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REGULAR REFLECTION IN SELF-SIMILAR POTENTIAL FLOW AND THE SONIC CRITERION

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Abstract

Reflection of a shock from a solid wedge is a classical problem in gas dynamics. Depending on the parameters either a regular or a irregular (Mach-type) reflection results. We construct regular reflection as an exact self-similar solution for potential flow. For some upstream Mach numbers M_I and isentropic coefficients γ, a solution exists for all wedge angles θ allowed by the *sonic criterion*. This demonstrates that, at least for potential flow, weaker criteria are false.

AMS Subject Classification: 76H05; 75M10

Keywords: shock, regular reflection, sonic criterion, potential flow

1 Introduction

The reflection problem

Reflection of an incident shock from a solid wedge is a classical problem of gas dynamics. It has been studied extensively by Ernst Mach [25, 20] and John von Neumann [26], as well as many other engineers and mathematicians.

Most commonly, reflection is studied in *steady* inviscid compressible flow, for example when shocks in a nozzle are reflected from the walls. The reflections can be classified roughly into *regular* and *irregular reflections*; see [1] for a more detailed discussion. In

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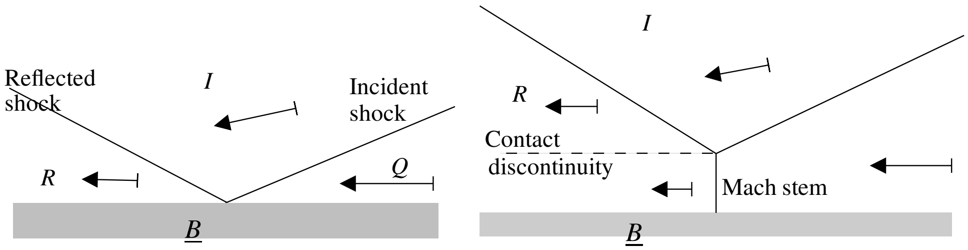


Figure 1. Left: regular reflection, right: single Mach reflection (oversimplified). If the reflection point is steady, the sonic criterion corresponds to $M_Q > 1$.

either type, an *incident shock* Q impinges on a solid surface $\underline{B}$ (see Figure 1). In regular reflection (RR), Q reaches a *reflection point* on the surface, continuing as the *reflected shock* R (see Figure 1 left).

In *irregular reflections* (IRR), incident and reflected shock are connected by a more or less complex interaction pattern which in turn connects to the solid surface by a third shock, called *Mach stem*. The most important irregular reflections are double, complex and single Mach reflection (DMR, CMR, SMR); various additional types have been proposed [15, 17, 18]. Figure 1 right shows an (oversimplified) version of single Mach reflection.

The reflection problem has several parameters. For polytropic gas it is sufficient to consider the isentropic coefficient γ as well as L_Q and L_I, the Mach numbers in the Q resp. I regions. The incident shock cannot exist unless $L_Q > 1$. L_Q and $L_I < L_Q$ determine the incident shock (not all L_I may admit a matching reflected shock).

In Mach reflection, the Mach stem, reflected and incident shock appear to meet in a *triple point*. In general this is possible only if they are joined by a contact discontinuity (slip line); for some parameter values it is not possible at all. In fact for certain values RR is not possible either. This is called the *von Neumann paradox*; it is perhaps the most famous of the many problems arising in reflection. Many ideas have been proposed towards the resolution of the paradox (see e.g. [15, 27, 13, 17, 18]); no single explanation has been accepted widely so far.

However, this article is concerned with a different question: it is natural to ask which parameters cause a RR and which yield IRR. Of course both sides of Figure 1 are perfectly valid stationary solutions, so the question has to be phrased more carefully. For example:

1. Which of the two is dynamically stable (e.g. asymptotically stable as a stationary solution of the time-dependent problem)?
2. Which of the two is structurally stable under perturbations like downstream nozzles, wall curvature or roughness, interaction with other flow patterns, perturbation of the upstream flow to non-constant with curved incident shock, viscosity, heat conduction, boundary layers, noise, slow relaxation to thermal equilibrum and other kinetic

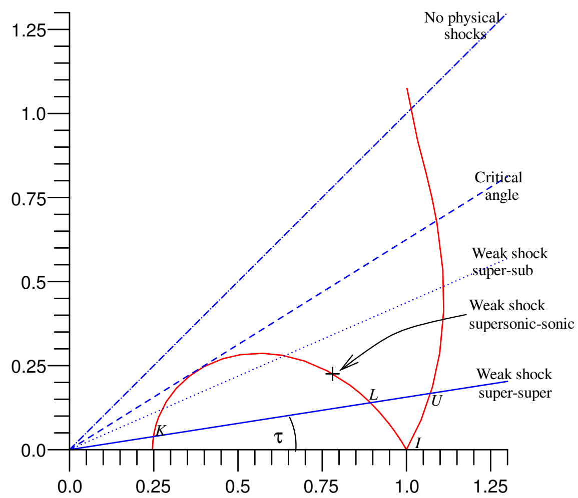


Figure 2. A given upstream velocity (I) with possible downstream velocities (red curve) for steady shocks with varying normal. The U shock is unphysical; K is the strong and L the weak shock. Shocks cannot turn velocities by more than the critical angle τ_*.

effects, dissociation etc.

It is not clear whether these questions are really any better than the original one — perhaps both sides of Figure 1 are stable. If so, then the new questions would merely fail in a less obvious way, as stability is harder to check than existence. But let us assume for the sake of the argument that the vague problem "does RR or IRR occur" can be expressed in some way as a precise mathematical question that selects exactly one of the two choices.

Among the criteria that have been proposed (see [1, Section 1.5]), three are most important. The first criterion, called *detachment criterion*, states that RR occurs whenever a reflected shock exists. Clearly RR is not possible without a reflected shock, so this is the weakest possible criterion.

The velocity $\vec{v}_I$ in the I region of Figure 4 forms an angle τ with $\underline{B}$; the reflected shock must turn this velocity by τ so that $\vec{v}_R$ is parallel to the wall, satisfying a slip boundary condition.

Given the I region data and γ, let the reflected shock be steady and pass through the reflection point, but vary its angle. This yields a one-parameter family of velocities $\vec{v}_R$, forming a curve called *shock polar* (see Figure 2). For *physical* shocks there is a maximum angle τ_* between downstream and upstream velocity. τ_* is determined by the upstream state.

If the angle τ between wall and $\vec{v}_I$ region of Figure 4 right is bigger than τ_*, no reflected shock exists. If $\tau = \tau_*$, there is exactly one reflected shock. For $\tau < \tau_*$ however there are *two*, called *weak reflection* and *strong reflection*. We encounter another one of the major issues in reflection: which of these two should occur? [11] have discussed this question in a related problem.

The flow in the R region can be supersonic or subsonic. If it is supersonic, then waves in the R region cannot travel towards the reflection point. If it is subsonic, however, they can reach it and interact with it, potentially altering the reflection type. This motivates the second criterion, called *sonic criterion*: RR occurs exactly if there is a reflected shock with supersonic R region, i.e. Mach number $L_R > 1$.

On the shock polar (Figure 2), $+$ indicates the point where $M_R = 1$; velocities right of it are supersonic, left of it subsonic. Hence there is an angle τ_+ so that for $\tau < \tau_+$ the weak reflection L has $L_R > 1$. For $\tau > \tau_+$ however it has $L_R < 1$. The strong reflection K is *always* subsonic in the R region — so the sonic criterion has a pleasant property: only the weak reflection is allowed, solving the uniqueness problem. Moreover since $\tau_+ < \tau_*$, the sonic criterion is stronger than the detachment criterion.

The third criterion is motivated by studying what happens when the parameters L_I, L_Q are varied so that a transition from RR to IRR occurs. One might suspect that the pressure in the reflection point in the R, S regions is continuous and does not jump during transition. Then the pressure behind the reflected shock in RR and the pressure behind the Mach stem in IRR, a shock approximately straight and perpendicular to the wall, must be equal at transition. There is a very limited set of L_I, L_Q, γ for which this happens; the *von Neumann criterion* (sometimes called *mechanical equilibrum criterion*) states that the transition can occur only at those parameters.

The von Neumann criterion has various problems. Most importantly, for weak incident shocks the pressure behind the Mach stem never matches the pressure below the reflected shock, so RR should occur in all cases, contradicting observations.

Self-similar reflection

Reflection can also be studied in *self-similar* (sometimes called *quasi-steady* or *pseudo-steady*) flow. In fact this is advantageous: for finding stationary solutions, choosing boundary conditions that yield well-posedness, in particular uniqueness, can be rather subtle, as evident from the awkward phrasing of the RR-or-IRR question above. For initial-value problems, on the other hand, uniqueness is expected[1] — or at least a necessary property of any interesting model equation. Moreover, self-similar flow patterns occur naturally in various reflection experiments.

In self-similar flow, density and velocity are functions of $\xi = x/t$ and $\eta = y/t$ rather than x, y. To produce a reflection, we consider the horizontal *upstream wall* $\hat{A}$ and the *downstream*

[1] [7, 6] raise doubt about the Cauchy problem for the Euler equations, but at least for potential flow the author expects uniqueness to hold.

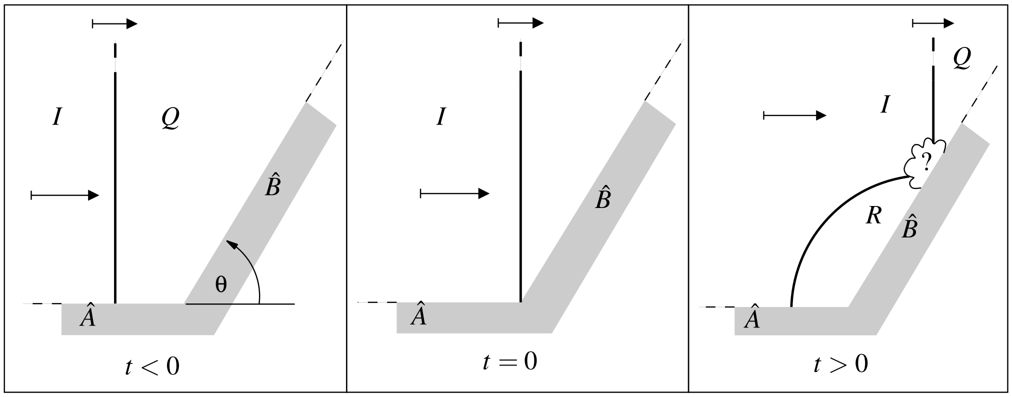


Figure 3. Self-similar reflection of a straight vertical shock in a convex corner. Different "?" patterns occur depending on corner angle and other parameters.

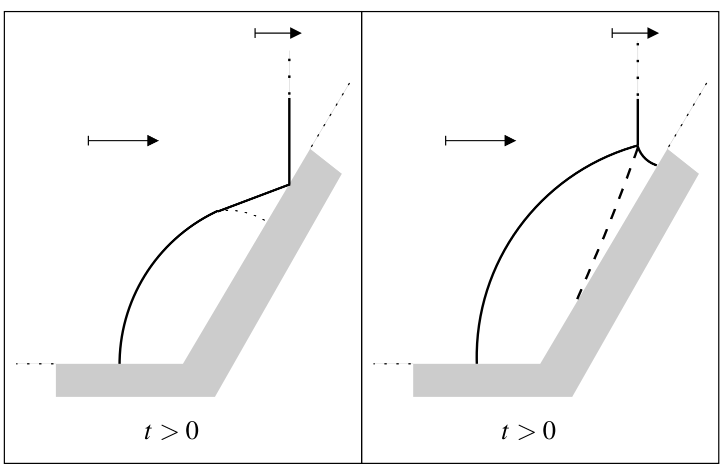


Figure 4. Left: regular reflection. The dotted arc separates a region of constant velocity (above) from a nontrivial region. Self-similar potential flow changes type from hyperbolic (above) to parabolic to elliptic across the arc. Right: single Mach reflection.

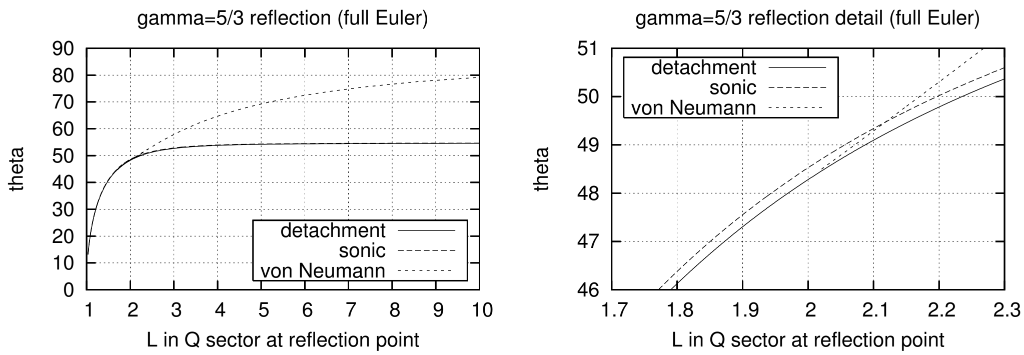


Figure 5. Left: transition angles predicted by each criterion (sonic and detachment almost coincide); right: detail.

wall $\hat{B}$ (see Figure 3), meeting in the origin and enclosing an angle $180^\circ - \theta$. For $t < 0$ a vertical incident shock approaches the corner from the left, reaching it at $t = 0$; for $t > 0$ it continues along $\hat{B}$, while a complex pattern is reflected back from the corner. For regular reflection, the incident and reflected shock meet in a point $\vec{\xi}$. An observer travelling in the reflection point will observe a flow expanding at a constant rate, approaching a local RR as in Figure 1 left as $t \uparrow +\infty$.

To understand self-similarity intuitively, focus on the corner between the two walls in Figure 3 right. $t \uparrow \infty$ corresponds to zooming into the corner whereas $t \downarrow 0$ corresponds to zooming infinitely far away from the corner.

The three transition criteria discussed for steady reflection specify angles θ_d (detachment), θ_s (sonic) and θ_N (von Neumann), depending on γ and L_Q, so that RR occurs for larger θ whereas IRR occurs for smaller θ. (Here, L_Q is the Q region Mach number as seen by an observer traveling in the intersection point of incident shock and $\hat{B}$ (= reflection point, in the RR case); of course an observer stationary in the corner will perceive a different velocity in the Q region.) Note that $\theta_d \leq \theta_s, \theta_N$ always. Figure 5 compares the criteria in the case of monatomic gas ($\gamma = 5/3$).

It has also been proposed that the correct criterion may not be the same in steady and self-similar flow (see below), or that there may be bistable cases where RR and IRR can both occur (see [16, 19]).

Nevertheless, it seems that there is an overall preference for the sonic criterion in the scientific community, at least for self-similar reflection.

Numerical and physical experiments are hampered by various difficulties and have not been able to select the correct criterion. For example numerical dissipation or physical viscosity smear the shocks and cause boundary layers that interact with the reflection pattern and can cause "spurious Mach stems" [28]. Moreover, θ_d and θ_s are only fractions of a degree apart (see Figure 5 right), a resolution that even sophisticated experiments (e.g. [24]) have been unable to reach. To quote [1]: "For this reason it is almost impossible to distinguish experimentally between the sonic and detachment criteria."

Constructing exact solutions of most genuinely multi-dimensional flow problems is infeasible or restricted to severely simplified equations. Moreover it would be prohibitively expensive if it could only confirm results that have already been obtained many orders of magnitude faster by numerical or physical experiments, unless the certainty of mathematical proof is needed. Regular reflection appears to be the first instance where rigorous analysis might make a genuine contribution by answering a problem that could not be resolved unambiguously by other techniques.

Results

In this article, using techniques developed in [11], regular reflection is constructed as a self-similar solution of compressible potential flow, with polytropic (γ-law) gas. While classical regular/Mach reflection studies vertical incident shocks, we consider the non-vertical cases too (these may not arise from any $t < 0$ flow), including cases where $\theta > \frac{\pi}{2}$.

Most importantly, for some values of γ and upstream Mach number M_I, in particular $\gamma = 5/3$ and $M_I = 1$, *every* θ near θ_s can be covered. This shows rigorously that criteria *stronger* than the sonic criterion are false, at least for potential flow with this choice of parameters.

As discussed above, there is some tendency to believe that regular reflection does not persist beyond the sonic criterion; ongoing work aims to show this rigorously, at least under mild assumptions. This would rule out the *weaker* criteria as well, in particular the detachment criterion, hence prove that sonic is correct. The problem of weak vs. strong reflection (see above) would vanish as well.

However, for now the success is qualified: potential flow lacks contact discontinuities, so *after* the transition to (say) SMR the flow pattern must be *qualitatively* different from the full Euler flow. It is still possible that the two models may have different transition criteria (however, the author believes that this is not the case).

Although some genuinely multi-dimensional exact solutions have been constructed for steady Euler flow, self-similar Euler flow is an open and inherently rather difficult problem. But again, it seems unlikely that numerical or experimental techniques will yield a clear — let alone universally accepted — answer soon, so rigorous analysis would be very valuable.

