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EXISTENCE OF SQUARE-MEAN ALMOST PERIODIC SOLUTIONS TO SOME STOCHASTIC HYPERBOLIC DIFFERENTIAL EQUATIONS WITH INFINITE DELAY

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Abstract

In this paper, we make extensive use of the well-known Krasnoselskii fixed point theorem to obtain the existence of square-mean almost periodic solutions to some classes of hyperbolic stochastic evolution equations with infinite delay. Next, the existence of square-mean almost periodic solutions to not only the heat equation but also to a boundary value problem with infinite delay arising in control systems are studied.

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1 Introduction

Let $(\mathbb{H}, \|\cdot\|, \langle\cdot,\cdot\rangle)$ be a real Hilbert space which is separable and let $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete probability space equipped with a normal filtration $\{\mathcal{F}_t : t \in \mathbb{R}\}$, that is, a right-continuous, increasing family of sub σ-algebras of $\mathcal{F}$.

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Throughout the rest of the paper, if $\mathcal{A} : D(\mathcal{A}) \subset \mathbb{H} \mapsto \mathbb{H}$ is a linear operator, we then define the operator $A : D(A) \subset L^2(\Omega, \mathbb{H}) \mapsto L^2(\Omega, \mathbb{H})$ as follows: $X \in D(A)$ and $AX = Y$ if and only if $X, Y \in L^2(\Omega, \mathbb{H})$ and $\mathcal{A}X(\omega) = Y(\omega)$ for all $\omega \in \Omega$.

Let $\mathcal{A} : D(\mathcal{A}) \subset \mathbb{H} \mapsto \mathbb{H}$ be a sectorial linear operator. For $\alpha \in (0,1)$, let $\mathbb{H}_\alpha$ denote the intermediate Banach space between $D(\mathcal{A})$ and $\mathbb{H}$. Examples of those $\mathbb{H}_\alpha$ include, among others, the fractional spaces $D((-\mathcal{A})^\alpha)$, the real interpolation spaces $D_\mathcal{A}(\alpha, \infty)$ due to Lions and Peetre, and the Hölder spaces $D_\mathcal{A}(\alpha)$, which coincide with the continuous interpolation spaces that both Da Prato and Grisvard introduced in the literature.

In this paper we study the existence of a square-mean mild solution for the following classes of stochastic hyperbolic evolution equations with infinite delay in the form

$$\begin{aligned} d\Big[X(\omega,t) + f_1(t, X_t(\omega))\Big] &= \Big[\mathcal{A}X(\omega,t) + f_2(t, X_t(\omega))\Big]dt \\ &+ f_3(t, X_t(\omega))d\mathbb{W}(\omega,t), \text{ for all } t \in \mathbb{R}, \ \omega \in \Omega, \end{aligned} \tag{1.1}$$

where $\mathcal{A} : D(\mathcal{A}) \subset \mathbb{H} \to \mathbb{H}$ is a sectorial linear operator whose corresponding analytic semigroup is hyperbolic, that is, $\sigma(\mathcal{A}) \cap i\mathbb{R} = \emptyset$, the history $X_t : (-\infty, 0] \mapsto \mathbb{H}$ defined by $X_t(\tau) = X(t + \tau)$ belongs to some abstract phase space $\mathfrak{B}$, which is defined axiomatically, and $f_1 : \mathbb{R} \times \mathfrak{B} \to \mathbb{H}_\beta$ $(0 < \alpha < \frac{1}{2} < \beta < 1)$ and $f_i(i = 2, 3) : \mathbb{R} \times \mathfrak{B} \to \mathbb{H}$ are jointly continuous functions.

To analyze Eq. (1.1), our strategy will consist of studying the existence of square-mean almost periodic solutions to the corresponding class of stochastic differential equations of the form

$$d\Big[X(t) + F_1(t, X_t)\Big] = \Big[AX(t) + F_2(t, X_t)\Big]dt + F_3(t, X_t)d\mathbb{W}(t) \text{ for all } t \in \mathbb{R}, \tag{1.2}$$

where $A : D(A) \subset L^2(\Omega, \mathbb{H}) \to L^2(\Omega, \mathbb{H})$ is a sectorial linear operator whose corresponding analytic semigroup $(T(t))_{t \geq 0}$ is hyperbolic, that is, $\sigma(A) \cap i\mathbb{R} = \emptyset$, the functions defined by $F_1 : \mathbb{R} \times L^2(\Omega, \mathfrak{B}) \to L^2(\Omega, \mathbb{H}_\beta)$ $(0 < \alpha < \frac{1}{2} < \beta < 1)$, $F_i(i = 2, 3) : \mathbb{R} \times L^2(\Omega, \mathfrak{B}) \to L^2(\Omega, \mathbb{H})$ are jointly continuous satisfying some additional assumptions, and $\mathbb{W}(t)$ is a $\mathbb{R}$-valued Brownian motion with the real number line as time parameter.

The literature related to functional differential equations with infinite delay on Banach spaces is vast, we refer the reader for instance to the following papers [9, 10, 11, 12, 17, 18, 19, 23]. The existence of almost periodic (respectively, periodic) solutions to autonomous stochastic differential equations has been studied by many authors, see, e.g., [2], [8], and [22] and the references therein. Though the existence of square-mean almost periodic solutions to Eq. (1.2) in the case when A is sectorial is an important topic with some interesting applications, which is still an untreated question and constitutes the main motivation of the present paper. Among other things, we will make extensive use of the method of analytic semigroups associated with sectorial operators and the well-known Krasnoselskii fixed point theorem to derive sufficient conditions for the existence of a square-mean almost periodic solution to Eq. (1.2). To illustrate our abstract results, the existence of square-mean almost periodic solutions to not only the heat equation but also a boundary value problem with infinite delay arising in control systems are studied.

2 Square-Mean Almost periodic Stochastic Processes

For details on this section, we refer the reader to [2, 4] and the references therein. In this paper, we assume that $(\Omega, \mathcal{F}, \mathbf{P})$ is a probability space. Let $\mathbb{W}$ be a Brownian motion on $\mathbb{R}$. It is worth mentioning that $\mathbb{W}$ can be obtained as follows: let $\{\mathbb{W}_i(t),\, t \in \mathbb{R}_+\},\;\; i = 1,2$, be independent $\mathbb{R}$-valued Brownian motions, then

$$\mathbb{W}(t) = \begin{cases} \mathbb{W}_1(t) & \text{if } t \geq 0, \\ \\ \mathbb{W}_2(-t) & \text{if } t \leq 0, \end{cases}$$

is a Brownian motion with the real number line as time parameter. We then let $\mathcal{F}_t = \sigma\{\mathbb{W}(s),\, s \leq t\}$.

Let $(\mathbb{B}, \|\cdot\|)$ be a Banach space. This setting requires the following preliminary definitions.

Definition 2.1. A stochastic process $X : \mathbb{R} \to L^2(\Omega;\mathbb{B})$ is said to be continuous whenever

$$\lim_{t \to s} \mathbf{E}\Big\|X(t) - X(s)\Big\|^2 = 0.$$

Definition 2.2. A continuous stochastic process $X : \mathbb{R} \to L^2(\Omega;\mathbb{B})$ is said to be square-mean almost periodic if for each $\varepsilon > 0$ there exists $l(\varepsilon) > 0$ such that any interval of length $l(\varepsilon)$ contains at least a number τ for which

$$\sup_{t \in \mathbb{R}} \mathbf{E}\Big\|X(t+\tau) - X(t)\Big\|^2 < \varepsilon.$$

The collection of all stochastic processes $X : \mathbb{R} \to L^2(\Omega;\mathbb{B})$ which are square-mean almost periodic is then denoted by $AP(\mathbb{R};L^2(\Omega;\mathbb{B}))$.

The next lemma provides some properties of square-mean almost periodic processes.

Lemma 2.3. *If X belongs to $AP(\mathbb{R};L^2(\Omega;\mathbb{B}))$, then*

(i) the mapping $t \to \mathbf{E}\Big\|X(t)\Big\|^2$ is uniformly continuous;

(ii) there exists a constant $M > 0$ such that $\mathbf{E}\Big\|X(t)\Big\|^2 \leq M$, for all $t \in \mathbb{R}$.

Let $\mathbf{CUB}(\mathbb{R}; L^2(\Omega;\mathbb{B}))$ denote the collection of all stochastic processes $X : \mathbb{R} \mapsto L^2(\Omega;\mathbb{B})$, which are continuous and uniformly bounded. It is then easy to check that $\mathbf{CUB}(\mathbb{R}; L^2(\Omega;\mathbb{B}))$ is a Banach space when it is equipped with the norm:

$$\Big\|X\Big\|_\infty = \sup_{t \in \mathbb{R}} \Big(\mathbf{E}\|X(t)\|^2\Big)^{\frac{1}{2}}.$$

Lemma 2.4. $AP(\mathbb{R};L^2(\Omega;\mathbb{B})) \subset \mathbf{CUB}(\mathbb{R};L^2(\Omega;\mathbb{B}))$ *is a closed subspace.*

In view of the above, the space $AP(\mathbb{R};L^2(\Omega;\mathbb{B}))$ of square-mean almost periodic processes equipped with the norm $\Big\|\cdot\Big\|_\infty$ is a Banach space.

Let $(\mathbb{B}_1,\Big\|\cdot\Big\|_{\mathbb{B}_1})$ and $(\mathbb{B}_2,\Big\|\cdot\Big\|_{\mathbb{B}_2})$ be Banach spaces and let $L^2(\Omega;\mathbb{B}_1)$ and $L^2(\Omega;\mathbb{B}_2)$ be their corresponding L^2-spaces, respectively.

Definition 2.5. A function $F:\mathbb{R}\times L^2(\Omega;\mathbb{B}_1)\to L^2(\Omega;\mathbb{B}_2))$, $(t,Y)\mapsto F(t,Y)$, which is jointly continuous, is said to be square-mean almost periodic in $t\in\mathbb{R}$ uniformly in $Y\in\mathbb{K}$ where $\mathbb{K}\subset L^2(\Omega;\mathbb{B}_1)$ is any compact subset if for any $\varepsilon>0$, there exists $l(\varepsilon,\mathbb{K})>0$ such that any interval of length $l(\varepsilon,\mathbb{K})$ contains at least a number τ for which

$$\sup_{t\in\mathbb{R}}\mathbf{E}\Big\|F(t+\tau,Y)-F(t,Y)\Big\|^2_{\mathbb{B}_2}<\varepsilon$$

for each stochastic process $Y:\mathbb{R}\to\mathbb{K}$.

