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ON THE BIFURCATION OF PERIODIC ORBITS IN DISCONTINUOUS SYSTEMS

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Abstract

We study behaviour of T–periodic trajectories in time-perturbed discontinuous systems, that transversally cross the discontinuity boundary. Sufficient condition of Melnikov-type is stated, under which the original piecewise C^1–orbit persists. Applications of derived result are also given to discontinuous planar systems.

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1 Introduction

Discontinuous systems occur in many physical applications. They describe behaviour of particles before and after collision with a rigid wall, motion of a body on oscillating belt

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(so-called dry-friction oscillator), switching in electric circuit because of the presence of a diode or transistor. But they can be found also in biology, medicine and optimal control theory (for references see [3, 5, 11, 12]). These non-smooth differential equations are characterized by existence of discontinuity boundary which divides phase space to two or more parts. Recently in [1], homoclinic orbits were investigated via Melnikov method [7]. It was assumed that the system has a homoclinic solution to a hyperbolic equilibrium and the sufficient condition was stated under which small perturbation of original system has a solution close to the homoclinic one. Similar method is used in our paper for the periodic bifurcation.

The plan of this paper is as follows. In Section 2, we introduce basic assumptions and the setting of our problem. In Section 3, firstly we show the existence of Poincaré mapping in a neighbourhood of the T–periodic trajectory of unperturbed system. Then properties of the mapping are used to determine a sufficient condition for the existence of T–periodic orbit of perturbed system close to original one. A geometric meaning of a nondegeneracy assumption on the periodic trajectory is explained in Section 4. Finally, in Section 5, applications to planar discontinuous systems are given. Firstly, we consider a piecewise nonlinear equation, then we proceed to piecewise linear one. In both cases, sufficient bifurcation conditions are derived for concrete values of parameters.

2 Preliminaries

Let $\Omega \subset \mathbb{R}^n$ be a open set in $\mathbb{R}^n$ and $h(x)$ be a C^r–function on $\bar{\Omega}$, with $r \geq 2$. We set $\Omega_\pm := \{x \in \Omega \mid \pm h(x) > 0\}$, $\Omega_0 := \{x \in \Omega \mid h(x) = 0\}$. Let $f_\pm \in C^r(\bar{\Omega})$ and $g \in C^r(\bar{\Omega} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^k)$, i.e. the derivative of $f_\pm$ and g are continuous up to the r-th order, respectively. Furthermore, we suppose that g is T–periodic in $t \in \mathbb{R}$. Let $\varepsilon \in \mathbb{R}$, $\alpha \in \mathbb{R}$, $\mu \in \mathbb{R}^k, k \geq 1$ are parameters and $\langle \cdot, \cdot \rangle$ denote inner product in $\mathbb{R}^n$.

We say that a function $x(t)$ is a solution of the equation

$$\dot{x} = f_\pm(x) + \varepsilon g(x, t+\alpha, \varepsilon, \mu), \quad x \in \bar{\Omega}_\pm, \tag{2.1}$$

if it is continuous, piecewise C^1, satisfies equation (2.1) on $\Omega_\pm$ and, moreover, the following holds: if for some t_0 we have $x(t_0) \in \Omega_0$, then there exists $r > 0$ such that for any $t \in (t_0 - r, t_0)$ we have $x(t) \in \Omega_+$, and for any $t \in (t_0, t_0 + r)$ we have $x(t) \in \Omega_-$.

We assume (see Figure 1)

H1) For $\varepsilon = 0$ equation (2.1) has a T–periodic solution $\gamma(t)$ which has a starting point $x_0 \in \Omega_+$ and consists of three branches

$$\gamma(t) = \begin{cases} \gamma_1(t) & \text{if} \quad t \in [0, t_1] \\ \gamma_2(t) & \text{if} \quad t \in [t_1, t_2] \\ \gamma_3(t) & \text{if} \quad t \in [t_2, T] \end{cases} \tag{2.2}$$

where $\gamma_1(t) \in \Omega_+$ for $t \in [0, t_1)$, $\gamma_2(t) \in \Omega_-$ for $t \in (t_1, t_2)$ and $\gamma_3(t) \in \Omega_+$ for $t \in (t_2, T]$ and

$$\begin{aligned} x_1 &:= \gamma_1(t_1) = \gamma_2(t_1) && \in \Omega_0, \\ x_2 &:= \gamma_2(t_2) = \gamma_3(t_2) && \in \Omega_0, \\ x_0 &:= \gamma_3(T) = \gamma_1(0) && \in \Omega_+. \end{aligned} \tag{2.3}$$

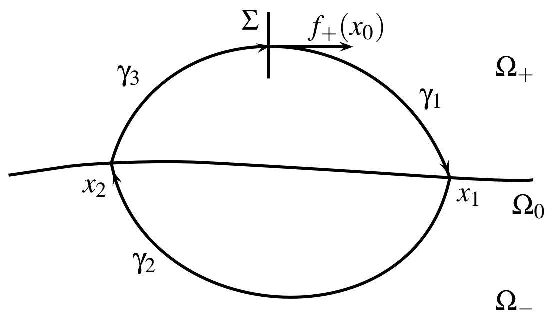

Figure 1. Used notation

H2) Moreover, we also assume that

$$\mathrm{D}h(x_1)f_\pm(x_1) < 0 \quad \text{and} \quad \mathrm{D}h(x_2)f_\pm(x_2) > 0.$$

3 Existence of periodic orbits close to $\gamma(t)$

Let $x_+(\tau,\xi)(t,\varepsilon,\mu,\alpha)$ denote a solution of initial value problem (3.1) which consists of equation

$$\dot{x} = f_+(x) + \varepsilon g(x, t+\alpha, \varepsilon, \mu) \tag{3.1}$$

and initial condition $x_+(\tau,\xi)(\tau,\varepsilon,\mu,\alpha) = \xi$, and $x_-(\tau,\xi)(t,\varepsilon,\mu,\alpha)$ denote a solution of similar problem (3.2) consisting of equation

$$\dot{x} = f_-(x) + \varepsilon g(x, t+\alpha, \varepsilon, \mu) \tag{3.2}$$

and condition $x_-(\tau,\xi)(\tau,\varepsilon,\mu,\alpha) = \xi$.

For simplicity, we suppose that $f_\pm$ and g are extended C^r–smoothly on $\mathbb{R}^n$ and $\mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^k$, respectively, with uniformly bounded derivatives up to the r-th order.

Using Implicit Function Theorem [6] we show that there are some trajectories in the neighbourhood of $\gamma(t)$ and then we select periodic ones from these.