Here is the precise result:

Theorem 1.1. *Consider potential flow, as discussed in Section 1. Consider a wall* $\hat{A} = (-\infty,0) \times \{0\}$ *(see Figure 6), a second wall ray* $\hat{B}$ *at a clockwise angle* $180^\circ - \theta$ *from* $\hat{A}$*, and an incident shock Q, at a clockwise angle* $180^\circ - \beta_Q$ *from* $\hat{A}$*, meeting* $\hat{B}$ *in the reflection*

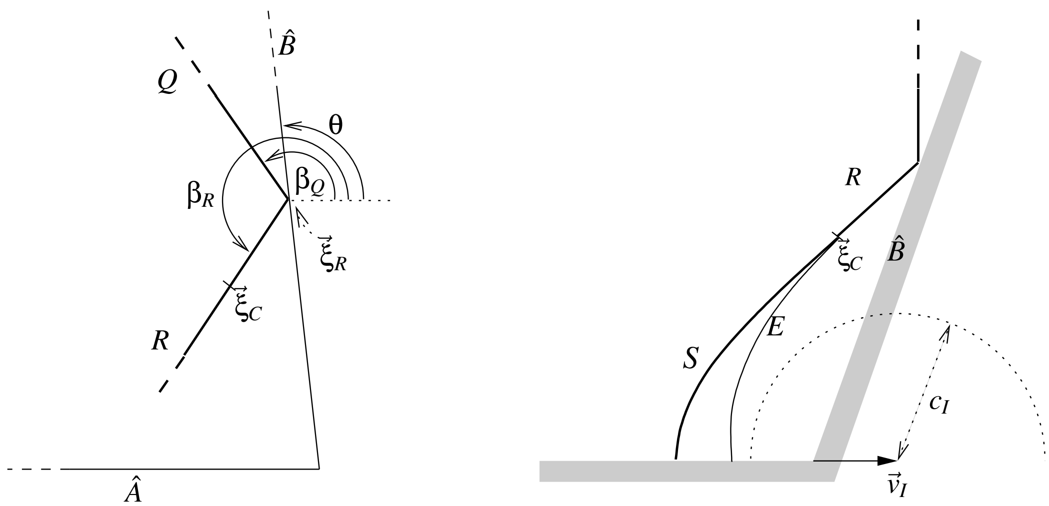


Figure 6. Left: a local RR pattern; right: the curved portion S of the reflected shock has $L_d \leq 1$, hence must be left of the envelope E, which bounds it away from the dotted circle and from $\hat{B}$.

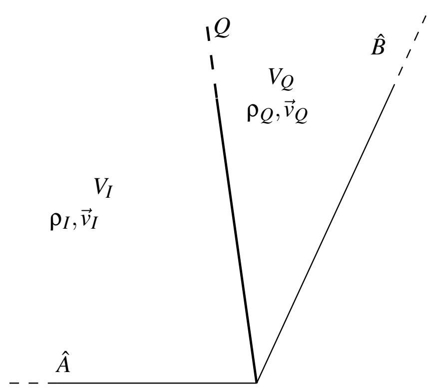


Figure 7. The initial data is constant in each of two sectors that are separated by the incident shock Q

point $\vec{\xi}_R$. Assume that there is a corresponding reflected shock R in $\vec{\xi}_R$, emanating down and left (or vertically down). Define

$$V_I := \{(x,y) \in \mathbb{R}^2 : y > 0,\ -\infty < x < y\cot(\beta_Q)\}$$
$$V_Q := \{(x,y) \in \mathbb{R}^2 : y > 0,\ y\cot(\beta_Q) < x < y\cot\theta\},$$
$$V := \{(x,y) \in \mathbb{R}^2 : y > 0,\ -\infty < x < y\cot\theta\}$$

(see Figure 7).

1. *Assume the* sonic criterion *holds: $L_R > 1$ in $\vec{\xi}_R$ in the sector below R.*
2. *Assume that*
$$|\vec{v}_I \cdot \vec{n}_B| \leq c_I \tag{1.1}$$
3. *Envelope condition: of the two[2] points on the R shock with $L_d = 1$, let $\vec{\xi}_C^{(0)}$ be the one closer to $\vec{\xi}_R$. Consider shocks with upstream data $\vec{v}_I, \rho_I$ that go from $\vec{\xi}_C^{(0)}$ counterclockwise and satisfy $L_d \leq 1$ in every point. Assume that all such shocks reach $\hat{A}$ before meeting $\hat{B}$ or the circle with center $\vec{v}_I$ and radius c_I.*

Then there exists a weak[3] solution $\phi = \phi(t,x,y) \in C^{0,1}([0,\infty) \times \overline{V})$ of

$$\text{unsteady potential flow} \qquad \text{for } t > 0,\ \vec{x} \in V, \tag{1.2}$$
$$\nabla\phi \cdot \vec{n} = 0 \qquad \text{on } \partial V, \tag{1.3}$$
$$\rho = \rho_I, \quad \nabla\phi = \vec{v}_I \qquad \text{for } t = 0,\ \vec{x} \in V_I, \tag{1.4}$$
$$\rho = \rho_Q, \quad \nabla\phi = \vec{v}_Q \qquad \text{for } t = 0,\ \vec{x} \in V_Q. \tag{1.5}$$

Of course existence by itself merely validates that potential flow has interesting solutions. In addition, detailed results about the structure of the weak solution can be obtained (see Remark 2.28); most importantly, the flow patterns are of RR type.

Remark 1.2. By weak solution we mean that

$$\nabla\phi(0,\vec{x}) = \vec{v}_I \qquad \text{for a.e. } \vec{x} \in V_I \tag{1.6}$$
$$\nabla\phi(0,\vec{x}) = \vec{v}_Q \qquad \text{for a.e. } \vec{x} \in V_Q \tag{1.7}$$

and

$$\int_\Omega \rho\vartheta_t + \rho\nabla\phi \cdot \nabla\vartheta\, d\vec{x}\, dt + \int_{V_I} \vartheta(0,\vec{x})\rho_I d\vec{x} + \int_{V_Q} \vartheta(0,\vec{x})\rho_Q d\vec{x} = 0$$

for all test functions $\vartheta \in C_c^\infty(\overline{\Omega})$.

(For $\phi \in C^{0,1}(\overline{\Omega})$, the velocity $\nabla\phi$ is a.e. well-defined on $\{0\} \times V$, but ϕ_t and hence ρ may not be well-defined.)

[2] see Section 1
[3] see Remark 1.2

Remark 1.3. Condition (1.1) and the envelope condition are merely technical. The envelope condition is needed in some cases to prove the shock does not vanish (which is never observed in numerics); none of the other estimates requires it. Both conditions can probably be removed by future research.

Related work on constructing exact solutions

In recent years multi-dimensional compressible inviscid flow has received renewed attention, after several recent breakthroughs brought the theory of one-dimensional compressible flow to a satisfactory state [14, 2, 23, 3].

[11] (see also [10, 9]) studies supersonic flow onto a solid wedge. For sufficiently sharp wedges, the steady solution consists of a straight shock on each side of the wedge, emanating downstream and separating two constant-state regions. In inviscid models this shock wave must keep the downstream velocity tangential to the wedge surface (slip condition). As for regular reflection, there are two different shocks for each (small) wedge angle, a *weak* and a *strong* shock. The weak shock is more commonly observed, but no mathematical argument was known to favor it prior to [11]. In that article, an exact solution was constructed for a wedge at rest in stagnant air, accelerated instantaneously to (sufficiently high) supersonic speed at time 0. The resulting flow pattern is self-similar and has a *weak* shock at the wedge tip.

Many of the techniques in [11] are essential in the present article.

The most closely related work, and so far the only other paper that proves global existence of some nontrivial time-dependent solution of potential flow is [5]: using different techniques, they construct exact solutions for regular reflection, assuming sufficiently blunt wedges ($\theta \approx \frac{\pi}{2}$).

Some prior work studies reflection and other problems for simplified models of gas dynamics. [4] consider regular reflection for the unsteady transonic small disturbance equation as model. [30] studies the same problem for the pressure-gradient system. The monographs [29, 21] compute various self-similar flows numerically and present some analysis and simplified models.

Potential flow

Here we briefly present derivation and elementary results for potential flow. More information can be found in [11].

Consider the isentropic Euler equations of compressible gas dynamics in d space dimensions:

$$\rho_t + \nabla \cdot (\rho \vec{v}) = 0 \tag{1.8}$$

$$(\rho \vec{v})_t + \sum_{i=1}^{d} (\rho v^i \vec{v})_{x^i} + \nabla(p(\rho)) = 0, \tag{1.9}$$

Hereafter, ∇ denotes the gradient with respect either to the space coordinates $\vec{x} = (x^1, x^2, \cdots, x^d)$ or the similarity coordinates $t^{-1}\vec{x}$. $\vec{v} = (v^1, v^2, \cdots, v^d)$ is the velocity of the gas, ρ the density, $p(\rho)$ pressure. In this article we consider only polytropic pressure laws (γ-laws) with $\gamma \geq 1$:

$$p(\rho) = \frac{c_0^2 \rho_0}{\gamma} \left(\frac{\rho}{\rho_0} \right)^{\gamma} \tag{1.10}$$

(here c_0 is the sound speed at density ρ_0).

For smooth solutions, substituting (1.8) into (1.9) yields the simpler form

$$\vec{v}_t + \vec{v} \cdot \nabla^T \vec{v} + \nabla(\pi(\rho)) = 0. \tag{1.11}$$

Here π is defined as

$$\pi(\rho) = c_0^2 \cdot \begin{cases} \frac{(\rho/\rho_0)^{\gamma-1} - 1}{\gamma - 1}, & \gamma > 1 \\ \log(\rho/\rho_0), & \gamma = 1. \end{cases}$$

This π is C^∞ in $\rho \in (0, \infty)$ and $\gamma \in [1, \infty)$ and has the property

$$\pi_\rho = \frac{p_\rho}{\rho}.$$

If we assume *irrotationality*

$$v^i_j = v^j_i$$

(where $i, j = 1, \dots, d$), then the Euler equations are reduced to potential flow:

$$\vec{v} = \nabla_{\vec{x}} \phi$$

for some scalar *potential*[4] function ϕ. For smooth flows, substituting this into (1.11) yields, for $i = 1, \dots, d$,

$$0 = \phi_{it} + \nabla \phi_i \cdot \nabla \phi + \pi(\rho)_i = \big(\phi_t + \frac{|\nabla \phi|^2}{2} + \pi(\rho) \big)_i.$$

Thus, for some constant A,

$$\rho = \pi^{-1}(A - \phi_t - \frac{|\nabla \phi|^2}{2}). \tag{1.12}$$

[4]We consider simply connected domains; otherwise ϕ might be multivalued.

Substituting this into (1.8) yields a single second-order quasilinear hyperbolic equation, the *potential flow* equation, for a scalar field ϕ:

$$\big(\rho(\phi_t,|\nabla\phi|)\big)_t + \nabla\cdot\big(\rho(\phi_t,|\nabla\phi|)\nabla\phi\big) = 0. \tag{1.13}$$

Henceforth we omit the arguments of ρ. Moreover we eliminate A with the substitution

$$A \leftarrow 0, \qquad \phi(t,\vec{x}) \leftarrow \phi(t,\vec{x}) - tA$$

(so that $\phi_t \leftarrow \phi_t - A$). Hence we use

$$\rho = \pi^{-1}(-\phi_t - \frac{1}{2}|\nabla\phi|^2) \tag{1.14}$$

from now on.

Using $c^2 = p_\rho$ and

$$(\pi^{-1})' = (\pi_\rho)^{-1} = (\frac{p_\rho}{\rho})^{-1} = \frac{\rho}{c^2} \tag{1.15}$$

the equation can also be written in nondivergence form:

$$\phi_{tt} + 2\nabla\phi_t\cdot\nabla\phi + \sum_{i,j=1}^{d}\phi_i\phi_j\phi_{ij} - c^2\Delta\phi = 0 \tag{1.16}$$

(1.16) is hyperbolic (as long as $c > 0$). For polytropic pressure law the local sound speed c is given by

$$c^2 = c_0^2 + (\gamma-1)(-\phi_t - \frac{1}{2}|\nabla\phi|^2). \tag{1.17}$$

Our initial data is self-similar: it is constant along rays emanating from $\vec{x} = (0,0)$. Our domain V is self-similar too: it is a union of rays emanating from $(t,x,y) = (0,0,0)$. In any such situation it is expected — and confirmed by numerical results — that the solution is self-similar as well, i.e. that $\rho,\vec{v}$ are constant along rays $\vec{x} = t\vec{\xi}$ emanating from the origin. Self-similarity corresponds to the ansatz

$$\phi(t,\vec{x}) := t\psi(\vec{\xi}), \qquad \vec{\xi} := t^{-1}\vec{x}. \tag{1.18}$$

Clearly, $\phi \in C^{0,1}(\Omega)$ if and only if $\psi \in C^{0,1}(\complement W)$. This choice yields

$$\vec{v}(t,\vec{x}) = \nabla\phi(t,\vec{x}) = \nabla\psi(t^{-1}\vec{x}),$$
$$\rho(t,\vec{x}) = \pi^{-1}(-\phi_t - \frac{1}{2}|\nabla\phi|^2) = \pi^{-1}(-\psi + \vec{\xi}\cdot\nabla\psi - \frac{1}{2}|\nabla\psi|^2).$$

The expression for ρ can be made more pleasant (and independent of $\vec{\xi}$) by using

$$\chi(\vec{\xi}) := \psi(\vec{\xi}) - \frac{1}{2}|\vec{\xi}|^2;$$

this yields

$$\rho = \pi^{-1}(-\chi - \frac{1}{2}|\nabla\chi|^2). \tag{1.19}$$

$\nabla\chi = \nabla\psi - \vec{\xi}$ is called *pseudo-velocity*.

(1.13) then reduces to

$$\nabla \cdot (\rho\nabla\chi) + 2\rho = 0 \tag{1.20}$$

(or $+d\rho$, in d dimensions) which holds in a distributional sense. For smooth solutions we obtain the non-divergence form

$$(c^2 I - \nabla\chi\nabla\chi^T) : \nabla^2\chi = (c^2 - \chi_\xi^2)\chi_{\xi\xi} - 2\chi_\xi\chi_\eta\chi_{\xi\eta} + (c^2 - \chi_\eta^2)\chi_{\eta\eta} = |\nabla\chi|^2 - 2c^2 \tag{1.21}$$

Another convenient form is

$$(c^2 I - \nabla\chi\nabla\chi^T) : \nabla^2\psi = (c^2 - \chi_\xi^2)\psi_{\xi\xi} - 2\chi_\xi\chi_\eta\psi_{\xi\eta} + (c^2 - \chi_\eta^2)\psi_{\eta\eta} = 0. \tag{1.22}$$

Here, (1.17) for polytropic pressure law yields

$$c^2 = c_0^2 + (\gamma - 1)(-\chi - \frac{1}{2}|\nabla\chi|^2) \tag{1.23}$$

Remark 1.4. (1.20) inherits a number of symmetries from (1.8), (1.9):

1. It is invariant under rotation.
2. It is invariant under reflection.
3. It is invariant under translation in $\vec{\xi}$, which is not as trivial as translation in $\vec{x}$: it corresponds to the Galilean transformation $\vec{v} \leftarrow \vec{v} + \vec{v}_0$, $\vec{x} \leftarrow \vec{x} - \vec{v}_0 t$ (with constant $\vec{v}_0 \in \mathbb{R}^d$) in $(t, \vec{x})$ coordinates. This is sometimes called *change of inertial frame*.

(1.21) is a PDE of mixed type. The type is determined by the *(local) pseudo-Mach number*

$$L := \frac{|\nabla\chi|}{c}, \tag{1.24}$$

with $0 \le L < 1$ for elliptic (pseudo-subsonic), $L = 1$ for parabolic (pseudo-sonic), $L > 1$ for hyperbolic (pseudo-supersonic) regions.

While velocity $\vec{v}$ is motion relative to space coordinates $\vec{x}$, pseudo-velocity

$$\vec{z} := \nabla\chi$$

is motion relative to similarity coordinates $\vec{\xi}$ *at time* $t = 1$.

The simplest class of solutions of (1.21) are the *constant-state solutions*: ψ affine in $\vec{\xi}$, hence $\vec{v}$, ρ and c constant. They are elliptic in a circle centered in $\vec{\xi} = \vec{v}$ with radius c, parabolic on the boundary of that circle and hyperbolic outside.

If we study a function called (e.g.) $\tilde{\chi}$, then $\tilde{\psi}$, $\tilde{\rho}$, $\tilde{L}$ etc. will refer to the quantities computed from it as ψ, ρ, L are computed from χ (e.g. $\tilde{\psi} = \tilde{\chi} + \frac{1}{2}|\vec{\xi}|^2$). We will tacitly use this notation from now on.

Potential flow shocks

Consider a ball U and a simple smooth curve S so that $U = U^u \cup S \cup U^d$ where U^u, U^d are open, connected, and S, U^u, U^d disjoint. Consider $\chi : U \to \mathbb{R}$ so that $\chi = \chi^{u,d}$ in $U^{u,d}$ where $\chi^{u,d} \in C^2(\overline{U^{u,d}})$.

χ is a weak solution of (1.20) if and only if it is a strong solution in each point of U_- and U_+ and if it satisfies the following conditions in each point of S:

$$\chi^u = \chi^d, \tag{1.25}$$

$$\vec{n} \cdot (\rho^u \nabla \chi^u - \rho^d \nabla \chi^d) = 0 \tag{1.26}$$

Here $\vec{n}$ is a normal to S.

(1.25) and (1.26) are the *Rankine-Hugoniot* conditions for self-similar potential flow shocks. They do not depend on $\vec{\xi}$ or on the shock speed explicitly; these quantities are hidden by the use of χ rather than ψ. The Rankine-Hugoniot conditions are derived in the same way as those for the full Euler equations (see [12, Section 3.4.1]).

Note that (1.25) is equivalent to

$$\psi^u = \psi^d. \tag{1.27}$$

Taking the tangential derivative of (1.25) resp. (1.27) yields

$$\frac{\partial \chi^u}{\partial t} = \frac{\partial \chi^d}{\partial t}, \tag{1.28}$$

$$\frac{\partial \psi^u}{\partial t} = \frac{\partial \psi^d}{\partial t}. \tag{1.29}$$

The shock relations imply that the tangential velocity is continuous across shocks.