Theorem 2.6. *Let $F:\mathbb{R}\times L^2(\Omega;\mathbb{B}_1)\to L^2(\Omega;\mathbb{B}_2)$, $(t,Y)\mapsto F(t,Y)$ be a square-mean almost periodic process in $t\in\mathbb{R}$ uniformly in $Y\in\mathbb{K}$, where $\mathbb{K}\subset L^2(\Omega;\mathbb{B}_1)$ is compact. Suppose that F is Lipschitz in the following sense:*

$$\mathbf{E}\Big\|F(t,Y)-F(t,Z)\Big\|^2_{\mathbb{B}_2}\le M\,\mathbf{E}\Big\|Y-Z\Big\|^2_{\mathbb{B}_1}$$

for all $Y,Z\in L^2(\Omega;\mathbb{B}_1)$ and for each $t\in\mathbb{R}$, where $M>0$. Then for any square-mean almost periodic process $\Phi:\mathbb{R}\to L^2(\Omega;\mathbb{B}_1)$, the stochastic process $t\mapsto F(t,\Phi(t))$ is square-mean almost periodic.

The present setting requires the following composition of square-mean almost periodic processes.

Theorem 2.7. *Let $F:\mathbb{R}\times L^2(\Omega;\mathbb{B}_1)\to L^2(\Omega;\mathbb{B}_2)$, $(t,Y)\mapsto F(t,Y)$ be a square-mean almost periodic process in $t\in\mathbb{R}$ uniformly in $Y\in K$, where $K\subset L^2(\Omega;\mathbb{B}_1)$ is any compact subset. Suppose that $F(t,\cdot)$ is uniformly continuous on bounded subsets $K'\subset L^2(\Omega;\mathbb{B}_1)$ in the following sense: for all $\varepsilon>0$ there exists $\delta_\varepsilon>0$ such that $X,Y\in K'$ and $\mathbf{E}\Big\|X-Y\Big\|^2_{\mathbb{B}_1}<\delta_\varepsilon$, then*

$$\mathbf{E}\Big\|F(t,Y)-F(t,Z)\Big\|^2_{\mathbb{B}_2}<\varepsilon,\ \ \forall t\in\mathbb{R}.$$

Then for any square-mean almost periodic process $\Phi:\mathbb{R}\to L^2(\Omega;\mathbb{B}_1)$, the stochastic process $t\mapsto F(t,\Phi(t))$ is square-mean almost periodic.

Proof. Since $\Phi:\mathbb{R}\to L^2(\Omega;\mathbb{B}_1)$ is a square-mean almost periodic process, for all $\varepsilon>0$ there exists $l_\varepsilon>0$ such that every interval of length $l_\varepsilon>0$ contains a τ with the property that

$$\mathbf{E}\Big\|\Phi(t+\tau)-\Phi(t)\Big\|^2_{\mathbb{B}_1}<\varepsilon,\ \ \forall t\in\mathbb{R}. \tag{2.1}$$

In addition, $\Phi:\mathbb{R}\to L^2(\Omega;\mathbb{B}_1)$ is bounded, that is, $\sup_{t\in\mathbb{R}}\mathbf{E}\Big\|\Phi(t)\Big\|^2_{\mathbb{B}_1}<\infty$. Let $K''\subset L^2(\Omega;\mathbb{B}_1)$ be a bounded subset such that $\Phi(t)\in K''$ for all $t\in\mathbb{R}$.

Now

$$\begin{aligned}\mathbf{E}\Big\|F(t+\tau,\Phi(t+\tau))-F(t,\Phi(t))\Big\|_{\mathbb{B}_2}^2 &\leq \mathbf{E}\Big\|F(t+\tau,\Phi(t+\tau))-F(t+\tau,\Phi(t))\Big\|_{\mathbb{B}_2}^2\\ &+ \mathbf{E}\Big\|F(t+\tau,\Phi(t))-F(t,\Phi(t))\Big\|_{\mathbb{B}_2}^2.\end{aligned}$$

Taking into account Eq. (2.1) (take $\delta_\varepsilon = \varepsilon$) and using the uniform continuity of F on bounded subsets of $L^2(\Omega;\mathbb{B}_1)$ it follows that

$$\sup_{t\in\mathbb{R}}\mathbf{E}\Big\|F(t+\tau,\Phi(t+\tau))-F(t+\tau,\Phi(t))\Big\|_{\mathbb{B}_2}^2 < \frac{\varepsilon}{2}. \tag{2.2}$$

Similarly, using the square-mean almost periodicity of F it follows that

$$\sup_{t\in\mathbb{R}}\mathbf{E}\Big\|F(t+\tau,\Phi(t))-F(t,\Phi(t))\Big\|_{\mathbb{B}_2}^2 < \frac{\varepsilon}{2}. \tag{2.3}$$

Combing Eq. (2.2) and Eq. (2.3) one obtains that

$$\sup_{t\in\mathbb{R}}\mathbf{E}\Big\|F(t+\tau,\Phi(t+\tau))-F(t,\Phi(t))\Big\|_{\mathbb{B}_2}^2 < \varepsilon,$$

and hence the stochastic process $t\mapsto F(t,\Phi(t))$ is square-mean almost periodic. □

3 Sectorial Linear Operators

In this section, we introduce some notations and collect some preliminary results from Diagana [7] that will be used later. If $\mathcal{A}$ is a linear operator on $\mathbb{H}$, then $\rho(\mathcal{A})$, $\sigma(\mathcal{A})$, $D(\mathcal{A})$, $\ker(\mathcal{A})$, $R(\mathcal{A})$ stand for the resolvent set, spectrum, domain, kernel, and range of $\mathcal{A}$. If $\mathbb{B}_1,\mathbb{B}_2$ are Banach spaces, then the notation $B(\mathbb{B}_1,\mathbb{B}_2)$ stands for the Banach space of bounded linear operators from $\mathbb{B}_1$ into $\mathbb{B}_2$. When $\mathbb{B}_1=\mathbb{B}_2$, this is simply denoted $B(\mathbb{B}_1)$.

Definition 3.1. A linear operator $\mathcal{A}: D(\mathcal{A})\subset\mathbb{H}\to\mathbb{H}$ (not necessarily densely defined) is said to be sectorial if the following hold: there exist constants $\zeta\in\mathbb{R}$, $\theta\in\left(\frac{\pi}{2},\pi\right)$, and $M>0$ such that $S_{\theta,\zeta}\subset\rho(\mathcal{A})$,

$$\begin{aligned}&S_{\theta,\zeta} := \{\lambda\in\mathbb{C}:\lambda\neq\zeta,\ |\arg(\lambda-\zeta)|<\theta\}\\ &\text{and } \|R(\lambda,\mathcal{A})\|\leq\frac{M}{|\lambda-\zeta|},\ \lambda\in S_{\theta,\zeta}\end{aligned}$$

where $R(\lambda,\mathcal{A})=(\lambda I-\mathcal{A})^{-1}$ for each $\lambda\in\rho(\mathcal{A})$.

Remark 3.2. If the operator $\mathcal{A}$ is sectorial, then it generates an analytic semigroup $(T(t))_{t\geq 0}$, which maps $(0,\infty)$ into $B(\mathbb{H})$ and such that there exist constants M_0, $M_1>0$ such that

$$\|T(t)\|\leq M_0e^{\zeta t},\ t>0 \tag{3.1}$$
$$\|t(\mathcal{A}-\zeta I)T(t)\|\leq M_1e^{\zeta t},\ t>0 \tag{3.2}$$

Definition 3.3. A semigroup $(T(t))_{t\geq 0}$ is hyperbolic, that is, there exist a projection P and constants M, $\delta > 0$ such that $T(t)$ commutes with P, $\mathrm{Ker}(P)$ is invariant with respect $T(t)$, $T(t) : R(Q) \to R(Q)$ is invertible, and

$$\|T(t)Px\| \leq Me^{-\delta t}\|x\|, \ t > 0, \tag{3.3}$$

$$\|T(t)Qx\| \leq Me^{\delta t}\|x\|, \ t \leq 0, \tag{3.4}$$

where $Q := I - P$ and, for $t \leq 0$, $T(t) := (T(-t))^{-1}$.

Recall that the analytic semigroup $(T(t))_{t\geq 0}$ associated with the linear operator $\mathcal{A}$ is hyperbolic if and if $\sigma(\mathcal{A}) \cap i\mathbb{R} = \emptyset$.

Definition 3.4. Let $\alpha \in (0,1)$. A Banach space $(\mathbb{H}_\alpha, \|\cdot\|_\alpha)$ is said to be an intermediate space between $D(\mathcal{A})$ and $\mathbb{H}$, or a space of class $\mathcal{J}_\alpha$, if $D(\mathcal{A}) \subset \mathbb{H}_\alpha \subset \mathbb{H}$ and there is a constant $c > 0$ such that

$$\|x\|_\alpha \leq c\|x\|^{1-\alpha}\|x\|^\alpha_{[D(\mathcal{A})]}, \ x \in D(\mathcal{A}), \tag{3.5}$$

where $\|\cdot\|_{[D(\mathcal{A})]}$ is the graph norm of $\mathcal{A}$.

Here, $\|u\|_{[D(\mathcal{A})]} = \|u\| + \|\mathcal{A}u\|$, for each $u \in D(\mathcal{A})$

Concrete examples of $\mathbb{H}_\alpha$ include $D((-\mathcal{A})^\alpha)$ for $\alpha \in (0,1)$, the domains of the fractional powers of $\mathcal{A}$, the real interpolation spaces $D_{\mathcal{A}}(\alpha, \infty)$, $\alpha \in (0,1)$, defined as the space of all $x \in \mathbb{H}$ such that

$$[x]_\alpha = \sup_{0 \leq t \leq 1} \|t^{1-\alpha}(\mathcal{A} - \zeta I)\, e^{-\zeta t}T(t)x\| < \infty,$$

with the norm

$$\|x\|_\alpha = \|x\| + [x]_\alpha,$$

and the abstract Holder spaces $D_{\mathcal{A}}(\alpha) := \overline{D(\mathcal{A})}^{\|\cdot\|_\alpha}$.

Lemma 3.5. *[6, 7] For the hyperbolic analytic semigroup $(T(t))_{t\geq 0}$, there exist constants $C(\alpha) > 0, \delta > 0, M(\alpha) > 0$, and $\gamma > 0$ such that*

$$\|T(t)Qx\|_\alpha \leq c(\alpha)e^{\delta t}\|x\| \text{ for } t \leq 0, \tag{3.6}$$

$$\|T(t)Px\|_\alpha \leq M(\alpha)t^{-\alpha}e^{-\gamma t}\|x\| \text{ for } t > 0. \tag{3.7}$$

The next Lemma is crucial for the rest of the paper. A version of it in a general Banach space is due to Diagana [6, 7].

