} F_\alpha^-(x^-,x^1,x^+) &= a_\alpha^- \min_{\alpha\in[0,1]}\left\{x_\alpha^-\left(K_\alpha^- - x_\alpha^-\right), x_\alpha^+\left(K_\alpha^+ - x_\alpha^+\right)\right\} \\ F^1(x^-,x^1,x^+) &= a^1x^1\left(K^1 - x^1\right) \\ F_\alpha^+(x^-,x^1,x^+) &= a_\alpha^+ \max_{\alpha\in[0,1]}\left\{x_\alpha^-\left(K_\alpha^- - x_\alpha^-\right), x_\alpha^+\left(K_\alpha^+ - x_\alpha^+\right)\right\} \end{aligned}$$

are piecewise quadratic functions of α and the more general characterization result 3.1 is necessarily used. Locally, the FIVP is equivalent to the problem

$$\left\{\begin{array}{c} (x_\alpha^-)' = F_\alpha^-(t,x^-,x^1,x^+) \\ (x_\alpha^+)' = F_\alpha^+(t,x^-,x^1,x^+) \\ x^-(t_0) = x_0^-, x^1(t_0) = x_0^1, x^+(t_0) = x_0^+ \end{array}\right. . \tag{4.2}$$

It is easy to see that the only solution we have locally on some interval $[0,p]$ is the solution according to case (i) of differentiability, since $x_0 \in \mathbb{R}$. Let $p = \inf\{t > 0 : x_\alpha^-(t)\left(K_\alpha^- - x_\alpha^-(t)\right) = x_\alpha^+(t)(K_\alpha^+ - x_\alpha^+(t))\}$. In our numerical simulation we have obtained $p \approx 0.407$. We observe that if we use now p and $x_0 = x(p)$ as initial values for a new initial value problem

$$x' = ax(K \ominus_g x), x(p) = x_0$$

two solutions will emerge according to Theorem 2.5 characterized by the systems (4.2) and (4.3),

$$\left\{\begin{array}{c} (x_\alpha^-)' = F_\alpha^-(t,x^-,x^1,x^+) \\ (x_\alpha^+)' = F_\alpha^+(t,x^-,x^1,x^+) \\ x^-(t_0) = x_0^-, x^1(t_0) = x_0^1, x^+(t_0) = x_0^+ \end{array}\right. . \tag{4.3}$$

One is differentiable as in (i) and another differentiable as in (ii) case of Definition 1.1. Let us remark also, that we can paste together the solutions and we obtain two solutions on the $[0,1]$ interval. Let us observe that the most realistic model appears to be the one which starts with case (i) of differentiability on $[0,p]$ and continues with case (ii) of differentiability on $[p,1]$. The solution which is (i) differentiable is the Hukuhara solution on $[0,1]$ and it is unrealistic from practical point of view since the population may increase according to that solution much more than the environment's capacity. In Fig. 1 these solutions are shown, where x^i represents the lower and upper endpoints of the 0-level set of the solution according to case (i) of differentiability, x^{ii} is the 0 level set of the solution as in case (ii) of differentiability. $x^1$is the common 1-level set of the two solutions. This example also allows us to underline once more the practical superiority of the strongly generalized differentiability with respect to the Hukuhara derivative case. Several other practical examples were also proposed in [1].

Figure 1. Solutions of the fuzzy logistic equation.

Example 3 As an example of a fuzzy differential equation with deviated argument we consider the so-called pantograph equation. It is an equation that appears as a model for several phenomena including cell-growth [23], [31]. Consider the Fuzzy differential equation with modified argument

$$x'(t) = ax(t) + bx(qt), x(0) = x_0 \tag{4.4}$$

where $0 \leq q \leq 1$ and $0 < a, b \in \mathbb{R}$ and $x_0 \in \mathbb{R}_{\mathcal{T}}$ is a nontrivial triangular fuzzy number ($\underline{x} < x^1 < \bar{x}$).