Define $(z_u^x, z_u^y) := \vec{z}_u := \nabla \chi^u$ and $(v_u^x, v_u^y) := \vec{v}_u := \nabla \psi^u$. Abbreviate $z_u^t := \vec{z}_u \cdot \vec{t}$, $z_u^n := \vec{z}_u \cdot \vec{n}$, and same for v instead of z. Same definitions for d instead of u. We can restate the shock relations as

$$\rho_u z_u^n = \rho_d z_d^n, \tag{1.30}$$

$$z_u^t = z_d^t. \tag{1.31}$$

Using the last relation, we often write z^t without distinction.

The *shock speed* is $\sigma = \vec{\xi}\cdot\vec{n}$, where $\vec{\xi}$ is any point on the shock. A shock is *steady* in a point if its tangent passes through the origin. We can restate (1.30) as

$$\rho_u v_u^n - \rho_d v_d^n = \sigma(\rho_u - \rho_d)$$

which is a more familiar form.

We focus on $\rho_u, \rho_d > 0$ from now on, which will be the case in all circumstances. If $\rho_u = \rho_d$ in a point, we say the shock *vanishes*; in this case $z_d^n = z_u^n$ in that point, by (1.31). In all other cases z_d^n, z_u^n must have equal sign by (1.31); we fix $\vec{n}$ so that $z_d^n, z_u^n > 0$. This means the normal points *downstream*. The shock is *admissible* if and only if $\rho_u \le \rho_d$ which is equivalent to $z_u^n \ge z_d^n$.

A shock is called *pseudo-normal* in a point $\vec{\xi}$ if $z^t = 0$ there. For $\vec{\xi} = 0$, this means that the shock is *normal* ($v^t = 0$), but for $\vec{\xi} \neq 0$ normal and pseudo-normal are not always equivalent.

It is good to keep in mind that for a *straight* shock, ρ_d and $\vec{v}_d$ are constant if ρ_u and $\vec{v}_u$ are. Obviously $\vec{z}_d$ may vary in this case.

We will need two detailed results.

Proposition 1.5. *Consider a fixed point on a shock with upstream density ρ_u and pseudo-velocity $\vec{z}_u$ held fixed while we vary the normal. Define $\beta := \measuredangle(\vec{z}_u, \vec{n})$. ρ_d is strictly decreasing in $|\beta|$, whereas $L_d, |\vec{z}_d|$ are strictly increasing. c_d is strictly decreasing for $\gamma > 1$, constant otherwise. Moreover*

$$(\partial_\beta \vec{v}_d)\cdot\vec{n} = (\partial_\beta \vec{z}_d)\cdot\vec{n} = z^t\Big(\frac{\partial z_d^n}{\partial z_u^n} - 1\Big), \tag{1.32}$$

$$(\partial_\beta \vec{v}_d)\cdot\vec{t} = (\partial_\beta \vec{z}_d)\cdot\vec{t} = z_d^n - z_u^n. \tag{1.33}$$

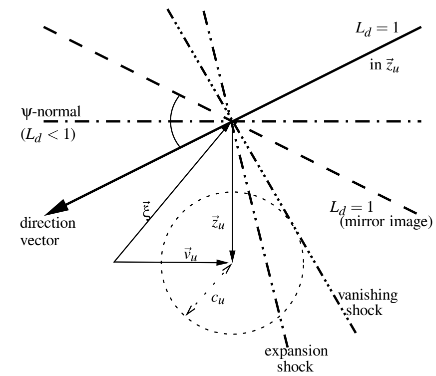
If $\vec{z}_u = (z_u^x, 0)$ with $z_u^x > 0$, then z_d^x is increasing in $|\beta|$.

Proof. This is [11, Proposition 2.5.1]. □

Proposition 1.6. *Consider a straight shock with $v_u^x = 0$, $v_u^y < 0$ and downstream normal $\vec{n} = (\sin\beta, -\cos\beta)$ through $\vec{\xi} = (0,\eta)$. For every $\beta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ there is a unique $\eta = \eta_0^* \in \mathbb{R}$ so that $v_d^y = 0$. η_0^* and the corresponding downstream data are analytic functions of β. η_0^* is strictly increasing in $|\beta|$.*

For the shock passing through $(0,\eta_0^*)$, let $\vec{\xi}_L^*$ and $\vec{\xi}_R^*$ be the two points with $L_d = \sqrt{1-\varepsilon}$. These points are analytic functions of β. L_u^n, ρ_d and z_u^n are increasing functions[5] of β; v_d^x

[5] All of these are independent of the location along the (straight) shock.



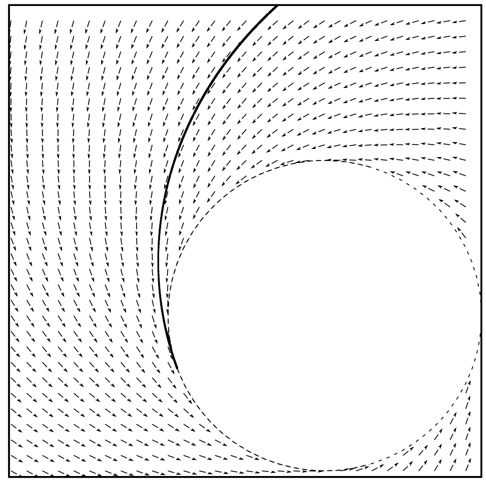


Figure 8. Left: through each $\vec{\xi}$ farther than c_u from $\vec{v}_u$ there are exactly two straight shocks (solid, dashed) with $L_d = 1$, mirror images of each other. The shocks with $L_d \leq 1$ are between them (indicated by arc left of $\vec{\xi}$). The solid lines define the direction field whose integral curves are "envelopes". Right: no shock with $L_d \leq 1$ can approach $\vec{v}_u$ faster (in counterclockwise direction) than the counterclockwise envelope.

and L_d^n *are decreasing functions of* β. *For* $\beta \in [0, \frac{\pi}{2})$, η_L^* *is a strictly decreasing function of* β *with range* $(\underline{\eta}_L^*, \overline{\eta}_0^*]$, *where* $\overline{\eta}_0^*$ *is the* η_0^* *for* $\beta = 0$, *and* $\underline{\eta}_L^*$ *is some* negative *constant.*

Proof. This is [11, Proposition 2.6.2]. □

Envelope

Many techniques in this paper are similar to the construction in [11]; Section 4.2 in loc.cit. is a good overview. However, in [11, Proposition 4.11.1], a lower bound for the shock strength is obtained by a delicate argument using the density. Although this argument would reproduce the results of [5] (namely RR existence for $\theta \approx \frac{\pi}{2}$), it cannot prove the main new contribution of this paper: existence (at least in some cases like $\gamma = 5/3, M_I = 1$) of RR for $\theta \approx \theta_s$ (with $\theta > \theta_s$), where θ_s is the smallest θ allowed by the sonic criterion (see Section 1).

For this goal, a new idea is needed: as we will show, the curved portion S of the reflected shock in Figure 4 left has an elliptic region of potential flow on its right (downstream) side, hence downstream pseudo-Mach number $L_d \leq 1$ everywhere. Such a shock cannot vanish until it reaches the circle of radius c_I around $\vec{v}_I$; moreover $L_d \leq 1$ is a constraint on the possible shock tangents, so that the shock cannot reach the circle quickly. It is bounded away from the circle by the *envelope*:

Definition 1.7. Given constant upstream velocity $\vec{v}_u$ and sound speed c_u. Consider a shock through a point $\vec{\xi}$ with $|\vec{z}_u| = |\vec{v}_u - \vec{\xi}| > c_u$. As shown in Proposition 1.5, L_d is strictly increasing in $|\beta|$ where $\beta = \measuredangle(\vec{z}_u, \vec{n}) \in (-\pi, \pi]$ is the counterclockwise angle from $\vec{z}_u$ to $\vec{n}$.

There are exactly two shock normals so that $L_d = 1$. They are mirror-images of each other under reflection across the line with tangent $\vec{z}_u$ through $\vec{\xi}$ (see Figure 8 left). Consider the one with $\beta > 0$; its tangent spans the solid line on Figure 8 left. The tangents for different $\vec{\xi}$ form a direction field. The *counterclockwise envelope* is defined to be a maximal integral curve of that direction field (see Figure 8 right).

We can parametrize the envelope (like other smooth shocks) in polar coordinates (r, ϕ) centered in $\vec{v}_u$, by a function $\phi \mapsto r^*(\phi)$ (because the shock relations do not admit shocks with a tangent passing through $\vec{v}_u$). The counterclockwise envelope satisfies an ODE of the form

$$\frac{\partial r^*}{\partial \phi}(\phi) = -f(r^*(\phi)) \tag{1.34}$$

for some analytic f.

We will not need the fact, but explicit formulas for f can be derived. For example for $\gamma > 1$,

$$f(r) = r\sqrt{\frac{1 - \frac{\gamma+1}{\gamma-1+2(r/c_u)^{-2}} \cdot \left(\frac{\gamma+1}{2+(\gamma-1)(r/c_u)^2}\right)^{\frac{2}{\gamma-1}}}{\frac{\gamma+1}{\gamma-1+2(r/c_u)^{-2}} - 1}} \tag{1.35}$$

Moreover it can be shown that the envelope always reaches the circle, meeting it in a point where the envelope is C^1, but not more regular, and tangent to the circle; it cannot be continued beyond that point.

Proposition 1.8. *Let some smooth shock be parametrized as $\phi \mapsto r(\phi)$; let the envelope be parametrized by $\phi \mapsto r^*(\phi)$. Assume that $L_d < 1$ in every point of the shock. If $r(\phi_0) \geq r^*(\phi_0)$ for some ϕ_0, then $r(\phi) > r^*(\phi)$ for $\phi > \phi_0$. If instead $L_d > 1$ in every point of the shock, then $r(\phi) < r^*(\phi)$ for $\phi > \phi_0$.*

Proof. Our discussion above can be restated as follows: $L_d < 1$ for the shock means $-\beta^* < \beta < \beta^*$ where β^* is the β for the envelope. Hence

$$|\frac{\partial r}{\partial \phi}| < f(r(\phi)).$$

In particular

$$\frac{\partial r}{\partial \phi} > -f(r(\phi)).$$

Since f is smooth, in particular Lipschitz, the invariant region theorem shows that the shock cannot meet the envelope for $\phi > \phi_0$. ☐

In Proposition 2.21 we will exploit this fact to bound the curved portion of the reflected shock away from the downstream wall and to ensure its uniform strength.

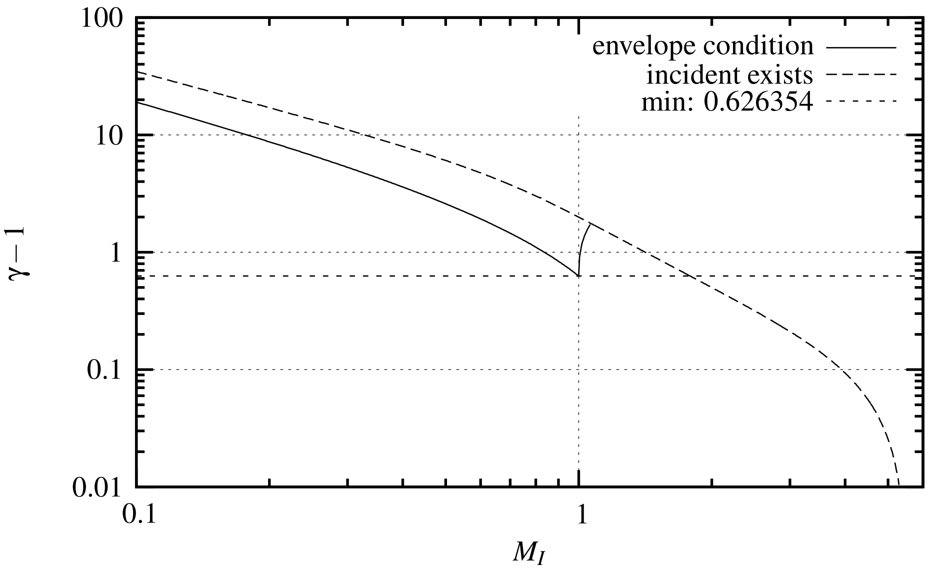


Figure 9. For $\beta_Q = 0$ (vertical incident shocks) and the set of M_I, γ enclosed below the dashed and above the solid line, solutions can be constructed for all $\theta \in (\theta_s, \frac{\pi}{2}]$.

Sonic criterion

We focus on the classical case of vertical incident shocks. In some cases, Theorem 1.1 allows us to construct a regular reflection pattern like Figure 4 left for every myrefsection:refl). As $\theta \downarrow \theta_s$, the dotted$\theta > \theta_s$ near θ_s, where θ_s is the smallest θ allowed by the sonic criterion (see Section parabolic arc in Figure 4 left approaches the reflection point.

To check whether the envelope condition is satisfied for a particular choice of θ and incident shock, it suffices to find the reflected shock and $\vec{\xi}_C$ on it (see Theorem 1.1) and to integrate the ODE (1.34) defining the envelope. Although the ODE is trivially separable, the resulting integral and nonlinear algebraic equation do not have an explicit solution except for special values of γ (see (1.35)). Numerical integration is needed to check whether the envelope meets $\hat{B}$ or the circle with center $\vec{v}_I$ and radius c_I before it meets $\hat{A}$.

In Figure 9, we consider arbitrary $\gamma \in [1, \infty)$ and $M_I \in (0, \infty)$ while fixing $\theta = \theta_s$. Values of γ and M_I above the dashed curve do not admit a vertical incident shock with zero velocity in the Q region (a similar phenomenon occurs in the full Euler equations). Values below both solid and dashed curve violate the envelope condtion. Values between solid and dashed curve do have an incident shock as well as a reflected shock that satisfies the envelope condition.

The smallest possible γ in that feasible region is $\gamma = 1.626354...$ with $M_I = 1$. In particular the monatomic gas case $\gamma = 5/3$ is covered, whereas $\gamma = 7/5$ or $\gamma = 4/3$ are not covered. (However, the latter values are also possible if we allow non-vertical incident shocks.) For

$\gamma = 5/3$, $M_I = 1$ we have $\theta_s = 55.4583...^\circ$; for $\theta = \theta_s$ the envelope meets $\hat{A}$ in the point $(-0.000012..., 0)$, just enough to avoid $\hat{B}$ and the circle.

While the proof of Theorem 1.1 itself is rigorous, checking the envelope condition is done numerically here, i.e. not a mathematical proof in the strict sense. However, the shock relations form a small system of nonlinear algebraic equations and the envelope is defined by (1.34), a scalar nonlinear ODE which is benign except for a mild singularity as $r \downarrow 1$. The numerical methods for these types of equations are well-understood and a complete convergence theory and error analysis is available — which is not at all the case for the full Euler or potential flow PDE. Another option is to study rigorous proofs in various asymptotic limits such as $M_I \downarrow 0$, $\gamma \uparrow \infty$. Moreover the envelope condition is most likely unnecessary since regular reflection up to $\theta = \theta_s$ is observed in numerics for many other values of γ and M_I as well. Since we expect that the condition will be eliminated by further research, it makes little sense to strive for absolute rigour at this point.

2 Construction of the flow

The elliptic region is constructed as follows: we define a function set $\mathcal{F}$ by imposing many constraints on a weighted Hölder space $C^{2,\alpha}_\beta$ (weighted to account for loss of regularity in the corners). An iteration $\mathcal{K} : \mathcal{F} \to C^{2,\alpha}_\beta$ is constructed so that its fixed points solve the PDE and boundary conditions for the elliptic region (see Remark 2.7). $\mathcal{F}$ and $\mathcal{K}$ depend on several parameters like γ, collected in a parameter vector λ. To show that $\mathcal{K}$ has a fixed point for all λ, we use Leray-Schauder degree theory.

Most of the effort is spent on showing that $\mathcal{K}$ does not have fixed points on $\partial\mathcal{F}$, which implies that $\mathcal{K}$ has the same Leray-Schauder degree for all λ. As $\partial\mathcal{F}$ is defined by constraints in the form of inequalities with continuous sides, this is achieved by showing that a fixed point satisfies the *strict* version of each inequality ($<$ instead of $\leq$).

A major technical difficulty are the parabolic arcs (dotted arc in Figure 4 left) where self-similar potential flow (1.22) degenerates from elliptic to parabolic. This problem has been solved in [11] (and, by different techniques, in [5]), by modifying the arc to be slightly elliptic, with boundary condition $L^2 = 1 - \varepsilon$, and obtaining estimates uniform in ε.

For a particular choice of λ the problem is much simpler (see Figure 18). In that case an explicit solution can be given and shown to be unique and have nonzero Leray-Schauder index. This implies that $\mathcal{K}$ has nonzero degree, hence at least one fixed point, for *every* λ. The fixed point is extended to a solution on the entire domain by adding the hyperbolic regions and interface shocks. Using the ε-uniform estimates as well as compactness, we can pass to the limit $\varepsilon \downarrow 0$ to obtain a solution of our problem.

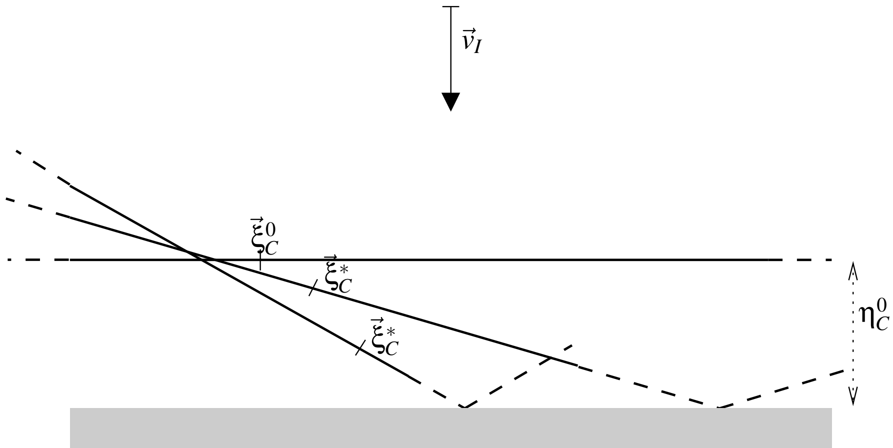


Figure 10. Perturbation from the trivial case of R parallel to the wall.

