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GROUND STATE SOLUTIONS FOR SINGULAR QUASILINEAR ELLIPTIC EQUATIONS

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Abstract

The existence of ground state solutions for the quasi-linear elliptic equation

$$-\mathrm{div}(|\nabla u|^{p-2}\nabla u) = \rho(x) f(u), \text{ in } \mathbf{R^N}$$

under suitable conditions is proved. We modify the method developed in [Z. Yang, Existence of positive entire solutions for singular and non-singular quasi-linear elliptic equation, J. Comput. Appl. Math. 197 (2006) 355-364] and extend the results of [A.Mohammed, Ground state solutions for singular semi-linear elliptic equations, Nonlinear Analysis(in press) and Teodora-Liliana Dinu, Entire solutions of sublinear elliptic equations in anisotropic media, J. Math. Anal. Appl. 322(2006), 382-392.]

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1 Introduction

In this paper, we are concerned with the existence of ground state solutions for the following singular quasilinear elliptic equation

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) = \rho(x) f(u), & \text{in } \mathbf{R^N}, \\ u > l, & \text{in } \mathbf{R^N}, \\ u(x) \to l, & \text{as } |x| \to \infty, \end{cases} \tag{1.1}$$

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where $N \geq 3$ and $l \geq 0$ is a real number.

When $p = 2$, these kinds of problems have been studied extensively by many authors in which $\mathbf{R}^N$ is replaced by a smooth bounded domain Ω with zero Dirichlet boundary condition (see [1]- [4]). Recently, the study of ground state solutions has received a lot of interest and numerous existence results have been established (see [5]- [16] and the references therein). Equations of (1.1) are mathematical models occurring in studies of the p-Laplace system, generalized reaction-diffusion theory, non-Newtonian fluid theory [19], non-Newtonian filtration [20] and the turbulent flow of a gas in porous medium [21]. In the non-Newtonian fluid theory, the quantity p is characteristic of the medium. Media with $p > 2$ are called dilatant fluids and those with $p < 2$ are called pseudoplastics. If $p = 2$, they are Newtonian fluids.

Recently, in [5] the author proved the existence of a ground state solution for the semilinear elliptic equation

$$\begin{cases} -\triangle u = f(x,u), & \text{in } \mathbf{R^N} \\ u > 0, & \text{in } \mathbf{R^N}, \\ u(x) \to 0, & \text{as } |x| \to \infty. \end{cases}$$

under suitable conditions on a locally H$\ddot{o}$lder continuous non-linearity $f(x,t)$. The non-linearity may exhibit a singularity as $t \to 0^+$.

In [17], Cirstea and Radulescu proved that the following problem

$$\begin{cases} -\triangle u = b(x)g(u), & \text{in } \mathbf{R^N} \\ u > 0, & \text{in } \mathbf{R^N}, \\ u(x) \to 0, & \text{as } |x| \to \infty, \end{cases} \tag{1.2}$$

admits a unique solution when g is bounded in a neighborhood of ∞, $\lim_{t\to 0^+} g(t)/t = \infty$, and $g(t)/(t+c)$ is decreasing for some constant $c > 0$.

In [20], Goncalves and Santos established the existence of a solution to (1.2) under the assumptions that $g(t)/t$ is decreasing, $\lim_{t\to 0^+} g(t)/t = \infty$ and $\lim_{t\to\infty} g(t)/t = 0$.

For $p > 1$, the existence and uniqueness of the positive solutions for quasilinear elliptic equation

$$\begin{cases} \mathrm{div}(|\nabla u|^{p-2}\nabla u) + \lambda f(u) = 0, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u(x) = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

with $\lambda > 0, p > 1, \Omega \subset \mathbf{R^N}, N \geq 2$ have been studied by many authors. When f is strictly increasing on $\mathbf{R}^+$, $f(0) = 0$, $\lim_{s\to 0^+} f(s)/s^{p-1} = 0$ and $f(s) \leq \alpha_1 + \alpha_2 s^{\mu}$, $0 < \mu < p-1$, $\alpha_1, \alpha_2 > 0$, it was shown in [22] that there exist at least two positive solutions for (1.3) when λ is sufficiently large.