Lemma 3.6. *[6, 7] Let $0 < \alpha < \beta < 1$. For the hyperbolic analytic semigroup $(T(t))_{t\geq 0}$, there exist constants $c > 0, \delta > 0$, and $\gamma > 0$ such that*

$$\|\mathcal{A}T(t)Qx\|_\alpha \leq n(\alpha,\beta)\, e^{\delta t}\|x\| \leq n'(\alpha,\beta)\, e^{\delta t}\|x\|_\beta, \text{ for } t \leq 0 \tag{3.8}$$

$$\|\mathcal{A}T(t)Px\|_\alpha \leq M(\alpha)\, t^{-\alpha}e^{-\gamma t}\|x\| \leq M'(\alpha)\, t^{-\alpha}e^{-\gamma t}\|x\|_\beta, \text{ for } t > 0. \tag{3.9}$$

4 The Phase Space $\mathfrak{B}_\alpha$ $(0 < \alpha < 1)$

In this work we will employ an axiomatic definition of the phase space $\mathfrak{B}_\alpha$ $(0 < \alpha < 1)$, which is similar to the one utilized in [12]. More precisely, $\mathfrak{B}_\alpha$ stands for a vector space of functions mapping $(-\infty, 0]$ into $\mathbb{H}_\alpha$ endowed with a seminorm $\|\cdot\|_{\mathfrak{B}_\alpha}$ such that the next assumptions hold.

(A) If $x : (-\infty, \sigma + a) \mapsto \mathbb{H}_\alpha$, $a > 0$, $\sigma \in \mathbb{R}$, is continuous on $[\sigma, \sigma + a)$ and $x_\sigma \in \mathfrak{B}_\alpha$, then for every $t \in [\sigma, \sigma + a)$ the following hold:

  (i) x_t is in $\mathfrak{B}_\alpha$;

  (ii) $\|x(t)\|_\alpha \leq H\|x_t\|_{\mathfrak{B}_\alpha}$;

  (iii) $\|x_t\|_{\mathfrak{B}_\alpha} \leq K(t-\sigma)\sup\{\|x(s)\|_\alpha : \sigma \leq s \leq t\} + M(t-\sigma)\|x_\sigma\|_{\mathfrak{B}_\alpha}$,

  where $H > 0$ is a constant; $K, M : [0, \infty) \mapsto [1, \infty)$, K is continuous, M is locally bounded and H, K, M are independent of $x(\cdot)$.

(A1) For the function $x(\cdot)$ appearing in (A), its corresponding history $t \to x_t$ is continuous from $[\sigma, \sigma + a)$ into $\mathfrak{B}_\alpha$.

(B) The space $\mathfrak{B}_\alpha$ is complete.

(C2) If $(y_n)_{n\in\mathbb{N}}$ is a uniformly bounded sequence in $C((-\infty, 0], \mathbb{H}_\alpha)$ given by functions with compact support and $y_n \to y$ in the compact-open topology, then $y \in \mathfrak{B}_\alpha$ and $\|y_n - y\|_{\mathfrak{B}_\alpha} \to 0$ as $n \to \infty$.

In what follows, we let $\mathfrak{B}_{0,\alpha} = \{y \in \mathfrak{B}_\alpha : y(0) = 0\}$.

Definition 4.1. Let $S(t) : \mathfrak{B}_\alpha \to \mathfrak{B}_\alpha$ be the C_0-semigroup defined by $S(t)y(\theta) = y(0)$ on $[-\infty, 0]$ and $S(t)y(\theta) = y(t + \theta)$ on $(-\infty, -t]$. The phase space $\mathfrak{B}_\alpha$ is called a fading memory if $\|S(t)y\|_{\mathfrak{B}_\alpha} \to 0$ as $t \to \infty$ for every $y \in \mathfrak{B}_{0,\alpha}$. Now, $\mathfrak{B}_\alpha$ is called uniform fading memory whenever $\|S(t)\|_{B(\mathfrak{B}_{0,\alpha})} \to 0$ as $t \to \infty$.

Remark 4.2. In this paper we suppose $\mathfrak{L} > 0$ is such that $\|y\|_{\mathfrak{B}_\alpha} \leq \mathfrak{L}\sup_{\theta\leq 0}\|y(\theta)\|_\alpha$ for each $y \in \mathfrak{B}_\alpha$ bounded continuous (see [12], Proposition 7.1.1) for details. Moreover, if $\mathfrak{B}_\alpha$ is a fading memory, we assume that $\max\{K(t), M(t)\} \leq \mathfrak{R}$ for all $t \geq 0$, (see [12], Proposition 7.1.5).

Remark 4.3. It is worth mentioning that in ([12], p. 190) it is shown that the phase $\mathfrak{B}_\alpha$ is a uniform fading memory space if and only if Axiom (C2) holds, the function $K(\cdot)$ is then bounded and $\lim_{t\to\infty} M(t) = 0$.

Example 4.4. The phase space $C_0 \times L^2(\rho, \mathbb{H}_\alpha)$.
Let $\rho : (-\infty, 0] \mapsto \mathbb{R}$ be a non-negative measurable function which satisfies the conditions $(g5)-(g6)$ in the terminology of [12]. Briefly, this means that ρ is locally integrable and there exists a non-negative locally bounded function γ on $(-\infty, 0]$ such that $\rho(\xi + \theta) \leq \gamma(\xi)\rho(\theta)$, for all $\xi \leq 0$ and $\theta \in (-\infty, 0) \setminus N_\xi$, where $N_\xi \subseteq (-\infty, 0)$ is a set whose Lebesgue measure is zero.

The space $\mathfrak{B}_\alpha = C_0 \times L^2(\rho, \mathbb{H}_\alpha)$ consists of all classes of functions $y : (-\infty, 0] \mapsto \mathbb{H}_\alpha$ such that y is continuous at 0, Lebsgue-measurable, and $\rho\|y\|_\alpha^2$ is Lebesgue integrable on $(-\infty, 0)$. The seminorm in $\mathfrak{B}_\alpha = C_0 \times L^2(\rho, \mathbb{H}_\alpha)$ is defined as follows:

$$\|y\|_{\mathfrak{B}_\alpha} := \|y(0)\|_\alpha + \Big(\int_{-\infty}^0 \rho(\theta) \|y(\theta)\|_\alpha^2 \, d\theta \Big)^{1/2}.$$

The space $\mathfrak{B}_\alpha = C_0 \times L^2(\rho, \mathbb{H}_\alpha)$ satisfies Axioms $(A), (A1)$, and (B) with $H = 1$, $M(t) = \gamma(-t)^{1/2}$ and $K(t) = 1 + \big(\int_{-t}^0 \rho(\theta) \, d\theta \big)^{1/2}$ for $t \geq 0$ (see [12],Theorem 1.3.8 for details). We also note that if the conditions $(g5) - (g7)$ of [12] hold, then $\mathfrak{B}_\alpha$ is a uniform fading memory.

5 Existence of Square-Mean Almost Periodic Solutions

This section is devoted to the existence and uniqueness of a square-mean almost periodic solution to the stochastic hyperbolic differential equation Eq. (1.2)

Definition 5.1. Let $\alpha \in (0,1)$. A continuous random function, $X : \mathbb{R} \to L^2(\Omega; \mathbb{H}_\alpha)$ is said to be a bounded solution of Eq.(1.2) provided that the function $s \to AT(t-s)PF_1(s, X_s)$ is integrable on $(-\infty, t)$, $s \to AT(t-s)QF_1(s, X_s)$ is integrable on (t, ∞) for each $t \in \mathbb{R}$, and

$$\begin{aligned} X(t) \;\; &= \;\; -F_1(t, X_t) - \int_{-\infty}^t AT(t-s)P\,F_1(s, X_s)\, ds + \int_t^\infty AT(t-s)Q\,F_1(s, X_s)\, ds \\ &\quad + \int_{-\infty}^t T(t-s)P\,F_2(s, X_s)\, ds - \int_t^\infty T(t-s)Q\,F_2(s, X_s)\, ds \\ &\quad + \int_{-\infty}^t T(t-s)P\,F_3(s, X_s)\, d\mathbb{W}(s) - \int_t^\infty T(t-s)Q\,F_3(s, X_s)\, d\mathbb{W}(s) \end{aligned}$$

for each $t \in \mathbb{R}$.

Throughout the rest of the paper we denote by Γ_1, Γ_2, Γ_3, Γ_4, Γ_5, and Γ_6 the nonlinear integral operators defined by

$$(\Gamma_1 X)(t) := \int_{-\infty}^t AT(t-s)P\,F_1(s, X_s)\, ds, \;\; (\Gamma_2 X)(t) := \int_t^\infty AT(t-s)Q\,F_1(s, X_s)\, ds,$$

$$(\Gamma_3 X)(t) := \int_{-\infty}^t T(t-s)P\,F_2(s, X_s)\, ds, \;\; (\Gamma_4 X)(t) := \int_t^\infty T(t-s)Q\,F_2(s, X_s)\, ds,$$

$$(\Gamma_5 X)(t) := \int_{-\infty}^t T(t-s)P\,F_3(s, X_s)\, d\mathbb{W}(s), \;\; (\Gamma_6 X)(t) := \int_t^\infty T(t-s)Q\,F_3(s, X_s)\, d\mathbb{W}(s).$$

To discuss the existence of square-mean almost periodic solution to Eq. (1.2) we need to set some assumptions on A, F_1, and $F_i (i = 2, 3)$. First of all, note that for $0 < \alpha < \beta < 1$, then

$$L^2(\Omega, \mathbb{H}_\beta) \hookrightarrow L^2(\Omega, \mathbb{H}_\alpha) \hookrightarrow L^2(\Omega; \mathbb{H})$$

are continuously embedded and hence there exist constants $k_1 > 0$, $k(\alpha) > 0$ such that

$$\mathbf{E}\|X\|^2 \leq k_1 \mathbf{E}\|X\|_\alpha^2 \;\text{ for each } X \in L^2(\Omega, \mathbb{H}_\alpha) \;\text{ and}$$

$$\mathbf{E}\|X\|_\alpha^2 \leq k(\alpha) \mathbf{E}\|X\|_\beta^2 \text{ for each } X \in L^2(\Omega, \mathbb{H}_\beta).$$

(H.1) The operator $\mathcal{A}$ is sectorial and generates a hyperbolic (analytic) semigroup $(T(t))_{t\geq 0}$.

