Communications in **M**athematical **A**nalysis
Volume 9, Number 1, pp. 32–47 (2010)
ISSN 1938-9787

www.commun-math-anal.org

BOUNDED AND PERIODIC SOLUTIONS OF A CLASS OF IMPULSIVE PERIODIC POPULATION EVOLUTION EQUATIONS OF VOLTERRA TYPE

JINRONG WANG *
Department of Mathematics
Guizhou University
Guiyang, 550025, China

W. WEI †
College of Science
Guizhou University
Guiyang, 550025, China

X. XIANG‡
Department of Mathematics
Guizhou University
Guiyang, 550025, China

(Communicated by Toka Diagana)

Abstract

This paper deals with a class of impulsive periodic population evolution equations of Volterra type on Banach space. By virtue of integral inequality of Gronwall type for piecewise continuous functions, the prior estimate on the *PC*-mild solutions is derived. The compactness of the new constructed *Poincaré* operator is shown. This allows us to apply Horn's fixed point theorem to prove the existence of T_0-periodic *PC*-mild solutions when *PC*-mild solutions are ultimate bounded. At last, an example is given for demonstration.

AMS Subject Classification: 45D05, 45N05.

Keywords: Impulsive, Integrodifferential equations, Volterra type, Periodic solutions, Existence.

*E-mail address: wjr9668@126.com
†E-mail address: wwei@gzu.edu.cn
‡E-mail address: xxl3621070@yahoo.com.cn

1 Introduction

For modeling the dynamics of an ecological system, Cushing [7] pointed out that it is necessary and important to consider models with periodic ecological parameters or perturbations which might be quite naturally exposed (for example, those due to seasonal effects of wheatear, food supply, mating habits, etc.). Particularly, the mathematical models of the periodic population evolution processes in a community, i.e., the periodic population evolution equations, are the theoretical bases for the quantitative demography to research population state and evolution law. The research of the population evolutions can help us to understand the dynamic characteristics of the population systems can provide us with a strict mathematical basis for the long-range and short-rang population forecast for the discovery of the population control law and for the decision of the population polices (See [4], [11], [16], [17], [26], [27]).

On the other hand, in order to describe dynamics of population subject to abrupt changes as well as other phenomena such as harvesting, diseases etc, some authors have used impulsive differential systems to describe the model since the last century. For the basic theory on impulsive differential equations on finite dimensional spaces, the reader can refer to Lakshmikantham's book, Liu's paper and Yang's book (see [9], [10], [25]). For the basic theory on impulsive differential equations on infinite dimensional spaces, the reader can refer to Ahmed's paper, Liu's papers and Xiang's (see [2], [3], [5], [13], [21], [22], [23], [24]).

As a result, it is necessary and important to investigate periodic population evolution systems with periodic perturbations. In this paper, we will study a class of generalized non-autonomous integrodifferential periodic population system with periodic impulsive perturbations which is governed by

$$\begin{cases} \frac{\partial}{\partial t}x(r,t)+a(t)\frac{\partial}{\partial r}x(r,t)=-a(t)\mu(r)x(r,t)-F_1(N(r,t))x(r,t)+F_2(r,t), \\ \qquad\qquad\qquad\qquad 0<r<r_m, t>0, t\neq\tau_k, k\in\mathbb{N}, \\ x(r,0)=x_0(r), \qquad\qquad 0\leq r\leq r_m, \\ x(0,t)=\beta\int_{r_1}^{r_2}k(r,t)h(r,t)x(r,t)dr,\ t>0,\ t\neq\tau_k, k\in\mathbb{N}, \\ \triangle x(r,t)=B_kx(r,t)+c_k, \qquad 0\leq r\leq r_m, t=\tau_k, k\in\mathbb{N}. \end{cases} \tag{1.1}$$

where t denotes time, r denotes age, r_m is the highest age ever attained by individuals of the population, the coefficient $a(t)$ is sufficiently smooth function, $x(r,t)=\frac{\partial N(r,t)}{\partial r}$ is called age density function, $N(r,t)$ denotes the amount of population aged less than r at time t, F_1 denotes nonnegative function which can describe the variation of $N(r,t)$, $x_0(r)$ denotes initial age density, $\mu(r)$ is the age-specific dead rate, the constant β is the specific fertility rate of females, $k(r,t)$ is the female ratio, and $h(r,t)$ is the fertility pattern satisfying $\int_{r_1}^{r_2}h(r,t)dr=1$, where $[r_1,r_2]$ denotes the fecundity period of female. $N(r,t)=\int_0^{r_m}x(r,t)dr$; $F_2(r,t)$ denotes that immigrant density. In general, the relationships among the $x(r,t)$, $F_2(r,t)$ and $N(r,t)$ are very complicated. Here, we only discuss the relationship between $F_2(r,t)$ and $x(r,t)$ satisfies $F_2(r,t)=\int_0^t g(t,s,x(r,s))ds$, where g is a real fucntion. Time sequence $0=\tau_0<\tau_1<\cdots<\tau_k\cdots$ and $\tau_k\to\infty$ as $k\to\infty$, $\Delta x(r,\tau_k)=x(r,\tau_k^+)-x(r,\tau_k)=B_kx(r,\tau_k)+c_k$ denote mutation of the population at time τ_k. For arbitrary $t\geq0$, there exist $T_0>0$ and $\delta\in\mathbb{N}$ such that $a(t+T_0)=a(t)$, $F_1(N(r,t+T_0))=F_1(N(r,t))$, $F_2(r,t+T_0)=F_2(r,t)$, $\tau_{k+\delta}=\tau_k+T_0$, $B_{k+\delta}=B_k$, $c_{k+\delta}=c_k$.

Then system (1.1) can be written as

$$\begin{cases} \dfrac{\partial}{\partial t}x(r,t)+a(t)\dfrac{\partial}{\partial r}x(r,t)+a(t)\mu(r)x(r,t)=f\left(t,x(r,t),\displaystyle\int_0^t g(t,s,x(r,s))ds\right), \\ \qquad\qquad\qquad\qquad\qquad\qquad 0<r<r_m, t>0, t\neq\tau_k, k\in\mathbb{N}, \\ x(r,0)=x_0(r), \qquad\qquad\qquad 0\leq r\leq r_m, \\ x(0,t)=\beta\int_{r_1}^{r_2}k(r,t)h(r,t)x(r,t)dr,\ t>0,\ t\neq\tau_k, k\in\mathbb{N}, \\ \triangle x(r,t)=B_kx(r,t)+c_k, \qquad\qquad 0\leq r\leq r_m, t=\tau_k, k\in\mathbb{N}. \end{cases} \tag{1.2}$$

Define $X=L^p(0,r_m)$, $1\leq p<\infty$, $A(t)\varphi(r)=-a(t)\left(\frac{d}{dr}\varphi(r)-\mu(r)\varphi(r)\right)$, for arbitrary $\varphi\in D(A)$, ($A(t)$ is called population evolution operator with time-varying) and $D(A)=\left\{\varphi\mid \varphi, A(t)\varphi\in X; \varphi(0)=\beta\int_{r_1}^{r_2}k(r)h(r)\varphi(r)dr\right\}$.