When $f : (0,\infty) \to (0,\infty)$ and $q : \mathbf{R^N} \to (0,\infty)$ are continuous functions, and

$$\int_1^{\infty} \Big(\int_0^u f(s)ds \Big)^{-1/p} du = \infty, \tag{1.4}$$

it has been shown in [23] that there exist entire radially symmetric solutions of the problem

$$\mathrm{div}(|\nabla u|^{p-2}\nabla u) = q(x)f(u), \quad x \in \mathbf{R^N}. \tag{1.5}$$

It was shown in [24] that problem

$$\mathrm{div}(|\nabla u|^{p-2}\nabla u)+f(x,u)=0, \quad x\in \mathbf{R^N}, \tag{1.6}$$

possesses infinitely many positive entire solutions. On the other hand, it was also shown in [25] that if $1<p<N$, $0\le\gamma<p-1$, and $q(x)\in C(\mathbf{R}^+)$ satisfies some suitable conditions, then problem

$$\mathrm{div}(|\nabla u|^{p-2}\nabla u)+q(x)u^{-\gamma}=0, \quad x\in \mathbf{R^N}, \tag{1.7}$$

has a positive entire solution.

In [26], the authors considered the existence of solutions of the singular quasilinear elliptic problem

$$\begin{cases} \text{-div}(|\nabla u|^{p-2}\nabla u)=a(x)g(u)+b(x)f(u), & \text{in } \Omega,\\ u>0, & \text{in } \Omega,\\ u=0, & \text{on } \partial\Omega, \end{cases} \tag{1.8}$$

where $\Omega\subset\mathbf{R^N}$ is a bounded domain with smooth boundary, $a,b:\overline{\Omega}\to[0,\infty)$ are Hölder continuous functions with exponent $\nu\in(0,1)$ and $p>1$. The authors also assumed that $a+b>0$ a.e in Ω and $f,g:(0,\infty)\to[0,\infty)$ are locally Lipschitz continuous functions.

Motivated by the above results, we investigate the existence of positive solutions to problem (1.1). We modify the method developed in [25]-[27] and extend the results of [25] and [27] to singular quasilinear elliptic equation.

2 Main Results

Throughout the paper, we assume that the variable potential $\rho(x)$ satisfies $\rho\in C^{0,\alpha}_{\mathrm{loc}}(\mathbf{R^N})(0<\alpha<1)$, $\rho>0$ and $\rho\neq 0$.

(ρ_1) For $\rho(x)\in C^{0,\alpha}_{\mathrm{loc}}(\mathbf{R^N})$, and $\Phi(r)=\max_{|x|=r}\rho(x)$

$$\int_0^\infty r^{1/(p-1)}\Phi^{1/(p-1)}(r)dr<\infty, \quad \text{if} \quad 1<p\le 2,$$

$$\int_0^\infty r^{\frac{(p-2)N+1}{p-1}}\Phi(r)dr<\infty, \quad \text{if} \quad 2\le p<\infty.$$

The nonlinearity function $f:\ (0,\infty)\to(0,\infty)$ satisfies $f\in C^{0,\alpha}_{\mathrm{loc}}(0,\infty)(0<\alpha<1)$ and has a sublinear growth, in the sense that

(f_1) the mapping $u\to f(u)/u^{p-1}$ is decreasing on $(0,\infty)$ and $\lim_{u\to\infty}f(u)/u^{p-1}=0$.

(f_2) f is increasing in $(0,\infty)$ and $\lim_{u\to 0}\frac{f(u)}{u^{p-1}}=+\infty$.

We point that condition (f_1) does not require that f is smooth at the origin. The standard example is $f(u)=u^q$, where $-\infty<q<1/(p-1)$. A nonlinearity function satisfying both (f_1) and (f_2) is $f(u)=u^q$ where $0<q<p-1$.

Theorem 2.1. *Assume that $l>0$ and assumption (ρ_1), (f_1) are fulfilled., then problem (1.1) has a solution.*

Proof. For any positive integer k we consider the following problem

$$\begin{cases} -\mathrm{div}(|\nabla u_k|^{p-2}\nabla u_k) = \rho(x)f(u_k), & \text{if} \quad |x|<k, \\ u_k > l, \quad \text{if} \quad |x|<k, \\ u_k(x) = l, \quad \text{if} \quad |x| = k. \end{cases} \tag{2.1}$$

Equivalently, the above boundary value problem can be rewritten into

$$\begin{cases} -\mathrm{div}(|\nabla v_k|^{p-2}\nabla v_k) = \rho(x)f(v_k+l), & \text{if} \quad |x|<k, \\ v_k(x) = 0, \quad \text{if} \quad |x| = k. \end{cases} \tag{2.2}$$

Since $f \in C(0,\infty)$ and $l>0$, it follows that the mapping $v \to \rho(x)f(v+l)$ is continuous in $[0,\infty)$. From

$$\rho(x)\frac{f(v+l)}{v^{p-1}} = \rho(x)\frac{f(v+l)}{(v+l)^{p-1}}\frac{(v+l)^{p-1}}{v^{p-1}},$$

by the positivity of ρ and (f_1) we deduce that the function $v \to \rho(x)\frac{f(v+l)}{v^{1/(p-1)}}$ is decreasing on $(0,\infty)$.