(H.2) Let $\alpha \in (0, \frac{1}{2})$. Then $\mathbb{H}_\alpha = D((-\mathcal{A})^\alpha)$, or $\mathbb{H}_\alpha = D_{\mathcal{A}}(\alpha, p)$, $1 \leq p \leq \infty$, or $\mathbb{H}_\alpha = D_{\mathcal{A}}(\alpha)$, or $\mathbb{H}_\alpha = [\mathbb{H}, D(\mathcal{A})]_\alpha$.

(H.3) Let $\alpha \in (0, \frac{1}{2})$ and $\alpha < \beta < 1$. The function $F_1 : \mathbb{R} \times L^2(\Omega; \mathfrak{B}_\alpha) \to L^2(\Omega, \mathbb{H}_\beta)$ is square-mean almost periodic in $t \in \mathbb{R}$ uniformly in $X \in O$ ($O \subset L^2(\Omega; \mathfrak{B}_\alpha)$ being any compact subset). Moreover, F is Lipschitz in the following sense: there exists $K > 0$ for which

$$\mathbf{E}\Big\|F_1(t,X) - F_1(t,Y)\Big\|_\beta^2 \leq K\,\mathbf{E}\Big\|X - Y\Big\|_{\mathfrak{B}_\alpha}^2$$

for all random variables $X, Y \in L^2(\Omega; \mathfrak{B}_\alpha)$ and $t \in \mathbb{R}$.

(H.4) The function $F_i (i = 2, 3) : \mathbb{R} \times L^2(\Omega; \mathfrak{B}_\alpha) \to L^2(\Omega, \mathbb{H})$ is square-mean almost periodic in $t \in \mathbb{R}$ uniformly in $X \in O_i$ ($O_i \subset L^2(\Omega; \mathfrak{B}_\alpha)$) being any compact subset). Moreover, $F_i(\mathbb{R} \times B)$ is precompact for each bounded subset B of $L^2(\Omega; \mathfrak{B}_\alpha)$, and locally uniformly continuous, that is, for each r, $\varepsilon > 0$, there is $\delta(r, \varepsilon)$ such that $\mathbf{E}\|F_i(t,X) - F_i(t,Y)\|_{[\mathrm{D}]}^2 \leq \varepsilon$ whenever $t \in \mathbb{R}$ and X, $Y \in L^2(\Omega; \mathfrak{B}_\alpha)$ with $\mathbf{E}\|X\|_{\mathfrak{B}_\alpha}^2 < r$, $\mathbf{E}\|Y\|_{\mathfrak{B}_\alpha}^2 < r$ and $\mathbf{E}\|X - Y\|_{\mathfrak{B}_\alpha}^2 < \delta$. Moreover, for any $\varepsilon > 0$, there is $a > 0$ such that $\mathbf{E}\|F_i(t,X)\|_{[\mathrm{D}]}^2 \leq \varepsilon \mathbf{E}\|X\|_{\mathfrak{B}_\alpha}^2$ for all $t \in \mathbb{R}$ and $X \in L^2(\Omega; \mathfrak{B}_\alpha)$ with $\mathbf{E}\|X\|_{\mathfrak{B}_\alpha}^2 \geq a$.

The main result of the present paper will be based upon the use of the well-known fixed point theorem of Krasnoselskii given as follows:

Theorem 5.2. *Let C be a closed bounded convex subset of a Banach space $\mathcal{B}$. Suppose the (possibly nonlinear) operators L and M map C into $\mathcal{B}$ satisfying*

(a) for all $u, v \in C$, then $Lu + Mv \in C$;

(b) the operator L is a contraction;

(c) the operator M is continuous and $M(C)$ is contained in a compact set.

Then there exists $u \in C$ such that $u = Lu + Mu$.

To prove the main result (Theorem 5.8) we need the following lemmas.

Lemma 5.3. *Under assumptions* (H.1)-(H.2)-(H.3)*, the integral operators Γ_1 and Γ_2 defined above map $AP(\mathbb{R}; L^2(\Omega, \mathbb{H}_\alpha))$ into itself.*

Proof. The proof for the square-mean almost periodicity of $\Gamma_2 X$ is similar to that of $\Gamma_1 X$ and hence will be omitted.
Let $X \in AP(\mathbb{R}; L^2(\Omega; \mathbb{H}_\alpha))$. Clearly, $X_t \in AP(\mathbb{R}; L^2(\Omega; \mathfrak{B}_\alpha))$. Setting $\Psi_1(t) = F_1(t, X_t)$ and using Theorem 2.6, it follows that $\Psi_1 \in AP(\mathbb{R}; L^2(\Omega; \mathbb{H}_\beta))$.

We can now show that $\Gamma_1 X \in AP(\mathbb{R}; L^2(\Omega; \mathbb{H}_\alpha))$. Indeed, since $\Psi_1 \in AP(\mathbb{R}; L^2(\Omega; \mathbb{H}_\beta))$, for every $\varepsilon > 0$ there exists $l(\varepsilon) > 0$ such that for all ξ there is $t \in [\xi, \xi + l(\varepsilon)]$ with the property:

$$\mathbf{E}\|\Psi_1 X(t + \tau) - \Psi_1 X(t)\|_\beta^2 < \nu^2 \varepsilon \text{ for each } t \in \mathbb{R},$$

where $\nu = \dfrac{\gamma^{1-\alpha}}{M'(\alpha)\Gamma(1-\alpha)}$ with $\Gamma(\cdot)$ being the classical gamma function.
Now, the estimate in Eq. (3.9) yields

$$\begin{aligned}
&\mathbf{E}\|\Gamma_1 X(t+\tau) - \Gamma_1 X(t)\|_\alpha^2 \\
&\leq \mathbf{E}\Big(\int_0^\infty \|AT(s)P[\Psi_1(t-s+\tau) - \Psi_1(t-s)]\|_\alpha \, ds\Big)^2 \\
&\leq M'(\alpha)^2 \Big(\int_0^\infty s^{-\alpha} e^{-\gamma s}\, ds\Big)\Big(\int_0^\infty s^{-\alpha} e^{-\gamma s} \mathbf{E}\|\Psi_1(t-s+\tau) - \Psi_1(t-s)]\|_\beta^2 \, ds\Big) \\
&\leq \Big(\frac{M'(\alpha)\Gamma(1-\alpha)}{\gamma^{1-\alpha}}\Big)^2 \sup_{t\in\mathbb{R}} \mathbf{E}\|\Psi_1(t+\tau) - \Psi_1(t)\|_\beta^2 \, ds \\
&< \varepsilon
\end{aligned}$$

for each $t \in \mathbb{R}$, and hence $\Gamma_1 X \in AP(\mathbb{R}; L^2(\Omega; \mathbb{H}_\alpha))$. □

Lemma 5.4. *Under assumptions* (H.1)-(H.2)-(H.3)*, the integral operators* Γ_3 *and* Γ_4 *defined above map* $AP(\mathbb{R}; L^2(\Omega; \mathbb{H}_\alpha))$ *into itself.*

Proof. The proof for the square-mean almost periodicity of $\Gamma_4 X$ is similar to that of $\Gamma_3 X$ and hence will be omitted. Note, however, that for $\Gamma_4 X$, we make use of Eq. (3.6) rather than Eq. (3.7).

Let $X \in AP(\mathbb{R}; L^2(\Omega, \mathbb{H}_\alpha))$. Clearly, $X_t \in AP(\mathbb{R}; L^2(\Omega, \mathfrak{B}_\alpha))$. Setting $\Psi_2(t) = F_2(t, X_t)$ and using Theorem 2.7 it follows that $\Phi \in AP(\mathbb{R}; L^2(\Omega, \mathbb{H}))$.

We now show that $\Gamma_3 X \in AP(\mathbb{R}; L^2(\Omega, \mathbb{H}_\alpha))$. Indeed, since $\Phi \in AP(\mathbb{R}; L^2(\Omega, \mathbb{H}))$, for every $\varepsilon > 0$ there exists $l(\varepsilon) > 0$ such that for all ξ there is $\tau \in [\xi, \xi + l(\varepsilon)]$ with

$$\mathbf{E}\|\Psi_2(t+\tau) - \Psi_2(t)\|_{[\mathrm{D}]}^2 < \mu^2 \cdot \varepsilon \text{ for each } t \in \mathbb{R},$$

where $\mu = \dfrac{\gamma^{1-\alpha}}{M(\alpha)\Gamma(1-\alpha)}$.
Now using the expression

$$(\Gamma_3 X)(t+\tau) - (\Gamma_3 X)(t) = \int_0^\infty T(s)P[\Psi_2(t-s+\tau) - \Psi_2(t-s)]\, ds$$

and Eq. (3.7) it easily follows that

$$\mathbf{E}\|(\Gamma_3 X)(t+\tau) - (\Gamma_3 X)(t)\|_\alpha^2 < \varepsilon \text{ for each } t \in \mathbb{R},$$

and hence, $\Gamma_3 X \in AP(\mathbb{R}; L^2(\Omega; \mathbb{H}_\alpha))$. □

Lemma 5.5. *Under assumptions* (H.1)-(H.2)-(H.3)*, the integral operators* Γ_5 *and* Γ_6 *defined above map* $AP(\mathbb{R}; L^2(\Omega; \mathbb{H}_\alpha))$ *into itself.*

Proof. Let $X \in AP(\mathbb{R}; L^2(\Omega; \mathbb{H}_\alpha))$. Clearly, $X_t \in AP(\mathbb{R}; L^2(\Omega; \mathfrak{B}_\alpha))$. Setting $\Psi_3(t) = F_3(t, X_t)$ and using Theorem 2.7 it follows that $\Psi_3 \in AP(\mathbb{R}; L^2(\Omega; \mathbb{H}))$. We claim that $\Gamma_5 X \in AP(\mathbb{R}; L^2(\Omega; \mathbb{H}_\alpha))$. Indeed, since $\Psi_3 \in AP(\mathbb{R}; L^2(\Omega; \mathbb{H}))$, for every $\varepsilon > 0$ there exists $l(\varepsilon) > 0$ such that for all ξ there is $\tau \in [\xi, \xi + l(\varepsilon)]$ with

$$\mathbf{E}\|\Psi_3(t+\tau) - \Psi_3(t)\|_{[\mathrm{D}]}^2 < \zeta \cdot \varepsilon \text{ for each } t \in \mathbb{R}, \tag{5.1}$$

where $\zeta = \dfrac{1}{2c^2 \cdot K(\alpha,\gamma,\delta,\Gamma)}$.
Now using the expression

$$(\Gamma_5 X)(t+\tau) - (\Gamma_5 X)(t) = \int_0^\infty T(s)P[\Psi_3(t-s+\tau) - \Psi_3(t-s)]\, dW(s),$$