Parameter set and definitions

Instead of working in the setting of Theorem 1.1, it will be convenient to choose parameters in a different way.

Choose $\rho_I, c_I > 0$. Note that we may fix ρ_0 and c_0 in the pressure law (1.10) separately; however, given these constants (and γ), every other c is a function of ρ only (and vice versa).

Let $\varepsilon \geq 0$ be sufficiently small for the following. Consider a vertical downward velocity $\vec{v}_I$ onto a solid wall $\underline{B}$ (see Figure 10). According to Proposition 1.6, there is exactly one straight shock with upstream velocity $\vec{v}_u = \vec{v}_I$ and sound speed $c_u = c_I$ so that $\vec{v}_d = 0$; that shock is horizontal. Let $\eta^0_C > 0$ be its vertical coordinate. Of the two points on that shock with $L_d = \sqrt{1-\varepsilon}$, let $\vec{\xi}^0_C = (\xi^0_C, \eta^0_C)$ be the right one. By the same proposition, the shock belongs to a smooth one-parameter family of shocks, each called *R shock*, parametrized by $\eta^*_C \in (0, \eta^0_C]$, so that $v^y_d = 0$ and so that $\vec{\xi}^*_C = (\xi^*_C, \eta^*_C)$ is the right $L_d = \sqrt{1-\varepsilon}$ point. Define $M^y_I \in [-1, 0)$ to be v^y_I / c_I in these coordinates. Note that (1.1) rules out $M^y_I < -1$. Let $\vec{v}_R = \vec{v}_d$ be the downstream velocity of the R shock.

It is not clear whether there is an incident shock Q matching each reflected shock R. In fact for $\eta^*_C = \eta^0_C$, the R shock does not even meet $\underline{B}$, so clearly there is no RR. However, for the construction of the elliptic region, a Q shock or reflection pattern are not needed.

To complete the situation of Theorem 1.1, a wall $\hat{A}$ is needed. To satisfy the slip boundary condition $(\vec{v}_I - \vec{\xi}) \cdot \vec{n} = 0$ on $\hat{A}$, necessarily the extension of $\hat{A}$ to a line has to pass through $\vec{v}_I$. We fix $\hat{A}$ by choosing $\vec{\xi}_{AB}$ on $\underline{B}$.

Let E be the counterclockwise envelope starting in $\vec{\xi}_C$. If E meets $\underline{B}$ before it meets the circle with center $\vec{v}_I$ and radius c_I (Figure 11 left), let $\vec{\xi}_{EB}$ be that point. Otherwise (Figure

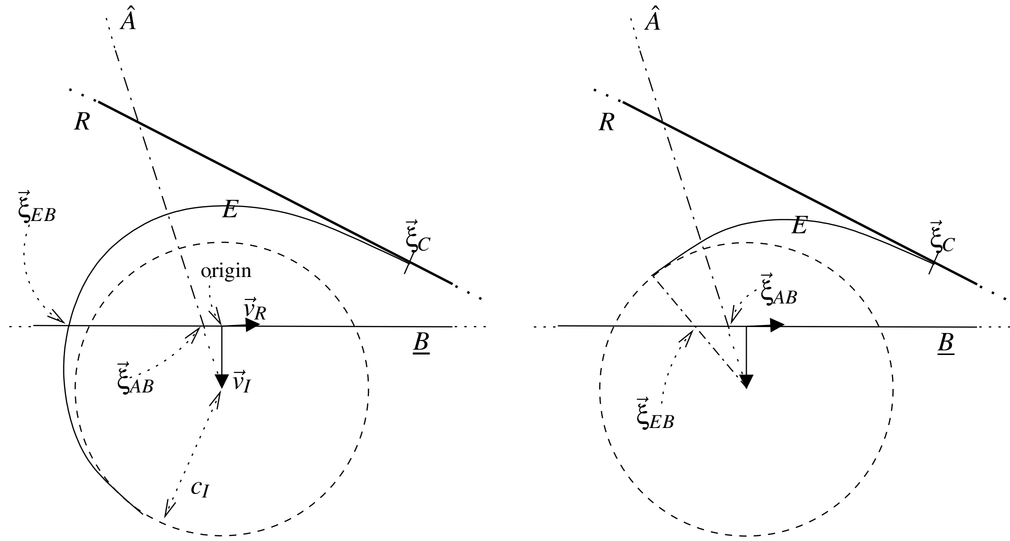


Figure 11. $\hat{A}$ is chosen so that (1) E reaches it before $\underline{B}$ or the dashed circle, and (2) it forms an angle $\leq 90°$ with R.

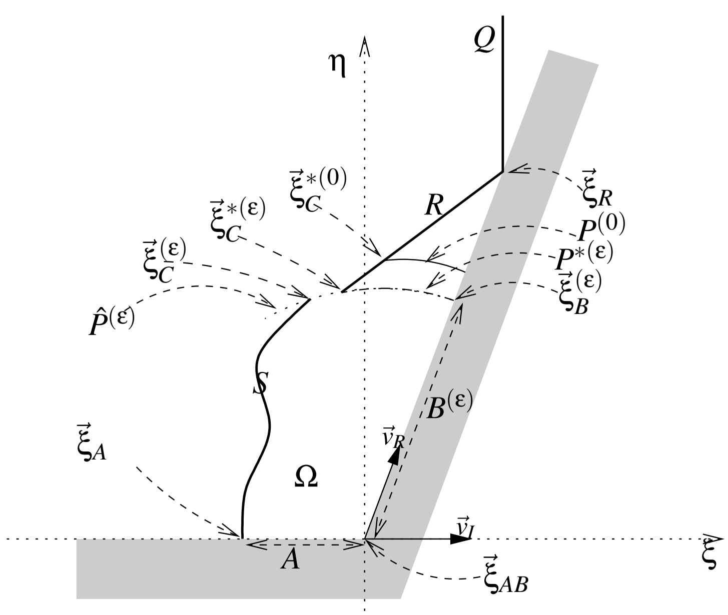


Figure 12. To avoid degeneracy, we impose a "slightly elliptic" boundary condition, $L^2 = 1 - \varepsilon$ for $\varepsilon > 0$, on $P^{(\varepsilon)}$. The shock S is free, along with the endpoints $\vec{\xi}_A$ and $\vec{\xi}_C^{(\varepsilon)}$ which may slide freely on $\hat{A}$ resp. $\hat{P}^{(\varepsilon)}$. But for fixed points $\vec{\xi}_C^{(\varepsilon)}$ can be shown to be close to $\vec{\xi}^{*(\varepsilon)}$, hence to $\vec{\xi}^{*(0)}$.

11 right) take the line through $\vec{v}_I$ and the meeting point of E and circle, and let $\vec{\xi}_{EB}$ be its intersection with $\underline{B}$. We allow

$$\xi_{AB} \in (\xi_{EB}, v_R^x] \tag{2.1}$$

(and $\eta_{AB} = 0$ obviously). This constraint ensures that (1) the envelope meets $\hat{A}$ first, while (2) R and A form a sharp or right angle.

Given $\vec{\xi}_{AB}$ we let $\hat{B}$ be the part of $\underline{B}$ *right* of $\vec{\xi}_{AB}$. $\hat{A}$ is the half-line upwards starting in $\vec{\xi}_{AB}$ whose extension passes through $\vec{v}_I$. Let $\vec{n}_A$ be the unit normal of A pointing left, $\vec{n}_B$ the unit normal of $\hat{B}$ pointing down. Let $\vec{n}_R$ be the downstream (hence downwards) unit normal of the R shock. For each $\vec{n}_?$, $\vec{t}_?$ is always the corresponding unit tangent in *counterclockwise* direction.

Remark 2.1. Every local RR pattern that satisfies the conditions of Theorem 1.1 is covered by the parameter ranges defined above.

ρ_I and $\vec{v}_I$ define a potential ψ^I for the I region:

$$\psi^I(\vec{\xi}) = -\pi(\rho_I) - \frac{|\vec{v}_I|^2}{2} + \vec{v}_I \cdot \vec{\xi}.$$

Similar potentials ψ^R and ψ^Q (if an incident shock Q exists) are defined by $\rho_R, \vec{v}_R$ and $\rho_Q, \vec{v}_Q$.

Now we use Remark 1.4: invariance under translation. Translation in self-similar coordinates corresponds to a change of inertial frame, i.e. to adding a constant velocity to all $\vec{v}, \vec{\xi}$. Moreover we may rotate by Galilean invariance. This changes Figure 11 to Figure 12 which has the coordinates in which we originally posed the self-similar reflection problem.

Let $P^{*(\varepsilon)}$ be the circle arc centered in $\vec{v}_R$ with radius $c_R \cdot \sqrt{1-\varepsilon}$ (see Figure 12, where the coordinates have been changed), passing from $\vec{\xi}_B^{(\varepsilon)}$ on $\hat{B}$ counterclockwise to $\vec{\xi}_C^{*(\varepsilon)}$ on R, excluding the endpoints. (We omit the superscript ε if it is clear from the context.) $\vec{\xi}_C^*$ will be called the *expected* corner location. Let $B^{(\varepsilon)}$ be the part of $\hat{B}$ from $\vec{\xi}_{AB}$ to $\vec{\xi}_B$ (excluding the endpoints).

Take $\vec{n}_R$, $\vec{n}_Q$ to be the downstream unit normals of the shocks R, Q ($\vec{n}_R$ points towards $\hat{B}$). Let $\vec{n}_A, \vec{n}_B$ be outer unit normals of $\hat{A}, \hat{B}$, i.e. pointing away from the gas-filled sector V enclosed by $\hat{B}, \hat{A}$.

We choose an extended arc $\hat{P}$ that overshoots $\vec{\xi}_C^*$ by an angle $\delta_{\hat{P}} > 0$, which we choose continuous in $\gamma, \xi_{AB}, \eta_C^*$. The particular $\delta_{\hat{P}}$ is not important, but it may not depend on ε, and $\hat{P}$ may not have a horizontal tangent in Figure 14 coordinates.

P^*, $\hat{P}$, and later P, are called *quasi-parabolic arc* (or *parabolic arcs*, by abuse of terminology, or short *arcs*).

Parameter set The Definitions 2.2, 2.5 and 2.6 use many constants and other objects that will be fixed later on. In all of these cases, an upper (or lower) bound for each constant is found. Whenever we say "for sufficiently small constants" (etc.), we mean that bounds for them are adjusted. To avoid circularity, it is necessary to specify which bounds may depend on the values of which other bounds. In the following list, bounds on a constant may only depend on bounds on *preceding* constants.

$$\begin{aligned}&\delta_{\hat{P}}, C_L, C_\eta, \delta_{SB}, \delta_{Cc}, \delta_{P\sigma}, \delta_{Pn}, \delta_d, \delta_\rho, \delta_{Lb},\\ &\quad C_{Pt}, C_{vtR}, C_{vnA}, C_{Sn}, \delta_{vtA}, \delta_{vnB}, \delta_o, C_d, \varepsilon, C_C, r_I, \alpha, \beta.\end{aligned} \tag{2.2}$$

The constants C_C, r_I, α, β may depend on ε itself, not just on an upper bound. r_I may also depend on ψ. The reader may convince himself that the remainder of the paper does respect this order.

The parameters γ, η_C^* and ξ_{AB} used in Leray-Schauder degree arguments will be restricted to compact sets below so that any constant that can be chosen continuous in them might as well be taken independent of them. Dependence on other parameters like ρ_I will not be pointed out explicitly.

Constants $\delta_?$ as well as $\alpha, \beta, r_I, \varepsilon$ are meant to be small and positive, constants $C_?$ are meant to be large and finite.

Definition 2.2. For the purposes of degree theory we define a restricted parameter set

$$\Lambda := \left\{\lambda = (\gamma, \eta_C^*, \xi_{AB}) : \gamma \in [1, \overline{\gamma}],\ \eta_C^* \in [\underline{\eta}_C^*, \overline{\eta}_C^*], \xi_{AB} \in [\underline{\xi}_{AB}, \overline{\xi}_{AB}]\right\}$$

where it is important that ξ_{AB} and v_R^x are the values in the coordinates of Figure 10 and Figure 11; clearly their values are entirely different in any other coordinate system we use. $\overline{\gamma} \in [1, \infty)$ is an arbitrary constant. Moreover,

$$\overline{\eta}_C^* := \eta_C^0 - \begin{cases} 0, & \gamma = 1, \\ C_\eta \cdot \varepsilon^{1/2}, & \gamma > 1, \end{cases} \tag{2.3}$$

and

$$\overline{\xi}_{AB} := v_R^x - \begin{cases} 0, & \gamma = 1, \\ C_\xi \cdot \varepsilon^{1/2}, & \gamma > 1, \end{cases} \tag{2.4}$$

where C_ξ, C_η (to be determined in Proposition 2.19) do not depend on ε or λ. $\underline{\eta}_C^*$ is a constant satisfying $0 < \underline{\eta}_C^* < \overline{\eta}_C^*$. Finally, $\underline{\xi}_{AB} \in (\xi_{EB}, \overline{\xi}_{AB}]$ may depend on γ and η_C^*.

Proposition 2.3. Λ *contains* $(\gamma, \eta_C^*, \xi_{AB}) = (1, \eta_C^0, v_R^x)$ *and is path-connected, for* ε *sufficiently small (depending on* C_η, C_ξ*) and* $\underline{\xi}_{AB}$ *sufficiently close to* ξ_{EB}.

Proof. 1. We note that the interval $(\xi_{EB}, v_R^x]$ has boundaries that are continuous functions of λ.

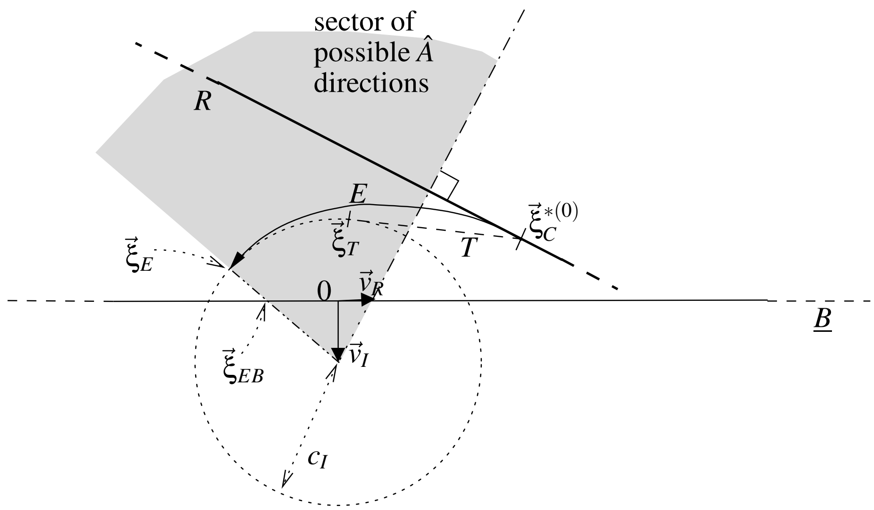


Figure 13. The shaded sector consists of all $\hat{A}$ rays that (1) form a sharp angle with R, while (2) meeting E before E meets $\underline{B}$ or the circle.

2. The interval is always nonempty: consider the coordinate system and setting of Figure 11, extended in Figure 13. Consider a line T through $\vec{\xi}_C^{*(0)}$ that touches the (upper half of the) circle with center $\vec{v}_I$ and radius c_I in a point $\vec{\xi}_T$. T can be considered a zero-strength shock (velocity $\vec{v}_I$, density ρ_I on both sides), with $L_d = 1$ in $\vec{\xi}_T$ and $L_d > 1$ elsewhere. Hence Proposition 1.8 applies: let $\phi \mapsto r(\phi)$ parametrize the line segment from $\vec{\xi}_C^{*(0)}$ to $\vec{\xi}_T$; let $\phi \mapsto r_E(\phi)$ parametrize E. Then $r_E(\phi) > r(\phi) > c_I$ on the interior of the corresponding ϕ interval, so the envelope E cannot touch the circle right of $\vec{\xi}_T$. Moreover, since we have assumed that $M_I^y \leq 1$ (restriction (1.1)), that means the circle either meets or intersects $\underline{B}$. If E meets $\underline{B}$ before it meets the circle, then necessarily it meets the part of $\underline{B}$ *left* of the circle first.

   On the other hand, the extremal choice $\xi_{AB} = v_R^x$ for $\hat{A}$ corresponds to (a segment of) the line through $\vec{v}_I$ and $\vec{v}_R$ (right side of the shaded sector in Figure 13), which is perpendicular to R. Its intersection with the circle is necessarily right of $\vec{\xi}_T$. Thus: if E meets the circle before it meets $\underline{B}$, then $\xi_{EB} < v_R^x$ necessarily. If E meets $\underline{B}$ before the circle, then it must meet it left of the origin, so $\xi_{EB} < 0 < v_R^x$. Either way the interval $(\xi_{EB}, v_R^x]$ is nonempty.

   Threfore the interval $(\underline{\xi}_{AB}, v_R^x - C_\xi \cdot \varepsilon^{1/2}]$ is also nonempty, if ε is sufficiently small (depending on C_ξ) and $\underline{\xi}_{AB}$ sufficiently close to ξ_{EB}.