By $\lim_{v\to\infty} f(v+l)/(v+l)^{p-1} = 0$ and $f \in C(0,\infty)$, we can get that there exists $M>0$ such that $f(v+l) \le M(v+l)^{p-1}$ for all $v \ge 0$. Then

$$\rho(x)f(v+l) \le \|\rho\|_{L^\infty(B(0,k))}M(v+l)^{p-1}$$

for all $v \ge 0$.

We have

$$a_0(x) = \lim_{v\to 0}\frac{\rho(x)f(v+l)}{v^{p-1}} = \infty;$$

and

$$a_\infty(x) = \lim_{v\to\infty}\frac{\rho(x)f(v+l)}{v^{p-1}} = \lim_{v\to\infty}\rho(x)\frac{f(v+l)}{(v+l)^{p-1}}\frac{(v+l)^{p-1}}{v^{p-1}} = 0;$$

thus by [22], problem (2.2) has a solution v_k which is positive in $|x|<K$. Then the maximum principle implies that $l \le u_k \le u_{k+1}$ in $\mathbf{R^N}$.

Now, we prove the existence of a continuous function $v : \mathbf{R^N} \to \mathbf{R}$, $v > l$, such that $u_k \le v$ in $\mathbf{R^N}$.

Firstly, we construct a positive radial symmetric function w such that

$$-\mathrm{div}(|\nabla w|^{p-2}\nabla w) = \Phi(r)(r=|x|), \quad \text{in} \quad \mathbf{R^N},$$

and $\lim_{r\to\infty} w(r) = 0$. A straightforward computation shows that

$$w(r) = K - \int_0^r [\xi^{1-N}\int_0^\xi \sigma^{N-1}\Phi(\sigma)d\sigma]^{1/(p-1)}d\xi,$$

where $K = \int_0^{+\infty}[\xi^{1-N}\int_0^\xi \sigma^{N-1}\Phi(\sigma)d\sigma]^{1/(p-1)}d\xi$. Then we prove that K is finite.

Case I. $1 < p < 2$, in this case, since $1 \leq \frac{1}{p-1} < \infty$, by the Hardy inequality, we have

$$\begin{aligned}
&\int_0^{+\infty}[\xi^{1-N}\int_0^{\xi}\sigma^{N-1}\Phi(\sigma)d\sigma]^{1/(p-1)}d\xi \\
&= \int_0^{+\infty}\xi^{-\frac{N-1}{p-1}}[\int_0^{\xi}\sigma^{N-1}\Phi(\sigma)d\sigma]^{1/(p-1)}d\xi \\
&\leq [\frac{1}{p-1}(\frac{N-1}{p-1})^{-1}]^{1/(p-1)}\int_0^{+\infty}\xi^{-\frac{N-1}{p-1}}[\xi\xi^{N-1}\Phi(\xi)]^{1/(p-1)}d\xi \\
&= (\frac{1}{N-1})^{\frac{1}{p-1}}\int_0^{+\infty}\xi^{1/(p-1)}\Phi^{1/(p-1)}(\xi)d\xi < \infty.
\end{aligned}$$

Case II. For $2 \leq p < +\infty$, then $1 \leq p-1$, $0 < \frac{1}{p-1} \leq 1$.

Set

$$\int_0^{\xi}\sigma^{N-1}\Phi(\sigma)d\sigma \leq 1, \text{ for } \xi > 0,$$

or

$$\int_0^{\xi}\sigma^{N-1}\Phi(\sigma)d\sigma > 1, \text{ for } \xi > 0.$$

In the first case, when

$$[\int_0^{\xi}\sigma^{N-1}\Phi(\sigma)d\sigma]^{1/(p-1)} \leq 1,$$

we can get that

$$\int_0^r \xi^{\frac{1-N}{p-1}}[\int_0^{\xi}\sigma^{N-1}\Phi(\sigma)d\sigma]^{1/(p-1)}d\xi \leq \int_0^r \xi^{\frac{1-N}{p-1}}d\xi = \lim_{\varepsilon\to 0}\frac{p-1}{p-N}\xi^{\frac{p-N}{p-1}}|_{\varepsilon}^{r} = \frac{p-1}{p-N}\lim_{\varepsilon\to 0}(r^{\frac{p-N}{p-1}} - \varepsilon^{\frac{p-N}{p-1}})$$

is finite as $r \to \infty$ and $N > p$.

