Communications in **M**athematical **A**nalysis
Volume 9, Number 1, pp. 48–65 (2010)
ISSN 1938-9787

www.commun-math-anal.org

THE CONTINUOUS WAVELET TRANSFORM ASSOCIATED WITH A DIFFERENTIAL-DIFFERENCE OPERATOR AND APPLICATIONS

H. MEJJAOLI*
Department of Mathematics
Faculty Sciences of Tunis
Tunis, 2092, TUNISIA

N. SRAIEB†
Department of Mathematics
Faculty of sciences of GABES
Gabes, 6072, TUNISIA

(Communicated by Vladimir Rabinovich)

Abstract

In this paper we consider a class of singular differential-difference operators on the real line. We define and study the continuous wavelet transform associated with these operators. We prove a Plancherel formula, an inversion formula and a weak uncertainty principle for it. As applications, we establish Calderón's formulas and give practical real inversion formula for the generalized continuous wavelet transform. At the end of the paper, analogous of Heisenberg's inequality for the generalized continuous wavelet transform on Chébli-Trimèche hypergroups and a special case of Jacobi-Dunkl operator are proved.

AMS Subject Classification:35C80,51F15,43A32.

Keywords: Differential-difference operator, generalized continuous wavelet transform, reproducing kernel inversion formulas, practical real inversion formulas.

1 Introduction

In this paper, we consider the first-order singular differential-difference operator Λ on $\mathbb{R}$ introduced by M.A. Mourou and K. Trimèche in [8] defined for a function f of class C^1 on $\mathbb{R}$ by

$$\Lambda f = \frac{df}{dx} + \frac{A'(x)}{A(x)}\left(\frac{f(x)-f(-x)}{2}\right),$$

*E-mail address: hatem.mejjaoli@ipest.rnu.tn
†E-mail address: nadia.sraieb@fsg.rnu.tn

where

$$A(x) = |x|^{2\alpha+1} B(x), \quad \alpha > -\frac{1}{2},$$

B being a positive C^∞ even function on $\mathbb{R}$. We suppose in addition that

(i) A is increasing on $[0,\infty[$;

(ii) There exists a constant $\delta > 0$ such that the function $e^{\delta x}\dfrac{B'(x)}{B(x)}$ is bounded for large $x \in [0,\infty[$ together with its derivatives.

This operator extend the usual partial derivatives by additional reflection terms and give generalizations of many analytic structures like the exponential function, the Fourier transform and the convolution product (cf. [8]).

For $A(x) = |x|^{2\alpha+1}$, $\alpha > -\frac{1}{2}$, we regain the differential-difference operator

$$D_\alpha f = \frac{df}{dx} + \left(\alpha + \frac{1}{2}\right) \frac{f(x) - f(-x)}{x},$$

which is referred to as the Dunkl operator of index $\alpha + \frac{1}{2}$ associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$ (cf. [4]).

For $A(x) = \left(\dfrac{\sinh|x|}{\cosh x}\right)^{2\alpha+1}$, $\alpha > -\frac{1}{2}$, we regain the differential-difference operator

$$l_\alpha f = \frac{df}{dx} + \Big[(2\alpha+1)(\coth x - \tanh x)\Big] \frac{f(x) - f(-x)}{x},$$

which is referred to as the special case of Jacobi-Dunkl operator on $\mathbb{R}$ (cf. [1, 2]).

For a function f of class C^1 on $\mathbb{R}$ even, we have

$$\Lambda^2 f = \mathcal{L} f,$$

with

$$\mathcal{L} = \frac{d^2}{dx^2} + \frac{A'(x)}{A(x)} \frac{d}{dx} \tag{1.1}$$

(cf. [11]).

In this paper, we are interested in generalized wavelets and the associated generalized continuous wavelet transform. More precisely, we give here general reconstruction formulas. We refer the readers for the classical case of the wavelet transform to [3, 10].

The contents of the paper are as follows:

In §2 we recall some basic harmonic analysis results related to the differential-difference operator Λ on $\mathbb{R}$. In §3, we introduce and study the generalized wavelet and the generalized continuous wavelet transform associated with the operator Λ. Thus, some results (Plancherel's formula, an inversion formula...) are given. We finish this section by proving the weak uncertainty principle for this transform. In §4 we give some applications. Firstly we study the approximative concentration of the generalized continuous wavelet transform. We prove the reproducing Calderón's formulas. We give also practical real inversion formulas using the theory of reproducing kernels. Finally an uncertainty principle of Heisenberg type for the generalized continuous wavelet transform on Chébli-Trimèche hypergroups and a special case of Jacobi-Dunkl operator are proved.

2 Preliminaries

In this section we recall some facts about harmonic analysis related to the differential-difference operator Λ. We cite here, as briefly as possible, only those properties actually required for the discussion. For more details we refer to [8, 9]. For each $\lambda \in \mathbb{C}$ the differential-difference equation

$$\Lambda u = \lambda u, \quad u(0) = 1,$$

admits a unique C^∞ solution on $\mathbb{R}$, denoted Φ_λ given by

$$\Phi_\lambda(x) = \begin{cases} \varphi_{i\lambda}(x) + \frac{1}{\lambda}\frac{d}{dx}\varphi_{i\lambda}(x) & \text{if } \lambda \neq 0, \\ 1 & \text{if } \lambda = 0, \end{cases}$$

where for each $z \in \mathbb{C}$, φ_z designates the solution of the differential equation

$$\mathcal{L}u = -z^2 u, \quad u(0) = 1$$

$\mathcal{L}$ being the second-order singular differential operator on $\mathbb{R}$ defined by (1.1). Moreover, $\Phi_\lambda(x)$ is entire in λ. Recently M.A. Mourou in [9] have proved for all $\lambda, x \in \mathbb{R}$,

$$|\Phi_{i\lambda}(x)| \leq 1. \tag{2.1}$$

Notations. We denote by

- $S(\mathbb{R})$ the space of C^∞-functions g on $\mathbb{R}$ which are rapidly decreasing together with their derivatives.
- For a Borel positive measure μ on $\mathbb{R}$, and $1 \leq p \leq \infty$, we write $L^p_\mu(\mathbb{R})$ for the Lebesgue space equiped with the norm $\|\cdot\|_{p,\mu}$ defined by

$$\|f\|_{p,\mu} = \left(\int_{\mathbb{R}} |f(x)|^p \, d\mu(x) \right)^{1/p}, \quad \text{if } p < \infty,$$

  and $\|f\|_{\infty,\mu} = \operatorname{ess\,sup}_{x\in\mathbb{R}} |f(x)|$. When $\mu = w(x)dx$, with w a nonnegative function on $\mathbb{R}$, we replace the μ in the norms by w.

The generalized Fourier transform of a function $f \in L^1_A(\mathbb{R})$ is defined by

$$\mathcal{F}f(\lambda) = \int_{\mathbb{R}} f(x)\Phi_{-i\lambda}(x)A(x)dx, \quad \lambda \in \mathbb{R}.$$

Many of the important properties of Fourier transforms on locally compact abelian groups are proved to hold true for $\mathcal{F}$.