Lemma 3.1. *Assume* H1) *and* H2). *Then there exist* $\varepsilon_3, r_3 > 0$ *and a Poincaré mapping*

$$P(\cdot,\varepsilon,\mu,\alpha) : U \to \Sigma$$

for all fixed $\varepsilon \in (-\varepsilon_3, \varepsilon_3)$, $\mu \in \mathbb{R}^k$, $\alpha \in \mathbb{R}$ *where* $\Sigma = \{x \in \mathbb{R}^n \mid \langle x - x_0, f_+(x_0)\rangle = 0\}$, $U = \Sigma \cap B(x_0, r_3)$ *and* $B(x,r)$ *is the ball of radius r and center in x. Moreover, P is* C^r*–smooth in all arguments.*

Proof. We denote $\mathcal{A}(\tau,\xi,t,\varepsilon,\mu,\alpha) = h(x_+(\tau,\xi)(t,\varepsilon,\mu,\alpha))$. Since

$$\begin{aligned}&\mathcal{A}(0,x_0,t_1,0,\mu,\alpha) = 0\\&\mathcal{A}_t(0,x_0,t_1,0,\mu,\alpha) = \mathrm{D}h(x_1)f_+(x_1) < 0\end{aligned}$$

Implicit Function Theorem gives the existence of $\tau_1, r_1, \delta_1, \varepsilon_1 > 0$ and C^r–function

$$t_1(\cdot,\cdot,\cdot,\cdot,\cdot) : (-\tau_1,\tau_1)\times B(x_0,r_1)\times(-\varepsilon_1,\varepsilon_1)\times\mathbb{R}^k\times\mathbb{R}\to(t_1-\delta_1,t_1+\delta_1)$$

such that $\mathcal{A}(\tau,\xi,t,\varepsilon,\mu,\alpha)=0$ for $\tau\in(-\tau_1,\tau_1)$, $\xi\in B(x_0,r_1)\subset\Omega_+$, $\varepsilon\in(-\varepsilon_1,\varepsilon_1)$, $\mu\in\mathbb{R}^k$, $\alpha\in\mathbb{R}$ and $t\in(t_1-\delta_1,t_1+\delta_1)$ if and only if $t=t_1(\tau,\xi,\varepsilon,\mu,\alpha)$.

Next we set

$$\mathcal{B}(\tau,\xi,t,\varepsilon,\mu,\alpha)=h(x_-(t_1(\tau,\xi,\varepsilon,\mu,\alpha),x_+(\tau,\xi)(t_1(\tau,\xi,\varepsilon,\mu,\alpha),\varepsilon,\mu,\alpha))(t,\varepsilon,\mu,\alpha)).$$

Then

$$\begin{aligned}&\mathcal{B}(0,x_0,t_2,0,\mu,\alpha)=0\\&\mathcal{B}_t(0,x_0,t_2,0,\mu,\alpha)=\mathrm{D}h(x_2)f_-(x_2)>0\end{aligned}$$

hence Implicit Function Theorem implies that there exist $\tau_2, r_2, \delta_2, \varepsilon_2 > 0$ and C^r–function

$$t_2(\cdot,\cdot,\cdot,\cdot,\cdot) : (-\tau_2,\tau_2)\times B(x_0,r_2)\times(-\varepsilon_2,\varepsilon_2)\times\mathbb{R}^k\times\mathbb{R}\to(t_2-\delta_2,t_2+\delta_2)$$

such that $\mathcal{B}(\tau,\xi,t,\varepsilon,\mu,\alpha)=0$ for $\tau\in(-\tau_2,\tau_2)$, $\xi\in B(x_0,r_2)\subset\Omega_+,\varepsilon\in(-\varepsilon_2,\varepsilon_2)$, $\mu\in\mathbb{R}^k$, $\alpha\in\mathbb{R}$ and $t\in(t_2-\delta_2,t_2+\delta_2)$ if and only if $t=t_2(\tau,\xi,\varepsilon,\mu,\alpha)$.

Once more time we use Implicit Function Theorem on function $\mathcal{C}$ defined as follows

$$\begin{aligned}\mathcal{C}(\tau,\xi,t,\varepsilon,\mu,\alpha)=&\langle x_+(t_2(\tau,\xi,\varepsilon,\mu,\alpha),x_-(t_1(\tau,\xi,\varepsilon,\mu,\alpha),x_+(\tau,\xi)(t_1(\tau,\xi,\varepsilon,\mu,\alpha),\varepsilon,\mu,\alpha))\\&(t_2(\tau,\xi,\varepsilon,\mu,\alpha),\varepsilon,\mu,\alpha))(t,\varepsilon,\mu,\alpha)-x_0,f_+(x_0)\rangle\end{aligned}$$

Since

$$\begin{aligned}&\mathcal{C}(0,x_0,T,0,\mu,\alpha)=0\\&\mathcal{C}_t(0,x_0,T,0,\mu,\alpha)=\|f_+(x_0)\|^2>0\end{aligned}$$

then there exist $\tau_3, r_3, \delta_3, \varepsilon_3 > 0$ and C^r–function

$$t_3(\cdot,\cdot,\cdot,\cdot,\cdot) : (-\tau_3,\tau_3)\times B(x_0,r_3)\times(-\varepsilon_3,\varepsilon_3)\times\mathbb{R}^k\times\mathbb{R}\to(T-\delta_3,T+\delta_3)$$

such that $\mathcal{C}(\tau,\xi,t,\varepsilon,\mu,\alpha)=0$ for $\tau\in(-\tau_3,\tau_3)$, $\xi\in B(x_0,r_3)\subset\Omega_+,\varepsilon\in(-\varepsilon_3,\varepsilon_3)$, $\mu\in\mathbb{R}^k$, $\alpha\in\mathbb{R}$ and $t\in(T-\delta_3,T+\delta_3)$ if and only if $t=t_3(\tau,\xi,\varepsilon,\mu,\alpha)$.
Moreover $t_1(0,x_0,0,\mu,\alpha)=t_1$, $t_2(0,x_0,0,\mu,\alpha)=t_2$ and $t_3(0,x_0,0,\mu,\alpha)=T$.
Now we can define the Poincaré mapping from lemma's statement

$$\begin{aligned}P(\xi,\varepsilon,\mu,\alpha)=&x_+(t_2(0,\xi,\varepsilon,\mu,\alpha),x_-(t_1(0,\xi,\varepsilon,\mu,\alpha),x_+(0,\xi)(t_1(0,\xi,\varepsilon,\mu,\alpha),\varepsilon,\mu,\alpha))\\&(t_2(0,\xi,\varepsilon,\mu,\alpha),\varepsilon,\mu,\alpha))(t_3(0,\xi,\varepsilon,\mu,\alpha),\varepsilon,\mu,\alpha)\end{aligned}$$

It is obvious that P maps U to Σ. □

Our aim is to find T–periodic orbits, which is the reason for solving the following system

$$\begin{aligned}P(\xi,\varepsilon,\mu,\alpha)&=\xi\\t_3(0,\xi,\varepsilon)&=T\end{aligned}$$

for ξ and ε sufficiently close to x_0 and 0, respectively. This problem can be reduced to one equation

$$\xi - \widetilde{P}(\xi, \varepsilon, \mu, \alpha) = 0 \tag{3.3}$$

where

$$\begin{aligned}\widetilde{P}(\xi,\varepsilon,\mu,\alpha) =& x_+(t_2(0,\xi,\varepsilon,\mu,\alpha), x_-(t_1(0,\xi,\varepsilon,\mu,\alpha), x_+(0,\xi)(t_1(0,\xi,\varepsilon,\mu,\alpha),\varepsilon,\mu,\alpha))\\ &(t_2(0,\xi,\varepsilon,\mu,\alpha),\varepsilon,\mu,\alpha))(T,\varepsilon,\mu,\alpha)\end{aligned} \tag{3.4}$$

It is easy to see that $(\xi,\varepsilon,\mu,\alpha) = (x_0,0,\mu,\alpha)$ solves (3.3). However, Implicit Function Theorem can not be used here, what is proved in the next lemma (see [9, 10]).