Let $x(\cdot)(r)=x(\cdot,r)$, $f\left(t,x(r,t),\int_0^t g(t,s,x(r,s))ds\right)=f\left(t,x(t),\int_0^t g(t,s,x(s))ds\right)(r)$, then the model (1.2) can be abstracted the following integrodifferential impulsive periodic system with time-varying generating operators:

$$\begin{cases} \dot{x}(t)=A(t)x(t)+f\left(t,x,\int_0^t g(t,s,x)ds\right), & t\neq\tau_k, \\ \Delta x(t)=B_kx(t)+c_k, & t=\tau_k. \end{cases} \tag{1.3}$$

in Banach space X. Suppose $\{A(t),\ t\in[0,T_0]\}$ is a family of closed densely defined linear unbounded operators on X and the resolvent of the unbounded operator $A(t)$ is compact. f is a measurable function from $[0,\infty)\times X\times X$ to X, g is a continuous function from $[0,\infty)\times[0,\infty)\times X$ to X. For arbitrary $t\geq 0$, there exist $T_0>0$ and $\delta\in\mathbb{N}$ such that $A(t+T_0)=A(t)$, $f(t+T_0,x,y)=f(t,x,y)$, $g(t+T_0,s+T_0,x)=g(t,s,x)$, $\tau_{k+\delta}=\tau_k+T_0$, $B_{k+\delta}=B_k$, $c_{k+\delta}=c_k$.

Here, we use Horn's fixed point theorem to obtain the existence of periodic solution for integrodifferential impulsive periodic system (1.3). First, by virtue of impulsive evolution operator corresponding to homogeneous linear impulsive system, we construct a new *Poincaré* operator P for integrodifferential impulsive periodic system (1.3), then overcome some difficulties to show the continuity and compactness of *Poincaré* operator P which are very important. By virtue of the integral inequality of Gronwall type for piecewise continuous functions, the prior estimate of *PC*-mild solutions is given. Therefore, the existence of T_0-periodic *PC*-mild solutions for integrodifferential impulsive periodic system (1.3) when *PC*-mild solutions are ultimate bounded is shown.

This paper is organized as follows. In section 2, some properties of impulsive evolution operator corresponding to homogeneous linear impulsive periodic system are recalled. In section 3, by the integral inequality of Gronwall type for piecewise continuous functions and existence and continuous dependence of *PC*-mild solutions for integrodifferential impulsive periodic system (1.3) are given. In section 4, the new *Poincaré* operator P is constructed and the relation between T_0-periodic *PC*-mild solution and the fixed point of *Poincaré* operator P is given. After the continuity and compactness of *Poincaré* operator P are shown, the existence of T_0-periodic *PC*-mild solutions for integrodifferential impulsive periodic system (1.3) is established by virtue of Horn's fixed point theorem when *PC*-mild solutions are ultimate bounded. At last, an example is given to demonstrate the applicability of our result.

2 Preliminaries

Let X be a Banach space. $£(X)$ denotes the space of linear operators in X; $£_b(X)$ denotes the space of bounded linear operators in X. $£_b(X)$ is the Banach space with the usual supremum norm. Define $\widetilde{D}$=$\{\tau_1,\cdots,\tau_\delta\} \subset [0,T_0]$, where $\delta \in \mathbb{N}$ denotes the number of impulsive points between $[0,T_0]$. We introduce $PC([0,T_0];X) \equiv \{x : [0,T_0] \to X \mid x$ is continuous at $t \in [0,T_0]\backslash\widetilde{D}$, x is continuous from left and has right hand limits at $t \in \widetilde{D}\}$ and $PC^1([0,T_0];X) \equiv \{x \in PC([0,T_0];X) \mid \dot{x} \in PC([0,T_0];X)\}$. Set

$$\|x\|_{PC} = \max\left\{\sup_{t\in[0,T_0]}\|x(t+0)\|,\ \sup_{t\in[0,T_0]}\|x(t-0)\|\right\} \quad \text{and} \quad \|x\|_{PC^1} = \|x\|_{PC} + \|\dot{x}\|_{PC}.$$

It can be seen that endowed with the norm $\|\cdot\|_{PC}$ ($\|\cdot\|_{PC^1}$), $PC([0,T_0];X)$ $\big(PC^1([0,T_0];X)\big)$ is a Banach space.

For homogeneous linear impulsive periodic system with time-varying generating operators

$$\begin{cases} \dot{x}(t) = A(t)x(t), & t \neq \tau_k, \\ \Delta x(\tau_k) = B_k x(\tau_k), & t = \tau_k. \end{cases} \tag{2.1}$$

on Banach space X, where $\triangle x(\tau_k) = x(\tau_k^+) - x(\tau_k^-)$, $\{A(t), t \in [0,T_0]\}$ is a family of closed densely defined linear unbounded operators on X satisfying the following assumption.

Assumption [A1] (See [1], p.158) For $t \in [0,T_0]$ one has

(P_1) The domain $D(A(t)) = D$ is independent of t and is dense in X.

(P_2) For $t \geq 0$, the resolvent $R(\lambda,A(t)) = (\lambda I - A(t))^{-1}$ exists for all λ with $Re\lambda \leq 0$, and there is a constant M independent of λ and t such that

$$\|R(\lambda,A(t))\| \leq M(1+|\lambda|)^{-1} \quad \text{for} \quad Re\lambda \leq 0.$$

(P_3) There exist constants $L > 0$ and $0 < \alpha \leq 1$ such that

$$\big\|\big(A(t)-A(\theta)\big)A^{-1}(\tau)\big\| \leq L|t-\theta|^{\alpha} \quad \text{for} \quad t,\theta,\tau \in [0,T_0].$$

Lemma 2.1 (See [1], p.159) Under the assumption [A1], the Cauchy problem

$$\dot{x}(t) + A(t)x(t) = 0,\ t \in (0,T_0] \quad \text{with} \quad x(0) = \bar{x} \tag{2.2}$$

has a unique evolution system $\{U(t,\theta) \mid 0 \leq \theta \leq t \leq T_0\}$ in X satisfying the following properties:

(1) $U(t,\theta) \in £_b(X)$ for $0 \leq \theta \leq t \leq T_0$.

(2) $U(t,r)U(r,\theta) = U(t,\theta)$ for $0 \leq \theta \leq r \leq t \leq T_0$.

(3) $U(\cdot,\cdot)x \in C(\Delta,X)$ for $x \in X$, $\Delta = \{(t,\theta) \in [0,T_0]\times[0,T_0] \mid 0 \leq \theta \leq t \leq T_0\}$.

(4) For $0 \leq \theta < t \leq T_0$, $U(t,\theta)$: $X \longrightarrow D$ and $t \longrightarrow U(t,\theta)$ is strongly differentiable in X. The derivative $\frac{\partial}{\partial t}U(t,\theta) \in £_b(X)$ and it is strongly continuous on $0 \leq \theta < t \leq T_0$.