Theorem 2.1. *The generalized Fourier transform $\mathcal{F}$ is a bijection from $\mathcal{S}(\mathbb{R})$ onto itself.*

Theorem 2.2. i*) Plancherel formula: There is an even positive tempered measure σ (and only one) on $\mathbb{R}$ such that for all $f \in \mathcal{S}(\mathbb{R})$,*

$$\int_{\mathbb{R}} |f(x)|^2 A(x)dx = \int_{\mathbb{R}} |\mathcal{F}f(\lambda)|^2 d\sigma(\lambda).$$

ii*)Plancherel theorem: The generalized Fourier transform $\mathcal{F}$ extends uniquely to a unitary isomorphism from $L^2_A(\mathbb{R})$ onto $L^2_\sigma(\mathbb{R})$.*

iii) *Inversion formula: Let f be a function in* $L^1_A(\mathbb{R})$*, such that* $\mathcal{F}(f) \in L^1_\sigma(\mathbb{R})$*. Then the inverse transform of f is given by*

$$\mathcal{F}^{-1}(f)(x) = \int_{\mathbb{R}} f(\lambda)\Phi_{i\lambda}(x)d\sigma(\lambda), \quad a.e.\ x \in \mathbb{R}.$$

The measure σ is called the spectral measure associated with the differential-difference operator Λ. Under our assumptions on the function A, it is known that the spectral measure σ takes the form

$$d\sigma(\lambda) = \frac{d\lambda}{|c(|\lambda|)|^2}, \quad \lambda \in \mathbb{R},$$

where $c(s)$ is a continuous function on $]0,\infty[$ such that

$$c(s)^{-1} \sim k_1\, s^{\alpha+\frac{1}{2}}, \quad \text{as } s \to \infty,$$

$$c(s)^{-1} \sim k_2\, s^{\alpha+\frac{1}{2}}, \quad \text{as } s \to 0,$$

for some $k_1\, k_2 \in \mathbb{C}$.

In the Dunkl operator case corresponding to $A(x) = |x|^{2\alpha+1}$, $\alpha > -\frac{1}{2}$, the spectral measure σ is given by

$$d\sigma(\lambda) = \frac{|\lambda|^{2\alpha+1}d\lambda}{2^{2\alpha+2}\,(\Gamma(\alpha+1))^2}.$$

In the special case of Jacobi-Dunkl operator corresponding to $A(x) = \left(\frac{\sinh|x|}{\cosh x}\right)^{2\alpha+1}$, $\alpha > -\frac{1}{2}$, the spectral measure σ is given by

$$d\sigma(\lambda) = \frac{d\lambda}{8\pi\,|c(|\lambda|)|^2},$$

where

$$c(\mu) = \frac{\Gamma(\alpha+1)\Gamma(i\mu)}{2^{i\mu}\Gamma(\frac{i\mu}{2})\Gamma(\alpha+1+\frac{i\mu}{2})},\ \mu \in \mathbb{C}\backslash\{i\mathbb{N}\}.$$

Definition 2.3. Let y be in $\mathbb{R}$. The generalized translation operator $f \mapsto \tau_y f$ is defined on $L^2_A(\mathbb{R})$ by

$$\mathcal{F}(\tau_y f)(x) = \Phi_{ix}(y)\mathcal{F}(f)(x), \quad \text{a.e. } x \in \mathbb{R}. \tag{2.2}$$

Using the generalized translation operator, we define the generalized convolution product of functions as follows.

Definition 2.4. The generalized convolution product of f and g in $\mathcal{S}(\mathbb{R})$ is the function $f * g$ defined by

$$\forall x \in \mathbb{R}, \quad f*g(x) = \int_{\mathbb{R}} \tau_x f(-y)g(y)A(y)dy. \tag{2.3}$$

The generalized convolution $*$ satisfies the following properties:

Proposition 2.5. i) *Let* $f, g \in \mathcal{S}(\mathbb{R})$*. Then*

$$\mathcal{F}(f*g) = \mathcal{F}(f)\mathcal{F}(g).$$

ii) *Let* $f, g \in L^2_A(\mathbb{R})$*. Then* $f*g$ *belongs to* $L^2_A(\mathbb{R})$ *if and only if* $\mathcal{F}(f)\mathcal{F}(g)$ *belongs to* $L^2_\sigma(\mathbb{R})$ *and we have*

$$\mathcal{F}(f*g) = \mathcal{F}(f)\mathcal{F}(g), \quad \text{in the } L^2\text{-case}$$

iii) *Let $f, g \in L^2_A(\mathbb{R})$. Then*

$$\int_{\mathbb{R}} |f * g(x)|^2 A(x)dx = \int_{\mathbb{R}} |\mathcal{F}(f)(\xi)|^2 |\mathcal{F}(g)(\xi)|^2 d\sigma(\xi),$$

where both sides are finite or infinite.

Proof. The proof uses the same idea as in Propositions II.7 of [11]. □

3 The generalized continuous wavelet transform

Using the harmonic analysis associated with the operator Λ we define and study in this section the generalized wavelet and the generalized continuous wavelet transform.

Definition 3.1. A generalized wavelet on $\mathbb{R}$ is a measurable function h on $\mathbb{R}$ satisfying for almost all $x \in \mathbb{R}$, the condition

$$0 < C_h = \int_0^{\infty} |\mathcal{F}(h)(\lambda x)|^2 \frac{d\lambda}{\lambda} < +\infty. \tag{3.1}$$

Example 3.2. Let E_t, $t > 0$, the heat kernel is defined on $\mathbb{R}$ by

$$\forall x \in \mathbb{R}, \quad E_t(x) = \mathcal{F}^{-1}(e^{-t\lambda^2})(x). \tag{3.2}$$

The function $h(x) = -\frac{d}{dt}E_t(x)$ is a generalized wavelet on $\mathbb{R}$ in $\mathcal{S}(\mathbb{R})$, and we have $C_h = \frac{1}{8t^2}$.

Proposition 3.3. i) *Let $a > 0$ and g be a function in $L^2_\sigma(\mathbb{R})$. Then the function $\lambda \mapsto g(a\lambda)$ belongs to $L^2_\sigma(\mathbb{R})$ and we have*

$$||g(a.)||_{2,\sigma} \leq \frac{k(a)}{\sqrt{a}} ||g||_{2,\sigma},$$

where

$$k(a) = \sup_{\lambda \in \mathbb{R}} \frac{|c(|\lambda|)|}{|c(\frac{|\lambda|}{a})|}.$$

ii) *For all $a > 0$, the dilatation operator D_a is a topological automorphism of $L^2_\sigma(\mathbb{R})$, where D_a is defined by*

$$\forall x \in \mathbb{R}, \quad D_a(f)(x) = \frac{1}{\sqrt{a}} f(\frac{x}{a}).$$

Proof. i) We have

$$||g(a.)||^2_{2,\sigma} = \int_{\mathbb{R}} |g(a\lambda)|^2 d\sigma(\lambda)$$

$$||g(a.)||^2_{2,\sigma} \leq \frac{k^2(a)}{a} ||g||^2_{2,\sigma}.$$

ii) We deduce the result from i). □

Proposition 3.4. *Let $a>0$ and h be a generalized wavelet in $L^2_A(\mathbb{R})$. Then there exists a function h_a in $L^2_A(\mathbb{R})$ such that*

$$\forall y\in\mathbb{R},\ \mathcal{F}(h_a)(y)=\mathcal{F}(h)(ay). \tag{3.3}$$

This function is given by the relation

$$h_a=\frac{1}{\sqrt{a}}\mathcal{F}^{-1}\circ D_{a^{-1}}\circ\mathcal{F}(h) \tag{3.4}$$

and satisfies

$$||h_a||_{2,A}\leq\frac{k(a)}{\sqrt{a}}||h||_{2,A}. \tag{3.5}$$

Proof. The proof is immediately from Proposition 3.3 □

Let $a>0$ and h be in $L^2_A(\mathbb{R})$. We consider the family $h_{a,x}$, $x\in\mathbb{R}$, of functions on $\mathbb{R}$ in $L^2_A(\mathbb{R})$ defined by

$$h_{a,x}(y)=\frac{a^{\frac{1}{2}}}{k(a)}\tau_x h_a(-y),\ y\in\mathbb{R}, \tag{3.6}$$

where τ_x, $x\in\mathbb{R}$, are the generalized translation operators given by (2.2).