Lemma 3.2. *$\widetilde{P}_\xi(x_0,0,\mu,\alpha)$ has eigenvalue 1 with corresponding eigenvector $f(x_0)$, i.e. $\widetilde{P}_\xi(x_0,0,\mu,\alpha)f(x_0) = f(x_0)$.*

Proof. Let V be a sufficiently small neighbourhood of 0. Then

$$\begin{aligned}x_+(0,x_+(0,x_0)(t,0,\mu,\alpha))(t_1(0,x_+(0,x_0)(t,0,\mu,\alpha),0,\mu,\alpha),0,\mu,\alpha) =\\ x_+(0,x_0)(t+t_1(0,x_+(0,x_0)(t,0,\mu,\alpha),0,\mu,\alpha),0,\mu,\alpha)\end{aligned} \tag{3.5}$$

for any $t \in V$, where the left-hand side of (3.5) is from Ω_0 and the right-hand side is a point of $\gamma(t)$. Thereafter the sum $t + t_1(0,x_+(0,x_0)(t,0,\mu,\alpha),0,\mu,\alpha) = t_1$, i.e. it is constant for all $t \in V$. Similarly

$$\begin{aligned}&x_-(t_1(0,x_+(0,x_0)(t,0,\mu,\alpha),0,\mu,\alpha), x_+(0,x_+(0,x_0)(t,0,\mu,\alpha))(t_1(0,x_+(0,x_0)\\ &\qquad(t,0,\mu,\alpha),0,\mu,\alpha),0,\mu,\alpha))(t_2(0,x_+(0,x_0)(t,0,\mu,\alpha),0,\mu,\alpha),0,\mu,\alpha)\\ &= x_-(t_1(0,x_+(0,x_0)(t,0,\mu,\alpha),0,\mu,\alpha), x_+(0,x_0)(t_1,0,\mu,\alpha))\\ &\qquad(t_2(0,x_+(0,x_0)(t,0,\mu,\alpha),0,\mu,\alpha),0,\mu,\alpha)\\ &= x_-(t_1 - t, x_+(0,x_0)(t_1,0,\mu,\alpha))(t_2(0,x_+(0,x_0)(t,0,\mu,\alpha),0,\mu,\alpha),0,\mu,\alpha)\\ &= x_-(t_1,x_1)(t_2(0,x_+(0,x_0)(t,0,\mu,\alpha),0,\mu,\alpha)+t,0,\mu,\alpha)\end{aligned}$$

and we obtain $t + t_2(0,x_+(0,x_0)(t,0,\mu,\alpha),0,\mu,\alpha) = t_2$ for all $t \in V$.
With these results we can derive

$$\begin{aligned}&\widetilde{P}(x_+(0,x_0)(t,0,\mu,\alpha),0,\mu,\alpha)\\ &\quad= x_+(t_2(0,x_+(0,x_0)(t,0,\mu,\alpha),0,\mu,\alpha), x_-(t_1(0,x_+(0,x_0)(t,0,\mu,\alpha),0,\mu,\alpha),\\ &\qquad\quad x_+(0,x_+(0,x_0)(t,0,\mu,\alpha))(t_1(0,x_+(0,x_0)(t,0,\mu,\alpha),0,\mu,\alpha),0,\mu,\alpha))\\ &\qquad\quad(t_2(0,x_+(0,x_0)(t,0,\mu,\alpha),0,\mu,\alpha),0,\mu,\alpha))(T,0,\mu,\alpha)\\ &\quad= x_+(t_2(0,x_+(0,x_0)(t,0,\mu,\alpha),0,\mu,\alpha),x_2)(T,0,\mu,\alpha)\\ &\quad= x_+(t_2 - t, x_2)(T,0,\mu,\alpha) = x_+(t_2,x_2)(T+t,0,\mu,\alpha)\end{aligned}$$

and finally

$$\begin{aligned}\widetilde{P}_\xi(x_0,0,\mu,\alpha)f(x_0) &= \frac{d}{dt}\left[\widetilde{P}(x_+(0,x_0)(t,0,\mu,\alpha),0,\mu,\alpha)\right]_{t=0}\\ &= \frac{d}{dt}\left[x_+(t_2,x_2)(T+t,0,\mu,\alpha)\right]_{t=0}\\ &= f(x_+(t_2,x_2)(T+t,0,\mu,\alpha))|_{t=0} = f(x_0)\end{aligned}$$

□

In the next step we construct the linearization $\widetilde{P}_\xi(x_0,0,\mu,\alpha)$ that will be important in further work.

Differentiating (3.1) with respect to ξ at the point $(\tau,\xi,\varepsilon)=(0,x_0,0)$ we have

$$\begin{aligned}\dot{x}_{+\xi}(0,x_0)(t,0,\mu,\alpha) &= \mathrm{D}f_+(x_+(0,x_0)(t,0,\mu,\alpha))x_{+\xi}(0,x_0)(t,0,\mu,\alpha)\\ &= \mathrm{D}f_+(\gamma(t))x_{+\xi}(0,x_0)(t,0,\mu,\alpha)\\ x_{+\xi}(0,x_0)(0,0,\mu,\alpha) &= \mathbb{I}\end{aligned}$$

where $\mathbb{I}$ denotes $n\times n$ identity matrix. Denote by $X_1(t)$ the matrix solution satisfying this linearized equation on $[0,t_1]$, i.e.

$$\begin{aligned}\dot{X}_1(t) &= \mathrm{D}f_+(\gamma(t))X_1(t)\\ X_1(0) &= \mathbb{I}\end{aligned} \tag{3.6}$$

So $x_{+\xi}(0,x_0)(t,0,\mu,\alpha)=X_1(t)$.
By differentiation (3.1) with respect to τ at the same point we get

$$\begin{aligned}\dot{x}_{+\tau}(0,x_0)(t,0,\mu,\alpha) &= \mathrm{D}f_+(x_+(0,x_0)(t,0,\mu,\alpha))x_{+\tau}(0,x_0)(t,0,\mu,\alpha)\\ x_{+\tau}(0,x_0)(0,0,\mu,\alpha) &= -f_+(x_+(0,x_0)(0,0,\mu,\alpha))\end{aligned}$$

Hence

$$x_{+\tau}(0,x_0)(t,0,\mu,\alpha) = -X_1(t)f_+(x_0)$$

for $t\in[0,t_1]$. Derivative of (3.1) with respect to ε at $(0,x_0,0)$ will be useful, too. We have equation

$$\begin{aligned}\dot{x}_{+\varepsilon}(0,x_0)(t,0,\mu,\alpha) &= \mathrm{D}f_+(x_+(0,x_0)(t,0,\mu,\alpha))x_{+\varepsilon}(0,x_0)(t,0,\mu,\alpha)\\ &\quad + g(x_+(0,x_0)(t,0,\mu,\alpha),t+\alpha,0,\mu)\\ x_{+\varepsilon}(0,x_0)(0,0,\mu,\alpha) &= 0\end{aligned}$$

which solved by variation of constants gives equality

$$x_{+\varepsilon}(0,x_0)(t,0,\mu,\alpha) = \int_0^t X_1(t)X_1^{-1}(s)g(\gamma(s),s+\alpha,0,\mu)ds$$

holding on $[0,t_1]$.