Moreover,

$$\begin{aligned}
\frac{\partial}{\partial t}U(t,\theta) &= -A(t)U(t,\theta) \quad \text{for} \quad 0 \le \theta < t \le T_0, \\
\left\|\frac{\partial}{\partial t}U(t,\theta)\right\|_{\pounds_b(X)} &= \|A(t)U(t,\theta)\|_{\pounds_b(X)} \le \frac{C}{t-\theta}, \\
\left\|A(t)U(t,\theta)A(\theta)^{-1}\right\|_{\pounds_b(X)} &\le C \quad \text{for} \quad 0 \le \theta \le t \le T_0.
\end{aligned}$$

(5) For every $v \in D$ and $t \in (0,T_0], U(t,\theta)v$ is differentiable with respect to θ on $0 \le \theta \le t \le T_0$

$$\frac{\partial}{\partial \theta}U(t,\theta)v = U(t,\theta)A(\theta)v.$$

And, for each $\bar{x} \in X$, the Cauchy problem (2.2) has a unique classical solution $x \in C^1([0,T_0];X)$ given by

$$x(t) = U(t,0)\bar{x}, \quad t \in [0,T_0].$$

In addition to assumption [A1], we introduce the following assumptions.

Assumption [A2] There exits $T_0 > 0$ such that $A(t+T_0) = A(t)$ for $t \in [0,T_0]$.

Assumption [A3] For $t \ge 0$, the resolvent $R(\lambda, A(t))$ is compact.

Then we have

Lemma 2.2 Assumptions [A1], [A2] and [A3] hold. Then evolution system $\{U(t,\theta) \mid 0 \le \theta \le t \le T_0\}$ in X also satisfying the following two properties:

(6) $U(t+T_0, \theta+T_0) = U(t,\theta)$ for $0 \le \theta \le t \le T_0$;

(7) $U(t,\theta)$ is compact operator for $0 \le \theta < t \le T_0$.

In order to introduce a impulsive evolution operator and give it's properties, we need the following assumption.

Assumption [B] For each $k \in \mathbb{Z}_0^+$, $B_k \in \pounds_b(X)$, there exists $\delta \in \mathbb{N}$ such that $\tau_{k+\delta} = \tau_k + T_0$ and $B_{k+\delta} = B_k$.

Consider the following Cauchy problem

$$\begin{cases} \dot{x}(t) = A(t)x(t), & t \in [0,T_0]\backslash\widetilde{D}, \\ \Delta x(\tau_k) = B_k x(\tau_k), & k = 1,2,\cdots,\delta, \\ x(0) = \bar{x}. \end{cases} \tag{2.3}$$

For every $\bar{x} \in X$, D is an invariant subspace of B_k, using Lemma 2.1, step by step, one can verify that the Cauchy problem (2.3) has a unique classical solution $x \in PC^1([0,T_0];X)$ represented by $x(t) = \mathbf{S}(t,0)\bar{x}$ where $\mathbf{S}(\cdot,\cdot) : \Delta \longrightarrow \pounds(X)$ given by

$$\mathbf{S}(t,\theta) = \begin{cases} U(t,\theta), & \tau_{k-1} \le \theta \le t \le \tau_k, \\ U(t,\tau_k^+)(I+B_k)U(\tau_k,\theta), & \tau_{k-1} \le \theta < \tau_k < t \le \tau_{k+1}, \\ U(t,\tau_k^+)\left[\prod_{\theta<\tau_j<t}(I+B_j)U(\tau_j,\tau_{j-1}^+)\right](I+B_i)U(\tau_i,\theta), & \\ \qquad \tau_{i-1} \le \theta < \tau_i \le \cdots < \tau_k < t \le \tau_{k+1}. & \end{cases} \tag{2.4}$$

The operator $\{\mathbf{S}(t,\theta),(t,\theta)\in\Delta\}$ is called impulsive evolution operator associated with $\{B_k;\tau_k\}_{k=1}^{\infty}$.

The following lemma on the properties of the impulsive evolution operator $\{\mathbf{S}(t,\theta),(t,\theta)\in\Delta\}$ associated with $\{B_k;\tau_k\}_{k=1}^{\infty}$ are widely used in this paper.

Lemma 2.3 (See Lemma 1 of [18]) Assumptions [A1], [A2], [A3] and [B] hold. Impulsive evolution operator $\{\mathbf{S}(t,\theta),(t,\theta)\in\Delta\}$ has the following properties:

(1) For $0\le\theta\le t\le T_0$, $\mathbf{S}(t,\theta)\in \pounds_b(X)$, i.e., $\sup_{0\le\theta\le t\le T_0}\|\mathbf{S}(t,\theta)\|\le M_{T_0}$, $M_{T_0}>0$.

(2) For $0\le\theta<r<t\le T_0$, $r\neq\tau_k$, $\mathbf{S}(t,\theta)=\mathbf{S}(t,r)\mathbf{S}(r,\theta)$.

(3) For $0\le\theta\le t\le T_0$ and $N\in Z_0^+$, $\mathbf{S}(t+NT_0,\theta+NT_0)=\mathbf{S}(t,\theta)$.

(4) For $0\le t\le T_0$ and $M\in Z_0^+$, $\mathbf{S}(MT_0+t,0)=\mathbf{S}(t,0)\left[\mathbf{S}(T_0,0)\right]^M$.

(5) $\mathbf{S}(t,\theta)$ is compact operator for $0\le\theta<t\le T_0$.

3 Integral inequalities of Gronwall type and Existence and continuous dependence of the Solutions

In order to derive the estimate of PC-mild solutions, we introduce the following integral inequalities of Gronwall type for piecewise continuous functions which is widely used in sequel.

Lemma 3.1 (See Theorem 2 of [6]) Let $t\ge t_0\ge 0$ the following inequality hold

$$x(t)\le a(t)+\int_{t_0}^{t}b(t,\theta)x(\theta)d\theta+\int_{t_0}^{t}\left(\int_{t_0}^{\theta}k(t,\theta,s)u(s)ds\right)d\theta+\sum_{t_0<\tau_k<t}\beta_k(t)u(\tau_k), \tag{3.1}$$

where x, $a\in PC([t_0,\infty),\mathbb{R}^+)$, a is nondecreasing, $b(t,\theta)$ and $k(t,\theta,s)$ are continuous and nonnegative functions for $t,\theta,s\ge t_0$ and are nondecreasing with respect to $t,\beta_k(t)(k\in\mathbb{N})$ are nondecreasing for $t\ge t_0$.

Then, for $t\ge t_0$, the following inequality is valid:

$$x(t)\le a(t)\prod_{t_0<\tau_k<t}(1+\beta_k(t))\exp\left(\int_{t_0}^{t}b(t,\theta)d\theta+\int_{t_0}^{t}\int_{t_0}^{\theta}k(t,\theta,s)dsd\theta\right).$$

Now, we consider the following integrodifferential impulsive periodic system with time-varying generating operators

$$\begin{cases}\dot{x}(t)=A(t)x(t)+f\left(t,x(t),\int_0^t g(t,s,x(s))ds\right), & t\neq\tau_k,\\ \Delta x(t)=B_kx(t)+c_k, & t=\tau_k.\end{cases} \tag{3.2}$$

and the associated Cauchy problem

$$\begin{cases}\dot{x}(t)=A(t)x(t)+f\left(t,x(t),\int_0^t g(t,s,x(s))ds\right), & t\in[0,T_0]\backslash\widetilde{D},\\ \Delta x(\tau_k)=B_kx(\tau_k)+c_k, & k=1,2,\cdots,\delta,\\ x(0)=\bar{x}.\end{cases} \tag{3.3}$$

We can introduce the PC-mild solution of the Cauchy problem (3.3).