In the second case,

$$[\int_0^{\xi}\sigma^{N-1}\Phi(\sigma)d\sigma]^{1/(p-1)} \leq \int_0^{\xi}\sigma^{N-1}\Phi(\sigma)d\sigma,$$

for $\xi \geq 0$, then

$$\int_0^r \xi^{\frac{1-N}{p-1}}[\int_0^{\xi}\sigma^{N-1}\Phi(\sigma)d\sigma]^{1/(p-1)}d\xi \leq \int_0^r \xi^{\frac{1-N}{p-1}}\int_0^{\xi}\sigma^{N-1}\Phi(\sigma)d\sigma d\xi.$$

Integration by parts shows that

$$\begin{aligned}
&\int_0^r \xi^{\frac{1-N}{p-1}}\int_0^{\xi}\sigma^{N-1}\Phi(\sigma)d\sigma d\xi \\
&= -\frac{p-1}{N-p}\int_0^r \frac{d}{d\xi}\xi^{\frac{p-N}{p-1}}\int_0^{\xi}\sigma^{N-1}\Phi(\sigma)d\sigma d\xi \\
&= \frac{p-1}{N-p}(-r^{\frac{p-N}{p-1}}\int_0^r \sigma^{N-1}\Phi(\sigma)d\sigma + \int_0^r \xi^{\frac{(p-2)N+1}{p-1}}\Phi(\xi)d\xi).
\end{aligned}$$

Using L'Hopital's rule, we have

$$\begin{aligned}
&\lim_{r\to\infty}[-r^{\frac{p-N}{p-1}}\int_0^r \sigma^{N-1}\Phi(\sigma)d\sigma+\int_0^r \xi^{\frac{(p-2)N+1}{p-1}}\Phi(\xi)d\xi]\\
&=\lim_{r\to\infty}\frac{-\int_0^r \sigma^{N-1}\Phi(\sigma)d\sigma+r^{\frac{N-p}{p-1}}\int_0^r \xi^{\frac{(p-2)N+1}{p-1}}\Phi(\xi)d\xi}{r^{\frac{N-p}{p-1}}}\\
&=\lim_{r\to\infty}\int_0^r \xi^{\frac{(p-2)N+1}{p-1}}\Phi(\xi)d\xi\\
&=\int_0^\infty \xi^{\frac{(p-2)N+1}{p-1}}\Phi(\xi)d\xi<\infty.
\end{aligned}$$

Moreover, w is decreasing and satisfies $0<w(r)<k$ for all $r>0$. Let $v>l$, we define the following function

$$w(r)=m^{-1}\int_0^{v(r)-l} t/f^{\frac{1}{p-1}}(t+l)dt,$$

in which $m>0$ is chosen such that

$$1<m\le\int_0^m \frac{t}{f^{\frac{1}{p-1}}(t+l)}dt.$$

Next, by L'Hopital's rule we have

$$\lim_{x\to\infty}\frac{\int_0^x \frac{t}{f^{\frac{1}{p-1}}(t+l)}dt}{x}=\lim_{x\to\infty}\frac{x}{f^{\frac{1}{p-1}}(x+l)}=\lim_{x\to\infty}\frac{(x+l)^{\frac{1}{p-1}}}{f(x+l)}(\frac{x}{x+l})^{\frac{1}{p-1}}=\infty.$$

This means that there exists $x_1>0$ such that $\int_0^x \frac{t}{f^{\frac{1}{p-1}}(t)}dt\ge Kx$ for all $x\ge x_1$. It follows that for any $m\ge x_1$ we have

$$Km\le\int_0^m \frac{t}{f^{\frac{1}{p-1}}(t)}dt.$$

Since w is decreasing, we can get that v is a decreasing function. Then

$$\int_0^{v(r)-l}\frac{t}{f^{\frac{1}{p-1}}(t+l)}dt\le\int_0^{v(0)-l}\frac{t}{f^{\frac{1}{p-1}}(t+l)}dt=mv(0)=mK\le\int_0^m\frac{t}{f^{\frac{1}{p-1}}(t+l)}dt.$$

It follows that $v(r)\le m+l$ for all $r>0$.