First intersection point on Ω_0 fulfills

$$h(x_+(\tau,\xi)(t_1(\tau,\xi,\varepsilon,\mu,\alpha),\varepsilon,\mu,\alpha))=0$$

for all (τ,ξ,ε) sufficiently close to $(0,x_0,0)$ and μ, α.

$$\begin{aligned}\mathrm{D}h(x_1)(X_1(t_1)+f_+(x_1)t_{1\xi}(0,x_0,0,\mu,\alpha)) &= 0\\ t_{1\xi}(0,x_0,0,\mu,\alpha) &= -\frac{\mathrm{D}h(x_1)X_1(t_1)}{\mathrm{D}h(x_1)f_+(x_1)}\end{aligned}$$

$$\begin{aligned}\mathrm{D}h(x_1)(-X_1(t_1)f_+(x_0)+f_+(x_1)t_{1\tau}(0,x_0,0,\mu,\alpha)) &= 0\\ t_{1\tau}(0,x_0,0,\mu,\alpha) &= \frac{\mathrm{D}h(x_1)X_1(t_1)f_+(x_0)}{\mathrm{D}h(x_1)f_+(x_1)}\end{aligned}$$

and

$$\mathrm{D}h(x_1)\left(f_+(x_1)t_{1\varepsilon}(0,x_0,0,\mu,\alpha)+\int_0^{t_1} X_1(t_1)X_1^{-1}(s)g(\gamma(s),s+\alpha,0,\mu)ds\right)=0$$
$$t_{1\varepsilon}(0,x_0,0,\mu,\alpha)=-\frac{\mathrm{D}h(x_1)\int_0^{t_1} X_1(t_1)X_1^{-1}(s)g(\gamma(s),s+\alpha,0,\mu)ds}{\mathrm{D}h(x_1)f_+(x_1)}$$

Differentiating (3.2) with respect to ξ, τ and ε at the point $(\tau,\xi,\varepsilon)=(t_1,x_1,0)$, respectively, we obtain

$$\begin{aligned}\dot{x}_{-\xi}(t_1,x_1)(t,0,\mu,\alpha)&=\mathrm{D}f_-(x_-(t_1,x_1)(t,0,\mu,\alpha))x_{-\xi}(t_1,x_1)(t,0,\mu,\alpha)\\ x_{-\xi}(t_1,x_1)(t_1,0,\mu,\alpha)&=\mathbb{I}\end{aligned}$$

$$\begin{aligned}\dot{x}_{-\tau}(t_1,x_1)(t,0,\mu,\alpha)&=\mathrm{D}f_-(x_-(t_1,x_1)(t,0,\mu,\alpha))x_{-\tau}(t_1,x_1)(t,0,\mu,\alpha)\\ x_{-\tau}(t_1,x_1)(t_1,0,\mu,\alpha)&=-f_-(x_-(t_1,x_1)(t_1,0,\mu,\alpha))\end{aligned}$$

and

$$\begin{aligned}\dot{x}_{-\varepsilon}(t_1,x_1)(t,0,\mu,\alpha)&=\mathrm{D}f_-(x_-(t_1,x_1)(t,0,\mu,\alpha))x_{-\varepsilon}(t_1,x_1)(t,0,\mu,\alpha)\\ &\quad+g(x_-(t_1,x_1)(t,0,\mu,\alpha),t+\alpha,0,\mu)\\ x_{-\varepsilon}(t_1,x_1)(t_1,0,\mu,\alpha)&=0\end{aligned}$$

Using matrix solution $X_2(t)$ of the first equation satisfying

$$\begin{aligned}\dot{X}_2(t)&=\mathrm{D}f_-(\gamma(t))X_2(t)\\ X_2(t_1)&=\mathbb{I}\end{aligned}\tag{3.7}$$

i.e. $x_{-\xi}(t_1,x_1)(t,0,\mu,\alpha)=X_2(t)$, we can write the other two solutions as

$$\begin{aligned}x_{-\tau}(t_1,x_1)(t,0,\mu,\alpha)&=-X_2(t)f_-(x_1)\\ x_{-\varepsilon}(t_1,x_1)(t,0,\mu,\alpha)&=\int_{t_1}^{t} X_2(t)X_2^{-1}(s)g(\gamma(s),s+\alpha,0,\mu)ds\end{aligned}$$

for $t\in[t_1,t_2]$.

Second intersection point is characterized by

$$h(x_-(t_1(\tau,\xi,\varepsilon,\mu,\alpha),x_+(\tau,\xi)(t_1(\tau,\xi,\varepsilon,\mu,\alpha),\varepsilon,\mu,\alpha))(t_2(\tau,\xi,\varepsilon,\mu,\alpha),\varepsilon,\mu,\alpha))=0$$

From that we derive

$$\begin{aligned}\mathrm{D}h(x_2)(x_{-\tau}(t_1,x_1)(t_2,0,\mu,\alpha)t_{1\xi}(0,x_0,0,\mu,\alpha)+x_{-\xi}(t_1,x_1)(t_2,0,\mu,\alpha)\\ \times[x_{+\xi}(0,x_0)(t_1,0,\mu,\alpha)+x_{+t}(0,x_0)(t_1,0,\mu,\alpha)t_{1\xi}(0,x_0,0,\mu,\alpha)]\\ +x_{-t}(t_1,x_1)(t_2,0,\mu,\alpha)t_{2\xi}(0,x_0,0,\mu,\alpha))=0\end{aligned}$$

$$t_{2\xi}(0,x_0,0,\mu,\alpha)=-\frac{\mathrm{D}h(x_2)X_2(t_2)S_1X_1(t_1)}{\mathrm{D}h(x_2)f_-(x_2)}$$

$$\begin{aligned}&\mathrm{D}h(x_2)(x_{-\tau}(t_1,x_1)(t_2,0,\mu,\alpha)t_{1\tau}(0,x_0,0,\mu,\alpha)+x_{-\xi}(t_1,x_1)(t_2,0,\mu,\alpha)\\&\qquad\times[x_{+\tau}(0,x_0)(t_1,0,\mu,\alpha)+x_{+t}(0,x_0)(t_1,0,\mu,\alpha)t_{1\tau}(0,x_0,0,\mu,\alpha)]\\&\qquad\qquad+x_{-t}(t_1,x_1)(t_2,0,\mu,\alpha)t_{2\tau}(0,x_0,0,\mu,\alpha))=0\end{aligned}$$

$$t_{2\tau}(0,x_0,0,\mu,\alpha)=\frac{\mathrm{D}h(x_2)X_2(t_2)S_1X_1(t_1)f_+(x_0)}{\mathrm{D}h(x_2)f_-(x_2)}$$

and

$$\begin{aligned}&\mathrm{D}h(x_2)(x_{-\tau}(t_1,x_1)(t_2,0,\mu,\alpha)t_{1\varepsilon}(0,x_0,0,\mu,\alpha)+x_{-\xi}(t_1,x_1)(t_2,0,\mu,\alpha)\\&\qquad\times[x_{+\varepsilon}(0,x_0)(t_1,0,\mu,\alpha)+x_{+t}(0,x_0)(t_1,0,\mu,\alpha)t_{1\varepsilon}(0,x_0,0,\mu,\alpha)]\\&\qquad\qquad+x_{-t}(t_1,x_1)(t_2,0,\mu,\alpha)t_{2\varepsilon}(0,x_0,0,\mu,\alpha)+x_{-\varepsilon}(t_1,x_1)(t_2,0,\mu,\alpha))=0\end{aligned}$$

$$\begin{aligned}t_{2\varepsilon}(0,x_0,0,\mu,\alpha)=-\frac{\mathrm{D}h(x_2)}{\mathrm{D}h(x_2)f_-(x_2)}&\Big(X_2(t_2)S_1\int_0^{t_1}X_1(t_1)X_1^{-1}(s)g(\gamma(s),s+\alpha,0,\mu)ds\\&+\int_{t_1}^{t_2}X_2(t_2)X_2^{-1}(s)g(\gamma(s),s+\alpha,0,\mu)ds\Big)\end{aligned}$$

where

$$S_1=\mathbb{I}+\frac{(f_-(x_1)-f_+(x_1))\mathrm{D}h(x_1)}{\mathrm{D}h(x_1)f_+(x_1)} \tag{3.8}$$

is so-called saltation matrix [13].