Proposition 3.5. *Let h be a generalized wavelet on $\mathbb{R}$ in $L^2_A(\mathbb{R})$. The function $h_{a,x}$, is a generalized wavelet on $\mathbb{R}$ in $L^2_A(\mathbb{R})$, and we have*

$$C_{h_{a,x}}\leq\frac{a^{\frac{1}{2}}}{k(a)}C_h.$$

Proof. From (2.2) and (3.3), we have

$$\mathcal{F}(h_{a,x})(y)=\frac{a^{\frac{1}{2}}}{k(a)}\Phi_{iy}(x)\mathcal{F}(h)(ay),\quad y\in\mathbb{R}.$$

Thus from (3.1) and (2.1)

$$C_{h_{a,x}}\leq\frac{a^{\frac{1}{2}}}{k(a)}\int_0^\infty|\mathcal{F}(h)(a\lambda y)|^2\frac{d\lambda}{\lambda}=\frac{a^{\frac{1}{2}}}{k(a)}C_h,$$

which gives the desired result. □

Notations. We denote by
$X^p_{A,k}$, $p\in[1,\infty]$ the space of measurable functions f on $\mathbb{R}^*_+\times\mathbb{R}$ with respect to the measure $d\lambda_{A,k}(a,x)=\dfrac{A(x)k^2(a)dxda}{a^2}$ such that

$$\begin{aligned}\|f\|_{p,\lambda_{A,k}} &= \left(\int_{\mathbb{R}^*_+\times\mathbb{R}}|f(a,x)|^p d\lambda_{A,k}(a,x)\right)^{\frac{1}{p}}<\infty;\ 1\leq p<\infty,\\ \|f\|_{\infty,\lambda_{A,k}} &= \operatorname*{ess\,sup}_{(a,x)\in\mathbb{R}^*_+\times\mathbb{R}}|f(a,x)|<\infty.\end{aligned}$$

$C_b(\mathbb{R}^*_+\times\mathbb{R})$ the space of continuous bounded functions on $\mathbb{R}^*_+\times\mathbb{R}$.

Definition 3.6. Let h be a generalized wavelet on $\mathbb{R}$ in $L^2_A(\mathbb{R})$. The generalized continuous wavelet transform S_h on $\mathbb{R}$ is defined for regular functions f on $\mathbb{R}$ by

$$S_h(f)(a,x) = \int_{\mathbb{R}} f(y)\overline{h_{a,x}(y)}A(y)dy,\ a>0,\ x\in\mathbb{R}. \tag{3.7}$$

This transform can also be written in the form

$$S_h(f)(a,x) = \frac{a^{\frac{1}{2}}}{k(a)} f * \overline{h_a}(x), \tag{3.8}$$

where $*$ is the generalized convolution product given by (2.3).

Remark 3.7. i) From Plancherel formula for $\mathcal{F}$ we get the following result

$$S_h(f)(a,x) = \frac{a^{\frac{1}{2}}}{k(a)} \mathcal{F}^{-1}\Big(\mathcal{F}(f)(\xi)\mathcal{F}(\overline{h})(a\xi)\Big)(x). \tag{3.9}$$

ii) Let h be a generalized wavelet. Then, for all f in $L^2_A(\mathbb{R})$ we have

$$\|S_h f\|_{\infty,\lambda_{A,k}} \leq \|f\|_{2,A}\|h\|_{2,A}. \tag{3.10}$$

Proposition 3.8. *(Covariance property) Let h be a generalized wavelet on $\mathbb{R}$ in $L^2_A(\mathbb{R})$. The generalized continuous wavelet transform S_h is a bounded linear operator from $L^2_A(\mathbb{R})$ into $C_b(\mathbb{R}^*_+\times\mathbb{R})$ possessing the following covariance property:*
for $f\in L^2_A(\mathbb{R})$ and $(a,x)\in\mathbb{R}^*_+\times\mathbb{R}$ arbitrary

$$S_h(\tau_{-x_0}f)(a,x) = \tau_{-x_0}(S_h f(a,.))(x),\ x_0\in\mathbb{R}. \tag{3.11}$$

Theorem 3.9. *(Plancherel formula for S_h) Let h be a generalized wavelet on $\mathbb{R}$ in $L^2_A(\mathbb{R})$. For all f in $L^2_A(\mathbb{R})$ we have*

$$\int_{\mathbb{R}} |f(x)|^2 A(x)dx = \frac{1}{C_h}\int_0^\infty \int_{\mathbb{R}} |S_h(f)(a,x)|^2 d\lambda_{A,k}(a,x). \tag{3.12}$$

Proof. From (3.3) and Proposition 2.5 iii), we obtain

$$\begin{aligned}\frac{1}{C_h}\int_0^\infty\int_{\mathbb{R}} |S_h f(a,x)|^2 A(x)dx\frac{k^2(a)da}{a^2} &= \frac{1}{C_h}\int_0^\infty\int_{\mathbb{R}} |f*h_a(x)|^2A(x)dx\frac{da}{a}\\ &= \int_{\mathbb{R}} |\mathcal{F}(f)(x)|^2\Big(\frac{1}{C_h}\int_0^\infty |\mathcal{F}(h)(ax)|^2\frac{da}{a}\Big)d\sigma(x).\end{aligned}$$

But from (3.1), we have for almost all $x\in\mathbb{R}$,

$$\frac{1}{C_h}\int_0^\infty |\mathcal{F}(h)(ax)|^2\frac{da}{a} = 1.$$

Then we deduce the assertion. □

Corollary 3.10. *(Parseval relation) Let h be a generalized wavelet on $\mathbb{R}$ in $L^2_A(\mathbb{R})$ and f_1, f_2 in $L^2_A(\mathbb{R})$. Then, we have*

$$\int_{\mathbb{R}} f_1(x)\overline{f_2(x)}A(x)dx = \frac{1}{C_h}\int_0^\infty\int_{\mathbb{R}} S_h(f_1)(a,x)\overline{S_h(f_2)(a,x)}d\lambda_{A,k}(a,x). \tag{3.13}$$

Theorem 3.11. *(Inversion formula for S_h) Let h be a generalized wavelet on $\mathbb{R}$ in $L^2_A(\mathbb{R})$. For all f in $L^1_A(\mathbb{R})$ (resp. $L^2_A(\mathbb{R})$) such that $\mathcal{F}(f)$ belongs to $L^1_\sigma(\mathbb{R})$ (resp. $L^1_\sigma(\mathbb{R})\bigcap L^\infty_\sigma(\mathbb{R})$) we have*

$$f(y) = \frac{1}{C_h}\int_0^\infty \int_{\mathbb{R}} S_h(f)(a,x)h_{a,y}(x)d\lambda_{A,k}(a,x),\ a.e., \tag{3.14}$$

where for each $y \in \mathbb{R}$, both the inner integral and the outer integral are absolutely convergent, but possible not the double integral.

Proof. The proof uses the same idea as for Theorem IV.6 of [11]. □

Corollary 3.12. *(Reproducing Kernel). Let h be a generalized wavelet on $\mathbb{R}$ in $L^2_A(\mathbb{R})$. Then, $S_h(L^2_A(\mathbb{R}))$ is a reproducing kernel Hilbert space in $X^2_{A,k}$ with kernel function*

$$\mathcal{K}_h(a',x';a,x) := \frac{1}{C_h}\int_{\mathbb{R}} h_{a',x'}(y)\overline{h_{a,x}(y)}A(y)dy. \tag{3.15}$$

The kernel is pointwise bounded:

$$|\mathcal{K}_h(a',x';a,x)| \leq \frac{||h||^2_{2,A}}{C_h};\ \forall (a',x'),\, (a,x) \in \mathbb{R}^* \times \mathbb{R}. \tag{3.16}$$

Proof. The result is obtained from the inversion formula (3.14) and the integral expression (3.7) of S_h. □

Next, we show the weak uncertainty principle for the generalized continuous wavelet transform.