From $w(r)\to 0$ as $r\to 0$, we deduce that $v(r)\to l$ as $r\to\infty$. By the choice of v we have

$$\nabla w=\frac{1}{m}\frac{v-l}{(f(v))^{\frac{1}{p-1}}}\nabla v,\ \ |\nabla w|^{p-2}\nabla w=\frac{1}{m^{p-1}}\frac{(v-l)^{p-1}}{f(v)}|\nabla v|^{p-2}\nabla v$$

$$\begin{aligned}
\operatorname{div}(|\nabla w|^{p-2}\nabla w)&=\tfrac{1}{m^{p-1}}\tfrac{(v-l)^{p-1}}{f(v)}\operatorname{div}(|\nabla v|^{p-2}\nabla v)+\tfrac{1}{m^{p-1}}(\tfrac{(v-l)^{p-1}}{f(v)})'|\nabla v|^p\\
&>(\tfrac{v-l}{m})^{p-1}\tfrac{1}{f(v)}\operatorname{div}(|\nabla v|^{p-2}\nabla v);
\end{aligned}$$

$$\begin{aligned}
\operatorname{div}(|\nabla v|^{p-2}\nabla v)\ &<\tfrac{m^{p-1}f(v)}{(v-l)^{p-1}}\operatorname{div}(|\nabla w|^{p-2}\nabla w)\\
&=\tfrac{m}{(v-l)^{p-1}}f(v)\Phi(r)\le -f(v)\Phi(r).
\end{aligned}\tag{2.3}$$

By (2.1), (2.3) and the hypothesis (f_1), we obtain that $u_k(x) \leq v(x)$ for each $|x| \leq k$ and so, for all $x \in \mathbf{R^N}$.

In conclusion

$$u_1 \leq u_2 \leq \cdots \leq u_k \leq u_{k+1} \leq \cdots \leq v,$$

with $v(x) \to l$ as $|x| \to \infty$. Thus, there exists a function $u \leq v$ such that $u_k \to u$ pointwise in $\mathbf{R^N}$. In particular, this shows that $u > l$ in $\mathbf{R^N}$ and $u(x) \to l$ as $|x| \to \infty$.

A standard bootstrap argument shows that u is a solution of problem (1.1). □

When $l = 0$ our result is as following.

Theorem 2.2. *Assume that $l = 0$ and assumption (ρ_1), (f_1) and (f_2) are fulfilled. Then problem (1.1) has a solution.*

Proof. Since f is an increasing positive function on $(0,\infty)$, the limit $\lim_{u\to 0} f(u)$ exists and is finite, so f can be extended by continuity to the origin. Consider the Dirichlet problem

$$\begin{cases} -\mathrm{div}(|\nabla u_k|^{p-2}\nabla u_k) = \rho(x) f(u_k), & \text{if} \quad |x| < k, \\ u_k(x) = 0, \quad \text{if} \quad |x| = k. \end{cases} \tag{2.4}$$

Using the same arguments as in case $l > 0$ we deduce that $\rho(x)f(u)$ is continuous in $[0,\infty)$. and $\rho(x)\frac{f(u)}{u^{p-1}}$ is decreasing on $(0,\infty)$. On the other hand, we use both hypothesis (f_1) and (f_2). Hence $f(u) \leq f(1)$ if $u \leq 1$ and $f(u)/u^{p-1} \leq f(1)$ if $u \geq 1$. Therefore $f(u) \leq f(1)(u^{p-1}+1)$, for all $u \geq 0$. The existence of a solution for (2.4) follows from [22]. These conditions are direct consequences of our assumptions $\lim_{u\to\infty} f(u)/u^{p-1} = 0$ and $\lim_{u\to 0} f(u)/u^{p-1} = +\infty$. Define $u_k(x) = 0$ for $|x| > K$. Using the same arguments as the case $l > 0$, we obtain $u_k \leq u_{k+1}$ in $\mathbf{R^N}$.