Definition 3.1 A function $x \in PC([0,T_0];X)$ is said to be a *PC*-mild solution of the Cauchy problem (3.3) corresponding to the initial value $\bar{x} \in X$ if x satisfies the following integral equation

$$x(t) = \mathbf{S}(t,0)\bar{x} + \int_0^t \mathbf{S}(t,\theta) f\left(\theta, x(\theta), \int_0^\theta g(\theta,s,x(s))ds\right) d\theta + \sum_{0\le\tau_k<t} \mathbf{S}\left(t,\tau_k^+\right) c_k$$

for $t \in [0,T_0]$.

Now, we introduce the T_0-periodic *PC*-mild solution of system (3.2).

Definition 3.2 A function $x \in PC([0,+\infty);X)$ is said to be a T_0-periodic *PC*-mild solution of system (3.2) if it is a *PC*-mild solution of Cauchy problem (3.3) corresponding to some $\bar{x}$ and $x(t+T_0) = x(t)$ for $t \ge 0$.

We make the following assumptions.

Assumption [C] For each $k \in \mathbb{Z}_0^+$ and $c_k \in X$, there exists $\delta \in \mathbb{N}$ such that $c_{k+\delta} = c_k$.

Assumption [F]

[F1]: $f : [0,\infty) \times X \times X \to X$ is measurable for $t \ge 0$ and for any x_1, x_2, $y_1, y_2 \in X$ satisfying $\|x_1\|$, $\|x_2\|$, $\|y_1\|$, $\|y_2\| \le \rho$ there exists a positive constant $L_f(\rho) > 0$ such that

$$\|f(t,x_1,y_1) - f(t,x_2,y_2)\| \le L_f(\rho)(\|x_1 - x_2\| + \|y_1 - y_2\|).$$

[F2]: There exists a positive constant $M_f > 0$ such that

$$\|f(t,x,y)\| \le M_f(1 + \|x\| + \|y\|) \text{ for all } x,y \in X.$$

[F3]: $f(t,x,y)$ is T_0-periodic in t, i.e., $f(t+T_0,x,y) = f(t,x,y), t \ge 0$.

Assumption [G]

[G1]: $g : [0,\infty) \times [0,\infty) \times X \to X$ is continuous for $t \ge s \ge 0$ and for any x, $y \in X$ satisfying $\|x\|$, $\|y\| \le \rho$ there exists a positive constant $L_g(\rho) > 0$ such that

$$\|g(t,s,x) - g(t,s,y)\| \le L_g(\rho)\|x - y\|.$$

[G2]: There exists a positive constant $M_g > 0$ such that

$$\|g(t,s,x)\| \le M_g(1 + \|x\|) \text{ for all } x \in X.$$

[G3]: $g(t,s,x)$ are T_0-periodic in t and s, i.e., $g(t+T_0, s+T_0, x) = g(t,s,x),$ for $t \ge s \ge 0$ and $\int_0^{T_0} g(t,s,x)ds = 0,$ for $t \ge s \ge 0,\ x \in X$.

Now we present the existence and continuous dependence of *PC*-mild solutions for system (3.3).

Theorem 3.1 Assumptions [A1], [F1], [F2], [G1] and [G2] hold, and for each $k \in \mathbb{Z}_0^+$, $B_k \in \pounds_b(X)$. Then system (3.3) has a unique *PC*-mild solution given by

$$x(t,\bar{x}) = \mathbf{S}(t,0)\bar{x} + \int_0^t \mathbf{S}(t,\theta) f\left(\theta, x(\theta), \int_0^\theta g(\theta,s,x(s))ds\right) d\theta + \sum_{0\leq \tau_k < t} \mathbf{S}\left(t,\tau_k^+\right) c_k.$$

Further, the *PC*-mild solutions of system (3.3) depend continuously on the initial conditions, i.e., for any number $\varepsilon > 0$, there exists a number $\sigma > 0$ such that for $\|\bar{x}-\bar{y}\| < \sigma$ the inequality

$$\|x(t,\bar{x}) - x(t,\bar{y})\| < \varepsilon$$

holds for $t \in [0,T_0]$.

Proof. In order to make the process clear we divide it into four steps.

Step 1, we consider the following integro-differential equation without impulse

$$\begin{cases} \dot{x}\,(t) = A(t)x(t) + f\left(t,x,\int_0^t g(t,s,x)ds\right),\ t \in [0,\tau], \\ x(0) = \bar{x} \in X. \end{cases} \tag{3.4}$$

In order to obtain the local existence of mild solution for system (3.4), we only need to set up the framework for use of the contraction mapping theorem. Consider the ball given by

$$\mathfrak{B} = \{x \in C([0,t_1];X) \mid \|x(t)-\bar{x}\| \leq 1, 0 \leq t \leq t_1\}$$

where t_1 would be chosen, and $\|x(t)\| \leq 1 + \|\bar{x}\| = \bar{\rho}$, $0 \leq t \leq t_1$. $\mathfrak{B} \subseteq C([0,t_1],X)$ is a closed convex set. Define a map $\mathbf{Q}$ on $\mathfrak{B}$ given by

$$(\mathbf{Q}x)(t) = U(t,0)\bar{x} + \int_0^t U(t,\theta) f\left(\theta, x(\theta), \int_0^\theta g(\theta,s,x(s))ds\right) d\theta.$$

Under the assumptions [A1], [F1], [F2], [G1], [G2] and Lemma 3.1, one can verify that map $\mathbf{Q}$ is a contraction map on $\mathfrak{B}$ with chosen $t_1 > 0$. This means that system (3.4) has a unique mild solution $x \in C([0,t_1];X)$ given by

$$x(t) = U(t,0)\bar{x} + \int_0^t U(t,\theta) f\left(\theta, x(\theta), \int_0^\theta g(\theta,s,x(s))ds\right) d\theta \text{ on } [0,t_1].$$

Again, using the Lemma 3.1, [G1] and [G2], we can obtain the a priori estimate of the mild solutions for system (3.4) and present the global existence of mild solutions.