Proposition 3.13. *Let h be a generalized wavelet such that $\|h\|_{2,A} = 1$. Suppose that $\|f\|_{2,A} = 1$. Then, for $U \subset \mathbb{R}^*_+ \times \mathbb{R}$ and $\varepsilon > 0$ satisfying*

$$\int\int_U |\mathcal{S}_h f(a,x)|^2 d\lambda_{A,k}(a,x) \geq 1-\varepsilon$$

we have,

$$\lambda_{A,k}(U) \geq 1-\varepsilon.$$

Proof. From the relation (3.10) we deduce that

$$\|\mathcal{S}_h f\|_{\infty,\lambda_{A,k}} \leq 1.$$

Thus,

$$1-\varepsilon \ \leq\ \int\int_U |\mathcal{S}_h f(a,x)|^2 d\lambda_{A,k}(a,x) \leq \|\mathcal{S}_h f\|^2_{\infty,\lambda_{A,k}}\lambda_{A,k}(U) \leq \lambda_{A,k}(U).$$

Which completes the proof. □

Proposition 3.14. *Let h be a generalized wavelet such that $\|h\|_{2,A} = 1$. Let $f \in L^2_A(\mathbb{R})$, and $p \in [2,\infty[$. Then,*

$$\int_{\mathbb{R}^*_+\times\mathbb{R}} |\mathcal{S}_h f(a,x)|^p d\lambda_{A,k}(a,x) \leq C_h\|f\|^p_{2,A}. \tag{3.17}$$

Proof. Using (3.10) and Theorem 3.9, the result follows by applying the Riesz-Thorin interpolation theorem. □

As a consequence of the inequality (3.17), we deduce that if the generalized continuous wavelet transform is essentially supported in a set $U \subset \mathbb{R}_+^* \times \mathbb{R}$ (for example, when $S_h f = (\lambda_{A,k}(U))^{-\frac{1}{2}}\chi_U$), then $\lambda_{A,k}(U) \geq C_h$.

In the follow we assume that $\Lambda \in \{D_\alpha, l_\alpha\}$. The characterization of $L_A^p(\mathbb{R})$ for $1 < p < \infty$ by the mean of the generalized continuous wavelet transform is given by the following theorem. A function $h \in L_A^1(\mathbb{R}) \bigcap L_A^2(\mathbb{R})$ is said to satisfy (H_1) if

$$\int_{|y|\geq 2|x|} |\tau_x h(y) - h(y)| A(y) dy \leq C, \quad |x| > 0.$$

In the rest of this section, we assume that the generalized wavelet $h \in L_A^1(\mathbb{R}) \bigcap L_A^2(\mathbb{R})$, even and satisfy (H_1).

Theorem 3.15. *Let h be a generalized wavelet. Then the generalized continuous wavelet transform S_h is a linear bounded operator*

$$\begin{array}{lll} L_A^p(\mathbb{R}) & \longrightarrow & L^2(\mathbb{R}_+^*, \frac{k^2(a)da}{a^2}) \times L_A^p(\mathbb{R}) \\ f & \longmapsto & S_h(f), \end{array}$$

moreover, for all function f in $L_A^p(\mathbb{R})$ ($1 < p < \infty$)

$$||f||_{p,A} \simeq \Big(\int_{-\infty}^{\infty} \Big(\int_0^{\infty} |S_h(f)(a,x)|^2 \frac{k^2(a)da}{a^2}\Big)^{\frac{p}{2}} A(x)dx\Big)^{\frac{1}{p}}. \tag{3.18}$$

Proof. Let denote by $W_{A,k}^p$ the space $L^2(\mathbb{R}_+^*, \frac{k^2(a)da}{a^2}) \times L_A^p(\mathbb{R})$ associated to the norm

$$||g||_{W_{A,k}^p} = \Big(\int_{-\infty}^{\infty} \Big(\int_0^{\infty} |g(a,x)|^2 \frac{k^2(a)da}{a^2}\Big)^{\frac{p}{2}} A(x)dx\Big)^{\frac{1}{p}}.$$

Consider the case $p = 2$, from Plancherel's theorem

$$||S_h(f)||_{W_{A,k}^2} = \sqrt{C_h}||f||_{2,A}.$$

Then the singular integral theorem applied to operators with values in $L^2(\mathbb{R}_+^*, \frac{k^2(a)da}{a^2})$, leads to the following inequality

$$||S_h(f)||_{W_{A,k}^p} \leq A_p ||f||_{p,A} \quad 1 < p \leq 2$$

where the constant A_p depends only on p and h. By duality, the inequality is also valid for $1 < p < \infty$. This demonstrates that

$$\Big(\int_{-\infty}^{\infty} \Big(\int_0^{\infty} |S_h(f)(a,x)|^2 \frac{k^2(a)da}{a^2}\Big)^{\frac{p}{2}} A(x)dx\Big)^{\frac{1}{p}} \leq A_p ||f||_{p,A}. \tag{3.19}$$

Conversely, let $f \in L_A^2(\mathbb{R}) \bigcap L_A^p(\mathbb{R})$. For all $g \in L_A^2(\mathbb{R}) \bigcap L_A^{p'}(\mathbb{R})$ the generalized wavelet transform S_h being an isometry from $L_A^2(\mathbb{R})$ to $L^2(\mathbb{R}_+^*, \frac{k^2(a)da}{a^2}) \times L_A^2(\mathbb{R})$, we have

$$\int_{\mathbb{R}} f(x)\overline{g(x)}A(x)dx = \frac{1}{C_h}\int_0^{\infty}\int_{\mathbb{R}} S_h(f)(a,x)\overline{S_h(g)(a,x)}d\lambda_{A,k}(a,x). \tag{3.20}$$

By Schwarz's and Hölder inequality, and using (3.19)

$$\begin{aligned}\int_{\mathbb{R}} f(x)\overline{g(x)}A(x)dx &\leq \frac{1}{C_h}\int_{\mathbb{R}}\Big(\int_0^\infty |S_h(f)(a,x)|^2\frac{k^2(a)da}{a^2}\Big)^{\frac{1}{2}}\times \\ &\quad \Big(\int_0^\infty |S_h(g)(a,x)|^2\frac{k^2(a)da}{a^2}\Big)^{\frac{1}{2}}A(x)dx \\ &\leq \frac{1}{C_h}\Big[\int_{\mathbb{R}}\Big(\int_0^\infty |S_h(f)(a,x)|^2\frac{k^2(a)da}{a^2}\Big)^{\frac{p}{2}}A(x)dx\Big]^{\frac{1}{p}}\times \\ &\quad \Big[\int_{\mathbb{R}}\Big(\int_0^\infty |S_h(g)(a,x)|^2\frac{k^2(a)da}{a^2}\Big)^{\frac{p'}{2}}A(x)dx\Big]^{\frac{1}{p'}} \\ &\leq C\Big[\int_{\mathbb{R}}\Big(\int_0^\infty |S_h(f)(a,x)|^2\frac{k^2(a)da}{a^2}\Big)^{\frac{p}{2}}A(x)dx\Big]^{\frac{1}{p}}||g||_{p',A}.\end{aligned}$$

By density the inequality is valid for all $g \in L_A^{p'}(\mathbb{R})$, remember that:

$$||f||_{p,A} = \sup_{g\in L_A^{p'}(\mathbb{R})} \frac{\int_{\mathbb{R}} f(x)\overline{g(x)}A(x)dx}{||g||_{p',A}}$$

then we have

$$||f||_{p,A} \leq C\Big(\int_{-\infty}^{\infty}\Big(\int_0^\infty |S_h(f)(a,x)|^2\frac{k^2(a)da}{a^2}\Big)^{\frac{p}{2}}A(x)dx\Big)^{\frac{1}{p}}.$$

□

Corollary 3.16. *Let consider $f \in L_A^p(\mathbb{R})$, $g \in L_A^{p'}(\mathbb{R})$ with $1 < p < \infty$ and $\frac{1}{p}+\frac{1}{p'} = 1$. If h is a real generalized wavelet. Then*

$$\int_{\mathbb{R}} f(x)\overline{g(x)}A(x)dx = \frac{1}{C_h}\int_{\mathbb{R}}\int_0^\infty S_h(f)(a,x)\overline{S_h(g)(a,x)}\, d\lambda_{A,k}(a,x). \tag{3.21}$$

The integrals of the right hand side have to be taken in the sense of distributions.