3. Finally, we show that the special $\lambda = (1, \eta_C^0, v_R^x)$ can be connected by paths in Λ to all other λ: it connects to any $(1, \eta_C^*, \xi_{AB})$ with $\eta_C^* \in [\underline{\eta}_C^*, \eta_C^0)$ and $\xi_{AB} \in [\underline{\xi}_{AB}, v_R^x]$. These include $(1, \eta_C^0 - C_\eta \cdot \varepsilon^{1/2}, v_R^x - C_\xi \cdot \varepsilon^{1/2})$ which connects to any $(\gamma, \eta_C^0 - C_\eta \cdot \varepsilon^{1/2}, v_R^x - C_\xi \varepsilon^{1/2})$ with $\gamma > 1$. This point, in turn, connects to any $(\gamma, \eta_C^*, \xi_{AB})$ with $\eta_C^* \in [\underline{\eta}_C^*, \overline{\eta}_C^*]$ and $\xi_{AB} \in [\underline{\xi}_{AB}, \overline{\xi}_{AB}]$. Hence Λ is path-connected.

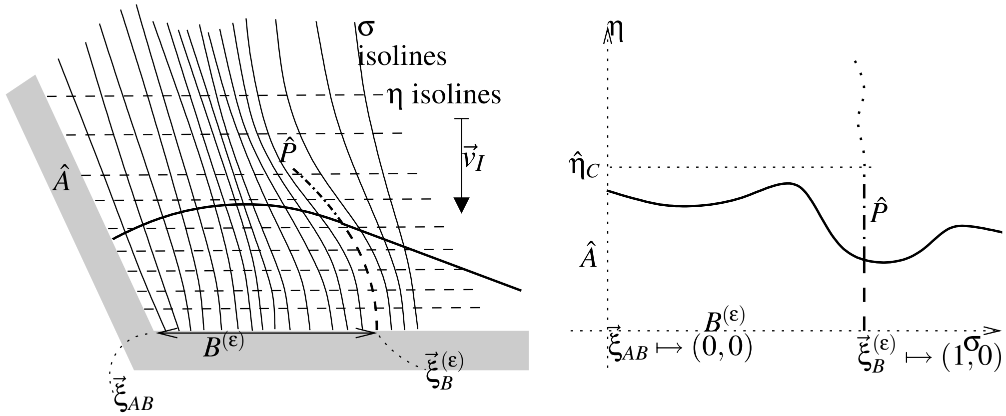


Figure 14. Transformation to "onion" coordinates (σ, η)

□

Function set and iteration

Definition 2.4. Let $U \subset \mathbb{R}^n$ open nonempty bounded with ∂U uniformly Lipschitz. Let $F \subset \partial U$. For $k \in \mathbb{N}_0$, $\alpha \in [0,1]$ and $\beta \in (-\infty, k+\alpha]$ we define the *weighted Hölder space* $C^{k,\alpha}_\beta(U,F)$ as the set of $u \in C^{k,\alpha}(\overline{U} - F)$ so that

$$\|u\|_{C^{k,\alpha}_\beta(U,F)} := \sup_{r>0} r^{k+\alpha-\beta} \|u\|_{C^{k,\alpha}(\overline{U} - B_r(F))}$$

is finite.

Definition 2.5. For sufficiently small $\delta_{\hat{P}} > 0$, there is a function $b \in C^2(\overline{V})$ with $b, |\nabla b| \le 1$ so that $b = 0$ on $\hat{P}^{(0)}$, $b > 0$ elsewhere, $b_n = 0$ on $\hat{A}$ and $\hat{B}$, and so that b depends continuously on the parameters λ but is independent of ε. From now on we fix a particular b.

Proof. The construction is straightforward. $\delta_{\hat{P}}$ is taken so small that $\hat{P}^{(0)}$ does not meet $\hat{A} \cup \hat{B} \cup \{\vec{\xi}_{AB}\}$ except in $\vec{\xi}^{(0)}_B$. □

Definition 2.6.

Onion coordinates

Rotate Figure 12 so that $\hat{B}$ is the positive horizontal axis (see Figure 14 left), then shift horizontally so that $\vec{v}_I$ is vertical (Remark 1.4). Define new coordinates $(\sigma, \eta) \in \mathbb{R}^2$ (see Figure 14 right) so that

1. the coordinate change from (ξ, η) to (σ, η) is C^∞ with C^∞ inverse,

2. $B^{(\varepsilon)}$ maps to $(0,1)\times\{0\}$,
3. $\hat{A}$ maps to $\{0\}\times(0,\infty)$,
4. $\hat{P}^{(\varepsilon)}$ maps to $\{1\}\times(0,\hat{\eta}_C)$ (where $\hat{\eta}_C$ is the η coordinate of the upper endpoint of $\hat{P}^{(\varepsilon)}$),
5. $\vec{\xi}_B^{(\varepsilon)}$ maps to $(1,0)$,
6. $\vec{\xi}_{AB}$ maps to $(0,0)$.

We require that the change of coordinates and its inverse depend continuously (in the C^∞ topology) on $\lambda\in\Lambda$. The construction is straightforward.

Here and in what follows, we will use the weighted Hölder spaces $C_\beta^{2,\alpha}(\overline{U})$, as in Definition 2.4. The domain U is either $[0,1]^2$ with $F=\{(0,0),(1,1)\}$, or $\overline{\Omega}$ with $F=\{\vec{\xi}_{AB},\vec{\xi}_C\}$ (to be defined). For the shock parametrization we use $U=[0,1]$ with $F=\{0,1\}$, or (in Figure 14 left coordinates) $U=[\xi_A,\xi_C]$ with $F=\{\xi_C\}$; for $U=P$ we use $F=\{\vec{\xi}_C\}$, and for $U=A$ or $U=B$ we take $F=\{\vec{\xi}_{AB}\}$. We omit F as it will be clear from the context. $\beta\in(1,2)$ and $\alpha\in(0,\beta-1]$ will be determined later. $C_\beta^{2,\alpha}$ are Banach spaces so that standard functional analysis applies. Moreover, $C_\beta^{2,\alpha}(\overline{\Omega})$ is continuously embedded in $C^1(\overline{\Omega})$, so we have C^1 regularity in the corners as well, which is crucial.

Free boundary fit

Let $\overline{\mathcal{F}}$ be the set of functions $\psi\in C_\beta^{2,\alpha}([0,1]^2)$ that satisfy all of the many conditions explained below. Require

$$\|\psi\|_{C_\beta^{2,\alpha}([0,1]^2)}\le C_C(\varepsilon). \tag{2.5}$$

The curves of constant σ (isolines) in the (ξ,η) coordinate plane are nowhere horizontal, since the other coordinate is η. Moreover $\psi^I_\eta=v^y_I<0$ and $\psi^I_\xi=v^x_I=0$, so for all $\sigma\in[0,1]$ there is a unique point $(\xi,s(\sigma))$ on the isoline so that

$$\psi^I(\xi,s(\sigma))=\psi(\sigma,1). \tag{2.6}$$

We define another coordinate transform by first mapping $(\sigma,\zeta)\in[0,1]$ to (σ,η) with $\eta=s(\sigma)\zeta$ and then mapping to $\vec{\xi}$ with the previous coordinate transform.

Let $\vec{\xi}_A$ resp. $\vec{\xi}_C$ be the $\vec{\xi}$ coordinates for the (σ,ζ) plane points $(0,1)$ and $(1,1)$. Let S be the $\vec{\xi}$ plane curve for $(0,1)\times\{1\}$ (it is the graph of s, with endpoints $\vec{\xi}_A$ and $\vec{\xi}_C$). Define P resp. A resp. Ω to be the image of $\{1\}\times(0,1)$ resp. $\{0\}\times(0,1)$ resp. $(0,1)\times(0,1)$.

Require shock-wall separation:

$$d(S,B)\ge\delta_{SB}>0. \tag{2.7}$$

(2.7) ensures that the map from (σ,ζ) to $\vec{\xi}$ is a well-defined change of coordinates, uniformly nondegenerate (depending on δ_{SB} and C_C), with $C^{2,\alpha}_\beta([0,1]^2)$ resp. $C^{2,\alpha}_\beta(\overline{\Omega})$ regularity. It is clear now that $\partial\Omega$ is the union of the disjoint sets S, P, A, B, and $\{\vec{\xi}_C,\vec{\xi}_B,\vec{\xi}_A,\vec{\xi}_{AB}\}$.

Require: corner close to target:

$$|\eta_C-\eta_C^*|\le\varepsilon^{1/2}, \tag{2.8}$$

We require ε to be so small that $\vec{\xi}_C\in\hat{P}$.

For later use we define $\eta_C^\pm:=\eta_C^*\pm\varepsilon^{1/2}$ and let $\xi_C^\pm$ be so that $\vec{\xi}_C^\pm\in\hat{P}_C$.

Corner cone:

$$\sup_{\vec{\xi},\vec{\xi}'\in\overline{\Omega}}\measuredangle(\vec{\xi}-\vec{\xi}_C,\vec{\xi}'-\vec{\xi}_C)\le\pi-\delta_{Cc}. \tag{2.9}$$

($\measuredangle(\vec{x},\vec{y})$ is the counterclockwise angle from $\vec{x}$ to $\vec{y}$.)

Iteration

Here we change to the coordinates of Figure 12 for the remainder of the definition.

Shock strength/density: require that

$$-\chi-\frac{1}{2}|\nabla\chi|^2>0, \tag{2.10}$$

so that ρ is well-defined (see (1.19)), and require

$$\min_{\overline{\Omega}}\rho\ge\rho_I+\delta_\rho. \tag{2.11}$$

Pseudo-Mach number bound: require

$$L^2\le1-\delta_{Lb}\cdot b\qquad\text{in }\overline{\Omega}, \tag{2.12}$$

(Note that L is well-defined because by (2.11) $\rho>0$, so $c>0$.) $b=0$ on $\hat{P}_C^{(0)}$ which has distance $\ge\frac{\varepsilon}{3}$ (for sufficiently small ε) from $\overline{\Omega}$, so (2.12) implies

$$L^2\le1-\frac{1}{3}|\nabla b|_{L^\infty}\delta_{Lb}\cdot\varepsilon\le1-\frac{1}{3}\delta_{Lb}\cdot\varepsilon\qquad\text{in }\overline{\Omega}, \tag{2.13}$$

Require: there is[6] a function $\hat{\psi}\in C^{2,\alpha}_\beta(\overline{\Omega})$ with the following properties:

[6] $\hat{\psi}$ is the product of an iteration step with input ψ. We will ensure in Proposition 2.10 that $\hat{\psi}$ is unique and continuously dependent on ψ.

1. ψ close to $\hat{\psi}$:

$$\|\psi-\hat{\psi}\|_{C^{2,\alpha}_{\beta}([0,1]^2)} \le r_I(\psi) \tag{2.14}$$

where $r_I \in C(\overline{\mathcal{F}};(0,\infty))$ is a continuous function to be determined later.

2. Right away we require r_I to be so small that

$$-\hat{\chi}-\frac{1}{2}|\nabla\hat{\chi}|^2 > 0, \tag{2.15}$$

so that in particular $\hat{\rho}$ is well-defined and positive. Moreover, require

$$\nabla\hat{\psi} \neq \vec{v}_I, \tag{2.16}$$

3. We require r_I to be so small that (using (2.13))

$$\big(c_0^2+(1-\gamma)(\chi+\frac{1}{2}|\nabla\hat{\chi}|^2)\big)I-\nabla\hat{\chi}^2 > 0, \tag{2.17}$$

i.e. is a (symmetric) positive definite matrix.

4. Let $\mathcal{L}=\mathcal{L}(\psi,\hat{\psi})$ be defined in $\vec{\xi}$ coordinates as

$$\Big(\big(c_0^2+(1-\gamma)(\chi+\frac{1}{2}|\nabla\hat{\chi}|^2)\big)I-\nabla\hat{\chi}^2\Big):\nabla^2\hat{\psi}, \tag{2.18}$$

$$\frac{|\nabla\hat{\chi}|^2}{2}+\frac{(1-\varepsilon)\big((\gamma-1)\chi+c_0^2\big)}{2+(1-\varepsilon)(\gamma-1)}, \tag{2.19}$$

$$\big(\hat{\rho}\nabla\hat{\chi}-\rho_I\nabla\chi^I\big)\cdot\frac{\vec{v}_I-\nabla\hat{\psi}}{|\vec{v}_I-\nabla\hat{\psi}|}, \tag{2.20}$$

$$\nabla\hat{\psi}\cdot\vec{n}_A,\nabla\hat{\psi}\cdot\vec{n}_B\Big). \tag{2.21}$$

where the codomain is

$$Y := C^{0,\alpha}_{\beta-2}(\overline{\Omega})\times C^{1,\alpha}_{\beta-1}(\overline{S})\times C^{1,\alpha}_{\beta-1}(\overline{P})\times C^{1,\alpha}_{\beta-1}(\overline{A})\times C^{1,\alpha}_{\beta-1}(\overline{B}).$$

(2.20) is well-defined by (2.15) and (2.16). The other components have no singularities.

Note: $\nabla\psi\in C^{1,\alpha}_{\beta-1}$, so $|\nabla\chi|^2\in C^{1,\alpha}_{\beta-1}$, so

$$\Big(\big(c_0^2+(1-\gamma)(\chi+\frac{1}{2}|\nabla\hat{\chi}|^2)\big)I-\nabla\hat{\chi}^2\Big)\in C^{1,\alpha}_{\beta-1}\hookrightarrow C^{0,\beta-1}\hookrightarrow C^{0,\alpha}$$

($\alpha\le\beta-1$ as required above), and $\nabla^2\psi\in C^{0,\alpha}_{\beta-2}$, so (2.18) is $\in C^{0,\alpha}_{\beta-2}$. In the same way we check that (2.19), (2.20) and (2.21) are $C^{1,\alpha}_{\beta-1}$.

For $\hat{\psi}$ we use the $C^{2,\alpha}_{\beta}(\overline{\Omega})$ topology. We pull back $\hat{\psi}$ and the value of $\mathcal{L}$ to (σ,ζ) coordinates, via the coordinate transform defined by ψ (see above), so that we have a fixed domain $[0,1]^2$ for all Banach spaces. Then $\mathcal{L}$ is a nonlinear smooth map in the corresponding topologies.

Most importantly: require

$$\mathcal{L}(\psi,\hat{\psi})=0. \tag{2.22}$$

Other bounds

Require

$$\|\psi\|_{C^{0,1}(\overline{\Omega})} \leq C_L \tag{2.23}$$

where C_L may not depend on ε.

χ_t and χ_n on parabolic arc:

$$\max_{\overline{P}} c^{-1}\left|\frac{\partial\chi}{\partial t}\right| \leq C_{Pt}\cdot\varepsilon^{1/2}, \tag{2.24}$$

$$\max_{\overline{P}} c^{-1}\frac{\partial\chi}{\partial n} \leq -\delta_{Pn}. \tag{2.25}$$

We emphasize that δ_{Pt}, δ_{Pn} may depend *only* on λ, but not on ε (or ψ).

Velocity components:

$$\vec{v}\cdot\vec{n}_A \leq C_{vnA}\cdot\varepsilon^{1/2}, \qquad \text{in } \overline{\Omega}, \tag{2.26}$$

$$\vec{v}\cdot\vec{t}_R \leq \vec{v}_R\cdot\vec{t}_R + C_{vtR}\cdot\varepsilon^{1/2}, \qquad \text{in } \overline{\Omega}, \tag{2.27}$$

$$\vec{v}\cdot\vec{n}_B \leq \vec{v}_I\cdot\vec{n}_B - \delta_{vnB} \qquad \text{in } \overline{\Omega} \tag{2.28}$$

and

$$\vec{v}\cdot\vec{t}_A \leq \vec{v}_I\cdot\vec{t}_A - \delta_{vtA} \qquad \text{in } \overline{\Omega}. \tag{2.29}$$

Shock normal: Let $N \subset S^1$ (unit circle) be the set of $\vec{n}$ counterclockwise from $\vec{n}_R$ to $\vec{t}_A$. Then the shock normal satisfies

$$\sup_S d(\vec{n}, N) \leq C_{Sn}\cdot\varepsilon^{1/2}. \tag{2.30}$$

Set $\Sigma_1 := A$, $\Sigma_2 := S$, $\Sigma_3 := P$ and $\Sigma_4 := B$. Write the components (2.19), (2.21), (2.20) of $\mathcal{L}$ as

$$g^i(\vec{\xi}, \hat{\chi}(\vec{\xi}), \underbrace{\nabla\hat{\chi}(\vec{\xi})}_{=:\vec{p}}) \qquad (i = 1, \ldots, 4),$$

where the $\vec{\xi}$ dependence includes the dependence on $\chi(\vec{\xi})$ and $\nabla\chi(\vec{\xi})$.

g^2 has some singularities, but not on the set of $\vec{\xi}, \chi, \nabla\chi$ so that (2.28) and (2.11) (resp. (2.15) and (2.16)) are satisfied. That set is simply connected, so we can modify g^2 on its complement and extend it smoothly to $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^2$. The modification is chosen to depend smoothly on λ.

Require uniform obliqueness:

$$|g^i_{\vec{p}} \cdot \vec{n}| \geq \delta_o |g^i_{\vec{p}}| \qquad \forall \vec{\xi} \in \Sigma_i. \tag{2.31}$$

Functional independence in upper corners: for $i, j = 1, 4$ and for $i, j = 2, 3$ set

$$G := \begin{bmatrix} g^i_{p^1} & g^j_{p^1} \\ g^i_{p^2} & g^j_{p^2} \end{bmatrix},$$

regard it as a function of $\vec{\xi}$ (including the dependence on $\nabla\hat{\chi}(\xi)$) and require

$$\|G\|, \|G^{-1}\| \leq C_d \qquad \text{in } B_{\delta_d}(\vec{\xi}_C) \cap \overline{\Omega}. \tag{2.32}$$

Let $\overline{\mathcal{F}}$ be the set[7] of admissible functions so that all of these conditions are satisfied. Define $\mathcal{F}$ to be the set of admissible functions such that all of these conditions are satisfied with *strict* inequalities, i.e. replace $\leq, \geq$ by $<, >$, "increasing" by "strictly increasing" etc.