Step 2, for $t \in (\tau_k, \tau_{k+1}]$, consider Cauchy problem

$$\begin{cases} \dot{x}\,(t) = A(t)x(t) + f\left(t,x(t),\int_0^t g(t,s,x(s))ds\right), t \in (\tau_k,\tau_{k+1}], \\ x(\tau_k) = x_k \equiv (I+B_k)x(\tau_k) + c_k \in X. \end{cases} \tag{3.5}$$

By Step 1, Cauchy problem (3.5) also has a unique *PC*-mild solution

$$x(t) = U(t,\tau_k)x_k + \int_{\tau_k}^t U(t,\theta) f\left(\theta, x(\theta), \int_0^\theta g(\theta,s,x(s))ds\right) d\theta.$$

Step 3, combining the all of solutions on $(\tau_k,\tau_{k+1}]$ ($k = 1,\cdots,\delta$), one can obtain the *PC*-mild solution of Cauchy problem (3.3) given by

$$x(t,\bar{x}) = \mathbf{S}(t,0)\bar{x} + \int_0^t \mathbf{S}(t,\theta) f\left(\theta, x(\theta), \int_0^\theta g(\theta,s,x(s))ds\right) d\theta + \sum_{0\leq \tau_k < t} \mathbf{S}\left(t,\tau_k^+\right) c_k.$$

Step 4, for the continuous dependence of *PC*-mild solutions, by standard argument, one can complete it. □

4 Existence of Periodic Solutions

To establish the periodic solutions for the system (3.2), we define a *Poincaré* operator from X to X as following

$$\begin{aligned} P(\bar{x}) &= x(T_0,\bar{x}) \\ &= \mathbf{S}(T_0,0)\bar{x} + \int_0^{T_0} \mathbf{S}(T_0,\theta) f\left(\theta, x(\theta,\bar{x}), \int_0^{\theta} g(\theta,s,x(s,\bar{x}))ds\right) d\theta \\ &\quad + \sum_{0\le\tau_k<T_0} \mathbf{S}\left(T_0,\tau_k^+\right) c_k \end{aligned} \tag{4.1}$$

where $x(\cdot,\bar{x})$ denote the PC-mild solution of the Cauchy problem (3.3) corresponding to the initial value $x(0)=\bar{x}$, then, examine whether P has a fixed point.

We first note that a fixed point of P gives rise to a periodic solution.

Lemma 4.1 Assumptions [A1], [A2], [B], [C], [F1], [F2], [F3], [G1], [G2] and [G3] hold. System (3.2) has a T_0-periodic PC-mild solution if and only if P has a fixed point.

Proof. Suppose $x(\cdot)=x(\cdot+T_0)$, then $x(0)=x(T_0)=P(x(0))$. This implies that $x(0)$ is a fixed point of P. On the other hand, if $Px_0=x_0$, $x_0\in X$, then for the PC-mild solution $x(\cdot,x_0)$ of the Cauchy problem (3.3) corresponding to the initial value $x(0)=x_0$, we can define $y(\cdot)=x(\cdot+T_0,x_0)$, then $y(0)=x(T_0,x_0)=Px_0=x_0$. Now, for $t>0$, we can use the (2), (3) and (4) of Lemma 2.3 and assumptions [A2], [B], [C], [F3], and [G3] to arrive at

$$\begin{aligned} y(t) &= x(t+T_0,x_0) \\ &= \mathbf{S}(t,0)x(T_0) \\ &\quad + \int_0^t \mathbf{S}(t+T_0,\theta+T_0) f\left(\theta+T_0, x(\theta+T_0,x_0), \int_{T_0}^{\theta+T_0} g(\theta+T_0,s,x(s,x_0))ds\right) d\theta \\ &\quad + \sum_{T_0\le\tau_{k+\delta}<t+T_0} \mathbf{S}\left(t+T_0,\tau_{k+\delta}^+\right) c_{k+\delta} \\ &= \mathbf{S}(t,0)x(T_0) \\ &\quad + \int_0^t \mathbf{S}(t,\theta) f\left(\theta, x(\theta+T_0,x_0), \int_0^{\theta} g(\theta+T_0,s+T_0,x(s+T_0,x_0))ds\right) d\theta \\ &\quad + \sum_{T_0\le\tau_{k+\delta}<t+T_0} \mathbf{S}\left(t+T_0,\tau_{k+\delta}^+\right) c_{k+\delta} \\ &= \mathbf{S}(t,0)x(T_0) + \int_0^t \mathbf{S}(t,\theta) f\left(\theta, y(\theta,y(0)), \int_0^{\theta} g(\theta,s,y(s,y(0)))ds\right) d\theta \\ &\quad + \sum_{0\le\tau_k<t} \mathbf{S}\left(t,\tau_k^+\right) c_k. \\ &= \mathbf{S}(t,0)y(0) + \int_0^t \mathbf{S}(t,\theta) f\left(\theta, y(\theta,y(0), \int_0^{\theta} g(\theta,s,y(s,y(0)))ds\right) d\theta \\ &\quad + \sum_{0\le\tau_k<t} \mathbf{S}\left(t,\tau_k^+\right) c_k. \end{aligned} \tag{4.2}$$

This implies that $y(\cdot,y(0))$ is a PC-mild solution of Cauchy problem (3.3) with initial value $y(0)=x_0$. Thus the uniqueness implies that $x(\cdot,x_0)=y(\cdot,y(0))=x(\cdot+T_0,x_0)$, so that $x(\cdot,x_0)$ is a T_0-periodic. □

Next, we show that P defined by (4.1) is a continuous and compact operator.

Lemma 4.2 Under the assumptions of Theorem 3.1 and [A3], P is a continuous and compact operator.

Proof. (1) Show that P is a continuous operator on X.

Let $\bar{x}$, $\bar{y} \in \Xi \subset X$, where Ξ is a bounded subset of X. Suppose $x(\cdot,\bar{x})$ and $x(\cdot,\bar{y})$ are the PC-mild solutions of Cauchy problem (3.3) corresponding to the initial value $\bar{x}$ and $\bar{y} \in X$ respectively given by

$$\begin{aligned} x(t,\bar{x}) &= \mathbf{S}(t,0)\bar{x} + \int_0^t \mathbf{S}(t,\theta) f\left(\theta, x(\theta,\bar{x}), \int_0^\theta g(\theta,s,x(s,\bar{x}))ds\right)d\theta + \sum_{0\le\tau_k<t} \mathbf{S}\left(T_0,\tau_k^+\right)c_k; \\ x(t,\bar{y}) &= \mathbf{S}(t,0)\bar{y} + \int_0^t \mathbf{S}(t,\theta) f\left(\theta, x(\theta,\bar{y}), \int_0^\theta g(\theta,s,x(s,\bar{y}))ds\right)d\theta + \sum_{0\le\tau_k<t} \mathbf{S}\left(T_0,\tau_k^+\right)c_k. \end{aligned}$$

Thus, we obtain

$$\begin{aligned} \|x(t,\bar{x})\| \le\ & M_{T_0}\|\bar{x}\| + (1+M_gT_0)M_{T_0}M_fT_0 + M_{T_0}\sum_{0\le\tau_k<T_0}\|c_k\| + M_{T_0}M_f\int_0^t \|x(\theta,\bar{x})\|d\theta \\ & + M_{T_0}M_fM_g\int_0^t\int_0^\theta \|x(s,\bar{x})\|dsd\theta, \end{aligned}$$

and

$$\begin{aligned} \|x(t,\bar{y})\| \le\ & M_{T_0}\|\bar{y}\| + (1+M_gT_0)M_{T_0}M_fT_0 + M_{T_0}\sum_{0\le\tau_k<T_0}\|c_k\| + M_{T_0}M_f\int_0^t \|x(\theta,\bar{y})\|d\theta \\ & + M_{T_0}M_fM_g\int_0^t\int_0^\theta \|x(s,\bar{y})\|dsd\theta. \end{aligned}$$

By Lemma 3.1 and elementary computation, there exist constants M_1^* and $M_2^* > 0$ such that $\|x(t,\bar{x})\| \le M_1^*$ and $\|x(t,\bar{y})\| \le M_2^*$. Let $\rho = \max\{M_1^*, M_2^*\} > 0$, then $\|x(\cdot,\bar{x})\|$, $\|x(\cdot,\bar{y})\| \le \rho$ which imply that they are locally bounded.