Proof. Consider the bilinear form:

$$\begin{aligned} B: L_A^p(\mathbb{R})\times L_A^{p'}(\mathbb{R}) &\longrightarrow \mathbb{R} \\ (f,g) &\longmapsto \langle S_h(f), S_h(g)\rangle_{d\lambda_{A,k}(a,x)}.\end{aligned}$$

with $\langle .,.\rangle_{d\lambda_{A,k}(a,x)}$ defined by

$$\langle u,\chi\rangle_{d\lambda_{A,k}(a,x)} = \int_{\mathbb{R}}\int_0^\infty u(a,x)\chi(a,x)d\lambda_{A,k}(a,x), \quad (u,\chi)\in W_{A,k}^p\times W_{A,k}^{p'}.$$

By using twice Hölder's inequality we obtain

$$\begin{aligned}|B(f,g)| &= |\langle S_h(f), S_h(g)\rangle_{d\lambda_{A,k}(a,x)}| \\ &\leq \int_{\mathbb{R}}\Big(\int_0^\infty |S_h(f)(a,x)|^2\frac{k^2(a)da}{a^2}\Big)^{\frac{1}{2}}\Big(\int_0^\infty |S_h(g)(a,x)|^2\frac{k^2(a)da}{a^2}\Big)^{\frac{1}{2}}A(x)dx \\ &\leq \Big[\int_{\mathbb{R}}\Big(\int_0^\infty |S_h(f)(a,x)|^2\frac{k^2(a)da}{a^2}\Big)^{\frac{p}{2}}A(x)dx\Big]^{\frac{1}{p}}\times \\ &\quad \Big[\int_{\mathbb{R}}\Big(\int_0^\infty |S_h(g)(a,x)|^2\frac{k^2(a)da}{a^2}\Big)^{\frac{p'}{2}}A(x)dx\Big]^{\frac{1}{p'}}\end{aligned}$$

and by Theorem 3.15

$$|B(f,g)| \leq C||f||_{p,A}||g||_{p',A}. \tag{3.22}$$

Moreover for all $f \in L^2_A(\mathbb{R}) \bigcap L^p_A(\mathbb{R})$ and $g \in L^2_A(\mathbb{R}) \bigcap L^p_A(\mathbb{R})$ we have

$$B(f,g) = \langle S_h(f), S_h(g)\rangle_{d\lambda_{A,k}(a,x)} = C_h\langle f,g\rangle. \tag{3.23}$$

Equations (3.22-3.23) and the density of spaces $L^2_A(\mathbb{R}) \bigcap L^p_A(\mathbb{R})$ in $L^p_A(\mathbb{R})$ gives the result. □

Theorem 3.17. *(Reconstruction formula in $L^p_A(\mathbb{R})$) Let h be a real generalized wavelet. For all f in $L^p_A(\mathbb{R})$ ($1 < p < \infty$) we have*

$$f(y) = \frac{1}{C_h}\int_0^\infty \int_{\mathbb{R}} S_h(f)(a,x)h_{a,y}(x)d\lambda_{A,k}(a,x), \ a.e.. \tag{3.24}$$

The equality is valid in the L^p_A sense and the integrals of the right hand side have to be taken in the sense of distributions.

Proof. It is directly derived from Corollary 3.16. For all $g \in L^{p'}_A(\mathbb{R})$ replacing $S_h(g)$ by its definition in formula (3.21), and applying Fubini's theorem

$$\langle f,g\rangle_{d\lambda_{A,k}(a,x)} = \frac{1}{C_h}\int_{\mathbb{R}} g(y)\Big(\int_0^\infty \int_{\mathbb{R}} S_h(f)(a,x)h_{a,y}(x)d\lambda_{A,k}(a,x)\Big)dy.$$

□

Remark 3.18. We remark that the analogous of Theorem 3.15, Corollary 3.16 and Theorem 3.17 are true in the case of Chébli-Trimèche hypergroups.

4 Applications

4.1 Approximative concentration of generalized continuous wavelet transform

In the following theorem, we will show that the portion of the generalized continuous wavelet transform lying outside some sufficiently small set of finite measure cannot be arbitrarily too small. Then, in order to prove a concentration result of the generalized continuous wavelet transform, we need the following notations:

$P_R : X^2_{A,k} \longrightarrow X^2_{A,k}$ the orthogonal projection from $X^2_{A,k}$ onto $\mathcal{S}_h(L^2_A(\mathbb{R}))$.

$P_M : X^2_{A,k} \longrightarrow X^2_{A,k}$ the orthogonal projection from $X^2_{A,k}$ onto the subspace of function supported in the subset $M \subset \mathbb{R}\times\mathbb{R}$ with $\lambda_{A,k}(M) < \infty$.

We put

$$||P_MP_R|| = \sup\Big\{||P_MP_Rv||_{2,\lambda_{A,k}},\ v \in X^2_{A,k};\ ||v||_{2,\lambda_{A,k}} = 1\Big\}. \tag{4.1}$$

The main result of this subsection is the following.

Theorem 4.1. *(Concentration of $\mathcal{S}_hf$ in small sets) Let h be a generalized wavelet on $\mathbb{R}$ in $L^2_A(\mathbb{R})$ and $M \subset \mathbb{R}^*_+ \times \mathbb{R}$ with*

$$\frac{||h||_{2,A}}{\sqrt{C_h}}\sqrt{\lambda_{A,k}(M)} < 1.$$

Then, for all $f \in L^2_A(\mathbb{R})$ *we have*

$$\|\mathcal{S}_h f - \chi_M \mathcal{S}_h f\|_{2,\lambda_{A,k}} \geq \sqrt{C_h}\Big(1 - \frac{||h||_{2,A}}{\sqrt{C_h}}\sqrt{\lambda_{A,k}(M)}\Big)\|f\|_{2,A}. \tag{4.2}$$

Proof. From the definition of P_M and P_R we have

$$\|\mathcal{S}_h f - \chi_M \mathcal{S}_h f\|_{2,\lambda_{A,k}} = ||(I - P_M P_R)\mathcal{S}_h f||_{2,\lambda_{A,k}}.$$

Then, using Theorem 3.9 we get

$$\begin{aligned} \|\mathcal{S}_h f - \chi_M \mathcal{S}_h f\|_{2,\lambda_{A,k}} &\geq \|\mathcal{S}_h f\|_{2,\lambda_{A,k}}(1 - \|P_M P_R\|) \\ &\geq \sqrt{C_h}\|h\|_{2,A}\|f\|_{2,A}(1 - \|P_M P_R\|). \end{aligned} \tag{4.3}$$

As P_R is a projection onto a reproducing kernel Hilbert space, then, from Saitoh [10], P_R can be represented by

$$P_R F(y,\nu) = \int_{\mathbb{R}\times\mathbb{R}} F(y',\nu')\,\mathcal{K}_h(y',\nu';y,\nu)d\lambda_{A,k}(y',\nu'),$$

with $\mathcal{K}_h$ defined by (3.15). Hence, for $F \in X^2_{A,k}$ arbitrary, we have

$$P_M P_R F(y,\nu) = \int_{\mathbb{R}\times\mathbb{R}} \chi_M(y,\nu) F(y',\nu')\,\mathcal{K}_h(y',\nu';y,\nu)d\lambda_{A,k}(y',\nu')$$

and its Hilbert-Schmidt norm

$$\|P_M P_R\|_{HS} = \Big(\int_{\mathbb{R}^2\times\mathbb{R}^2} |\chi_M(y,\nu)|^2 |\mathcal{K}_h(y',\nu';y,\nu)|^2 d\lambda_{A,k}(y',\nu')d\lambda_{A,k}(y,\nu)\Big)^{\frac{1}{2}}.$$

By the Cauchy-Schwartz inequality we see that

$$\|P_M P_R\|_{HS} \geq \|P_M P_R\|. \tag{4.4}$$

On the other hand, from (3.16), Plancherel formula for the generalized Fourier transform and Fubini's theorem, it is easy to see that

$$\|P_M P_R\|_{HS} \leq \frac{||h||_{2,A}}{\sqrt{C_h}}\sqrt{\lambda_{A,k}(M)}. \tag{4.5}$$

Thus, from the relations (4.3),(4.4) and (4.5) we obtain the result. □

4.2 Calderón's formulas for S_h

In this paragraph we establish Calderón's formulas for the generalized wavelet transform on $\mathbb{R}$. We start with the following technical lemma.