It is easy to see that all the conditions in 2.5 are fulfilled and as a conclusion the equation 4.4 has two solutions on an interval $[0, p]$. Moreover, using the characterization result 3.1 we can reduce (4.4) to the functional differential equations

$$\begin{cases} (\underline{x})'(t) = a\underline{x}(t) + b\underline{x}(qt) \\ (x^1)'(t) = ax^1(t) + bx^1(qt) \\ (\overline{x})'(t) = a\overline{x}(t) + b\overline{x}(qt) \\ \underline{x}(0) = \underline{x_0}, x^1(0) = x_0^1, \overline{x}(0) = \overline{x_0} \end{cases}$$

and

$$\begin{cases} (\underline{x})'(t) = a\overline{x}(t) + b\overline{x}(qt) \\ (x^1)'(t) = ax^1(t) + bx^1(qt) \\ (\overline{x})'(t) = a\underline{x}(t) + b\underline{x}(qt) \\ \underline{x}(0) = \underline{x_0}, x^1(0) = x_0^1, \overline{x}(0) = \overline{x_0} \end{cases}.$$

Following [16], we can find the analytic solutions of both equations as a convergent power series $x^i = \sum_{n=0}^{\infty} \alpha_n t^n$, where $\alpha_0 = x_0$, $\alpha_{n+1} = \frac{a+bq^n}{n+1}\alpha_n, n \geq 1$ is the solution according to Definition 1.1, (i) of the equation [23], and $x^{ii} = \left(\sum_{n=0}^{\infty} a_n t^n, \sum_{n=0}^{\infty} b_n t^n, \sum_{n=0}^{\infty} c_n t^n\right)$, $a_0 = \underline{x_0}$,

Figure 2. Solutions of the fuzzy pantograph equation

$b_0 = x_0^1$, $c_0 = \overline{x_0}$ $a_{n+1} = \frac{a+bq^n}{n+1} c_n$, $b_{n+1} = \frac{a+bq^n}{n+1} b_n$, $c_{n+1} = \frac{a+bq^n}{n+1} a_n$, $n \geq 1$ is the solution according to Definition 1.1, (ii) of (4.4). In Fig. 2 these solutions are shown for the case when $a = b = 1$ and $q = \frac{1}{2}$, $x_0 = (1,4,7)$ in (4.4). x^i represents the lower and upper endpoints of the 0-level set of the solution according to case (i) of differentiability, x^{ii} is the 0-level set of the solution as in case (ii) of differentiability and $x^1$is the common 1-level set of the two solutions.

5 Corrections of Recent Results

In what follows we correct some results of [7], which have been propagated also to [21].

Remarks. 1) For the particular case $h(t) = t$ in Theorem 2.5 (and therefore $F := F(t,x(t))$) we obtain a correct version of the existence and uniqueness result stated by Theorem 6 in [7] cited also as Theorem 3.1 in [21]. Indeed, Theorem 6 in [7] asserts that given $F : T \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$ continuous and satisfying a Lipschitz condition in the second argument, the Cauchy problem

$$x'(t) = F(t,x(t)),\ x(t_0) = x_0 \tag{5.1}$$

has exactly two solutions in $T = [a,b]$. Unfortunately, this theorem fails if for example $x_0 \in \mathbb{R}$ and if $F(t,x(t)) \in \mathbb{R}_{\mathcal{F}} \setminus \mathbb{R}$ in a neighborhood of (t_0,x_0) (for such an example, it is enough to take $F(t,x(t)) = C \in \mathbb{R}_{\mathcal{F}} \setminus \mathbb{R}$, where C is a fixed fuzzy number). Indeed, let us assume that a solution, differentiable according to Definition 1.1, (ii), would exists under the assumptions above. Then, by Theorem 5 in [7], for $x(t) = [f_\alpha(t), g_a(t)]$ we have

$$x'(t) = [g'_\alpha(t), f'_\alpha(t)],$$

for any $t > t_0$. We easily obtain that the function $len(x(t)) = g_0(t) - f_0(t)$ is decreasing whenever x is assumed to be in the case of (ii)-differentiability in Definition 1.1. Since

$len(x(t_0)) = 0$ and since $len(x(t))$ is decreasing, we obtain $len(x(t)) < 0$ which is impossible for a correct fuzzy number valued function $x(t)$.