Finally we count derivatives of (3.1) with respect to ξ, τ and ε at $(\tau,\xi,\varepsilon)=(t_2,x_2,0)$ to obtain

$$\begin{aligned}\dot{x}_{+\xi}(t_2,x_2)(t,0,\mu,\alpha)&=\mathrm{D}f_+(x_+(t_2,x_2)(t,0,\mu,\alpha))x_{+\xi}(t_2,x_2)(t,0,\mu,\alpha)\\x_{+\xi}(t_2,x_2)(t_2,0,\mu,\alpha)&=\mathbb{I}\end{aligned}$$

$$\begin{aligned}\dot{x}_{+\tau}(t_2,x_2)(t,0,\mu,\alpha)&=\mathrm{D}f_+(x_+(t_2,x_2)(t,0,\mu,\alpha))x_{+\tau}(t_2,x_2)(t,0,\mu,\alpha)\\x_{+\tau}(t_2,x_2)(t_2,0,\mu,\alpha)&=-f_+(x_+(t_2,x_2)(t_2,0,\mu,\alpha))\end{aligned}$$

and

$$\begin{aligned}\dot{x}_{+\varepsilon}(t_2,x_2)(t,0,\mu,\alpha)&=\mathrm{D}f_+(x_+(t_2,x_2)(t,0,\mu,\alpha))x_{+\varepsilon}(t_2,x_2)(t,0,\mu,\alpha)\\&\quad+g(x_+(t_2,x_2)(t,0,\mu,\alpha),t+\alpha,0,\mu)\\x_{+\varepsilon}(t_2,x_2)(t_2,0,\mu,\alpha)&=0\end{aligned}$$

Matrix solution $X_3(t)$ for first equation that for $t\in[t_2,T]$ fulfills

$$\begin{aligned}\dot{X}_3(t)&=\mathrm{D}f_+(\gamma(t))X_3(t)\\X_3(t_2)&=\mathbb{I}\end{aligned} \tag{3.9}$$

i.e. $x_{+\xi}(t_2,x_2)(t,0,\mu,\alpha)=X_3(t)$, simplifies expressions for the other two solutions

$$\begin{aligned}x_{+\tau}(t_2,x_2)(t,0,\mu,\alpha)&=-X_3(t)f_+(x_2)\\x_{+\varepsilon}(t_2,x_2)(t,0,\mu,\alpha)&=\int_{t_2}^{t}X_3(t)X_3^{-1}(s)g(\gamma(s),s+\alpha,0,\mu)ds\end{aligned}$$

for $t\in[t_2,T]$.

Now we can state the following lemma

Lemma 3.3. *Let $\widetilde{P}(\xi,\varepsilon,\mu,\alpha)$ be defined by* (3.4). *Then*

$$\widetilde{P}_\xi(x_0,0,\mu,\alpha) = X_3(T)S_2X_2(t_2)S_1X_1(t_1) \tag{3.10}$$

$$\widetilde{P}_\varepsilon(x_0,0,\mu,\alpha) = \int_0^T A(s)g(\gamma(s),s+\alpha,0,\mu)ds \tag{3.11}$$

where $X_1(t)$, $X_2(t)$ and $X_3(t)$ are matrix solutions of corresponding linearized equations (3.6), (3.7) *and* (3.9)*, respectively. S_1 is the saltation matrix given by* (3.8)*, S_2 is a second saltation matrix given by*

$$S_2 = \mathbb{I} + \frac{(f_+(x_2)-f_-(x_2))\mathrm{D}h(x_2)}{\mathrm{D}h(x_2)f_-(x_2)} \tag{3.12}$$

and

$$A(t) = \begin{cases} X_3(T)S_2X_2(t_2)S_1X_1(t_1)X_1^{-1}(t) & \text{if } t\in[0,t_1) \\ X_3(T)S_2X_2(t_2)X_2^{-1}(t) & \text{if } t\in[t_1,t_2) \\ X_3(T)X_3^{-1}(t) & \text{if } t\in[t_2,T] \end{cases} \tag{3.13}$$

Proof. Direct differentiation of (3.4) and the use of previous results give statement of the lemma:

$$\begin{aligned}
\widetilde{P}_\xi(x_0,0,\mu,\alpha) &= x_{+\tau}(t_2,x_2)(T,0,\mu,\alpha)t_{2\xi}(0,x_0,0,\mu,\alpha) + x_{+\xi}(t_2,x_2)(T,0,\mu,\alpha) \\
&\quad\times[x_{-\tau}(t_1,x_1)(t_2,0,\mu,\alpha)t_{1\xi}(0,x_0,0,\mu,\alpha)+x_{-\xi}(t_1,x_1)(t_2,0,\mu,\alpha) \\
&\quad\times[x_{+\xi}(0,x_0)(t_1,0,\mu,\alpha)+x_{+t}(0,x_0)(t_1,0,\mu,\alpha)t_{1\xi}(0,x_0,0,\mu,\alpha)] \\
&\quad+x_{-t}(t_1,x_1)(t_2,0,\mu,\alpha)t_{2\xi}(0,x_0,0,\mu,\alpha)] \\
&= X_3(T)f_+(x_2)\frac{\mathrm{D}h(x_2)X_2(t_2)S_1X_1(t_1)}{\mathrm{D}h(x_2)f_-(x_2)} + X_3(T)\left[X_2(t_2)f_-(x_1)\frac{\mathrm{D}h(x_1)X_1(t_1)}{\mathrm{D}h(x_1)f_+(x_1)}\right. \\
&\quad\left.+X_2(t_2)\left[X_1(t_1)-f_+(x_1)\frac{\mathrm{D}h(x_1)X_1(t_1)}{\mathrm{D}h(x_1)f_+(x_1)}\right]-f_-(x_2)\frac{\mathrm{D}h(x_2)X_2(t_2)S_1X_1(t_1)}{\mathrm{D}h(x_2)f_-(x_2)}\right] \\
&= X_3(T)S_2X_2(t_2)S_1X_1(t_1)
\end{aligned}$$

(3.11) can be shown by the same way. □

We recall the following well-known result (cf. [8])

Lemma 3.4. *Let $X(t)$ be a fundamental matrix solution of equation $X' = UX$. Then $X(t)^{-1*}$ is a fundamental matrix solution of adjoint equation*

$$\left(X(t)^{-1*}\right)' = -U^*X(t)^{-1*}$$

We solve equation (3.3) via Lyapunov-Schmidt reduction.
As it was already shown in Lemma 3.2, dim $\mathcal{N}(\mathbb{I}-\widetilde{P}_\xi(x_0,0,\mu,\alpha)) \geq 1$. From now on we suppose that

H3) dim $\mathcal{N}(\mathbb{I}-\widetilde{P}_\xi(x_0,0,\mu,\alpha)) = 1$

and therefore codim $\mathcal{R}(\mathbb{I}-\widetilde{P}_\xi(x_0,0,\mu,\alpha))=1$. We denote

$$R_1=\mathcal{R}(\mathbb{I}-\widetilde{P}_\xi(x_0,0,\mu,\alpha)),\qquad R_2=\Big[\mathcal{R}(\mathbb{I}-\widetilde{P}_\xi(x_0,0,\mu,\alpha))\Big]^\perp$$

Then two linear projections are considered $\mathcal{P}:\mathbb{R}^n\to R_2$ and $\mathcal{Q}:\mathbb{R}^n\to R_1$ defined by

$$\mathcal{P}y=\frac{\langle y,\psi\rangle}{|\psi|^2}\psi$$
$$\mathcal{Q}y=(\mathbb{I}-\mathcal{P})y=y-\frac{\langle y,\psi\rangle}{|\psi|^2}\psi$$

where $\psi\in R_2$ is fixed. We assume that initial point ξ of perturbed periodic trajectory is an element of Σ. Equation (3.3) is equivalent to a couple of equations

$$\mathcal{Q}(\xi-\widetilde{P}(\xi,\varepsilon,\mu,\alpha))=0\qquad\qquad \mathcal{P}(\xi-\widetilde{P}(\xi,\varepsilon,\mu,\alpha))=0$$

The first equation can be solved via Implicit Function Theorem which gives the existence of $r_0,\varepsilon_0>0$ and a C^r–function $\xi:(-\varepsilon_0,\varepsilon_0)\times\mathbb{R}^k\times\mathbb{R}\to B(x_0,r_0)\cap\Sigma$ such that $\mathcal{Q}(\xi-\widetilde{P}(\xi,\varepsilon,\mu,\alpha))=0$ for $\varepsilon\in(-\varepsilon_0,\varepsilon_0)$, $\mu\in\mathbb{R}^k$, $\alpha\in\mathbb{R}$ and $\xi\in B(x_0,r_0)\cap\Sigma$ if and only if $\xi=\xi(\varepsilon,\mu,\alpha)$, moreover $\xi(0,\mu,\alpha)=x_0$.