Lemma 4.2. *Let h be a generalized wavelet on* $\mathbb{R}$ *in* $L^2_A(\mathbb{R})$ *such that* $\mathcal{F}(h) \in L^\infty_\sigma(\mathbb{R})$. *For* $0 < \varepsilon < \delta < \infty$, *we put:*

$$H_{\varepsilon,\delta}(x) := \frac{1}{C_h}\int_\varepsilon^\delta \widetilde{h_a} * \overline{h_a}(-x)\frac{da}{a}, \quad \textit{where } \widetilde{h}(x) = h(-x), \tag{4.6}$$

and

$$K_{\varepsilon,\delta}(x) := \frac{1}{C_h}\int_{\varepsilon}^{\delta} |\mathcal{F}(h)(ax)|^2 \frac{da}{a}. \tag{4.7}$$

Then

$$H_{\varepsilon,\delta} \in L^2_A(\mathbb{R}), \quad K_{\varepsilon,\delta} \in L^1_\sigma(\mathbb{R}) \cap L^\infty_\sigma(\mathbb{R}), \tag{4.8}$$

and

$$\mathcal{F}(H_{\varepsilon,\delta}) = K_{\varepsilon,\delta}.$$

Proof. Using Hölder's inequality for the measure $\frac{da}{a}$ we obtain

$$|H_{\varepsilon,\delta}(x)|^2 \le \frac{\log(\delta/\varepsilon)}{(C_h)^2}\int_{\varepsilon}^{\delta} |\widetilde{h_a} * \overline{h_a}(-x)|^2 \frac{da}{a}, \quad x \in \mathbb{R}.$$

So

$$\|H_{\varepsilon,\delta}\|^2_{2,A} \le \frac{\log(\delta/\varepsilon)}{(C_h)^2}\int_{\varepsilon}^{\delta}\Big(\int_{\mathbb{R}} |\widetilde{h_a} * \overline{h_a}(x)|^2 A(x)dx\Big)\frac{da}{a}.$$

From Proposition 2.5 iii) we have

$$\int_{\mathbb{R}} |\widetilde{h_a} * \overline{h_a}(x)|^2 A(x)dx = \int_{\mathbb{R}} |\mathcal{F}(h_a)(\xi)|^4 d\sigma(\xi).$$

Thus

$$\int_{\mathbb{R}} |\widetilde{h_a} * \overline{h_a}(x)|^2 A(x)dx \le \frac{k^2(a)}{a}\|\mathcal{F}(h)\|^2_{\infty,\sigma}\|h\|^2_{2,A}.$$

Hence

$$\|H_{\varepsilon,\delta}\|^2_{2,A} \le C(\varepsilon,\delta)\,\frac{\log(\delta/\varepsilon)}{(C_h)^2}\|\mathcal{F}(h)\|^2_{\infty,\sigma}\|h\|^2_{2,A} < \infty,$$

with

$$C(\varepsilon,\delta) = \int_{\varepsilon}^{\delta}\frac{k^2(a)da}{a^2}.$$

The second assertion in (4.8) is easily checked. Let us calculate $\mathcal{F}(H_{\varepsilon,\delta})$. From Theorem 2.2 iii) we get

$$\widetilde{h_a} * \overline{h_a}(-x) = \int_{\mathbb{R}} |\mathcal{F}(h)(a\xi)|^2 \Phi_{i\xi}(x)d\sigma(\xi).$$

So

$$H_{\varepsilon,\delta}(x) = \frac{1}{C_h}\int_{\varepsilon}^{\delta}\Big(\int_{\mathbb{R}} |\mathcal{F}(h)(a\xi)|^2 \Phi_{i\xi}(x)d\sigma(\xi)\Big)\frac{da}{a}.$$

Using Plancherel formula we see that

$$\frac{1}{C_h}\int_{\varepsilon}^{\delta}\Big(\int_{\mathbb{R}} |\mathcal{F}(h)(a\xi)|^2 \Phi_{i\xi}(x)d\sigma(\xi)\Big)\frac{da}{a} \le \frac{C(\varepsilon,\delta)}{C_h}\|h\|^2_{2,A} < \infty.$$

Hence applying Fubini's Theorem, we find that

$$H_{\varepsilon,\delta}(x) = \int_{\mathbb{R}} K_{\varepsilon,\delta}(\xi)\Phi_{i\xi}(x)d\sigma(\xi),$$

and lemma is proved. □

We can now state the main result of this paragraph.

Theorem 4.3. *Let g be a generalized wavelet on* $\mathbb{R}$ *in* $L^2_A(\mathbb{R})$ *such that* $\mathcal{F}(h) \in L^\infty_\sigma(\mathbb{R})$. *Then for* $f \in L^2_A(\mathbb{R})$ *and* $0 < \varepsilon < \delta < \infty$, *the function* $f_{\varepsilon,\delta}$ *given by*

$$f_{\varepsilon,\delta}(x) := \frac{1}{C_h}\int_\varepsilon^\delta \Big(\int_{\mathbb{R}} S_h f(a,y)\overline{h_{a,-x}(y)}A(y)dy\Big)\frac{k^2(a)da}{a^2}, \quad x \in \mathbb{R}, \tag{4.9}$$

belongs to $L^2_A(\mathbb{R})$ *and satisfies:*

$$\lim_{\varepsilon\to 0,\delta\to\infty} \|f_{\varepsilon,\delta} - f\|_{2,A} = 0.$$

Proof. It is easily to see that

$$f_{\varepsilon,\delta} = f * H_{\varepsilon,\delta},$$

where $H_{\varepsilon,\delta}$ is the function given by (4.6). It follows by Proposition 2.5 iii) and Lemma 4.2 that $f_{\varepsilon,\delta} \in L^2_A(\mathbb{R})$ and

$$\mathcal{F}(f_{\varepsilon,\delta}) = \mathcal{F}(f)K_{\varepsilon,\delta},$$

where $K_{\varepsilon,\delta}$ is the function given by (4.7). From this relation and Plancherel theorem we obtain

$$\|f_{\varepsilon,\delta} - f\|^2_{2,A} = \int_{\mathbb{R}} |\mathcal{F}(f)(\xi)|^2\Big(1 - K_{\varepsilon,\delta}(\xi)\Big)^2 d\sigma(\xi).$$

But by (3.1) we have

$$\lim_{\varepsilon\to 0,\delta\to\infty} K_{\varepsilon,\delta}(\xi) = 1, \quad \text{for all } \xi \in \mathbb{R}.$$

So, the relation (4.9) follows from the dominated convergence theorem. □

4.3 Practical real inversion formulas for S_h

In this paragraph we give practical real inversion formulas.

• Let $s \in \mathbb{R}$. We define the space $H^s_A(\mathbb{R})$ by

$$H^s_A(\mathbb{R}) := \Big\{f \in L^2_A(\mathbb{R}) : (1 + |\xi|^2)^{s/2}\mathcal{F}(f) \in L^2_\sigma(\mathbb{R})\Big\}.$$

The space $H^s_A(\mathbb{R})$ provided with the inner product

$$\langle f, g\rangle_{H^s_A} = \int_{\mathbb{R}} (1 + |\xi|^2)^s \mathcal{F}(f)(\xi)\overline{\mathcal{F}(g)}(\xi)d\sigma(\xi),$$

and the norm $\|f\|^2_{H^s_A} = \langle f, f\rangle_{H^s_A}$, is a Hilbert space.