[This is the end of Definition 2.6.]

The elliptic problem is solved by iteration; $\hat{\psi}$ is the new iterate, ψ the old one. $\mathcal{L}$ defines $\hat{\psi}$, as we show later. As always, the iteration is designed so that its fixed points solve the problem:

Remark 2.7. If $\hat{\psi} = \psi$, then (2.18), (2.20), (2.19), (2.21) and the definition of S yield

$$\begin{aligned} (c^2 I - \nabla\chi^2) : \nabla^2\psi &= 0 && \text{in } \overline{\Omega}, \\ \nabla\chi \cdot \vec{n} &= 0 && \text{on } \overline{A} \text{ and } \overline{B}, \\ \chi^I &= \chi && \text{and} \\ (\rho\nabla\chi - \rho_I \nabla\chi^I) \cdot \vec{n} &= 0 && \text{on } \overline{S}, \\ L &= \sqrt{1-\varepsilon} && \text{on } \overline{P} \end{aligned}$$

(we may take closures by regularity (2.5)).

Remark 2.8. Consider a coordinate system where $\vec{\xi}_{AB} = 0$. For any point on A or B, we can use even reflection of ψ across the corresponding boundary to obtain a new situation where the point is in the *interior*. (In $\vec{\xi}_A$ or $\vec{\xi}_B$, we obtain a new situation with a point at a shock resp. quasi-parabolic arc with an elliptic region on one side.) The boundary condition $\chi_n = \psi_n = 0$ (due to $\vec{\xi}_{AB} = 0$), for even reflection of ψ, implies that ψ is C^1 across the boundary; then necessarily it is also $C^{2,\alpha}$.

[7] The notation $\overline{\mathcal{F}}$ does not necessarily imply that $\overline{\mathcal{F}}$ is the closure of $\mathcal{F}$.

For fixed points $\psi = \hat{\psi}$, standard regularity theory immediately yields that the solution is locally analytic (even after reflection). The same technique applied to $\hat{\psi}$ and to solutions $\acute{\psi}$ of linearized equations (here ψ, $\hat{\psi}$ and $\acute{\psi}$ are reflected) yields $C^{2,\alpha}$ regularity. (The same argument applies to S extended by mirror reflection across $\hat{A}$.)

Proposition 2.9. *For sufficiently small ε (with bound depending only on C_{Pt}) and r_I (depending continuously and only on ψ, δ_{vx}):*

for all $\psi \in \overline{\mathcal{F}}$, $\mathcal{L}(\psi, \hat{\psi}')$ is well-defined for $\hat{\psi}'$ near ψ, and the Fréchet derivative $\partial\mathcal{L}/\partial\hat{\psi}'(\psi,\psi)$ (of $\mathcal{L}$ with respect to its second argument $\hat{\psi}'$, evaluated at $\hat{\psi}' = \psi$) is a linear isomorphism of $\mathcal{C}^{2,\alpha}_{\beta}$ onto Y.

Proof. The proof is almost identical to [11, Proposition 4.4.6]; the new corner between A, B is covered by [22, Theorem 1.4] in the same way as the other ones. □

Proposition 2.10. *r_I can be chosen so that $\hat{\psi}$ is unique and depends continuously on $\psi \in \overline{\mathcal{F}}$ (both in the $\mathcal{C}^{2,\alpha}_{\beta}$ topology) and λ.*

Proof. The proof is exactly the same as for [11, Proposition 4.4.7]. □

Proposition 2.11. *For ε and r_I sufficiently small: for all continuous paths $t \in [0,1] \mapsto \lambda(t)$ in Λ, $\bigcup_{t\in(0,1)} \big(\{t\} \times \mathcal{F}_{\lambda(t)}\big)$ is open and $\bigcup_{t\in[0,1]} \big(\{t\} \times \overline{\mathcal{F}}_{\lambda(t)}\big)$ is closed*[8] *in $[0,1] \times \mathcal{C}^{2,\alpha}_{\beta}([0,1]^2)$.*

Proof. All conditions on ψ in Definition 2.6 are inequalities which can be made scalar by taking a suitable supremum or infimum. Then their sides are continuous under $\mathcal{C}^{2,\alpha}_{\beta}([0,1]^2)$ changes to ψ which, by Proposition 2.10, means continuous in $\mathcal{C}^{2,\alpha}_{\beta}([0,1]^2)$ change to $\hat{\psi}$. (Most inequalities need only $\mathcal{C}^1([0,1]^2)$.)

1. Closedness: consider sequences (t_n, ψ_n) in $\bigcup_{t\in[0,1]} \big(\{t\} \times \overline{\mathcal{F}}_{\lambda(t)}\big)$ that converge to a limit (t, ψ).

   Let $\hat{\psi}_n$ be associated to ψ_n as in Definition 2.6. By continuity (Proposition 2.10), $(\hat{\psi}_n)$ converges to a limit $\hat{\psi}$ as well. By continuity of $\mathcal{L}$ in ψ, $\hat{\psi}$ and λ, we have $\mathcal{L}_{\lambda(t)}(\psi, \hat{\psi}) = 0$ as well.

   Let s_n be defined by ψ_n as in (2.6), with $s \leftarrow s_n$ and $\psi \leftarrow \psi_n$. Then by (2.6), (s_n) converges in $\mathcal{C}^{2,\alpha}_{\beta}[0,1]$ as well, to a limit s which satisfies (2.6) itself.

   Most conditions on ψ are nonstrict inequalities with continuous left- and right-hand side, so they are still satisfied by ψ. We check the strict inequalities explicitly and in order:

[8] We make no statement about $\overline{\mathcal{F}}$ being the closure of $\mathcal{F}$. It certainly contains the closure, but it could be bigger, for example if one of the inequalities in Definition 2.6 becomes nonstrict in the interior without being violated.

(2.10) is implied by (2.11).

(2.15) resp. (2.16) resp. (2.17) are implied by (2.14) resp. (2.28) resp. (2.13), by choosing r_I sufficiently small.

All inequalities are satisfied, so $\psi \in \overline{\mathcal{F}}$.

2. Openness:

   same proof, using that all inequalities are strict now, by definition of $\mathcal{F}$, hence preserved by sufficiently small perturbations.

□

Definition 2.12. Define $\mathcal{K} : \overline{\mathcal{F}} \to C^{2,\alpha}_{\beta}([0,1]^2)$ to map ψ into $\hat{\psi}$ as given in Definition 2.6, but pulled back to (σ,ζ) coordinates and the $[0,1]^2$ domain (see Definition 2.6) with the coordinate transform defined by ψ.

Regularity and compactness

Proposition 2.13. *For sufficiently small* $\alpha \in (0,1)$ *and* $\beta \in (1,2)$*, depending only on* C_d*,* $\delta_{Lb} \cdot \varepsilon$*,* δ_o*,* C_L*,* δ_{vx}*:*

1. *When parametrized in the coordinates of Figure 10,*

$$\|S\|_{C^{0,1}} \le C_{sL} \tag{2.33}$$

   and

$$\|S\|_{C^{2,\alpha}_{\beta}} \le C_s \tag{2.34}$$

   for $C_{sL} = C_{sL}(C_L, \delta_{vx})$ *and* $C_s = C_s(C_C, \delta_{vx})$*; the weight* β *is with respect to the endpoints* $\vec{\xi}_A, \vec{\xi}_C$*.*

2. *For a* fixed point ψ *of* $\mathcal{K}$*:*

   (a) (2.23) is strict for sufficiently large C_L*.*

   (b) (2.5) is strict for sufficiently large $C_C = C_C(C_d, \delta_{Lb} \cdot \varepsilon, C_L, \delta_o, \delta_{vtA}, \delta_d)$*.*

   (c) For $K \Subset \overline{\Omega} - \hat{P} - \{\vec{\xi}_B, \vec{\xi}_{AB}\}$ *and all* $k \ge 0$*,* $\alpha' \in (0,1)$*,*

$$\|\psi\|_{C^{k,\alpha'}(K)} \le C_{CK} \tag{2.35}$$

   where $C_{CK} = C_{CK}(d, C_L, \delta_o, \delta_{vtA})$ *is decreasing in* $d := d(K, \hat{P} \cup \{\vec{\xi}_{AB}\})$ *and* not *dependent on* ε*.*

   (d) ψ *is analytic in* $\overline{\Omega} - \{\vec{\xi}_{AB}, \vec{\xi}_C\}$*;* S *is analytic except in* $\vec{\xi}_C$*.*

3. *For sufficiently small $r_I > 0$, depending continuously and only on ψ, there are $\delta_\alpha, \delta_\beta > 0$ so that for all $\psi \in \mathcal{F}$,*

$$\|\hat{\psi}\|_{C^{2,\alpha+\delta_\alpha}_{\beta+\delta_\beta}(\overline{\Omega})} \le C_{\mathcal{K}} \tag{2.36}$$

Here, $C_{\mathcal{K}}, \delta_\alpha, \delta_\beta$ depend only on $C_d, \delta_{Lb} \cdot \varepsilon, \delta_o, C_L, \delta_{vx}$,

Proof. The proof is as the one for [11, Proposition 4.5.2], with obvious modifications. The only additional problem is the corner in $\vec{\xi}_{AB}$. This is very easy to treat with [11, Proposition 5.1.1] because of (2.32) and (2.31) for $\vec{\xi}_{AB}$. Note that the corner angle in $\vec{\xi}_{AB}$ is bounded away from π because of the restrictions on ξ_{AB} (see Section 2). □

Remark 2.14. (2.36) implies in particular that $\mathcal{K}$ is a compact map. $\psi \in C^{2,\alpha}_\beta([0,1]^2)$ is mapped continuously into $\hat{\psi} \in C^{2,\alpha+\delta_\alpha}_{\beta+\delta_\beta}(\overline{\Omega})$. The latter space is compactly embedded in $C^{2,\alpha}_\beta(\overline{\Omega})$. Pullback to $C^{2,\alpha}_\beta([0,1]^2)$ by the σ, ζ coordinates defined by ψ (not $\hat{\psi}$) may destroy the extra regularity, but preserves compactness.

Pseudo-Mach number control

Proposition 2.15. *For ε and δ_{Lb} sufficiently small, with bounds depending only on δ_ρ: if $\psi \in \overline{\mathcal{F}}$ is a fixed point of $\mathcal{K}$, then (2.12) is strict and*

$$L^2 < 1 - \varepsilon \qquad \text{in } \overline{\Omega} - \overline{P}. \tag{2.37}$$

Proof.

$$d(\overline{\Omega}, \hat{P}^{(0)}) \ge \frac{1}{3} \cdot \varepsilon,$$

for ε small enough. Remember from Definition 2.5 that $b = 0$ on $\hat{P}^{(0)}$. Therefore:

$$L^2 = 1 - \varepsilon < 1 - \|b\|_{C^{0,1}} \cdot d(P^{(\varepsilon)}, \hat{P}^{(0)}) \le 1 - \delta_{Lb} \cdot b \qquad \text{on } \overline{P}^{(\varepsilon)},$$

e.g. for $\delta_{Lb} \le 1$.

On the shock, we may use (2.11) combined with [11, Proposition 3.6.1] to rule out that $L^2 + \delta_{Lb} \cdot b$ has a maximum in a point where $L < 1$ and $L \ge 1 - \delta_{LS}$, with δ_{LS} as supplied by loc.cit. Here δ_{Lb} has to be chosen so that $|\delta_{Lb} \nabla b| \le \delta_{LS}$ is satisfied. (Now δ_{Lb} depends continuously on δ_ρ as well.)

In addition we can choose δ_{Lb} so small that $\delta_{Lb} \cdot b$ satisfies the preconditions of Theorem 1 and Theorem 2 in [8] (where it is called b). For Theorem 2 we use that $b_n = 0$ on $\hat{A}$ and on $\hat{B}$. Let $\delta_{L\Omega}$ be the δ from those theorems (it depends only and continuously on λ). Then $L^2 + \delta_{Lb} \cdot b$ cannot have a maximum in a point of $\Omega \cup A \cup B$ where $L^2 \ge 1 - \delta_{L\Omega}$.

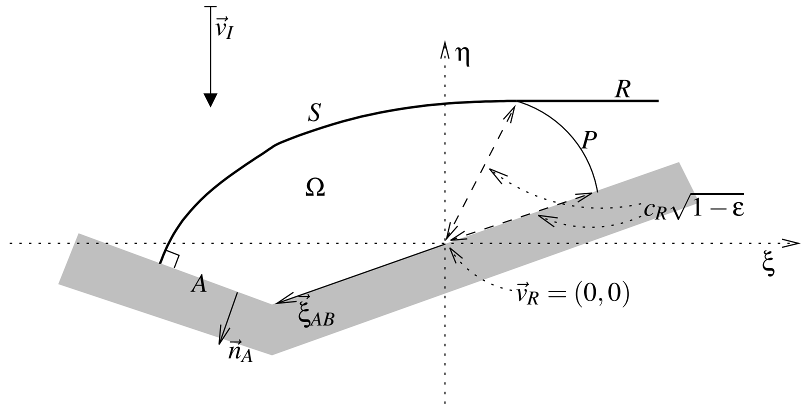


Figure 15. In this frame P is centered in $\vec{v}_R = 0$, R is horizontal and $\vec{v}_I$ is vertical.

In the corner between A, B, due to C^1 regularity the boundary conditions imply $\nabla\chi = 0$, so $L = 0$, so $L^2 + \delta_{Lb} \cdot b = \delta_{Lb} \cdot b < 1$ for δ_{Lb} sufficiently small.

In $\vec{\xi}_A$ we use that the shock is pseudo-normal (by the boundary condition $\nabla\chi \cdot \vec{n}_A = 0$ which implies $\nabla\chi \cdot \vec{t} = 0$ for the corresponding shock tangent $\vec{t}$ since S, A form a right angle), so $L_d = L_d^n$ which is uniformly bounded above away from 1 by a constant depending on δ_ρ, since (2.11) implies uniform shock strength.

Assume that (2.12) is not strict (or violated). Then $L^2 + \delta_{Lb} \cdot b$ has a maximum ≥ 1 somewhere. For δ_{Lb} sufficiently small (no new dependencies) that means L^2 has a maximum $\geq 1 - \min\{\delta_{L\Omega}, \delta_{LS}\}$ somewhere. But no matter where in $\overline{\Omega}$ this occurs, it contradicts one of the cases discussed above. Hence (2.12) is strict.

(2.37) can be shown in the same manner, by taking $b = 0$ instead, using the actual boundary condition $L = \sqrt{1-\varepsilon}$ on P and and considering $\varepsilon < \delta_{LS}, \delta_{L\Omega}$. □

Arc control and corner bounds

The discussion of parabolic arcs is very similar to [11, Sections 4.7 to 4.10]. For the convenience of the reader we restate the results using new notation and point out some differences in details.

A new choice of coordinates is convenient (see Figure 15): since self-similar potential flow is invariant under translations, we may translate so that $\vec{v}_R$ moves to the origin (all other velocities $\vec{v}$ and coordinates $\vec{\xi}$ have $\vec{v}_R$ subtracted), then rotate clockwise until R is horizontal. In this frame, $\vec{v}_I$ is vertical down and P is centered in $\vec{v}_R = 0$. This means ψ^R and χ^R are both constant on P, which simplifies certain calculations.

In polar coordinates (r,ϕ) with respect to the origin (center of P), P corresponds to $r = c_R \cdot \sqrt{1-\varepsilon}$.

Proposition 2.16. *If $C_{Pt} < \infty$ is sufficiently large, if $\delta_{Pn} > 0$ is sufficiently small, if ε is sufficiently small and $C_{Pv}, C_{P\rho}$ sufficiently large, with bounds depending only on C_{Pt}, then for any fixed point χ of $\mathcal{K}$, (2.24) and (2.25) are strict, and*

$$|\rho - \rho_R| \le C_{P\rho}\varepsilon^{1/2} \qquad \text{and} \tag{2.38}$$

$$|\vec{v} - \vec{v}_R| \le C_{Pv}\varepsilon^{1/2} \qquad \text{on } P. \tag{2.39}$$

Proof. The proof is as for [11, Proposition 4.8.1], with obvious modifications. □

If (2.8) is satisfied, but not in its strict version, then $\eta_C^* = \eta_C^+$ or $\eta_C^* = \eta_C^-$ (where $\vec{\xi}^{\pm}$ are as defined in Definition 2.6 after (2.8)). Each of these two cases must be ruled out.

Proposition 2.17. *For ε sufficiently small: for any fixed point $\psi \in \overline{\mathcal{F}}$ of $\mathcal{K}$, the lower bound in (2.8) is strict:*

$$\eta_C > \eta_C^-$$

Proof. Same as for [11, Proposition 4.10.1]. □

Proposition 2.18. *Consider $\eta_C = \eta_C^+$. For sufficiently small ε, there is an $a \ge 0$ so that*

1. *$\psi + a\xi$ does not have a local minimum (with respect to $\overline{\Omega}$) at $P \cup \{\vec{\xi}_B\}$, and*
2. *a shock through $\vec{\xi}_C^+$ with upstream data $\vec{v}_I$ and ρ_I and tangent $(1, \frac{a}{-v_I^y})$ has $v_d^y > 0$.*

Proof. This follows as in Propositions 4.10.2, 4.10.3 and 4.10.5 of [11]. □

Only the final upper bound requires some adaptation:

Proposition 2.19. *Let $\chi \in \overline{\mathcal{F}}$ be a fixed point of $\mathcal{K}$. For C_η sufficiently large and for $\varepsilon > 0$ sufficiently small, the upper part of (2.8) is strict:*

$$\eta_C < \eta_C^+.$$

Proof. Again, consider the coordinates of Figure 15.