Eq. (3.5), the arithmetic-geometric inequality, and Ito isometry we have

$$\begin{aligned}
&\mathbf{E}\|(\Gamma_5 X)(t+\tau) - (\Gamma_5 X)(t)\|_\alpha^2 \\
&\quad\leq c^2\,\mathbf{E}\Bigg\{(1-\alpha)\Big\|\int_0^\infty T(s)P[\Psi_3(t-s+\tau)-\Psi_3(t-s)]\,dW(s)\Big\| \\
&\qquad +\alpha\Big\|\int_0^\infty T(s)P[\Psi_3(t-s+\tau)-\Psi_3(t-s)]\,dW(s)\Big\|_{[D(A)]}\Bigg\}^2 \\
&\quad\leq c^2\,\mathbf{E}\Bigg\{\Big\|\int_0^\infty T(s)P[\Psi_3(t-s+\tau)-\Psi_3(t-s)]\,dW(s)\Big\| \\
&\qquad +\Big\|A\int_0^\infty T(s)P[\Psi_3(t-s+\tau)-\Psi_3(t-s)]\,dW(s)\Big\|\Bigg\}^2 \\
&\quad\leq 2c^2\Bigg\{\int_0^\infty \mathbf{E}\|T(s)P[\Psi_3(t-s+\tau)-\Psi_3(t-s)]\|^2\,ds \\
&\qquad +\int_0^\infty \mathbf{E}\|AT(s)P[\Psi_3(t-s+\tau)-\Psi_3(t-s)]\|^2\,ds\Bigg\}.
\end{aligned}$$

Now

$$\mathbf{E}\|T(s)P[\Psi_3(t-s+\tau)-\Psi_3(t-s)]\|^2 \leq M^2\, e^{-2\delta s}\mathbf{E}\|\Psi_3(t-s+\tau)-\Psi_3(t-s)\|^2$$

and

$$\mathbf{E}\|AT(s)P[\Psi_3(t-s+\tau)-\Psi_3(t-s)]\|^2 \leq M_1^2(\alpha)s^{-2\alpha}e^{-2\gamma s}\mathbf{E}\|\Psi_3(t-s+\tau)-\Psi_3(t-s)\|^2.$$

Hence,

$$\mathbf{E}\|(\Gamma_5 X)(t+\tau) - (\Gamma_5 X)(t)\|_\alpha^2 \leq 2c^2 \cdot K(\alpha,\gamma,\delta,\Gamma)\sup_{t\in\mathbb{R}}\mathbf{E}\|\Psi_3(t+\tau)-\Psi_3(t)\|^2.$$

where $K(\alpha,\gamma,\delta,\Gamma) = \dfrac{M^2}{2\delta} + \dfrac{M_1^2(\alpha)\Gamma(1-2\alpha)}{\gamma^{1-2\alpha}}$, and it follows from Eq. (5.1) that $\Gamma_5 X \in AP(\mathbb{R}; L^2(\Omega;\mathbb{H}_\alpha))$.

The proof for $\Gamma_6 X(\cdot)$ is similar to that of $\Gamma_5 X(\cdot)$ except that Eq. (3.6) and Eq. (3.8) are used instead of Eq. (3.7) and Eq. (3.9), respectively. □

Consider the nonlinear operator Ξ on the space $\Big(AP(\mathbb{R};L^2(\Omega;\mathbb{H}_\alpha)), \Big\|\cdot\Big\|_{\infty,\alpha}\Big)$ defined by

$$\Xi X = \Xi_1 X + \Xi_2 X \text{ for all } X \in AP(\mathbb{R};L^2(\Omega;\mathbb{H}_\alpha)),$$

where

$$
\begin{aligned}
(\Xi_1 X)(t) &= -F(t,X_t) - \int_{-\infty}^{t} AT(t-s)PF_1(s,X_s)\,ds + \int_t^{\infty} AT(t-s)QF_1(s,X_s)\,ds \\
(\Xi_2 X)(t) &= \int_{-\infty}^{t} T(t-s)PF_2(s,X_s)\,ds - \int_t^{\infty} T(t-s)QF_2(s,X_s)\,ds \\
&\quad + \int_{-\infty}^{t} T(t-s)PF_3(s,X_s)\,d\mathbb{W}(s) - \int_t^{\infty} T(t-s)QF_3(s,X_s)\,d\mathbb{W}(s).
\end{aligned}
$$

for each $t \in \mathbb{R}$.

In view of Lemma 5.3, Lemma 5.4, and Lemma 5.5, it follows that Ξ maps $AP(\mathbb{R};L^2(\Omega;\mathbb{H}_\alpha))$ into itself. In order to apply Krasnoselskii's fixed point theorem, we need to construct two mappings: a contraction map and a compact map.

Lemma 5.6. *The operator Ξ_1 is a contraction provided $K(\alpha,\beta,\delta) < 1$ for some constant $K(\alpha,\beta,\delta)$.*

Proof. Let X, $Y \in AP(\mathbb{R};L^2(\Omega;\mathbb{H}_\alpha))$. Using (H.1)-(H.4), we obtain

$$
\begin{aligned}
\mathbf{E}\left\|F_1(t,X_t) - F_1(t,Y_t)\right\|_\alpha^2 &\leq k(\alpha)K\mathbf{E}\left\|X_t - Y_t\right\|_{\mathfrak{B}_\alpha}^2 \\
&\leq k(\alpha)\cdot K \cdot \mathfrak{L}\left\|X - Y\right\|_{\infty,\alpha}^2,
\end{aligned}
$$

which yields

$$
\left\|F_1(\cdot,X_\cdot) - F_1(\cdot,Y_\cdot)\right\|_{\infty,\alpha} \leq k'(\alpha)\cdot K_1' \cdot \mathfrak{L}'\left\|X - Y\right\|_{\infty,\alpha}.
$$

Now for Γ_1 and Γ_2, we have the following evaluations

$$
\begin{aligned}
&\mathbf{E}\left\|(\Gamma_1 X)(t) - (\Gamma_1 Y)(t)\right\|_\alpha^2 \\
&\quad\leq \mathbf{E}\Big(\int_{-\infty}^{t}\left\|AT(t-s)P[F_1(s,X_s) - F_1(s,Y_s)]\right\|_\alpha ds\Big)^2 \\
&\quad\leq c^2\Big(\int_{-\infty}^{t}(t-s)^{-\alpha}e^{-\frac{\gamma}{2}(t-s)}\,ds\Big)\times \\
&\qquad\times\Big(\int_{-\infty}^{t}(t-s)^{-\alpha}e^{-\frac{\gamma}{2}(t-s)}\mathbf{E}\left\|F_1(s,X_s) - F_1(s,Y_s)\right\|_\alpha^2 ds\Big) \\
&\quad\leq c^2\,k(\alpha)\,K\cdot\mathfrak{L}\Big(\int_{-\infty}^{t}(t-s)^{-\alpha}e^{-\frac{\gamma}{2}(t-s)}\,ds\Big)^2\left\|X - Y\right\|_{\infty,\alpha}^2,
\end{aligned}
$$

and hence

$$
\left\|\Gamma_1 X - \Gamma_1 Y\right\|_{\infty,\alpha} \leq c\cdot k'(\alpha)\cdot K'\cdot\mathfrak{L}'\,\frac{2^{1-\alpha}\Gamma(1-\alpha))}{\gamma^{1-\alpha}}\left\|X - Y\right\|_{\infty,\alpha}.
$$

Similarly,

$$\begin{aligned}
&\mathbf{E}\Big\|(\Gamma_2 X)(t)-(\Gamma_2 Y)(t)\Big\|_\alpha^2\\
&\quad\leq \mathbf{E}\Big(\int_t^\infty \Big\|AT(t-s)Q[F_1(s,X_s)-F_1(s,Y_s)]\Big\|_\alpha ds\Big)^2\\
&\quad\leq c^2\Big(\int_t^\infty e^{-\delta(s-t)}\,ds\Big)\Big(\int_t^\infty e^{-\delta(s-t)}\mathbf{E}\|F_1(s,X_s)-F_1(s,Y_s)\|_\alpha^2\,ds\Big)\\
&\quad\leq c^2\,k(\alpha)\,K\cdot\mathfrak{L}\Big(\int_t^\infty e^{-\delta(s-t)}\,ds\Big)^2\Big\|X-Y\Big\|_{\infty,\alpha}^2,
\end{aligned}$$

and hence,

$$\Big\|\Gamma_2 X-\Gamma_2 Y\Big\|_{\infty,\alpha}\leq \frac{c\cdot k'(\alpha)\cdot K'\cdot\mathfrak{L}'}{\delta}\Big\|X-Y\Big\|_{\infty,\alpha}.$$

□

Lemma 5.7. *The nonlinear operator Ξ_2 is continuous. Moreover, its image is contained in a compact set.*

Proof. Let us consider the set $V=\big\{X\in AP(\mathbb{R};L^2(\Omega,\mathbb{H}_\alpha)):\|X\|_{\infty,\alpha}^2\leq R'\big\}$ for some fixed $R'>0$. For the continuity, let $X^n\in AP(\mathbb{R};L^2(\Omega,\mathbb{H}_\alpha))$ be a sequence which converges to some $X\in AP(\mathbb{R};L^2(\Omega,\mathbb{H}_\alpha))$, that is, $\|X^n-X\|_{\infty,\alpha}\to 0$ as $n\to\infty$. It follows from the approximations in Lemma 3.5 that

$$\begin{aligned}
&\mathbf{E}\Big\|\int_{-\infty}^t T(t-s)P[F_2(s,X_s^n)-F_2(s,X_s)]\,ds\Big\|_\alpha^2\\
&\qquad\leq \mathbf{E}\Big[\int_{-\infty}^t c(\alpha)(t-s)^{-\alpha}e^{-\frac{\delta}{2}(t-s)}\Big\|F_2(s,X_s^n)-F_2(s,X_s)\Big\|\,ds\Big]^2.
\end{aligned}$$

Now, using the continuity of F_2 and the Lebesgue Dominated Convergence Theorem we obtain that

$$\mathbf{E}\Big\|\int_{-\infty}^t T(t-s)P[F_2(s,X_s^n)-F_2(s,X_s)]\,ds\Big\|_\alpha^2\to 0\ \text{ as }\ n\to\infty.$$

By similar arguments, we can also show that

$$\mathbf{E}\Big\|\int_t^\infty T(t-s)Q[F_2(s,X_s^n)-F_2(s,X_s)]\,ds\Big\|_\alpha^2\to 0\ \text{ as }\ n\to\infty.$$