2) For the particular case $h(t) = t$ in Theorem 3.1 (and therefore $F := F(t, x(t))$) we obtain a correct version of the characterization result stated in Theorem 3.3. in [21]. It is easy to see that the proof of Theorem 3.3. in [21], uses the result in Theorem 6 of [7]. It is easy to observe that we may have situations when the solution in the sense of 1.1, (ii) of (3.1) does not exist (e.g. if $y_0 \in \mathbb{R}$) however the system (3.3) has a solution.

3) Also, in [7] a very interesting result is obtained (Theorem 8), which connects fuzzy differential inclusions to fuzzy differential equations with generalized differentiability. Since Theorem 8 in [7] assumes the existence of the solution for (ii)-differentiable case in Definition 1.1, in the light of Theorem 2.5 above (written for the particular choice $h(t) = t$), Theorem 8 in [7] should be slightly reformulated as follows.

Theorem 5.1. *Let $f : [0, a] \times \mathbb{R} \to \mathbb{R}$ be continuous and $F : [0, a] \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$, be the Zadeh's extension of f, i.e., $[F(t, x)]^{\alpha} = f(t, [x]^{a})$. If f is nonincreasing with respect to the second argument, using the derivative in Definition 1, case (ii), the fuzzy solution of (2) whenever it exists, coincides with the solution obtained via differential inclusions.*

Remark. Theorem 5.1 is another argument for the importance of the cases (iii) and (iv) in Definition 1.1. Indeed, the above stated theorem does not cover the case when $f(t, x)$ has not constant monotonicity. In these cases we will have to switch between the cases (i) and (ii) of differentiability in Definition 1.1, so the cases (iii) and (iv) in Definition 1.1 may become important as switch-points.

At the end of this note we will correct other two results in [7].

Thus, in Theorem 3 in [7], the authors claim that if F is continuous in $T = [a, b]$, then the function $G(t) = \int_a^t F(s)ds$ is differentiable according to Definition 1, (ii), and $G'(t) = F(t)$. But for any $h > 0$ sufficiently small, the H-differences $G(t) \ominus G(t+h)$ and $G(t-h) \ominus G(t)$ do not necessarily exist. Indeed, if we suppose, for example, that $G(t) \ominus G(t+h)$ exists, we easily get $G(t) = G(t+h) + \alpha(t)$ and $0_{\mathbb{R}_{\mathcal{F}}} = \int_t^{t+h} F(s)ds + \alpha(t)$, which is impossible for non-real fuzzy numbers (see e.g. Theorem 1, (ii), in [1]). Therefore, Theorem 3 in [7] is not valid and its correct version is the following.

Theorem 5.2. *Let $F : T \to \mathbb{R}_{\mathcal{F}}$ be continuous in $T = [a, b]$. Then the function $G(t) = -\int_t^b F(s)ds$ is differentiable according to Definition 1.1, (ii), and $G'(t) = F(t)$.*

Proof. For sufficiently small $h > 0$ we have

$$\begin{aligned} G(t) &= -\int_t^b F(s)ds = -\int_t^{t+h} F(s)ds - \int_{t+h}^b F(s)ds \\ &= -\int_t^{t+h} F(s)ds + G(t+h) \end{aligned}$$

and so the H-difference $G(t) \ominus G(t+h)$ exists for any $0 < h < b - t$. We obtain

$$\frac{G(t) \ominus G(t+h)}{(-h)} = \frac{1}{h} \int_t^{t+h} F(s)ds$$

and finally

$$\lim_{h \searrow 0} \frac{G(t) \ominus G(t+h)}{(-h)} = F(s).$$

Together with the symmetric case, the statement in Theorem 5.2 now easily follows. □