By Proposition 2.18, $\psi + a\xi$ cannot have a local minimum at $P \cup \{\vec{\xi}_B\}$. For $\eta_C = \eta_C^+$, we have $(\psi + a\xi)_\eta = \psi_\eta > 0$ in $\vec{\xi}_C$ by [11, (4.9.8)] (for sufficiently small ε), so the minimum cannot be in $\vec{\xi}_C$ either (note that the domain locally contains the ray downward from the corner).

On the shock (excluding endpoints): let $\xi \mapsto s(\xi)$ be a local parametrization of the shock . $\psi + a\xi = \psi^I + a\xi$, so

$$\partial_t(\psi + a\xi) = \partial_t(\psi^I + a\xi) = \vec{v}_I \cdot \vec{t} + \frac{a}{(1+s_\xi^2)^{1/2}} = \frac{v_I^y s_\xi + a}{(1+s_\xi^2)^{1/2}}.$$

For a local minimum at the shock we need $\partial_t(\psi + a\xi) = 0$, so

$$s_\xi = \frac{a}{-v_I^y}.$$

A *global* minimum, in particular $\leq \psi(\vec{\xi}_C) + a\xi_C$, additionally requires that $\vec{\xi}_C$ (as well as the rest of the shock) is on or below the tangent through the minimum point, because ψ^I and thus $\psi^I + a\xi$ are decreasing in η. By Proposition 2.18, the shock through $\vec{\xi}_C^+$ with that tangent has $v_d^y > 0$ for $\eta_C = \eta_C^+$. In the minimum point the tangent has same slope but is at least as high, so the shock speed is at least as high, so $v_d^y = \psi_\eta = (\psi + a\xi)_\eta$ there is at least as high, in particular > 0 too. But that contradicts a minimum (the ray vertically downwards from any shock point is locally contained in $\overline{\Omega}$, by (2.30)). Hence $\psi + a\xi$ cannot have a global minimum at the shock.

The equation (2.2.5) yields

$$(c^2 I - \nabla\chi^2) : \nabla^2(\psi + a\xi) = 0$$

($a\xi$ is linear), so the classical strong maximum principle rules out a minimum in the interior (unless $\psi + a\xi$ is constant, which means we are looking at the unperturbed solution which has $\eta_C = \eta_C^* < \eta_C^+$).

On B, the boundary condition $\psi_n = \chi_n = 0$ implies $(\psi + a\xi)_n = a\xi_n \geq 0$ (the slope of B in the frame of Figure 15) is always nonnegative), so the Hopf lemma rules out a minimum of $\psi + a\xi$ at B.

On $\overline{A}$ the boundary condition $\chi_n = 0$ yields $\psi_n = \vec{\xi} \cdot \vec{n} = \vec{\xi}_{AB} \cdot \vec{n}_A \geq 0$ (see Figure 15). This is actually $\psi_n > 0$, except in the special case where (in the notation of Definition 2.2) $\xi_{AB} = v_R^x$ which is allowed only if $\gamma = 1$ and $\eta_C^* = \eta_C^0$: the "unperturbed" case. In that case, the proof of Proposition 2.26 shows that only the unperturbed solution (Figure 18) can solve the problem. Its corner is exactly in the expected location, so that $\eta_C = \eta_C^* < \eta_C^+$. □

Velocity and shock normal control

Proposition 2.20. *If C_{vtR}, C_{vnA} are sufficiently large (bounds depending only on C_{Pt}), if C_{Sn} is sufficiently large (bound depending only on C_{vtR}, C_{vnA}), if ε is sufficiently small (bound depending only on C_{Sn}), and if δ_{Cc} is sufficiently small, then for any fixed point $\psi \in \overline{\mathcal{F}}$ of $\mathcal{K}$, the inequalities (2.27), (2.26), (2.30) and (2.9) are* strict. *Moreover*

$$|\chi_t| \geq \delta_{\chi t} \qquad \text{on } S \cap B_{\delta_d}(\vec{\xi}_C), \tag{2.40}$$

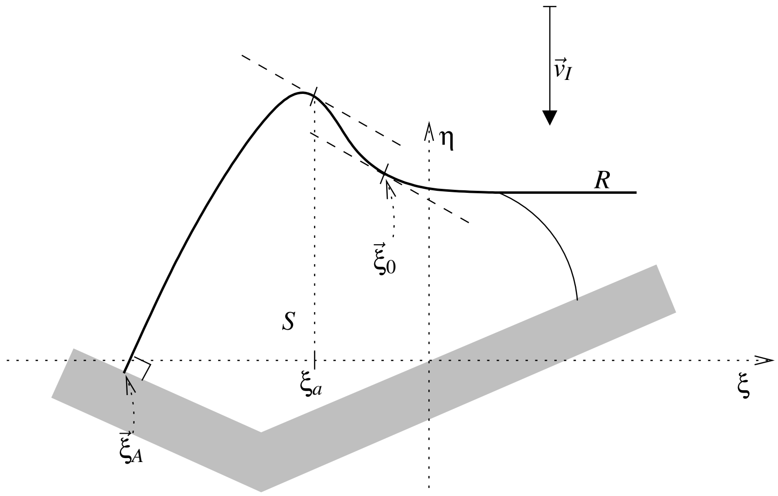


Figure 16. A maximum of v^x requires negative curvature, causing a contradiction

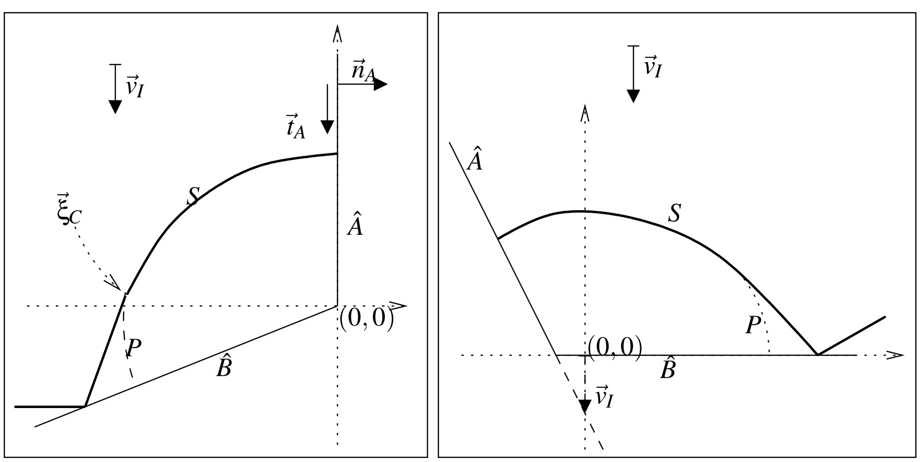


Figure 17. Left: mirror-reflect Figure 12 across $\hat{A}$ and rotate around the origin. Right: setting of Figure 14 left.

for some constants $\delta_{\chi t}, \delta_d > 0$.

Proof. 1. For (2.27): consider the coordinates of Figure 15 where $\vec{t}_R = (1,0)$. Let $\xi \mapsto s(\xi)$ parametrize S (the shock normal bounds (2.30) show that S is nowhere vertical in these coordinates,, for sufficiently small ε, bound depending on C_{Sn}). Assume that $\vec{v} \cdot \vec{t}_R = v^x$ attains a positive global maximum (with respect to $\overline{\Omega}$) in a point $\vec{\xi}_0$ at S (i.e. on the downstream side). Since $\vec{v}_I = (0, v_I^y)$ with $v_I^y < 0$, this means $n^x < 0$ in $\vec{\xi}_0$ (because $n^y < 0$), i.e. $s_\xi(\xi_0) < 0$ (see Figure 16).

$s_\xi(\xi_0)$ can be expressed as a continuous function of $v^x(\vec{\xi}_0)$ and $\vec{\xi}_0$. The set of possible $\vec{\xi}_0$ is contained in the set of possible shock locations which is pre-compact. Therefore if $v^x = C_{vtR} \cdot \varepsilon^{1/2}$ in $\vec{\xi}_0 \in S$, then

$$s_\xi(\xi_0) \leq -C_{s1} \cdot \varepsilon^{1/2} \tag{2.41}$$

where $C_{s1} = C_{s1}(C_{vtR}) > 0$ is uniformly increasing in C_{vtA}.

For a constant-state solution (2.27) is immediate. Otherwise, since S and ψ are analytic (Proposition 2.13), we can apply [11, Proposition 3.5.1] with $\vec{w} = (1,0)$, which yields that curvature $\kappa < 0$, i.e. $s_{\xi\xi} > 0$, in $\vec{\xi}_0$. Therefore $s_\xi(\xi) < s_\xi(\xi_0)$ for $\xi < \xi_0$ near ξ_0. On the other hand, $s_\xi \geq 0$ in $\vec{\xi}_A$ since the boundary condition $\chi_n = 0$ requires the shock to be perpendicular to the wall A; in particular $s_\xi(\xi_A) > 0 > s_\xi(\xi_0)$ by (2.41). (In this choice of coordinates, A is either vertical or has negative slope, since we require it to form right or sharp angles with R, by choice of ξ_{AB} in Section 2.)

Therefore we can pick $\xi_a \in (\xi_A, \xi_0)$ maximal so that $s_\xi(\xi_a) = s_\xi(\xi_0)$. Then $s_\xi(\xi) < s_\xi(\xi_0)$ for $\xi \in (\xi_a, \xi_0)$, so by integration

$$s(\xi_a) > s(\xi_0) + s_\xi(\xi_0) \cdot (\xi_a - \xi_0).$$

But that means the shock tangent in ξ_a is parallel to the one in ξ_0 but *higher*, so the shock speed $\sigma := \vec{\xi} \cdot \vec{n}$ is smaller. By [11, (2.4.19)], that means v_d^n is smaller, whereas v^t is the same (parallel tangents). $n^x < 0$, so v_d^x is *bigger*. Contradiction — we assumed that we have a *global* maximum of v^x in $\vec{\xi}_0$.

[11, Propositions 3.3.1 and 3.4.1] rule out local maxima of v^x in Ω and at B, where we use that χ is analytic and that $(1,0)$ is not vertical, i.e. not normal to B.

At A: if A is vertical, then the boundary condition requires $v^x = \xi_A < 0$; if A is not vertical, then $(1,0)$ is not normal, so [11, Proposition 3.3.1] applies again.

In $\vec{\xi}_{AB}$, the two boundary conditions combine to yield $\vec{v} = \vec{\xi}_{AB}$, so $v^x = \xi_{AB} < 0$.

In $\vec{\xi}_A$, $s_\xi \geq 0$ (see above) yields $v^x \leq 0$.

On $\overline{P}$ we can use (2.39) with $v_R^x = 0$, increasing C_{vtR} to $> C_{Pv}$ if necessary (this makes C_{vtR} depend on C_{Pt} as well).

All parts of $\overline{\Omega}$ are covered; (2.27) is strict.

2. For (2.26): consider the coordinates of Figure 17 left. There, $\vec{n}_A = (1,0)$, so we need to show $v^x = \vec{v} \cdot \vec{n}_A \leq C_{vnA} \cdot \varepsilon^{1/2}$. On $\overline{A}$, the boundary condition yields $v^x = 0$. B is never vertical, so [11, Proposition 3.4.1] rules out extrema of v^x at B. [11, Proposition 3.3.1] does not allow extrema in Ω. At P, (2.39) yields $v^x = v^x_R + O(\varepsilon^{1/2})$; note that $v^x_R < 0$ in these coordinates. At S, we can use the same curvature argument as for $\vec{v} \cdot \vec{t}_R$, except that we now use $s_\xi \geq 0$ in $\vec{\xi}_C$ rather than $\vec{\xi}_A$. Altogether we obtain a contradiction again, if C_{vnA} is sufficiently large, depending only and continuously on C_{Pt}.

3. Consider the coordinates of Figure 15. The slope s_ξ of some shock passing through a point $\vec{\xi}$ is uniquely determined by (and continuous in) $\vec{\xi}$ and v^x, with $\operatorname{sgn} s_\xi = -\operatorname{sgn} v^x$ (since $\vec{v}_I = (0, v^y_I)$, $v^y_I < 0$). The set of possible shock locations $\vec{\xi}$ is pre-compact, so (2.27) implies
$$\sup \measuredangle(\vec{n}, \vec{n}_R) < C_{Sn} \cdot \varepsilon^{1/2}$$
where $C_{Sn} = C_{Sn}(C_{vtR})$.

   Analogously we argue that (2.29) implies
$$\sup \measuredangle(\vec{t}_A, \vec{n}) < C_{Sn} \cdot \varepsilon^{1/2},$$
where $C_{Sn} = C_{Sn}(C_{vtR}, C_{vnA})$ now. (2.30) is strict with these choices.

4. These shock normal bounds also imply (2.9) is strict, for $\delta_{Cc} > 0$ and $\varepsilon > 0$ sufficiently small(er), with ε bound depending only on C_{Sn}.

5. Near each corner the shock normal bound bounds $\vec{n}$ away from the $\vec{\xi}$ direction, so $|\chi^I_t| \geq \delta_{\chi t}$ and therefore (2.40) for some $\delta_{\chi t}$.

□

Proposition 2.21. *1. If δ_{SB} is sufficiently small, then (2.7) is strict.*

2. There is a constant $\delta_{\rho S} > 0$ so that
$$\rho_d \geq \rho_I + \delta_{\rho S} \qquad \text{at } \overline{S} \tag{2.42}$$

Proof. 1. Consider the envelope E defined in Section 2. The parameter set Λ (see Definition 2.2) has been chosen so that for any $\lambda \in \Lambda$, E passes from $\vec{\xi}^{*(0)}_C$ to $\hat{A}$ without meeting $\hat{B}$ or the circle (with radius c_I centered in $\vec{v}_I$). Since Λ has also been chosen compact, E is in fact uniformly bounded away from $\hat{B}$ and the circle.

E starts in $\vec{\xi}^{*(0)}_C$; let E' be the counterclockwise envelope (Definition 1.7) starting in $\vec{\xi}_C$ instead. E, E' are solutions of an ODE (1.34), so they depend continuously on the initial point. Hence for $\vec{\xi}_C$ sufficiently close to $\vec{\xi}^{*(0)}_C$, i.e. by (2.8) for sufficiently small ε (with upper bound depending only on the choice of Λ), E' is also uniformly bounded away from $\hat{B}$ and the circle.

Now we can apply the argument displayed in Figure 6 right: $|\vec{\xi}-\vec{v}_I|$ is r in the polar coordinates used in Section 1. Let E' and the shock S be parametrized by $\phi \mapsto r_S(\phi)$ resp. $\phi \mapsto r_{E'}(\phi)$, with $\phi \in [\phi_C, \phi_A]$, ϕ_C corresponding to the ray from $\vec{v}_I$ through $\vec{\xi}_C$ and ϕ_A to the ray from $\vec{v}_i$ containing $\hat{A}$. $r_S(\phi_C) = r_{E'}(\phi_C)$ because S and E' both pass through $\vec{\xi}_C$. By (2.13), $L_d < 1$ at S. Therefore, Proposition 1.8 yields $r_S(\phi) > r_{E'}(\phi)$ for all $\phi > \phi_C$. Hence topologically S is separated from $\hat{B}$ and the circle by E', so it also has uniformly lower bounded distance from them. In particular (2.7) is strict, for sufficiently small δ_{SB} (depending only on the choice of Λ, but not on any other constant).

2. If S vanishes in some point $\vec{\xi}$, then $L_d = L_u = |\vec{\xi}-\vec{v}_I|/c_I$ which — since S has uniform distance from the circle — is uniformly bounded below away from 1. However, this contradicts (2.13). The shock cannot vanish; on the contrary, by continuity the shock has uniformly lower-bounded strength. That implies (2.42), for sufficiently small $\delta_{\rho S}$. (Again, it depends only on Λ, not on the choice of other constants.)

□

Proposition 2.22. *If δ_ρ and ε are sufficiently small (with bounds depending only on C_{Pt}), then for any fixed point $\psi \in \overline{\mathcal{F}}$ of $\mathcal{K}$, the inequality (2.11) is* strict.

Proof. By Proposition 2.13, ψ and hence s are analytic. Thus we may use [11, Proposition 3.2.1] which rules out minima of ρ in Ω and (using Remark 2.8) at A or B.

Consider the coordinates of Figure 12. In $\vec{\xi}_A$, the first shock condition is

$$\psi(\vec{\xi}_A) = \psi^I(\vec{\xi}_A) = -\pi(\rho_I) + v_I^x\big(\xi_A - \frac{1}{2}v_I^x\big).$$

(2.29) implies

$$\psi(\vec{\xi}_{AB}) \le \psi(\vec{\xi}_A) + (\underbrace{\xi_{AB}}_{=0} - \underbrace{\xi_A}_{<0})(v_I^x - \delta_{vtA}) = -\pi(\rho_I) + \underbrace{\delta_{vtA}\xi_A - \frac{1}{2}(v_I^x)^2}_{<0}.$$

So in $\vec{\xi}_{AB} = 0$, since $\nabla\chi = 0$ by boundary conditions on A, B and C^1 regularity:

$$\rho = \pi^{-1}(-\chi - \frac{1}{2}|\nabla\chi|^2) = \pi^{-1}(-\psi) = \rho_I + \delta_{\rho AB}$$

for some constant $\delta_{\rho AB} > 0$ depending only on the parameters λ; note that π is a strictly increasing function for any $\gamma \ge 1$. We can pick $\delta_\rho < \delta_{\rho AB}$ so that $\rho \le \rho_I + \delta_\rho$ is not possible in $\vec{\xi}_{AB}$.

On P we know ρ up to a small constant, by (2.38), so we can choose δ_ρ even smaller so that $\rho \le \rho_I + \delta_\rho$ is not possible at $\overline{P}$.

By (2.42), ρ at $\overline{S}$ is uniformly bounded below away from ρ_I. Hence, for δ_ρ sufficiently small, ρ cannot have a global minimum close to ρ_I at S.