By assumptions [F1], [F2], [G1] and [G2], we obtain

$$\begin{aligned} \|x(t,\bar{x}) - x(t,\bar{y})\| \le\ & \|\mathbf{S}(t,0)\|\|\bar{x}-\bar{y}\| \\ & + \int_0^t \|\mathbf{S}(t,\theta)\| \left\| f\left(\theta, x(\theta,\bar{x}), \int_0^\theta g(\theta,s,x(s,\bar{x}))ds\right)\right. \\ & \left. - f\left(\theta, x(\theta,\bar{y}), \int_0^\theta g(\theta,s,x(s,\bar{y}))ds\right)\right\| d\theta \\ \le\ & M_{T_0}\|\bar{x}-\bar{y}\| + M_{T_0}L_f(\rho)\int_0^t \|x(\theta,\bar{x}) - x(\theta,\bar{y})\|d\theta \\ & + M_{T_0}L_f(\rho)L_g(\rho)\int_0^t\int_0^\theta \|x(s,\bar{x}) - x(s,\bar{y})\|dsd\theta. \end{aligned}$$

By Lemma 3.1 and elementary computation again, there exists constant $M_3^* > 0$ such that

$$\|x(t,\bar{x}) - x(t,\bar{y})\| \le M_3^*\|\bar{x}-\bar{y}\|, \text{ for all } t \in [0,T_0],$$

which implies that

$$\|P(\bar{x})-P(\bar{y})\| = \left\|x(T_0,\bar{x})-x(T_0,\bar{y})\right\| \leq M_3^*\|\bar{x}-\bar{y}\|.$$

Hence, P is a continuous operator on X.

(2) Verify that P takes a bounded set into a precompact set in X.

Let Γ is a bounded subset of X. Define $K = P\Gamma = \{P(\bar{x}) \in X \mid \bar{x} \in \Gamma\}$.

For $0 < \varepsilon \leq T_0$, define

$$K_\varepsilon = P_\varepsilon\Gamma = \mathbf{S}(T_0, T_0-\varepsilon)\left\{x(T_0-\varepsilon,\bar{x}) \mid \bar{x} \in \Gamma\right\}.$$

Next, we show that K_ε is precompact in X. In fact, for $\bar{x} \in \Gamma$ fixed, we have

$$\begin{aligned}
&\|x(T_0-\varepsilon,\bar{x})\| \\
\leq\ & \|\mathbf{S}(T_0-\varepsilon,0)\bar{x}\| + \int_0^{T_0-\varepsilon}\left\|\mathbf{S}(T_0-\varepsilon,\theta)f\left(\theta, x(\theta,\bar{x}), \int_0^\theta g(\theta,s,x(s,\bar{x}))ds\right)\right\|d\theta \\
&+ \sum_{0\leq\tau_k<T_0-\varepsilon}\left\|\mathbf{S}\left(T_0-\varepsilon,\tau_k^+\right)c_k\right\| \\
\leq\ & M_{T_0}\|\bar{x}\| + M_{T_0}M_fT_0(1+M_gT_0) + (1+M_gT_0)M_{T_0}M_fT_0\rho + M_{T_0}\sum_{0\leq\tau_k<T_0}\|c_k\|.
\end{aligned}$$

This implies that the set $\left\{x(T_0-\varepsilon,\bar{x}) \mid \bar{x} \in \Gamma\right\}$ is totally bounded.

By virtue of (5) of Lemma 2.3, $\mathbf{S}(T_0, T_0-\varepsilon)$ is a compact operator. Thus, K_ε is precompact in X.

On the other hand, for arbitrary $\bar{x} \in \Gamma$,

$$\begin{aligned}
P_\varepsilon(\bar{x}) &= \mathbf{S}(T_0,0)\bar{x} + \int_0^{T_0-\varepsilon}\mathbf{S}(T_0,\theta)f\left(\theta, x(\theta,\bar{x}), \int_0^\theta g(\theta,s,x(s,\bar{x}))ds\right)d\theta \\
&\quad + \sum_{0\leq\tau_k<T_0-\varepsilon}\mathbf{S}\left(T_0,\tau_k^+\right)c_k.
\end{aligned}$$

Thus, combined with (4.1), we have

$$\begin{aligned}
&\left\|P_\varepsilon(\bar{x})-P(\bar{x})\right\| \\
\leq\ & \left\|\int_0^{T_0-\varepsilon}\mathbf{S}(T_0,\theta)f\left(\theta, x(\theta,\bar{x}), \int_0^\theta g(\theta,s,x(s,\bar{x}))ds\right)d\theta\right. \\
&\left. -\int_0^{T_0}\mathbf{S}(T_0,\theta)f\left(\theta, x(\theta,\bar{x}), \int_0^\theta g(\theta,s,x(s,\bar{x}))ds\right)d\theta\right\| \\
&+\left\|\sum_{0\leq\tau_k<T_0-\varepsilon}\mathbf{S}\left(T_0,\tau_k^+\right)c_k - \sum_{0\leq\tau_k<T_0}\mathbf{S}\left(T_0,\tau_k^+\right)c_k\right\| \\
\leq\ & \int_{T_0-\varepsilon}^{T_0}\|\mathbf{S}(T_0,\theta)\|\left\|f\left(\theta, x(\theta,\bar{x}), \int_0^\theta g(\theta,s,x(s,\bar{x}))ds\right)\right\|d\theta + M_{T_0}\sum_{T_0-\varepsilon\leq\tau_k<T_0}\|c_k\| \\
\leq\ & M_{T_0}M_f(1+M_gT_0)(1+\rho)\varepsilon + M_{T_0}\sum_{T_0-\varepsilon\leq\tau_k<T_0}\|c_k\|.
\end{aligned}$$

It is showing that the set K can be approximated to an arbitrary degree of accuracy by a precompact set K_ε. Hence K itself is precompact set in X. That is, P takes a bounded set into a precompact set in X. As a result, P is a compact operator. □

After showing the continuity and compactness of operator P, we can follow and derive periodic PC-mild solutions for system (3.2). In the sequel, we define the following definitions. The following definitions are standard, we state them here for convenient references. Note that the uniform boundedness and uniform ultimate boundedness are not required to obtain the periodic PC-mild solutions here, so we only define the (locally) boundedness and ultimate boundedness.

Definition 4.1 We say that PC-mild solutions of Cauchy problem (3.3) are bounded if for each $B_1 > 0$, there is a $B_2 > 0$ such that $\|\bar{x}\| \leq B_1$ implies $\|x(t,\bar{x})\| \leq B_2$ for $t \geq 0$.

Definition 4.2 We say that PC-mild solutions of Cauchy problem (3.3) are locally bounded if for each $B_1 > 0$ and $k_0 > 0$, there is a $B_2 > 0$ such that $\|\bar{x}\| \leq B_1$ implies $\|x(t,\bar{x})\| \leq B_2$ for $0 \leq t \leq k_0$.

Definition 4.3 We say that PC-mild solutions of Cauchy problem (3.3) are ultimate bounded if there is a bound $B > 0$, such for each $B_3 > 0$, there is a $k > 0$ such that $\|\bar{x}\| \leq B_3$ and $t \geq k$ imply $\|x(t,\bar{x})\| \leq B$.