Then the second equation has the form

$$\begin{aligned}\mathcal{P}(\xi(\varepsilon,\mu,\alpha)-\widetilde{P}(\xi(\varepsilon,\mu,\alpha),\varepsilon,\mu,\alpha))&=0\\ \langle\xi(\varepsilon,\mu,\alpha)-\widetilde{P}(\xi(\varepsilon,\mu,\alpha),\varepsilon,\mu,\alpha),\psi\rangle&=0\end{aligned}\tag{3.14}$$

Again $\varepsilon=0$ solves this equation. Differentiation with respect to ε at 0 gives

$$\begin{aligned}&\Big\langle\xi_\varepsilon(0,\mu,\alpha)-\widetilde{P}_\xi(x_0,0,\mu,\alpha)\xi_\varepsilon(0,\mu,\alpha)-\widetilde{P}_\varepsilon(x_0,0,\mu,\alpha),\psi\Big\rangle\\ &=\Big\langle(\mathbb{I}-\widetilde{P}_\xi(x_0,0,\mu,\alpha))\xi_\varepsilon(0,\mu,\alpha)-\widetilde{P}_\varepsilon(x_0,0,\mu,\alpha),\psi\Big\rangle\\ &=\Big\langle(\mathbb{I}-\widetilde{P}_\xi(x_0,0,\mu,\alpha))\xi_\varepsilon(0,\mu,\alpha),\psi\Big\rangle-\Big\langle\widetilde{P}_\varepsilon(x_0,0,\mu,\alpha),\psi\Big\rangle\\ &=-\Big\langle\int_0^T A(s)g(\gamma(s),s+\alpha,0,\mu)ds,\psi\Big\rangle\\ &=-\int_0^T\langle A(s)g(\gamma(s),s+\alpha,0,\mu),\psi\rangle ds\\ &=-\int_0^T\langle g(\gamma(s),s+\alpha,0,\mu),A(s)^*\psi\rangle ds\end{aligned}$$

where

$$A(t)^*=\begin{cases}X_1^{-1*}(t)X_1(t_1)^*S_1^*X_2(t_2)^*S_2^*X_3(T)^* & \text{if}\quad t\in[0,t_1)\\ X_2^{-1*}(t)X_2(t_2)^*S_2^*X_3(T)^* & \text{if}\quad t\in[t_1,t_2)\\ X_3^{-1*}(t)X_3(T)^* & \text{if}\quad t\in[t_2,T]\end{cases}\tag{3.15}$$

Note that by Lemma 3.4, $A(t)^*$ solves adjoint equation

$$\begin{aligned}X'&=-f_+^*(\gamma(t))X &&\text{if}\quad 0<t<t_1\\ X'&=-f_-^*(\gamma(t))X &&\text{if}\quad t_1<t<t_2\\ X'&=-f_+^*(\gamma(t))X &&\text{if}\quad t_2<t<T\end{aligned}\tag{3.16}$$

Differentiation of (3.14) with respect to ε and α at $\varepsilon = 0$ gives

$$-\int_0^T \langle g_t(\gamma(s), s+\alpha, 0, \mu), A(s)^*\psi\rangle ds$$

All results together complete the next theorem.

Theorem 3.5. *Let conditions* H1), H2), H3) *hold,* $\gamma(t)$ *and* $A(t)^*$ *are defined by* (2.2) *and* (3.15)*, respectively, and* $\psi \in R_2$*. If there is* $(\mu_0, \alpha_0) \in \mathbb{R}^k \times \mathbb{R}$*,* $k \geq 1$ *such that*

$$\int_0^T \langle g(\gamma(t), t+\alpha_0, 0, \mu_0), A(t)^*\psi\rangle dt = 0 \tag{3.17}$$

$$\int_0^T \langle g_t(\gamma(t), t+\alpha_0, 0, \mu_0), A(t)^*\psi\rangle dt \neq 0 \tag{3.18}$$

then there exists a neighbourhood U *of the point* $(0,\mu_0)$ *in* $\mathbb{R}\times\mathbb{R}^k$ *and a* C^{r-1}*–function* $\alpha(\varepsilon,\mu)$*, with* $\alpha(0,\mu_0) = \alpha_0$*, such that perturbed equation* (2.1) *possesses a unique* T*–periodic piecewise* C^1*–smooth solution for each* $(\varepsilon,\mu) \in U$*.*

Proof. Let denote $\mathcal{D}(\varepsilon,\mu,\alpha) = \frac{d}{d\varepsilon}\langle \xi(\varepsilon,\mu,\alpha) - \widetilde{P}(\xi(\varepsilon,\mu,\alpha),\varepsilon,\mu,\alpha),\psi\rangle$. Then the assumptions (3.17) and (3.18) are fulfilled if and only if

$$\mathcal{D}(0,\mu_0,\alpha_0) = 0 \qquad\qquad \mathcal{D}_\varepsilon(0,\mu_0,\alpha_0) \neq 0$$

Implicit Function Theorem gives the existence of the function $\alpha(\varepsilon,\mu)$ from the statement of the theorem. It is obvious that equation (3.3) has a unique solution

$$\xi(\varepsilon,\mu,\alpha(\varepsilon,\mu)) - \widetilde{P}(\xi(\varepsilon,\mu,\alpha(\varepsilon,\mu)),\varepsilon,\mu,\alpha(\varepsilon,\mu)) = 0$$

which completes the proof. □

Remark 3.6. If g is discontinuous in x, i.e.

$$g(x,t,\varepsilon,\mu) = \begin{cases} g_+(x,t,\varepsilon,\mu) & \text{if } x \in \Omega_+ \\ g_-(x,t,\varepsilon,\mu) & \text{if } x \in \Omega_- \end{cases}$$

it is possible to show that Theorem 3.5 still holds. Of course, g has to be T–periodic in t.

4 Geometric interpretation of condition H3

Consider the linearization of unperturbed problem of (2.1) along $\gamma(t)$, given by

$$\dot{x} = \mathrm{D}f_\pm(\gamma(t))x \tag{4.1}$$

Then (4.1) splits into two unperturbed equations

$$\begin{aligned} \dot{x} &= \mathrm{D}f_+(\gamma(t))x \qquad && \text{if } t \in [0,t_1]\cup[t_2,T] \\ \dot{x} &= \mathrm{D}f_-(\gamma(t))x && \text{if } t \in (t_1,t_2) \end{aligned}$$

with impulsive conditions [9, 10, 13]

$$x(t_1+) = S_1 x(t_1-) \qquad x(t_2+) = S_2 x(t_2-)$$

where $x(t\pm) = \lim_{s\to t^\pm} x(s)$. We already know (from (3.6), (3.7), (3.9)) that they have the fundamental matrices $X_1(t)$ resp. $X_3(t)$ and $X_2(t)$ satisfying $X_1(0) = X_2(t_1) = X_3(t_2) = \mathbb{I}$. Consequently, the fundamental matrix solution of discontinuous variational equation (4.1) is given by

$$X(t) = \begin{cases} X_1(t) & \text{if } t \in [0,t_1) \\ X_2(t)S_1X_1(t_1) & \text{if } t \in [t_1,t_2) \\ X_3(t)S_2X_2(t_2)S_1X_1(t_1) & \text{if } t \in [t_2,T] \end{cases}$$

Then a T–periodic solution of (4.1) with an initial point ξ fulfills $\xi = X(T)\xi$ or equivalently $(\mathbb{I} - X(T))\xi = 0$.