We see that for sufficiently small δ_ρ and ε, depending continuously on C_{Pt} (and λ), (2.11) is *strict*. □

Proposition 2.23. *If δ_{vtA}, δ_{vnB} and ε are sufficiently small (δ_{vtA}, δ_{vnB} bounds depending only on δ_ρ, C_{Sn}, ε bound depending only on C_{Pt}), and if δ_{Cc} is sufficiently small, then for any fixed point $\psi \in \overline{\mathcal{F}}$ of $\mathcal{K}$, the inequalities (2.29) and (2.28) are* strict.

Proof. Consider the coordinates of Figure 17 right, where $\vec{v} \cdot \vec{n}_B = -v^y$. (2.11) implies that the shock is uniformly strong. By (2.30), the shock normal $\vec{n}$ is everywhere downwards and uniformly not horizontal. Thus $v^y > v^y_I + \delta_{vnB}$ at $\overline{S}$ for sufficiently small δ_{vnB}, depending only on δ_ρ and C_{Sn}.

[11, Proposition 3.3.1] rules out local maxima of v^y in Ω.

If v^y has a local maximum at A, then A must be horizontal ([11, Proposition 3.4.1]), but by construction it is not.

On $\overline{B}$ the boundary condition implies $0 = \chi_n = \chi_2$, so $v^y = \psi_2 = \eta_{AB} = 0$.

At $\overline{P}$ we can use (2.39) with $v^y_R = 0 > v^y_I$ to obtain $v^y > v^y_I$ if ε is small enough (depending on C_{Pt}).

Altogether we have that (2.28) is strict if δ_{vnB} is small enough.

The arguments for (2.29) are analogous, looking at Figure 17 left coordinates instead: the shock S is nowhere vertical (by (2.30)), so $v^y > v^y_I + \delta_{vtA}$ at $\overline{S}$ for sufficiently small δ_{vtA}. If B is not horizontal, then the direction $(0,1)$ is not perpendicular to it, so [11, Proposition 3.4.1] rules out a local v^y extremum at B; if B is horizontal, then $0 = \chi_n = \chi_2$, so $v^y = \psi_2 = \eta_{AB} = 0$ on it. A is always vertical, i.e. never perpendicular to $(0,1)$, so by [11, Proposition 3.4.1] no v^y extremum is possible at it. In $\vec{\xi}_{AB} = 0$, the boundary conditions combine to $\vec{v} = 0$, so $v^y = 0 > v^y_I + \delta_{vtA}$ if δ_{vtA} is small enough. At $\overline{P}$ we can use (2.39) again to obtain $v^y \geq v^y_R - C_{Pt} \cdot \varepsilon^{1/2} > v^y_I$ (using $v^y_R > v^y_I$ and for ε sufficiently small, with bound depending only on C_{Pt}). [11, Proposition 3.3.1] rules out interior extrema of v^y. Hence (2.29) is strict if δ_{vtA} is small enough. □

Fixed points

Proposition 2.24. *For δ_o sufficiently small, with bounds depending only on δ_ρ and C_L, for C_d resp. δ_d sufficiently large resp. small, with bounds depending only on δ_ρ and C_L, and for ε sufficiently small, with bounds depending only on C_{Pt}, C_L and δ_ρ:*

If $\chi \in \overline{\mathcal{F}}$ is a fixed point of $\mathcal{K}$, then (2.31) and (2.32) are strict.

Proof. Compared to [11, Proposition 4.13.1], the only new case is a corner between two walls, A and B. The corner angle is bounded away from 0 and π by constants depending only on the parameters λ. (Note that ξ_{AB} in Section 2 has been lower-bounded uniformly by $\underline{\xi}_{AB}$ in Definition 2.2, so that $\hat{A}, \hat{B}$ are uniformly not parallel.) $g_{\vec{p}}$ on A and B is their respective normal, so (2.32) is obvious. □

Proposition 2.25. *If the constants in (2.2) in Definition 2.6 are chosen sufficiently small resp. large: for any $\lambda \in \Lambda$, $\mathcal{K}_\lambda$ cannot have fixed points on $\overline{\mathcal{F}}_\lambda - \mathcal{F}_\lambda$.*

Proof. Let $\chi \in \overline{\mathcal{F}}$ be a fixed point of $\mathcal{K}$. We show that every inequality in the definition of $\overline{\mathcal{F}}$ is strict, so $\chi \in \mathcal{F}$.

(2.5) and (2.23) are strict by Proposition 2.13.

(2.7) is strict by Proposition 2.21.

(2.11) is strict by Proposition 2.22.

A fixed point satisfies $\psi = \hat{\psi}$, so $\|\psi - \hat{\psi}\| = r_I(\psi) > 0$ cannot be true. (2.14) is strict.

(2.12) strict is provided by Proposition 2.15.

Due to Proposition 2.15, $L^2 = 1 - \varepsilon$ on each point of $\overline{P}$, so we are in the situation of Section 2 and [11, Section 4.7 etc]. Proposition 2.16 shows that (2.24) and (2.25) are strict.

(2.26) and (2.27) are strict by Proposition 2.20.

(2.28) and (2.29) are strict by Proposition 2.23.

Propositions 2.17 and 2.19 rule out $\eta_C = \eta_C^* \pm \delta^{-1}\varepsilon$ if δ is small enough, so (2.8) is strict.

(2.9) is strict by Proposition 2.20.

(2.5) yields a trivial upper bound on the density in $\overline{\Omega}$, hence downstream at the shock.

(2.30) is strict by Proposition 2.20.

Proposition 2.24 shows that (2.31) and (2.32) are strict.

All inequalities are strict, so $\psi \in \mathcal{F}$. □

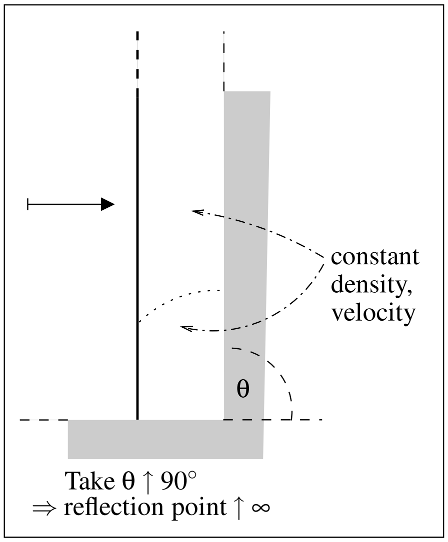


Figure 18. The unperturbed case: a straight vertical shock R. In this case there is no reflection point and no incident shock.

Existence of fixed points

We determine the Leray-Schauder degree of $\mathcal{K}$ on $\mathcal{F}$ for a particular choice of parameters λ: the unperturbed problem (see Figure 18), featuring a straight shock separating two constant-state regions ($\eta_C^* = \overline{\eta}_C^0$, $\xi_{AB} = v_R^x$ in the coordinates of Definition 2.2), for $\gamma = 1$.

Proposition 2.26. *For sufficiently small* ε*:*

For $\gamma = 1$*,* $\eta_C^* = \overline{\eta}_C^*$ *and* $\xi_{AB} = v_R^x$*,* $\mathcal{K}$ *has nonzero Leray-Schauder degree.*

Proof. We can use reflection across A (Remark 2.8) to obtain the problem of Propositions 4.14.1 and 4.14.3 in [11]. The resulting iteration $\mathcal{K}$ is almost the same as in loc.cit., except for minor differences in the coordinate transform from $(\sigma,\zeta) \in [0,1]^2$ (fixed domain) to $\vec{\xi}$ coordinates (see Definition 2.6 as compared to [11, Definition 4.4.3]). The proofs of [11, Propositions 4.14.1 and 4.14.3] carry over without any change to show that the present problem has nonzero Leray-Schauder degree. ☐

Proposition 2.27. *For sufficiently small resp. large constants in (2.2):* $\mathcal{K}$ *has a fixed point for all* $\lambda \in \Lambda$*.*

Proof. The proof is identical to the one of [11, Proposition 4.15.1], except for the definition of Λ (Definition 2.2); we use the known Leray-Schauder degree in $(\gamma, \eta_C^*, \xi_{AB}) = (1, \eta_C^0, v_R^x)$ from Proposition 2.26. ☐

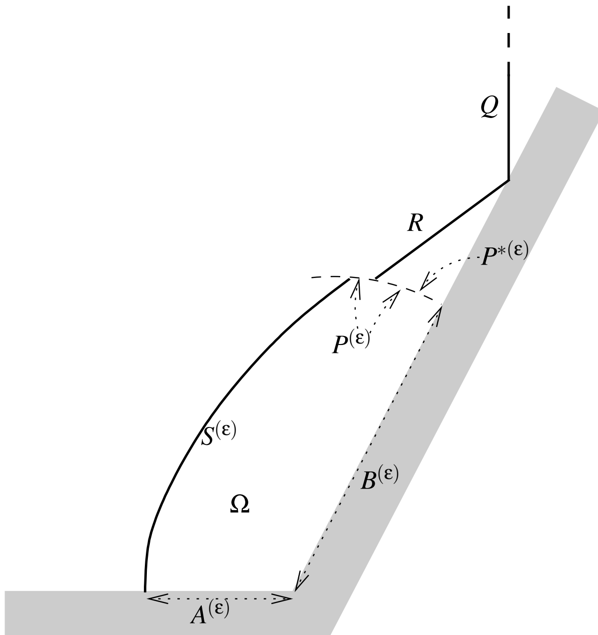


Figure 19. The expected and actual parabolic arc ($P^{*(\varepsilon)}$ and $P^{(\varepsilon)}$) differ by curve of length $O(\varepsilon^{1/2})$ (by (2.8))

Construction of the entire flow

Proof of Theorem 1.1. For all $\rho_I, c_I, M_I \in (0,\infty)$ and for each choice (in Definition 2.2) of $\bar{\gamma}$, $\underline{\eta}_C^*$ and $\underline{\xi}_{AB}$ we obtain a separate parameter set Λ. For sufficiently small constants in (2.2), Proposition 2.27 yields fixed points ψ for all $\lambda \in \Lambda$. Note that there is no lower bound on ε, except that α, β etc. may change as $\varepsilon \downarrow 0$.

By Definition 2.6, Remark 2.7, Proposition 2.16 and (2.35), the fixed points satisfy

$$
\begin{aligned}
(c^2 I - \nabla\chi^2) : \nabla^2\psi &= 0 && \text{in } \Omega^{(\varepsilon)}, && (2.43)\\
|\psi - \psi^R(\vec{\xi}_C^*)| &= O(\varepsilon^{1/2}) && \text{and} && (2.44)\\
|\rho - \rho_R| &= O(\varepsilon^{1/2}) && \text{and} && (2.45)\\
|\nabla\psi - \vec{v}_R| &= O(\varepsilon^{1/2}) && \text{on } P^{(\varepsilon)}, && (2.46)\\
\chi &= \chi^I && \text{and} && (2.47)\\
(\rho\nabla\chi - \rho_I\nabla\chi^I)\cdot\vec{n} &= 0 && \text{on } S, && (2.48)\\
\nabla\chi\cdot\vec{n} &= 0 && \text{on } A\cup B, && (2.49)\\
|\vec{\xi}_C - \vec{\xi}_C^{*(\varepsilon)}| &= O(\varepsilon^{1/2}) && && (2.50)
\end{aligned}
$$

where the O constants are independent of ε. For regularity, Proposition 2.13 yields

$$\|\psi\|_{C^{0,1}(\overline{\Omega}^{(\varepsilon)})} \leq C_1, \tag{2.51}$$

$$\|\psi\|_{C^{k,\alpha}(K\cap\overline{\Omega}^{(\varepsilon)})}, |S|_{C^{k,\alpha}(K\cap\overline{S}^{(\varepsilon)})} \leq C_2(d) \tag{2.52}$$

$$\text{where } d := d(K, \hat{P}^{(\varepsilon)} \cup \{\vec{\xi}_{AB}\}) > 0.$$

for constants C_1 and $C_2(d)$ independent of ε.

Now consider those parameter vectors λ that arise from the situtation in Theorem 1.1, i.e. so that there is an incident shock Q meeting R in a local regular reflection. We extend ψ from above to a function $\psi^{(\varepsilon)}$ defined on all of $\overline{V}$ as shown in Figure 19: set $\rho = \rho_R$, $\vec{v} = \vec{v}_R$ in the region enclosed by R shock, $\hat{B}$ and $P^{*(\varepsilon)}$; set $\rho = \rho_Q$, $\vec{v} = \vec{v}_{R,Q}$ in the region right of the Q shock and $\rho = \rho_I$, $\vec{v} = \vec{v}_I$ in the remaining area. In each of the four regions, $\psi^{(\varepsilon)}$ is a strong solution of self-similar potential flow, so we can multiply the divergence-form PDE [11, (2.2.3)] with any test function $\vartheta \in C_c^\infty(\overline{V})$ and integrate over all region to obtain a sum of boundary integrals of the type

$$\int_M \rho \nabla\chi \cdot \vec{n}\, ds$$

where M are various curves; $\nabla\chi$ and ρ are limits on one of the sides of M.

The symmetric difference of $P^{(\varepsilon)}$ and $P^{*(\varepsilon)}$ has length $O(\varepsilon^{1/2})$ (by (2.50), so since $\nabla\psi$ and ψ are bounded in each region (uniformly in ε, by (2.51)), the boundary integral over the difference contributes only $O(\varepsilon^{1/2})$. The difference of the integrals on each side of $P^{*(\varepsilon)} \cap P^{(\varepsilon)}$ are $O(\varepsilon^{1/2})$ due to (2.45) and (2.46). The integrals over A, B vanish due to (2.49). Finally, the integrals on each side of $S^{(\varepsilon)}$ cancel due to (2.47) and (2.48). Altogether:

$$\int_{\overline{V}} \rho^{(\varepsilon)} \nabla\chi^{(\varepsilon)} \cdot \nabla\vartheta - 2\rho^{(\varepsilon)}\vartheta\, d\vec{\xi} = O(\varepsilon^{1/2}). \tag{2.53}$$

$C^{k,\alpha}$ with $k+\alpha > 1$ is compactly embedded in $C^{0,1}$, so by (2.52) with a diagonalization argument, for every compact $K \subset \overline{V} - \{\vec{\xi}_{AB}\} - \overline{P}^{*(0)}$ we can find a sequence $(\varepsilon_k) \downarrow 0$ so that $\psi^{(\varepsilon_k)}$ converges to $\psi^{(0)}$ in $C^{0,1}(K)$. Moreover $\rho^{(\varepsilon)}$ and $\nabla\chi^{(\varepsilon)}$ are bounded on $\overline{V}$ uniformly in ε, so we may take $\varepsilon \downarrow 0$ in (2.53) to obtain

$$\int_V \rho^{(0)} \nabla\chi^{(0)} \cdot \nabla\vartheta - 2\rho^{(0)}\vartheta\, d\vec{\xi} = 0. \tag{2.54}$$

In addition, (2.47) and (2.44) combined with (2.51) show that

$$\psi^{(0)} \in C(\overline{V}) \tag{2.55}$$

Finally, by construction of $\psi^{(\varepsilon)}$,

$$\rho^{(0)}(s\vec{\xi}), \vec{v}^{(0)}(s\vec{\xi}) \to \begin{cases} \rho_I, \vec{v}_I, & \vec{\xi} \in V_I, \\ \rho_Q, \vec{v}_Q, & \vec{\xi} \in V_Q \end{cases} \qquad \text{as } s \to \infty, \tag{2.56}$$

i.e. their limits on rays to infinity are exactly as for the initial data in Figure 3. This means the limit approaches the initial data as $t \downarrow 0$.

(2.54), (2.49), (2.55) and (2.56) show that $\phi(t,\vec{x}) := \psi^{(0)}(t^{-1}\vec{x})$ defines a solution of (1.2), (1.3), (1.4) and (1.5).

By taking $\overline{\gamma} \uparrow \infty$, $\underline{\eta}_C^* \downarrow 0$ and $\underline{\xi}_{AB} \downarrow \xi_{EB}$, we obtain a solution for *every* $\gamma \in [1,\infty)$, $\eta_C^* \in [\eta_C^0, 0)$ and $\xi_{AB} \in (\xi_E B, v_R^x]$. (in the cases $\gamma > 1$ and $\eta_C^* = \eta_C^0$, we may use that $\overline{\eta}_C^*$ approaches η_C^0 as $\varepsilon \downarrow 0$).

As mentioned (Remark 2.1), this exhausts all cases covered by the conditions of Theorem 1.1. The proof is therefore complete. □

Remark 2.28. In addition to mere existence we obtain some structural information in the proof:

1. The solution has the structure shown in Figure 4 left, with pseudo-Mach number $L > 1$ in the I, R, Q regions, $L < 1$ in the elliptic region Ω.
2. The solution has constant density and velocity in each of the I, R, Q regions.
3. The solution is analytic everywhere except perhaps at $\overline{P}^{*(0)}$ and in $\vec{\xi}_{AB}$ and, of course, the shocks.
4. The curved shock is analytic away from $\hat{A}$ and $\overline{P}^{*(0)}$ and Lipschitz overall.
5. Density and velocity are bounded.

It is expected that density and velocity are at least continuous. However, the methods developed in [11] yield boundedness everywhere, but continuity only away from $\overline{P}^*$. Note that $\overline{P}^*$ can *not* be a classical shock with smooth data on each side, because the one-sided limit of L on the hyperbolic side R of P^* is $= 1$ everywhere (> 1 is needed for positive shock strength).

Some additional structural information:

1. The possible (downstream) normals of the curved shock are between $\vec{n}_R$ and $\vec{t}_A$ (counterclockwise).
2. The shocks are admissible and do not vanish anywhere.
3. In the elliptic region, $v^x < v_I^x$ and $v^y \geq 0$ (in Figure 4 left coordinates).
4. In the elliptic region, the density ρ is greater than ρ_I.

Additional information can be obtained from the inequalities in Definition 2.6.

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