We also need the following results as a reference.

Lemma 4.3 (See Theorem 3.1 of [14]) Locally boundedness and ultimate boundedness implies boundedness and ultimate boundedness.

Lemma 4.4 (See [8] or Lemma 3.1 of [12], Horn's Fixed Point Theorem) Let $E_0 \subset E_1 \subset E_2$ be convex subsets of Banach space X, with E_0 and E_2 compact subsets and E_1 open relative to E_2. Let P: $E_2 \to X$ be a continuous map such that for some integer m, one has

$$P^j(E_1) \subset E_2,\ 1 \leq j \leq m-1, \quad P^j(E_1) \subset E_0,\ m \leq j \leq 2m-1,$$

then P has a fixed point in E_0.

Theorem 4.1 Assumptions [A1], [A2], [A3], [B], [C], [F] and [G] hold. If the PC-mild solutions of Cauchy problem (3.3) are ultimate bounded, then system (3.2) has a T_0-periodic PC-mild solution.

Proof. By Theorem 3.1 and Definition 4.2, Cauchy problem (3.3) corresponding to the initial value $x(0) = \bar{x}$ has a PC-mild solution $x(\cdot,\bar{x})$ which is locally bounded. From ultimate boundedness and Lemma 4.3, $x(\cdot,\bar{x})$ is bounded. Next, let $B > 0$ be the bound in the definition of ultimate boundedness. Then by boundedness, there is a $B_1 > B$ such that $\|\bar{x}\| \leq B$ implies $\|x(t,\bar{x})\| \leq B_1$ for $t \geq 0$. Furthermore, there is a $B_2 > B_1$ such that $\|\bar{x}\| \leq B_1$ implies $\|x(t,\bar{x})\| \leq B_2$ for $t \geq 0$. Now, using ultimate boundedness again, there is a positive integer m such that $\|\bar{x}\| \leq B_1$ implies $\|x(t,\bar{x})\| \leq B$ for $t \geq (m-2)T_0$.

Define $y(\cdot,y(0)) = x(\cdot+T_0,\bar{x})$, then $y(0) = x(T_0,\bar{x}) = P(\bar{x})$. From (4.2) in Lemma 4.1, we obtain $P(y(0)) = y(T_0,y(0)) = x(2T_0,\bar{x})$. Thus, $P^2(\bar{x}) = P(P(\bar{x})) = P(y(0)) = x(2T_0,\bar{x})$. Suppose there exists integer $m-1$ such that $P^{m-1}(\bar{x}) = x((m-1)T_0,\bar{x})$. By induction we arrive at

$$P^m(\bar{x}) = P^{m-1}(P(\bar{x})) = P^{m-1}(y(0)) = y((m-1)T_0,y(0)) = x(mT_0,\bar{x}).$$

Thus, we obtain

$$\left\|P^{j-1}(\bar{x})\right\| = \left\|x\big((j-1)T_0,\bar{x}\big)\right\| < B_2,\ j = 1,2,\cdots,m-1 \text{ and } \|\bar{x}\| < B_1; \tag{4.3}$$

$$\left\|P^{j-1}(\bar{x})\right\| = \left\|x\big((j-1)T_0,\bar{x}\big)\right\| < B,\ j \geq m \text{ and } \|\bar{x}\| < B_1. \tag{4.4}$$

It comes from Lemma 4.2 that $P(\bar{x}) = x(T_0,\bar{x})$ on X is compact. Now let

$$\begin{aligned} H &= \{\bar{x} \in X\colon \|\bar{x}\| < B_2\},\ E_2 = \text{cl.}\big(\text{conv}(P(H))\big), \\ W &= \{\bar{x} \in X\colon \|\bar{x}\| < B_1\},\ E_1 = W \cap E_2, \\ G &= \{\bar{x} \in X\colon \|\bar{x}\| < B\},\ E_0 = \text{cl.}\big(\text{conv}(P(G))\big), \end{aligned}$$

where $\text{conv}(Y)$ is the convex hull of the set Y defined by

$$\text{conv}(Y) = \left\{\sum_{i=1}^{n} \lambda_i y_i \mid n \geq 1,\ y_i \in Y,\ \lambda_i,\ \sum_{i=1}^{n} \lambda_i = 1\right\},$$

and cl. denotes the closure. Then we see that $E_0 \subset E_1 \subset E_2$ are convex subset of X with E_0, E_2 compact subsets and E_1 open relative to E_2 and from (4.3) and (4.4), one has

$$P^j(E_1) \subset P^j(W) = PP^{j-1}(W) \subset P(H) \subset E_2,\ j = 1,2,\cdots,m-1;$$

$$P^j(E_1) \subset P^j(W) = PP^{j-1}(W) \subset P(G) \subset E_0,\ j = m,m+1,\cdots,2m-1.$$

We see that P: $E_2 \to X$ be continuous map continuous from Lemma 4.2. Consequently, from Horn's fixed point theorem, we know that the operator P has a fixed point $x_0 \in E_0 \subset X$. By Lemma 4.1, we know that the PC-mild solution $x(\cdot,x_0)$ of Cauchy problem (3.3) corresponding to the initial value $x(0) = x_0$, is just T_0-periodic. Therefore $x(\cdot,x_0)$ is a T_0-periodic PC-mild solution of system (3.2). This proves the theorem. □

At last, an application in impulsive periodic population evolution equation is given to illustrate our theory. Consider the following problem:

$$\begin{cases} \frac{\partial}{\partial t}x(r,t) + \sin t \frac{\partial}{\partial r}x(r,t) = -0.2\sin t x(r,t) + x(r,t) \\ \qquad\qquad + \int_0^t \psi(s)(1+\sin(t-s))\sqrt{x^2(r,s)+1}ds, \\ \qquad\qquad r \in \Omega = (0,1), t,s \in (0,2\pi] \setminus \{\frac{1}{2}\pi,\pi,\frac{3}{2}\pi\}, \\ \Delta x(r,\tau_i) = \begin{cases} 0.05Ix(r,\tau_i), i = 1, \\ -0.05Ix(r,\tau_i), i = 2, \\ 0.05Ix(r,\tau_i), i = 3, \end{cases} \quad r \in \Omega, \tau_i = \frac{i}{2}\pi, i = 1,2,3, \\ x(r,0) = x_0(r), \quad t > 0, \\ x(0,t) = \varphi_0(t), \quad t \in (0,2\pi) \setminus \{\frac{1}{2}\pi,\pi,\frac{3}{2}\pi\}, \\ x(r,0) = x(r,2\pi). \end{cases} \tag{4.5}$$

where I is identity operator.