The second result which needs a correction is Theorem 4 in [7]. It asserts that for F differentiable in $T=[a,b]$ according to Definition 1.1, (ii), assuming that the derivative F' is integrable over T, then for each $t\in T$ we have

$$F(t)=F(a)+\int_a^t F'(s)ds.$$

Since its statement was based on Theorem 3 in [7], it needs to be corrected as well. Let us remark that in general, the above equality is not true. Indeed, supposing that it is true and taking $t=a+h$, $h>0$ sufficiently small, it follows that the H-difference $F(a+h)\ominus F(a)$ exists. Also, by the assumption that F is differentiable according to Definition 1.1, (ii), $F(a)\ominus F(a+h)$ must exists and the simultaneous existence of these two H-differences easily imply that $F'(a)$ is real (see e.g. the reasonings in Remark 6, (1) and Theorem 7 in [2]). Consequently, the equality will not hold for the case when the values of F' belong to $\mathbb{R}_{\mathcal{F}}\setminus\mathbb{R}$.

The correct form of Theorem 4 in [7] is the following.

Theorem 5.3. *Let $F:T\to\mathbb{R}_{\mathcal{F}}$ be differentiable in $T=[a,b]$ according to Definition 1.1, (ii). Then the derivative F' is integrable over T and we have*

$$F(t)=F(b)-\int_t^b F'(s)ds,\forall t\in T. \tag{5.2}$$

Proof. First we prove that for F differentiable (everywhere in $[a,b]$) according to Definition 1.1, (ii), it follows that F' is integrable. This follows by Theorem 5 in [7]. Indeed, let us assume that $F:T\to\mathbb{R}_{\mathcal{F}}$ is differentiable according to Definition 1.1, (ii), and let $[F(t)]^a=[f_\alpha(t),g_\alpha(t)]$ denote the α-level sets of F, $\alpha\in[0,1]$. Then, by Theorem 5 in [7] we have that f_α and g_α are (real valued) differentiable functions and $[F'(t)]^\alpha=[g'_\alpha(t),f'_\alpha(t)]$. The functions $g'_\alpha(t)$ and $f'_\alpha(t)$ are integrable as real valued functions and therefore, by levelwise the Aumann integral exists and we have

$$\int_a^b [F'(t)]^\alpha dt=\left[\int_a^b g'_\alpha(t)dt,\int_a^b f'_\alpha(t)dt\right]. \tag{5.3}$$

The levelwise existence of the classical Aumann integral implies the existence of the Fuzzy Aumann integral (see [9]). Let us remark here that Theorem 5 in [7] is valid and its proof does not employ Theorem 4 in [7]. □

Proof. Continuing the proof of the relation in (5.2), by (5.3) we have for any $\alpha\in[0,1]$

$$-\left[\int_t^b F'(s)ds\right]^\alpha=(-1)\left[\int_t^b g'_\alpha(s)ds,\int_t^b f'_\alpha(s)ds\right]=\left[-\int_t^b f'_\alpha(s)ds,-\int_t^b g'_\alpha(s)ds\right]$$
$$=[f_\alpha(t)-f_\alpha(b),g_\alpha(t)-g_\alpha(b)].$$

This relation is equivalent to

$$[f_\alpha(b),g_\alpha(b)]-\left[\int_t^b F'(s)ds\right]^\alpha=[f_\alpha(t),g_\alpha(t)],\forall\alpha\in[0,1].$$

and finally we get

$$F(b) - \int_t^b F'(s)ds = F(t)$$

which proves Theorem 5.3. □

6 Conclusions

We have proved a very general result on existence and uniqueness of two solutions for fuzzy differential equations with modified argument, based on generalized differentiability. This generalizes an earlier existence result in [2] and corrects the existence result in [7] and the characterization result in [21]. Also characterizations of these solutions by ODEs were obtained. Real-world applications were presented in examples 2 and 3, consisting in applications to fuzzy pantograph equation that models the cell growth and to fuzzy logistic equation that models the population dynamics under uncertainty. At the end of the paper, for other two results in [7] concerning generalized differentiability we prove their correct statements.

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