Now one can easily conclde the following result

Proposition 4.1. *Condition* H3) *is equivalent to say that discontinuous variational equation* (4.1) *has the unique T–periodic solution up to a scalar multiple.*

5 Discontinuous planar systems

5.1 Piecewise nonlinear problems

In this section, we consider the following problem

$$\begin{aligned} &\left.\begin{aligned}\dot{x} &= \omega_1(y-\delta) + \varepsilon g_1(x,y,t+\alpha,\varepsilon,\mu) \\ \dot{y} &= -\omega_1 x + \varepsilon g_2(x,y,t+\alpha,\varepsilon,\mu)\end{aligned}\right\} \quad \text{for} \quad y > 0 \\ &\left.\begin{aligned}\dot{x} &= \eta x + \omega_2(y+\delta) \\ &\quad + \left[x^2 + (y+\delta)^2\right]\left[-ax - b(y+\delta)\right] + \varepsilon g_1(x,y,t+\alpha,\varepsilon,\mu) \\ \dot{y} &= -\omega_2 x + \eta(y+\delta) \\ &\quad + \left[x^2 + (y+\delta)^2\right]\left[bx - a(y+\delta)\right] + \varepsilon g_2(x,y,t+\alpha,\varepsilon,\mu)\end{aligned}\right\} \quad \text{for} \quad y < 0 \end{aligned} \tag{5.1}$$

with assumptions

$$\eta, \delta, \omega_1, \omega_2, \omega, a > 0, \quad b \in \mathbb{R}, \quad \omega_2 - \frac{\eta b}{a} > 0, \quad \frac{\eta}{a} > \delta^2 \tag{5.2}$$

For $\varepsilon = 0$ the first part of (5.1) can be easily solved via exponential matrix. For starting point $(x_0, y_0) = \left(0, \delta + \sqrt{\frac{\eta}{a}}\right)$ and $t \in [0, t_1]$ the solution is

$$\gamma_1(t) = \left(\sqrt{\frac{\eta}{a}} \sin \omega_1 t, \delta + \sqrt{\frac{\eta}{a}} \cos \omega_1 t\right) \tag{5.3}$$

t_1 and (x_1, y_1) are obtained from relations $h(\gamma_1(t_1)) = 0$ for $h(x,y) = y$ and $(x_1, y_1) = \gamma_1(t_1)$, respectively:

$$t_1 = \frac{1}{\omega_1} \arccos\left(-\sqrt{\frac{a}{\eta}}\delta\right)$$

$$(x_1, y_1) = \left(\sqrt{\frac{\eta}{a} - \delta^2}, 0\right)$$

After transformation $x = r\cos\theta$, $y + \delta = r\sin\theta$ in the second part of (5.1) with $\varepsilon = 0$, we have

$$\begin{aligned}\dot{r} &= \eta r - ar^3 \\ \dot{\theta} &= -\omega_2 + br^2\end{aligned}$$

from which, one can see, that the second part of (5.1) with $\varepsilon = 0$ possesses a stable limit cycle/circle with the center at $(0, -\delta)$ and radius $\sqrt{\frac{\eta}{a}}$, which intersects boundary $\Omega_0 = \{(x,y) \in \mathbb{R}^2 | y = 0\}$. Now it is obvious that (x_1, y_1) is a point of this cycle and direction of rotation remains the same as in $\Omega_+ = \{(x,y) \in \mathbb{R}^2 | y > 0\}$. Therefore $\gamma_2(t)$ is a part of the circle, given by

$$\begin{aligned}\gamma_2(t) = (&x_1 \cos\omega_3(t - t_1) + \delta\sin\omega_3(t - t_1), \\ &-\delta - x_1 \sin\omega_3(t - t_1) + \delta\cos\omega_3(t - t_1))\end{aligned} \tag{5.4}$$

for $t \in [t_1, t_2]$, where $\omega_3 = \omega_2 - \frac{\eta b}{a}$. Equation $h(\gamma_2(t_2)) = 0$ and symmetry of $\gamma_2(t)$ give couple of equations

$$\begin{aligned}x_1 \cos\omega_3(t_2 - t_1) + \delta\sin\omega_3(t_2 - t_1) &= -x_1 \\ -\delta - x_1 \sin\omega_3(t_2 - t_1) + \delta\cos\omega_3(t_2 - t_1) &= 0\end{aligned}$$

From them we obtain

$$t_2 = \frac{1}{\omega_3}\left(\pi + \operatorname{arccot}\frac{-\delta^2 + x_1^2}{2\delta x_1}\right) + t_1$$

(x_2, y_2) is a second intersection point of the limit cycle and Ω_0

$$(x_2, y_2) = \gamma_2(t_2) = \left(-\sqrt{\frac{\eta}{a} - \delta^2}, 0\right)$$

Then orbit $\gamma(t)$ continues in Ω_+ with

$$\begin{aligned}\gamma_3(t) = (&x_2 \cos\omega_1(t - t_2) - \delta\sin\omega_1(t - t_2), \\ &\delta - x_2 \sin\omega_1(t - t_2) - \delta\cos\omega_1(t - t_2))\end{aligned} \tag{5.5}$$

for $t \in [t_2, T]$. Period T is found from equation $\gamma_3(T) = (x_0, y_0)$

$$T = \frac{1}{\omega_1}\arccos\left(-\sqrt{\frac{a}{\eta}}\delta\right) + t_2$$

The next theorem is due to Diliberto (cf. [4, 14]) and it is used to find fundamental matrix solution of the variational equation

Theorem 5.1. *Let $\gamma(t)$ is the solution of the differential equation $\dot{x} = f(x)$, $x \in \mathbb{R}^2$. If $\gamma(0) = p$, $f(p) \neq 0$ then the variational equation along $\gamma(t)$*

$$\dot{V} = \mathrm{D}f(\gamma(t))V$$

has the fundamental matrix solution $\Phi(t)$ satisfying $\det\Phi(0) = \|f(p)\|^2$, given by

$$\Phi(t) = [f(\gamma(t)), V(t)]$$

where $[\lambda_1, \lambda_2]$ stands for a matrix with columns λ_1 and λ_2 and

$$\begin{gathered}
V(t) = a(t)f(\gamma(t)) + b(t)f^{\perp}(\gamma(t)) \\
a(t) = \int_0^t \left[2\kappa(\gamma(s))\|f(\gamma(s))\| + \operatorname{div} f^{\perp}(\gamma(s))\right] b(s)ds \\
b(s) = \frac{\|f(p)\|^2}{\|f(\gamma(t))\|^2} \mathrm{e}^{\int_0^t \operatorname{div} f(\gamma(s))ds} \\
\operatorname{div} f(x) = \frac{\partial f_1(x)}{\partial x_1} + \frac{\partial f_2(x)}{\partial x_2} \qquad \operatorname{div} f^{\perp}(x) = -\frac{\partial f_2(x)}{\partial x_1} + \frac{\partial f_1(x)}{\partial x_2} \\
\kappa(\gamma(t)) = \frac{1}{\|f(\gamma(t))\|^3} \left[f_1(\gamma(t))\dot{f}_2(\gamma(t)) - f_2(\gamma(t))\dot{f}_1(\gamma(t))\right]
\end{gathered}$$