Define $X = L^1(0,1)$, and $A(t)\varphi(r) = -\sin t\left(\frac{d}{dr}\varphi(r) - 0.2\varphi(r)\right)$, for arbiartry $\varphi \in D(A)$, and $D(A) = \{\varphi \mid \varphi, A(t)\varphi \in L^1(0,1); \varphi(0) = \varphi_0\}$. By virtue of Theorem 1 of [27], $A(t)(t >$

1) can determine a compact evolutionary process $\{U(t,\theta),\ t \geq \theta \geq 0\}$. Define $x(\cdot)(r) = x(r,\cdot)$, $\sin(\cdot)(r) = \sin(r,\cdot)$ and $f\left(\cdot, x(\cdot), \int_0^{\cdot} g(\cdot,s,x)ds\right)(r) = x(\cdot)(r) + \int_0^t \psi(s)(1+\sin(\cdot - s))\sqrt{x^2(\cdot)+1}ds(r)$ where $\psi(\cdot+2\pi) = \psi(\cdot) \in L^1_{loc}([0,+\infty);X)$ and $\int_0^{2\pi} \psi(s)(1+\sin(t-s))\sqrt{x^2(t)+1}ds = 0$,

$$B_i = \begin{cases} 0.05I, i=1, \\ -0.05I, i=2, \\ 0.05I, i=3. \end{cases}$$

Thus problem (4.5) can be rewritten as

$$\begin{cases} \dot{x}(t) = A(t)x(t) + f\left(t, x, \int_0^t g(t,s,x)ds\right), & t \in (0,2\pi] \setminus \{\frac{1}{2}\pi, \pi, \frac{3}{2}\pi\}, \\ \Delta x\left(\frac{i}{2}\pi\right) = B_i x\left(\frac{i}{2}\pi\right), & i = 1,\ 2,\ 3, \\ x(0) = x(2\pi). \end{cases} \tag{4.6}$$

If the *PC*-mild solutions of Cauchy problem (4.6) are ultimate bounded, then all the assumptions in Theorem 4.1 are met, our results can be used to system (4.5). That is, problem (4.5) has a 2π-periodic *PC*-mild solution $x_{2\pi}(\cdot,y) \in PC_{2\pi}\left([0+\infty);L^1(0,1)\right)$, where

$$PC_{2\pi}([0,+\infty);L^1(0,1)) \equiv \left\{x \in PC\left([0,+\infty);L^1(0,1)\right) \mid x(t) = x(t+2\pi), t \geq 0\right\};$$

Acknowledgments

The authors thanks the referees for their careful reading of the manuscript and insightful comments.
The authors acknowledge support from the National Natural Science Foundation of China (No.10961009), Introducing Talents Foundation for the Doctor of Guizhou University (2009, No.031) and Youth Teachers Natural Science Foundation of Guizhou University (2009, No.083).

References

[1] N. U. Ahmed, *Semigroup theory with applications to system and control*, Longman Scientific Technical, New York, 1991.

[2] N. U. Ahmed, Some remarks on the dynamics of impulsive systems in Banach space, *Mathematical Anal.* **8** (2001), pp. 261-274.

[3] N. U Ahmed, K. L. Teo and S. H. Hou, Nonlinear impulsive systems on infinite dimensional spaces, *Nonlinear Anal.* **54** (2003), pp 907-925.

[4] D. D. Bainov, P. S. Simeonov, *Impulsive differential equations: periodic solutions and applications*, New York, Longman Scientific and Technical Group. Limited, 1993.

[5] M. Benchohra, J. Henderson and S. K. Ntouyas, Impulsive differential equations and inclusions, *Contemporary Mathematics and Its Applications* Vol. 2, Hindawi Publishing Corporation, New York, 2006.

[6] D. D. Bainov, S. G. Hristova, Integral inequalities of Gronwall type for piecewise continuous functions, *J. Applied Mathematics and Stochastic Analysis* **10** (1997), pp 89-94.

[7] J. M. Cushing, Periodic time-dependent predator-prey system, *SIAM J. Appl. Math.* **32** (1977), pp 82-95.

[8] W. A. Horn, Some fixed point theorems for compact maps and flows in Banach spaces, *Transactions of AMS* **149** (1970), pp 391-404.

[9] V. Lakshmikantham, D. D. Bainov and P. S. Simeonov, *Theory of impulsive differential equations*, World Scientific, Singapore-London, 1989.

[10] Lishan Liu, Congxin Wu and Fei Guo, A unique solution of initial value problems for first-order impulsive integro-differential equations of mixed type in Banach spaces, *J. Math. Anal. Appl.* **275** (2002), pp 369-385.

[11] X. Liu, Impulsive stabilization and applications to population growth models, *Rocky Mountain J. Math.* **25** (1995), pp 381-395.

[12] J. Liu, Bounded and periodic solutions of finite delay evolution equations, *Nonlinear Analysis* **34** (1998), pp 101-111.

[13] J. Liu, Nonlinear impulsive evolution equations, *Dynamices of Continuous, Discrete and Impulsive Systems* **6** (1999), pp 77-85.

[14] J. Liu, Toshiki Naito, Nguyen Van Minh, Bounded and periodic solutions of infinite delay evolution equations, *J. Math. Anal. Appl.* **286** (2003), pp 705-712.

[15] J. Y. Park, Y. C. Kwunm J. M. Jeong, Existence of periodic solutions for delay evolution integrodifferential equations, *J. Math. and Comput. Model.* **65** (2004), pp 597-603.

[16] G. F. Webb, Logistic models of structured population growth, *Comp. and Maths. With Appls.* **12A** (1986), pp 527-529.

[17] K. Kunisch, W. Schappacher, G. F. Webb, Nonlinear age-dependent population dynamic with random diffusion, *Comp. and Maths. With Appls.* **11** (1985), pp 155-173.

[18] J. R. Wang, X. Xiang, and W. Wei, Linear impulsive periodic system with time-varying generating operators on Banach space, *Advances in Difference Equations* **2007** (2007), Article ID 26196, pp 1-16.

[19] P. Sattayatham, S. Tangmanee and Wei Wei, On periodic solution of nonlinear evolution equations in Banach spaces, *J. Math. Anal. Appl.* **276** (2002), pp 98-108.

[20] X. Xiang and N. U. Ahmed, Existence of periodic solutions of semilinear evolution equations with time lags, *Nonlinear Anal.* **18** (1992), pp 1063-1070.

[21] W. Wei, X. Xiang and Y. Peng, Nonlinear impulsive integro-differential equation of mixed type and optimal controls, *Optimization* **55** (2006), pp 141-156.

[22] X. Xiang and W. Wei, Mild solution for a class of nonlinear impulsive evolution inclusion on Banach space, *Southeast Asian Bulletin of Mathematics* **30** (2006), pp 367-376.

[23] X. Xiang, W. Wei and Y. Jiang, Strongly nonlinear impulsive system and necessary conditions of optimality, *Dynamics of Continuous, Discrete and Impulsive Systems* **12** (2005), pp 811-824.

[24] X. L. Yu, X. Xiang and W. Wei, Solution bundle for class of impulsive differential inclusions on Banach spaces, *J. Math. Anal. Appl.* **327** (2007), pp 220-232.

[25] T. Yang, *Impulsive control theory*, Springer-Verlag Berlin Heidelberg, 2001.

[26] S. H. Zhang, G. T. Zhu, The solution and asymptotity of nonlinear population evolution equation, *ACTA Mathematicae Applicate Sinica* **14** (1991), pp 289-295.

[27] S. H. Zhang, On the mild solution and weak solution of nonlinear population evolution equation, *Applied Mathematics A Journal of Chinese University* **5** (1991), pp 476-483.