• Let h be a generalized wavelet on $\mathbb{R}$ in $L^2_A(\mathbb{R})$ such that $\mathcal{F}(h) \in L^\infty_\sigma(\mathbb{R})$. It is easily to see that the generalized continuous wavelet transform S_h, is a bounded linear operator from $H^s_A(\mathbb{R})$, $s \geq 0$, into $L^2_A(\mathbb{R})$, and we have

$$\|S_h f(a,.)\|_{2,A} \leq C(a)\|\mathcal{F}(h)\|_{\infty,\sigma}\|f\|_{H^s_A}, \quad f \in H^s_A(\mathbb{R}).$$

• Let $\lambda > 0$, $s \geq 0$ and h be a generalized wavelet on $\mathbb{R}$ in $L^2_A(\mathbb{R})$ such that $\mathcal{F}(h)$ belongs to $L^\infty_\sigma(\mathbb{R})$. We define the Hilbert space $\mathcal{H}^{\lambda,s}_{h,A}(\mathbb{R})$ as the subspace of $H^s_A(\mathbb{R})$ with the inner product:

$$\langle f, g\rangle_{\mathcal{H}^{\lambda,s}_{h,A}} = \lambda\langle f, g\rangle_{H^s_A} + \langle S_h f(a,.), S_h g(a,.)\rangle_{2,A}, \quad f, g \in H^s_A(\mathbb{R}).$$

The norm associated to the inner product is define by:

$$\|f\|^2_{\mathcal{H}^{\lambda,s}_{h,A}} := \lambda\|f\|^2_{H^s_A} + \|S_h f(a,.)\|^2_{2,A}.$$

We start with the following fundamental theorem (cf. [10]).

Theorem 4.4. *Let H_K be a Hilbert space admitting the reproducing kernel $K(p,q)$ on a set E and H a Hilbert space. Let $L : H_K \to H$ be a bounded linear operator on H_K into H. For $r > 0$, introduce the inner product in H_K and call it H_{K_r} as*

$$\langle f_1, f_2\rangle_{H_{K_r}} = r\langle f_1, f_2\rangle_{H_K} + \langle Lf_1, Lf_2\rangle_H.$$

Then

i*) H_{K_r} is the Hilbert space with the reproducing kernel $K_r(p,q)$ on E and satisfying the equation*

$$K(.,q) = (rI + L^*L)K_r(.,q),$$

where L^* is the adjoint operator of $L : H_K \to H$.

ii*) For any $r > 0$ and for any g in H, the infinitum*

$$\inf_{f\in H_K} \left\{ r\|f\|^2_{H_K} + \|Lf - g\|^2_H \right\}$$

is attained by a unique function $f^*_{r,g}$ in H_K and this extremal function is given by

$$f^*_{r,g}(p) = \langle g, LK_r(.,p)\rangle_H. \tag{4.10}$$

We proceed as in [10] we prove the following results.

Proposition 4.5. *Let $s > \alpha + 1$ and h be a generalized wavelet on $\mathbb{R}$ in $L^2_A(\mathbb{R})$ such that $\mathcal{F}(h) \in L^\infty_\sigma(\mathbb{R})$. Then the Hilbert space $\mathcal{H}^{\lambda,s}_{h,A}(\mathbb{R})$ admits the following reproducing kernel*

$$\mathcal{W}_{\lambda,h}(x,y) = \int_{\mathbb{R}} \frac{\Phi_{ix}(\xi)\Phi_{-iy}(\xi)}{\lambda(1+|\xi|^2)^s + [\mathcal{F}(h)(a\xi)]^2} d\sigma(\xi).$$

Theorem 4.6. *Let $s > \alpha + 1$ and h be a generalized wavelet on $\mathbb{R}$ in $L^2_A(\mathbb{R})$ such that $\mathcal{F}(h) \in L^\infty_\sigma(\mathbb{R})$.*

i*) For any $g \in L^2_A(\mathbb{R})$ and for any $\lambda > 0$, the best approximate function $f^*_{\lambda,g}$ in the sense*

$$\inf_{f\in H^s_A(\mathbb{R})} \left\{ \lambda\|f\|^2_{H^s_A} + \|g - S_h f(a,.)\|^2_{2,A} \right\}$$

$$= \lambda\|f^*_{\lambda,g}\|^2_{H^s_A} + \|g - S_h f^*_{\lambda,g}(a,.)\|^2_{2,A}$$

exists uniquely and $f^*_{\lambda,g}$ is represented by

$$f^*_{\lambda,g}(a,x) = \int_{\mathbb{R}} g(y) Q_{\lambda,h}(x,y) A(y) dy,$$

where

$$Q_{\lambda,h}(x,y) = \int_{\mathbb{R}} \frac{\mathcal{F}(h)(a\xi)\Phi_{ix}(\xi)\Phi_{-iy}(\xi)}{\lambda(1+|\xi|^2)^s + [\mathcal{F}(h)(a\xi)]^2} d\sigma(\xi).$$

ii*) If we take $g = S_h f(a,.)$, then*

$$f^*_{\lambda,g} \to f \quad as \quad \lambda \to 0, \quad uniformly\ .$$

iii*) Let $\delta > 0$ and let g, g_δ satisfy $\|g - g_\delta\|_{2,A} \le \delta$. Then*

$$\|f^*_{\lambda,g} - f^*_{\lambda,g_\delta}\|_{H^s_A} \le \frac{\delta}{\sqrt{\lambda}}.$$

4.4 Uncertainty principles of Heisenberg Type

In this subsection firstly we will prove the Heisenberg inequality for the generalized continuous wavelet transform in special case of Jacobi-Dunkl operator.

Let $D_\alpha := [\frac{1}{2}, 1]$ if $\alpha \geq \frac{1}{2}$. For $-\frac{1}{4} < \alpha < \frac{1}{2}$, $D_\alpha :=]\frac{1}{2}, \frac{2(\alpha+1)}{3}]$.

In her article [5], R. Ma shows an analogous of Heisenberg's inequality for the Jacobi-Dunkl transform given by the following theorem.

Theorem 4.7. *(Uncertainty principle of Heisenberg type for $\mathcal{F}$)*
For $\alpha > -\frac{1}{4}$, assume $s,t > 0$ and $\gamma \in D_\alpha$. Then there exists a constant $C > 0$ such that

$$\left(\int_{\mathbb{R}} |x|^{2t} |\mathcal{F}(f)(x)|^2 d\sigma(x)\right)^{\frac{s}{s+t}} \left(\int_{\mathbb{R}} |y|^{2\gamma s} |f(y)|^2 A(y) dy\right)^{\frac{t}{s+t}} \geq C \|f\|_{2,A}^2 \tag{4.11}$$

for all $f \in L^2_A(\mathbb{R})$.

Theorem 4.8. *(Uncertainty principles of Heisenberg Type for S_h)*
Let h be a generalized wavelet on $\mathbb{R}$ in $L^2_A(\mathbb{R})$. For $\alpha > -\frac{1}{4}$, assume that $s,t > 0$ and $\gamma \in D_\alpha$. Then there exists a constant $C > 0$ such that

$$\left(\int_0^\infty \int_{\mathbb{R}} |x|^{2s\gamma} |S_h f(a,x)|^2 d\lambda_{A,k}(a,x)\right)^{\frac{t}{s+t}} \left(\int_{\mathbb{R}} |\xi|^{2t} |\mathcal{F}(f)(\xi)|^2 d\sigma(\xi)\right)^{\frac{s}{s+t}}$$

$$\geq C C_h^{\frac{t}{s+t}} \|f\|_{2,A}^2 \tag{4.12}$$

for all $f \in L^2_A(\mathbb{R})$.