For the term containing the Brownian motion $\mathbb{W}$, it is easy to see that there exists a constant $q(\alpha)>0$ such that

$$\begin{aligned}
&\mathbf{E}\Big\|\int_{-\infty}^t T(t-s)P[F_3(s,X_s^n)-F_3(s,X_s)]\,d\mathbb{W}(s)\Big\|_\alpha^2\\
&\qquad\leq q(\alpha)\int_{-\infty}^t (t-s)^{-2\alpha}e^{-\delta(t-s)}\mathbf{E}\Big\|F_3(s,X_s^n)-F_3(s,X_s)\Big\|^2 ds.
\end{aligned}$$

Now, using the continuity of F_3 and the Lebesgue Dominated Convergence Theorem we obtain that

$$\mathbf{E}\Big\|\int_{-\infty}^{t} T(t-s)P[F_3(s,X_s^n)-F_3(s,X_s)]\,d\mathbb{W}(s)\Big\|_\alpha^2 \to 0 \ \text{ as } \ n\to\infty\,.$$

By similar arguments, we can also show that

$$\mathbf{E}\Big\|\int_{t}^{\infty} T(s-t)Q[F_3(s,X_s^n)-F_3(s,X_s)]\,d\mathbb{W}(s)\Big\|_\alpha^2 \to 0 \ \text{ as } \ n\to\infty\,.$$

Therefore,

$$\Big\|\Xi_2 X^n - \Xi_2 X\Big\|_{\infty,\alpha} \to 0 \ \text{ as } \ n\to\infty.$$

We now show that Ξ_2 maps V into a compact set; in particular, we show that $\Xi_2(V)$ is an equicontinuous set. Indeed, let $\varepsilon>0$, $t_1<t_2$, and $X\in V$ be arbitrary.

Now

$$\begin{aligned}
&\mathbf{E}\Big\|(\Xi_2X)(t_2)-(\Xi_2X)(t_1)\Big\|_\alpha^2\\
&\qquad\leq\ 4\mathbf{E}\Big\|(\Gamma_3X)(t_2)-(\Gamma_3X)(t_1)\Big\|_\alpha^2+4\mathbf{E}\Big\|(\Gamma_4X)(t_2)-(\Gamma_4X)(t_1)\Big\|_\alpha^2\\
&\qquad\quad +4\mathbf{E}\Big\|(\Gamma_5X)(t_2)-(\Gamma_5X)(t_1)\Big\|_\alpha^2+4\mathbf{E}\Big\|(\Gamma_6X)(t_2)-(\Gamma_6X)(t_1)\Big\|_\alpha^2.
\end{aligned}$$

We have

$$\begin{aligned}
&\mathbf{E}\Big\|(\Gamma_3X)(t_2)-(\Gamma_3X)(t_1)\Big\|_\alpha^2\\
&\leq\ 2\mathbf{E}\Big\|\int_{t_1}^{t_2}T(t_2-s)P\Psi_2(s)\,ds\Big\|_\alpha^2+2\mathbf{E}\Big\|\int_{-\infty}^{t_1}[T(t_2-s)-T(t_1-s)]P\Psi_2(s)\,ds\Big\|_\alpha^2\\
&=\ 2\mathbf{E}\Big\|\int_{t_1}^{t_2}T(t_2-s)P\Psi_2(s)\,ds\Big\|_\alpha^2+2\mathbf{E}\Big\|\int_{-\infty}^{t_1}\Big(\int_{t_1}^{t_2}\frac{\partial T(\tau-s)}{\partial\tau}d\tau\Big)P\Psi_2(s)\,ds\Big\|_\alpha^2\\
&=\ 2\mathbf{E}\Big\|\int_{t_1}^{t_2}T(t_2-s)P\Psi_2(s)\,ds\Big\|_\alpha^2+2\mathbf{E}\Big\|\int_{-\infty}^{t_1}\Big(\int_{t_1}^{t_2}AT(\tau-s)P\Psi_2(s)\,d\tau\Big)\,ds\Big\|_\alpha^2\\
&=\ N_1+N_2.
\end{aligned}$$

Clearly,

$$\begin{aligned}
N_1\ &\leq\ \mathbf{E}\Big\{\int_{t_1}^{t_2}\Big\|T(t_2-s)P\Psi_2(s)\Big\|_\alpha ds\Big\}^2\\
&\leq\ M(\alpha)^2\mathbf{E}\Big\{\int_{t_1}^{t_2}(t_2-s)^{-\alpha}e^{-\gamma(t_2-s)}\Big\|\Psi_2(s)\Big\|\,ds\Big\}^2.
\end{aligned}$$

Similarly,

$$\begin{aligned}
N_2\ &\leq\ \mathbf{E}\Big\{\int_{-\infty}^{t_1}\Big(\int_{t_1}^{t_2}\Big\|AT(\tau-s)P\Psi_2(s)\Big\|_\alpha d\tau\Big)\,ds\Big\}^2\\
&\leq\ \mathbf{E}\Big\{\int_{-\infty}^{t_1}\Big(\int_{t_1}^{t_2}\Big\|AT(\tau-s)P\Psi_2(s)\Big\|_\alpha d\tau\Big)\,ds\Big\}^2\\
&\leq\ M(\alpha)^2\,\mathbf{E}\Big\{\int_{-\infty}^{t_1}\Big(\int_{t_1}^{t_2}(\tau-s)^{-\alpha}e^{-\gamma(\tau-s)}\Big\|\Psi_2(s)\Big\|\,d\tau\Big)\,ds\Big\}^2.
\end{aligned}$$

Therefore,

$$
\begin{aligned}
&\mathbf{E}\Big\|(\Gamma_3 X)(t_2)-(\Gamma_3 X)(t_1)\Big\|_\alpha^2\\
&\leq\ 2M(\alpha)^2\mathbf{E}\Big\{\int_{t_1}^{t_2}(t_2-s)^{-\alpha}e^{-\gamma(t_2-s)}\Big\|\Psi_2(s)\Big\|\,ds\Big\}^2\\
&\quad+2M(\alpha)^2\,\mathbf{E}\Big\{\int_{-\infty}^{t_1}\Big(\int_{t_1}^{t_2}(\tau-s)^{-\alpha}e^{-\gamma(\tau-s)}\Big\|\Psi_2(s)\Big\|\,d\tau\Big)\,ds\Big\}^2\\
&\leq\ K(\alpha,\gamma)\,(t_2-t_1)^2\sup_{t\in\mathbb{R}}\mathbf{E}\Big\|\Psi_2(t)\Big\|^2,
\end{aligned}
$$

where $K(\alpha,\gamma)$ is a positive constant.

Similar computations show that

$$
\begin{aligned}
&\mathbf{E}\Big\|(\Gamma_4 X)(t_2)-(\Gamma_4 X)(t_1)\Big\|_\alpha^2\\
&\leq\ 2M(\alpha)^2\mathbf{E}\Big\{\int_{t_1}^{t_2}e^{-\delta(s-t_1)}\Big\|\Psi_2(s)\Big\|\,ds\Big\}^2\\
&\quad+2M(\alpha)^2\,\mathbf{E}\Big\{\int_{t_2}^{\infty}\Big(\int_{t_1}^{t_2}e^{-\delta(s-\tau)}\Big\|\Psi_2(s)\Big\|\,d\tau\Big)\,ds\Big\}^2\\
&\leq\ K(\alpha,\delta)\,(t_2-t_1)^2\sup_{t\in\mathbb{R}}\mathbf{E}\Big\|\Psi_2(t)\Big\|^2,
\end{aligned}
$$

where $K(\alpha,\delta)$ is a positive constant.

Let us now evaluate $\Gamma_5 X$. We have

$$
\begin{aligned}
&\mathbf{E}\Big\|(\Gamma_5 X)(t_2)-(\Gamma_5 X)(t_1)\Big\|_\alpha^2\\
&\leq\ 2\mathbf{E}\Big\|\int_{t_1}^{t_2}T(t_2-s)P\Psi_3(s)\,d\mathbb{W}(s)\Big\|_\alpha^2\\
&\quad+2\mathbf{E}\Big\|\int_{-\infty}^{t_1}[T(t_2-s)-T(t_1-s)]P\Psi_3(s)\,d\mathbb{W}(s)\Big\|_\alpha^2\\
&=\ N_1'+N_2'.
\end{aligned}
$$

Let us start with the first term. By Ito isometry identity, we have

$$
\begin{aligned}
N_1' &\leq\ c^2\Big\{\int_{t_1}^{t_2}\mathbf{E}\Big\|T(t_2-s)P\Lambda(s)\Big\|^2ds+\int_{t_1}^{t_2}\mathbf{E}\Big\|AT(t_2-s)P\Psi_3(s)\Big\|^2ds\Big\}\\
&\leq\ 2c^2M_1'(\alpha)^2\int_{t_1}^{t_2}(t_2-s)^{-2\alpha}e^{-2\gamma(t_2-s)}\mathbf{E}\Big\|\Psi_3(s)\Big\|^2ds\Big\}.
\end{aligned}
$$

Similarly,

$$\begin{aligned}
N_2' &\leq c^2\Big\{\int_{-\infty}^{t_1}\mathbf{E}\Big\|[T(t_2-s)-T(t_1-s)]P\Psi_3(s)\Big\|^2 ds \\
&\quad+\int_{-\infty}^{t_1}\mathbf{E}\Big\|A[T(t_2-s)-T(t_1-s)]P\Psi_3(s)\Big\|^2 ds\Big\} \\
&\leq c^2\Big\{\int_{-\infty}^{t_1}\mathbf{E}\Big[\int_{t_1}^{t_2}\Big\|AT(\tau-s)P\Psi_3(s)\Big\|\,d\tau\Big]^2 ds \\
&\quad+\int_{-\infty}^{t_1}\Big\|A\Big\|^2\mathbf{E}\Big[\int_{t_1}^{t_2}\Big\|AT(\tau-s)P\Psi_3(s)\Big\|\,d\tau\Big]^2 ds\Big\} \\
&\leq c^2 M(\alpha)^2\Big\{\int_{-\infty}^{t_1}\mathbf{E}\Big[\int_{t_1}^{t_2}(\tau-s)^{-\alpha}e^{-\gamma(\tau-s)}\Big\|\Psi_3(s)\Big\|\,d\tau\Big]^2 ds \\
&\quad+\int_{-\infty}^{t_1}\mathbf{E}\Big[\int_{t_1}^{t_2}(\tau-s)^{-\alpha}e^{-\gamma(\tau-s)}\Big\|A\Psi_3(s)\Big\|\,d\tau\Big]^2 ds\Big\} \\
&\leq c^2 M(\alpha)^2\int_{-\infty}^{t_1}\mathbf{E}\Big[\int_{t_1}^{t_2}(\tau-s)^{-\alpha}e^{-\gamma(\tau-s)}\Big\|\Psi_3(s)\Big\|_{[\mathrm{D}]}\,d\tau\Big]^2 ds \\
&\leq K(\alpha,\gamma,\Gamma)(t_2-t_1)^2\sup_{t\in\mathbb{R}}\mathbf{E}\Big\|\Psi_3(t)\Big\|^2_{[\mathrm{D}]}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\mathbf{E}\Big\|(\Gamma_5X)(t_2)-(\Gamma_5X)(t_1)\Big\|^2_\alpha \\
&\quad\leq\Big[K(\alpha,\gamma)\,(t_2-t_1)+K(\alpha,\gamma,\Gamma)\,(t_2-t_1)^2\Big]\sup_{t\in\mathbb{R}}\mathbf{E}\Big\|\Psi_3(t)\Big\|^2_{[\mathrm{D}]},
\end{aligned}$$

where $K(\alpha,\gamma)$ and $K(\alpha,\gamma,\Gamma)$ are positive constants.