Next, we prove the existence of a continuous function $v : \mathbf{R^N} \to \mathbf{R}$ such that $u_k < v$ in $\mathbf{R^N}$. Using the same arguments as in case $l > 0$, we first construct a positive radially symmetric function w satisfying $-\mathrm{div}(|\nabla w|^{p-2}\nabla w) = \Phi(r)$ $(r = |x|)$ in $\mathbf{R^N}$ and $\lim_{r\to\infty} w(r) = 0$. We obtain

$$w(r) = K - \int_0^r [\xi^{1-N} \int_0^\xi \sigma^{N-1}\Phi(\sigma)d\sigma]^{1/(p-1)} d\xi,$$

where $K = \int_0^{+\infty} [\xi^{1-N} \int_0^\xi \sigma^{N-1}\Phi(\sigma)d\sigma]^{1/(p-1)} d\xi$. Using the same arguments as in case $l > 0$, we can prove that K is finite, and we have

$$w(r) < \begin{cases} (\frac{1}{N-1})^{\frac{1}{p-1}} \int_0^{+\infty} \xi^{\frac{1}{p-1}} \Phi^{\frac{1}{p-1}}(\xi)d\xi, & \text{if} \quad 1 < p \leq 2; \\ \int_0^{+\infty} \xi^{\frac{(p-2)N+1}{p-1}} \Phi^{\frac{1}{p-1}}(\xi)d\xi, & \text{if} \quad 2 < p \leq +\infty; \end{cases}$$

for all $r > 0$.

Let v be a positive function such that

$$w(r) = C^{-1} \int_0^{v(r)} \frac{t}{f^{\frac{1}{p-1}}(t)} dt,$$

where C is chosen such that

$$KC \leq \int_0^C \frac{t}{f^{\frac{1}{p-1}}(t)} dt.$$

We argue in what follows that we can find $C > 0$ with this property. Indeed, by L'Hopital's rule

$$\lim_{x\to\infty}\frac{\int_0^x \frac{t}{f^{\frac{1}{p-1}}(t)}dt}{x} = \lim_{x\to\infty}\left(\frac{x^{p-1}}{f(x)}\right)^{\frac{1}{p-1}} = +\infty.$$

This means that there exists $x_1 > 0$ such that $\int_0^x \frac{t}{f^{\frac{1}{p-1}}(t)}dt \geq Kx$, for all $x > x_1$. It follows that for any $C \geq x_1$ we have

$$Kx \leq \int_0^C \frac{t}{f^{\frac{1}{p-1}}(t)}dt.$$

On the other hand, since w is decreasing, we deduce that v is decreasing function, too. Hence

$$\int_0^C \frac{t}{f^{\frac{1}{p-1}}(t)}dt \leq \int_0^{v(0)} \frac{t}{f^{\frac{1}{p-1}}(t)}dt = C\cdot w(0) = C\cdot K \leq \int_0^C \frac{t}{f^{\frac{1}{p-1}}(t)}dt.$$

It follows that $v(r) \leq C$ for all $r > 0$.

From $w(r) \to 0$ as $r \to \infty$ we deduce that $v(r) \to 0$ as $r \to \infty$. By the choice of v we have

$$\nabla w = \frac{1}{C}\frac{v}{(f(v))^{\frac{1}{p-1}}}\nabla v,\ \ |\nabla w|^{p-2}\nabla w = \frac{v^{p-1}}{C^{p-1}}\frac{1}{f(v)}|\nabla v|^{p-2}\nabla v,$$

$$\operatorname{div}(|\nabla w|^{p-2}\nabla w) = \left(\tfrac{v}{c}\right)^{p-1}\tfrac{1}{f(v)}\operatorname{div}(|\nabla v|^{p-2}\nabla v) + \left(\tfrac{1}{c}\right)^{p-1}\left(\tfrac{v^{p-1}}{f(v)}\right)'|\nabla v|^p; \tag{2.5}$$

combining the fact that $f(u)/u^{p-1}$ is a decreasing function on $(0,\infty)$ with relation (2.5), we deduce that

$$\begin{aligned}\operatorname{div}(|\nabla v|^{p-2}\nabla v) \quad &< C^{p-1}\frac{f(v)}{v^{p-1}}\operatorname{div}(|\nabla w|^{p-2}\nabla w)\\ &= -C^{p-1}\frac{f(v)}{v^{p-1}}\Phi(r)\\ &\leq -f(v)\Phi(r),\end{aligned} \tag{2.6}$$

By (2.4) and (2.6) and using our hypothesis (f_2), we obtain that $u_k(x) \leq v(x)$ for each $|x| \leq K$ and so, for all $x \in \mathbf{R^N}$.

We have obtained a bounded increasing sequence

$$u_1 \leq u_2 \leq \cdots \leq u_k \leq u_{k+1} \leq \cdots \leq v.$$

with v vanishing at infinity. Thus, there exists a function $u \leq v$ such that $u_k \to u$ pointwise in $\mathbf{R^N}$. A standard bootstrap arguments implies that u is a solution of problem (1.1).

□

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