Proof. Let us assume the non-trivial case that both integrals on the left hand side of (4.12) are finite. We get from the admissibility condition (3.1) for h that

$$\int_0^\infty \int_{\mathbb{R}} |\xi|^{2t} |\mathcal{F}(h)(a\xi)|^2 |\mathcal{F}(f)(\xi)|^2 \frac{d\sigma(\xi) da}{a} = C_h \int_{\mathbb{R}} |\xi|^{2t} |\mathcal{F}(f)(\xi)|^2 d\sigma(\xi).$$

Using the relation (3.9), we obtain

$$\int_0^\infty \int_{\mathbb{R}} |\xi|^{2t} |\mathcal{F}(S_h f(a,.))(\xi)|^2 \frac{d\sigma(\xi) k^2(a) da}{a^2} = C_h \int_{\mathbb{R}} |\xi|^{2t} |\mathcal{F}(f)(\xi)|^2 d\sigma(\xi). \tag{4.13}$$

On the other hand, Theorem 4.7 implies that

$$\forall a > 0,\ \left(\int_{\mathbb{R}} |\xi|^{2t} |\mathcal{F}(S_h f(a,.))(\xi)|^2 d\sigma(\xi)\right)^{\frac{s}{s+t}} \left(\int_{\mathbb{R}} |x|^{2s\gamma} |S_h f(a,x)|^2 A(x) dx\right)^{\frac{t}{s+t}}$$

$$\geq C \int_{\mathbb{R}} |S_h f(a,x)|^2 A(x) dx.$$

Integrating with respect to $\dfrac{k^2(a)da}{a^2}$ we obtain,

$$\int_0^\infty \Big[\left(\int_{\mathbb{R}} |\xi|^{2t} |\mathcal{F}(S_h f(a,.))(\xi)|^2 d\sigma(\xi)\right)^{\frac{s}{s+t}} \left(\int_{\mathbb{R}} |x|^{2s\gamma} |S_h^D f(a,x)|^2 A(x) dx\right)^{\frac{t}{s+t}} \Big] \frac{k^2(a) da}{a^2}$$

$$\geq C\int_0^\infty\int_{\mathbb{R}}|S_hf(a,x)|^2A(x)dx\frac{k^2(a)da}{a^2}.$$

The left hand side of this inequality may be estimated from above using Cauchy Schwartz inequality. The right hand side can be rewritten by Plancherel formula for S_h. Therefore, from (4.13) we get,

$$\left(\int_0^\infty\int_{\mathbb{R}}|\xi|^{2t}|\mathcal{F}(S_hf(a,.))(\xi)|^2\frac{d\sigma(\xi)k^2(a)da}{a^2}\right)^{\frac{s}{s+t}}\left(\int_0^\infty\int_{\mathbb{R}}|x|^{2s\gamma}|S_hf(a,x)|^2d\lambda_{A,k}(a,x)\right)^{\frac{t}{s+t}}$$

$$= C_h^{\frac{s}{s+t}}\left(\int_{\mathbb{R}}|\xi|^{2t}|\mathcal{F}(f(\xi))|^2d\sigma(\xi)\right)^{\frac{s}{s+t}}\left(\int_0^\infty\int_{\mathbb{R}}|x|^{2s\gamma}|S_hf(a,x)|^2d\lambda_{A_{A,k}}(a,x)\right)^{\frac{t}{s+t}}$$

$$\geq CC_h||f||^2_{2,A}.$$

This proves the result. □

In this paragraph we establish Heisenberg inequality for the generalized continuous wavelet transform on Chébli-Trimèche hypergroups . We start with the following hypotheses.

A function f is said to satisfy the condition (H_2) if for some $\mu > 0$ and $x_0 > 0$,

$$\int_{x_0}^\infty x^{\omega(\mu)}|\zeta(x)|dx < \infty, \quad \text{where} \quad \zeta(x) = f(x) - \frac{\mu^2-\frac{1}{4}}{x^2},$$

and if ζ is bounded for $x > x_0$, here $\omega(\mu) = \mu + \frac{1}{2}$ if $\mu \geq \frac{1}{2}$ and $\omega(\mu) = 1$ otherwise.
In the sequel, we assume that A satisfies the condition (H_2) and there exists $\beta \in (-\frac{1}{2},\mu]$ such that $A(x) = O(|x|^{2\beta+1})(|x| \to \infty)$.

In her article [6], R. Ma shows an analogous of Heisenberg's inequality for the generalized Fourier transform given by the following theorem.

Theorem 4.9. *(Uncertainty principle of Heisenberg type for $\mathcal{F}$)*
Assume $s,t > 0$ and $\gamma \in D := [\frac{\beta+1}{\mu+1},1]$. Then there exists a constant $C > 0$ such that

$$\left(\int_{\mathbb{R}}|x|^{2t}|\mathcal{F}(f)(x)|^2d\sigma(x)\right)^{\frac{s}{s+t}}\left(\int_{\mathbb{R}}|y|^{2\gamma s}|f(y)|^2A(y)dy\right)^{\frac{t}{s+t}} \geq C\|f\|^2_{2,A} \tag{4.14}$$

for all $f \in L^2_{A,e}(\mathbb{R}) = \left\{f \in L^2_A(\mathbb{R}),\ f\,\text{even}\right\}$.

Theorem 4.10. *(Uncertainty principles of Heisenberg Type for S_h)*
Let h be a generalized wavelet on $\mathbb{R}$ in $L^2_A(\mathbb{R})$. Let $s,t > 0$ and $\gamma \in D$. Then there exists a constant $C > 0$ such that

$$\left(\int_0^\infty\int_{\mathbb{R}}|x|^{2s\gamma}|S_hf(a,x)|^2d\lambda(a,x)\right)^{\frac{t}{s+t}}\left(\int_{\mathbb{R}}|\xi|^{2t}|\mathcal{F}(f)(\xi)|^2d\sigma(\xi)\right)^{\frac{s}{s+t}} \geq CC_h^{\frac{t}{s+t}}\|f\|^2_{2,A} \tag{4.15}$$

for all $f \in L^2_{A,e}(\mathbb{R})$.

Proof. The proof uses the same ideas as Theorem 4.8. □

Remark 4.11. We note that the analogous result for the Dunkl operator is proved in [7].

Acknowledgments

The authors express their gratitude to Professor K. Trimèche for several valuable discussions, reading of the manuscript, and suggestions to improve it. The authors thank the referees for their careful reading of the manuscript and insightful comments.

References

[1] N. Ben Salem and A. Oueld Ahmed Salem, A convolution structure associated with the generalized operator on $\mathbb{R}$. *Ramanujan J.* **12** (2006), no. 3 pp 359–378.

[2] F. Chouchane, M. Mili and K. Trimèche, Positivity of the intertwining operator and harmonic analysis associated with the generalized operator on $\mathbb{R}$. *J. Anal. Appl. Vol.* **1** (2003), pp 387-412.

[3] A. Grossmann and J. Morlet, Decomposition of Hardy functions into square integrable wavelets of constant shape. *SIAM* **15** (1984), pp 723-736.

[4] C.F. Dunkl, Integral kernels with reflection group invariance. *Can. J. Math.* **43** (1991), pp 1213-1227.

[5] R. Ma, Heisenberg inequalities for Jacobi transforms. *J. Math. Anal. Appl.* **332** (2007), pp 155-163.

[6] R. Ma, Heisenberg uncertainty principle on Chébli-Trimèche hypergroups, *Pacific J. Math.* **235** N2 (2008), pp 289-296.

[7] H. Mejjaoli and N. Sraieb, Uncertainty principles for the continuous Dunkl Gabor transform and the Dunkl continuous wavelet transform. *Mediterr. j. math.* **5** (2008), pp 443-466.

[8] M.A. Mourou and K. Trimèche, Transmutation operators and Paley-Wiener associated with a singular differential-difference operator on the real line. *Analysis and Applications* **1**, No. 1 (2003), pp 43-70.

[9] M.A. Mourou, Hardy and Cowling-Price Theorems Associated with a Dunkl Type Differential-Difference Operator on the Real Line. *Communications in Mathematical Analysis* Proceedings of the **16**th Symposium of the Tunisian Mathematical Society Conf. 02, (2008), pp 56-62.

[10] S. Saitoh, One approach to some general integral transforms and its applications. *Int. Trans. Spec. Funct.* **3** (1995), pp 49-84.

[11] K. Trimèche, Inversion of the Lions transmutation operators using generalized wavelets. *App. Comput. Harm. Anal.* **4** (1997), pp 97-112.