Similar computations show

$$\mathbf{E}\Big\|(\Gamma_6X)(t_2)-(\Gamma_6X)(t_1)\Big\|^2_\alpha\leq\Big[K(\alpha,\delta)\,(t_2-t_1)+K(\alpha,\beta,\delta)\,(t_2-t_1)^2\Big]\sup_{t\in\mathbb{R}}\mathbf{E}\Big\|\Psi_3(t)\Big\|^2_{[\mathrm{D}]},$$

where $K(\alpha,\delta)$ and $K(\alpha,\beta,\delta)$ are positive constants.

From the theorem of Ascoli-Arzela, it follows that $\Xi_2(V)$ is contained in a compact set. The proof is complete. □

Theorem 5.8. *Suppose assumptions* (H.1)-(H.2)-(H.3)-(H.4) *hold and that* $K(\alpha,\beta,\delta)<1$, *the evolution equation Eq.* (1.2) *has a square-mean almost periodic solution X satisfying*

$$X=\Xi_1X+\Xi_2X.$$

Proof. Fix $\varepsilon>0$ and let i=2, 3. It follows from assumption (H.4) that there exists $r>0$ such that

$$\mathbf{E}\Big\|F_i(t,Y)\Big\|^2_{[\mathrm{D}]}\leq\varepsilon\mathbf{E}\Big\|Y\Big\|^2_{\mathfrak{B}_\alpha}\ \text{for all } t\in\mathbb{R}\text{ and } Y\in L^2(\Omega,\mathfrak{B}_\alpha)\text{ with }\mathbf{E}\Big\|Y\Big\|^2_{\mathfrak{B}_\alpha}>r.$$

Setting

$$M_i=\sup\Big\{\mathbf{E}\Big\|F_i(t,Y)\Big\|^2_{[\mathrm{D}]}:\ t\in\mathbb{R},\ \mathbf{E}\Big\|Y\Big\|^2_{\mathfrak{B}_\alpha}\leq r\Big\}.$$

Therefore,

$$\mathbf{E}\Big\|F_i(t,Y)\Big\|^2_{[\mathrm{D}]} \leq M_i + \varepsilon\mathbf{E}\Big\|Y\Big\|^2_{\mathfrak{B}_\alpha} \text{ for all } (t,Y) \in \mathbb{R}\times L^2(\Omega,\mathfrak{B}_\alpha)\,. \tag{5.2}$$

Also, using Lemma 3.5 , Lemma 3.6, Eq. (5.2), and assumption (H.4) we can show that

$$\begin{aligned}
&\mathbf{E}\Big\|(\Xi_1X)(t)+(\Xi_2Y)(t)\Big\|^2_\alpha \\
\leq\;& 12\Big[1+M(\alpha)^2\Big(\frac{\Gamma(1-\alpha)}{\gamma^{1-\alpha}}\Big)^2+\Big(\frac{n(\alpha,\beta)}{\delta}\Big)^2\Big]\Big(K_1\,\Big\|X\Big\|_{\infty,\alpha}+\sup_{t\in\mathbb{R}}\mathbf{E}\Big\|F_1(t,0)\Big\|^2\Big) \\
+\;& 8M(\alpha)^2\Bigg[\Big[\Big(\frac{\Gamma(1-\alpha)}{\gamma^{1-\alpha}}\Big)^2+\Big(\frac{C(\alpha)}{\delta}\Big)^2\Big]\Big(M_2+\varepsilon\Big\|Y\Big\|_{\infty,\alpha}\Big) \\
+\;& \Big[K(\alpha)\frac{\Gamma(1-2\alpha)}{(2\gamma)^{1-2\alpha}}+\frac{K(\alpha,\beta)}{2\delta}\Big]\Big(M_3+\varepsilon\Big\|Y\Big\|_{\infty,\alpha}\Big)\Bigg] \\
=\;& c_1(\alpha,\beta,\delta,\Gamma)\Big(K_1\,\Big\|X\Big\|_{\infty,\alpha}+\sup_{t\in\mathbb{R}}\mathbf{E}\Big\|F_1(t,0)\Big\|^2\Big)+c_2(\alpha,\beta,\delta,\Gamma)\Big(M_2+\varepsilon\Big\|Y\Big\|_{\infty,\alpha}\Big) \\
+\;& c_3(\alpha,\beta,\delta,\Gamma)\Big(M_3+\varepsilon\Big\|Y\Big\|_{\infty,\alpha}\Big).
\end{aligned}$$

Now, for ε, K_1 small enough, choose R such that

$$c_1(\alpha,\beta,\Gamma,\delta)\Big(K_1\,R+a\Big)+c_2(\alpha,\beta,\Gamma,\delta)\Big(M_2+\varepsilon R\Big)+c_3(\alpha,\beta,\Gamma,\delta)\Big(M_3+\varepsilon R\Big)\leq R$$

where $a=\sup_{t\in\mathbb{R}}\mathbf{E}\Big\|F_1(t,0)\Big\|^2$ and $c_i(\alpha,\beta,\Gamma,\delta)$ $(i=1,2,3)$ are constants depending on α, β , δ, and the classical gamma function Γ.

Let $W=\Big\{Z\in AP(\mathbb{R};L^2(\Omega;\mathbb{H}_\alpha):\Big\|Z\Big\|_{\infty,\alpha}\leq R\Big\}$. For X, $Y\in W$, we have

$$\mathbf{E}\Big\|(\Xi_1X)(t)+(\Xi_2Y)(t)\Big\|^2_\alpha\leq R\,.$$

Thus $(\Xi_1X)(t)+(\Xi_2Y)(t)\in W$. In view of Lemma 5.3, Lemma 5.4, Lemma 5.5, Lemma 5.6, and Lemma 5.7, the proof can be completed by using the Krasnoselskii's fixed point theorem (Theorem 5.2).

□

6 Examples

Throughout the rest of this paper, we suppose $0<\alpha<0.5<\beta<1$.

Example 6.1. Let $\mathcal{O}\subset\mathbb{R}^N$ $(N\geq 1)$ be an open bounded subset with regular boundary $\mathcal{O}'=\partial\mathcal{O}$ and let $\mathbb{H}=L^2(\mathcal{O})$ equipped with its natural topology. Here, for $\mu\in(0,1)$, we take $\mathbb{H}_\mu=\mathbb{H}^\mu_0(\mathcal{O})\cap\mathbb{H}^{2\mu}(\mathcal{O})$ equipped with its μ-norm $\|\cdot\|_\mu$ and we choose as a phase space, the one described in Example 4.4, that is, $\mathfrak{B}_\alpha=C_0\times L^2(\rho,\mathbb{H}_\alpha)$.

To illustrate our main result, we study the existence of square-mean almost periodic solutions to the stochastic heat equation with with infinite delay given by

$$\begin{cases} \partial\left[\varphi + F_1(t,\Phi_t)\right] = \left[\Delta\Phi + F_2(t,\Phi_t)\right]\partial t + F_3(t,\Phi_t)d\mathbb{W}(t), & \text{in } \mathbb{R}\times O \\ \\ \Phi = 0, & \text{on } \mathbb{R}\times O' \end{cases} \tag{6.1}$$

where $F_1 : \mathbb{R}\times L^2(\Omega;\mathfrak{B}_\alpha) \mapsto L^2(\Omega,\mathbb{H}_\beta)$ and $F_i(i=2,\,3) : \mathbb{R}\times L^2(\Omega,\mathfrak{B}_\alpha) \to L^2(\Omega,L^2(O))$ are square-mean almost periodic processes.

Define the linear operator

$$Au = \Delta u \ \text{ for all } \ u \in D(A)) = L^2(\Omega, H_0^1(O)\cap\mathbb{H}^2(O)).$$

Under previous assumptions, it is clear that the operators A is sectorial. Moreover, the analytic semigroup associated with A is hyperbolic.

We have

Theorem 6.2. *Under previous assumptions, then the heat equation Eq. (6.1) has a solution* $\Phi \in AP(\mathbb{R}, L^2(\Omega,\mathbb{H}_0^1(O)\cap\mathbb{H}^2(O)))$.

Example 6.3. Here, for $\mu \in (0,1)$, we take $\mathbb{H}_\mu = \mathbb{H}_0^\mu([0,\pi])\cap\mathbb{H}^{2\mu}([0,\pi])$ equipped with its μ-norm $\|\cdot\|_\mu$.

Define the linear operator A by:

$$D(A) := \{u \in L^2[0,\pi] : u'' \in L^2[0,\pi],\ u(0) = u(\pi) = 0\}, \ \text{and}, \ Au := u'', \ \forall u \in D(A).$$

Clearly, A is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq 0}$ on $L^2[0,\pi]$. Furthermore, A has a discrete spectrum with eigenvalues of the form $-n^2, n \in \mathbb{N}$, whose corresponding (normalized) eigenfunctions are given by: $z_n(\xi) := \sqrt{\frac{2}{\pi}}\sin(n\xi)$. In addition, the following properties hold:

(a) $\{z_n : n \in \mathbb{N}\}$ is an orthonormal basis for $L^2[0,\pi]$;

(b) For $u \in L^2[0,\pi]$, $T(t)u = \sum_{n=1}^{\infty} e^{-n^2 t}\langle u, z_n\rangle z_n$ and

$$Au = -\sum_{n=1}^{\infty} n^2\langle u, z_n\rangle z_n$$

for all $u \in D(A)$;

(c) It is possible to define the fractional power $(-A)^\alpha$, $0 < \alpha \leq 1$ of A, as a closed linear operator over its domain $D((-A)^\alpha)$. More precisely, the operator $(-A)^\alpha$: $D((-A)^\alpha) \subseteq L^2[0,\pi] \to L^2[0,\pi]$ is given by

$$(-A)^\alpha u = \sum_{n=1}^{\infty} n^{2\alpha}\langle u, z_n\rangle z_n,$$

for all $u \in D((-A)^{\alpha})$, where

$$D((-A)^{\alpha}) = \Big\{ u(\cdot) \in L^2[0,\pi] : \sum_{n=1}^{\infty} n^{2\alpha} \langle u, z_n \rangle z_n \in L^2[0,\pi] \Big\};$$

(e) If $\mathbb{H}_\mu$ denotes the space $D((-A)^{\mu})$ endowed with the graph norm $\|\cdot\|_\mu$, then $\mathbb{H}_\mu$ is a Banach space. Moreover, $\mathbb{H}_\mu \to \mathbb{H}_\nu$ is continuous for $0 < \nu \le \mu \le 1$ and there exist some constants $C_\mu, \delta_\mu > 0$ such that

$$\|T(t)\|_{B(\mathbb{H}_\mu, L^2[0,\pi])} \le \frac{C_\mu e^{-\delta_\mu t}}{t^{\mu}}$$

for $t > 0$.

Now, let $\lambda > 0$ and define the phase space

$$\mathfrak{B}_\alpha = \left\{ \phi \in C((-\infty,0];\mathbb{H}_\alpha) : \lim_{\theta \to -\infty} e^{-\lambda\theta} \phi(\theta) \text{ exists in } \mathbb{H}_\alpha \right\}.$$

The seminorm in $\mathfrak{B}_\alpha$ is defined as follows:

$$\left\|\phi\right\|_{\mathfrak{B}_\alpha} = \sup_{\theta \in (-\infty,0]} \left\{ e^{-\lambda\theta} \left\|\phi(\theta)\right\|_\alpha \right\}.$$

The space $\mathfrak{B}_\alpha$ satisfies Axioms (A), (A1), and (B) with $H = 1$, $K(t) = \max\{1, e^{\lambda t}\}$, and $M(t) = e^{\lambda t}$.

Consider the first-order boundary value problem

$$\begin{aligned}
\partial\Big[\Phi(t,\xi) + f_1(t,\Phi(t+\theta,\xi))\Big] &= \Big[\frac{\partial^2}{\partial\xi^2}\Phi(t,\xi) + \int_{-\infty}^{0}\int_0^{\pi} a_2(t,\xi,\theta) f_2(\Phi(t+\theta,\eta))\,d\eta\,d\theta\Big]\partial t \\
&\quad + \Big[\int_{-\infty}^{0}\int_0^{\pi} a_3(t,\xi,\theta) f_3(\Phi(t+\theta,\eta))\,d\eta\,d\theta\Big] d\mathbb{W}(t), \\
\Phi(t,0) &= \Phi(t,\pi) = 0, \qquad (6.2)
\end{aligned}$$

for $(t,\xi,\theta) \in \mathbb{R} \times [0,\pi] \times (-\infty,0]$.
Here,

(a) the function f_1 satisfies assumption (H.3);

(b) the function $a_i(t,\xi,\theta)$ $(i = 2,3)$ is nonnegative and almost periodic in $t \in \mathbb{R}$ and uniformly in $(\xi,\theta) \in [0,\pi] \times (-\infty,0]$ with $\frac{\partial^j}{\partial\zeta^j} a_i(t,\zeta,\theta)$, $j = 0,1$ are (Lebesgue) measurable with $a_i(\tau,\pi,\theta) = 0$, $a_i(t,0,\theta) = 0$ for every (t,θ) such that

$$N_1 := \sup_{t\in\mathbb{R}} \max \left\{ \int_0^{\pi} \left(\int_{-\infty}^{0} \frac{\partial^j}{\partial\zeta^j} a_i(t,\zeta,\theta)\,d\theta \right)^2 d\zeta : j = 0,2 \right\} < \infty;$$

(c) the function $f_i(\cdot)$ $(i=2,3)$ is continuous, $0 \le f_i(u(\theta,\xi)) \le \psi_i^{1/2}\Big(\Big\|u(\theta,\cdot)\Big\|_\alpha\Big)$, for $(\theta,\xi) \in (-\infty,0]\times[0,\pi]$, where $\psi_i(\cdot):(0,\infty)\to(0,\infty)$ is continuous, nondecreasing, concave, and

$$\lim_{x\to\infty}\frac{\psi_i(x)}{x}=0.$$

Note that equations of type Eq. (6.2) arise for instance in control systems described by abstract retarded functional-differential equations with feedback control governed by proportional integro-differential law, see [9, Examples 4.2] for details.

For $u \in L^2(\Omega;\mathfrak{B}_\alpha)$ and $(\theta,\xi)\in(-\infty,0]\times[0,\pi]$, let $u(\theta)(\xi)=u(\theta,\xi)$ and define $F_1 : \mathbb{R}\times L^2(\Omega;\mathfrak{B}_\alpha)\mapsto L^2(\Omega,\mathbb{H}_\beta)$ and $F_i(i=2,\,3):\mathbb{R}\times L^2(\Omega,\mathfrak{B}_\alpha)\mapsto L^2(\Omega,L^2[0,\pi])$ by setting

$$\begin{aligned} F_1(t,\Phi)(\xi) &:= f_1(t,\Phi(\theta)(\xi)) && (6.3)\\ F_i(t,\Phi)(\xi) &:= \int_{-\infty}^0\int_0^\pi a_i(t,\xi,\theta)f_i(\Phi(\theta)(\eta))d\eta\,d\theta. && (6.4)\end{aligned}$$

Let us now check that (H4) holds. Indeed,

$$\begin{aligned}
&\mathbf{E}\Big\|(-A)^\alpha F_i(t,\Phi)(\xi)\Big\|^2_{L^2[0,\pi]}\\
&= \mathbf{E}\sum_{n\ge1} n^{4\alpha}\|z_n\|^2_{L^2[0,\pi]}\Big|\langle F_i(t,\Phi),z_n\rangle\Big|^2\\
&\le \mathbf{E}\sum_{n\ge1} n^2\mathbf{E}\Big|\langle F_i(t,\Phi),z_n\rangle\Big|^2\\
&= \frac{2}{\pi}\sum_{n\ge1}\mathbf{E}\Big|\int_0^\pi F_i(t,\Phi)(\xi)\,n\sin(n\xi)d\xi\Big|^2\\
&= \mathbf{E}\sum_{n\ge1}\frac{1}{n^2}\Big|\int_0^\pi\frac{\partial^2}{\partial\xi^2}F_i(t,\Phi)(\xi)\,z_n(\xi)d\xi\Big|^2\\
&\le \frac{\pi^2}{6}\mathbf{E}\Big\|\frac{\partial^2}{\partial\xi^2}F_i(t,\Phi)\Big\|^2_{L^2[0,\pi]}\\
&= \frac{\pi^2}{6}\mathbf{E}\left[\int_0^\pi\left[\int_{-\infty}^0\frac{\partial^2}{\partial\xi^2}a_i(t,\xi,\theta)\int_0^\pi f_i(\Phi(t+\theta,\eta))\,d\eta\,d\theta\right]^2 d\xi\right]\\
&\le \frac{\pi^4}{6}\mathbf{E}\left[\int_0^\pi\left[\int_{-\infty}^0\frac{\partial^2}{\partial\xi^2}a_i(t,\xi,\theta)\psi_i^{1/2}\Big(\Big\|\Phi(\theta)(\cdot)\Big\|_\alpha\Big)\,d\theta\right]^2 d\xi\right]\\
&\le \frac{\pi^4}{6}\mathbf{E}\left[\int_0^\pi\left[\int_{-\infty}^0\frac{\partial^2}{\partial\xi^2}a_i(t,\xi,\theta)\psi_i^{1/2}\Big(e^{-\lambda\theta}\Big\|\Phi(\theta)(\cdot)\Big\|_\alpha\Big)\,d\theta\right]^2 d\xi\right]\\
&\le \frac{\pi^4}{6}\mathbf{E}\psi_i\Big(\Big\|\Phi\Big\|_{\mathfrak{B}_\alpha}\Big)\int_0^\pi\left[\int_{-\infty}^0\frac{\partial^2}{\partial\xi^2}a_i(t,\xi,\theta)\,d\theta\right]^2 d\xi\\
&\le \frac{\pi^4}{6}N_1\mathbf{E}\psi_i\Big(\Big\|\Phi\Big\|_{\mathfrak{B}_\alpha}\Big)
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{E}\Big\|F_i(t,\Phi)\Big\|^2_{L^2[0,\pi]} &= \mathbf{E}\left[\int_0^{\pi}\Big(\int_{-\infty}^{0}\int_0^{\pi} a_i(t,\xi,\theta) f_i(\Phi(t+\theta,\eta))\,d\eta\,d\theta\Big)^2 d\xi\right] \\
&\leq \pi^2\,\mathbf{E}\left[\int_0^{\pi}\Big(\int_{-\infty}^{0} a_i(t,\xi,\theta)\psi_i^{1/2}\Big(\Big\|\Phi(\theta)(\cdot)\Big\|_{\alpha}\Big)\,d\theta\Big)^2 d\xi\right] \\
&\leq \pi^2\,\mathbf{E}\left[\int_0^{\pi}\Big(\int_{-\infty}^{0} a_i(t,\xi,\theta)\psi_i^{1/2}\Big(e^{-\lambda\theta}\Big\|\Phi(\theta)(\cdot)\Big\|_{\alpha}\Big)\,d\theta\Big)^2 d\xi\right] \\
&\leq \pi^2 N_1\,\mathbf{E}\psi_i\Big(\Big\|\Phi\Big\|_{\mathfrak{B}_\alpha}\Big).
\end{aligned}$$

Thus, combining these two evaluations and using the concavity of ψ_i, we obtain

$$\mathbf{E}\Big\|F_i(t,\Phi)\Big\|^2_{[\mathrm{D}]} \leq \Big(\pi^2+\frac{\pi^4}{6}\Big) N_1\,\psi_i\Big(\mathbf{E}\Big\|\Phi\Big\|_{\mathfrak{B}_\alpha}\Big).$$

Theorem 6.4. *Under previous assumptions, then the system Eq. (6.2) has a square-mean almost periodic solution.*

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