Lemma 5.2. *For unperturbed system*

$$\begin{aligned}
&\begin{aligned} \dot{x} &= \omega_1(y - \delta) \\ \dot{y} &= -\omega_1 x \end{aligned} && \text{for} \quad y > 0 \\
&\begin{aligned} \dot{x} &= \eta x + \omega_2(y + \delta) + \left[x^2 + (y+\delta)^2\right][-ax - b(y+\delta)] \\ \dot{y} &= -\omega_2 x + \eta(y+\delta) + \left[x^2 + (y+\delta)^2\right][bx - a(y+\delta)] \end{aligned} && \text{for} \quad y < 0
\end{aligned} \tag{5.6}$$

we have, by fulfilled assumptions (5.2), corresponding fundamental matrices (see (3.6)–X_1, (3.7)–X_2 and (3.9)–X_3)

$$\begin{gathered}
X_1(t) = \begin{pmatrix} \cos\omega_1 t & \sin\omega_1 t \\ -\sin\omega_1 t & \cos\omega_1 t \end{pmatrix} \\
X_2(t) = \frac{a}{\eta}[\lambda_1, \lambda_2] \qquad X_3(t) = X_1(t - t_2) \\
\lambda_1 = \begin{pmatrix} U(-\delta x_1 + \delta x_1 W + x_1^2 \widetilde{W}) + V(\delta^2 + x_1^2 W - \delta x_1 \widetilde{W}) \\ U(-\delta^2 - x_1^2 W + \delta x_1 \widetilde{W}) + V(-\delta x_1 + \delta x_1 W + x_1^2 \widetilde{W}) \end{pmatrix} \\
\lambda_2 = \begin{pmatrix} U(x_1^2 + \delta^2 W + \delta x_1 \widetilde{W}) + V(-\delta x_1 + \delta x_1 W - \delta^2 \widetilde{W}) \\ U(\delta x_1 - \delta x_1 W + \delta^2 \widetilde{W}) + V(x_1^2 + \delta^2 W + \delta x_1 \widetilde{W}) \end{pmatrix} \\
U = \sin\omega_3(t - t_1) \qquad V = \cos\omega_3(t - t_1) \\
W = \mathrm{e}^{-2\eta(t-t_1)} \qquad \widetilde{W} = \frac{b}{a}(1 - W)
\end{gathered}$$

and saltation matrices (see (3.8)–S_1 and (3.12)–S_2)

$$S_1 = \begin{pmatrix} 1 & -\frac{\delta(\omega_1+\omega_3)}{\omega_1 x_1} \\ 0 & \frac{\omega_3}{\omega_1} \end{pmatrix} \qquad S_2 = \begin{pmatrix} 1 & -\frac{\delta(\omega_1+\omega_3)}{\omega_3 x_1} \\ 0 & \frac{\omega_1}{\omega_3} \end{pmatrix}$$

Proof. $X_1(t)$ and $X_3(t)$ are obtained easily because of the linearity of $f_+(x,y)$. Since

$$\begin{aligned} f_+(x_1,y_1) &= \begin{pmatrix} -\omega_1\delta \\ -\omega_1\sqrt{\frac{\eta}{a}-\delta^2} \end{pmatrix} \qquad & f_-(x_1,y_1) &= \begin{pmatrix} \omega_3\delta \\ -\omega_3\sqrt{\frac{\eta}{a}-\delta^2} \end{pmatrix} \\ f_+(x_2,y_2) &= \begin{pmatrix} -\omega_1\delta \\ \omega_1\sqrt{\frac{\eta}{a}-\delta^2} \end{pmatrix} \qquad & f_-(x_2,y_2) &= \begin{pmatrix} \omega_3\delta \\ \omega_3\sqrt{\frac{\eta}{a}-\delta^2} \end{pmatrix} \end{aligned} \tag{5.7}$$

so are the saltation matrices. Since (5.6) is 2–dimensional and one solution of the second part is already known - limit cycle, we can use Theorem 5.1. Then we have a matrix

$$\widetilde{X}_2(t) = \omega_3 \begin{pmatrix} -x_1U+\delta V & U(\delta W + x_1\widetilde{W}) + V(x_1W-\delta\widetilde{W}) \\ -\delta U - x_1V & U(-x_1W+\delta\widetilde{W}) + V(\delta W + x_1\widetilde{W}) \end{pmatrix}$$

$$\widetilde{X}_2^{-1}(t_1) = \frac{a}{\eta\omega_3}\begin{pmatrix} \delta & -x_1 \\ x_1 & \delta \end{pmatrix}$$

such that $\det\widetilde{X}_2(t_1) = \|f_-(x_1,y_1)\|^2 = \frac{\eta}{a}\omega_3^2$. If $X_2(t)$ has to satisfy (3.7) then $X_2(t) = \widetilde{X}_2(t)\widetilde{X}_2^{-1}(t_1)$. □

Now the following result can be stated.

Proposition 5.3. *Assuming* (5.2) *unperturbed system* (5.6) *has a T–periodic orbit to initial point* $(x_0,y_0) = \left(0, \delta + \sqrt{\frac{\eta}{a}}\right)$

$$\gamma(t) = \begin{cases} \gamma_1(t) & \text{if } t \in [0,t_1] \\ \gamma_2(t) & \text{if } t \in [t_1,t_2] \\ \gamma_3(t) & \text{if } t \in [t_2,T] \end{cases}$$

where parts $\gamma_1(t)$, $\gamma_2(t)$ and $\gamma_3(t)$ are given by (5.3), (5.4) *and* (5.5), *respectively. Moreover, conditions* H1), H2) *and* H3) *are satisfied.*

Proof. Condition H1) was already verified. Since $\nabla h(x,y) = (0,1)$ for all $(x,y) \in \mathbb{R}^2$ and (5.7) holds, condition H2) is also fulfilled.

Now suppose that dim $\mathcal{N}(\mathbb{I} - \widetilde{P}_\xi(x_0,0,\mu,\alpha)) > 1$. We recall that $f_+(x_0,y_0) \in \mathcal{N}(\mathbb{I} - \widetilde{P}_\xi(x_0,0,\mu,\alpha))$. Since $\mathcal{N}(\mathbb{I} - \widetilde{P}_\xi(x_0,0,\mu,\alpha))$ is linear, there is a vector

$$\bar{v} \in \mathcal{N}(\mathbb{I} - \widetilde{P}_\xi(x_0,0,\mu,\alpha))$$

such that $\langle \bar{v}, f_+(x_0,y_0)\rangle = 0$. Then we can write $\bar{v} = (0,v)^*$. Using (3.10) we look for its image by mapping $\widetilde{P}_\xi(x_0,0,\mu,\alpha)$

$$S_1X_1(t_1)\bar{v} = \frac{v}{\omega_1}\sqrt{\frac{a}{\eta}}\begin{pmatrix} \omega_1x_1 + \frac{\delta^2(\omega_1+\omega_3)}{x_1} \\ -\delta\omega_3 \end{pmatrix}$$

Next we have

$$X_2(t_2)S_1X_1(t_1)\bar{v} = \frac{v}{\omega_1}\sqrt{\frac{a}{\eta}}\begin{pmatrix} \frac{\delta^2}{x_1}(\omega_1+\omega_3) - x_1\omega_1Z - \delta\omega_1\widetilde{Z} \\ \delta(\omega_1+\omega_3) + \delta\omega_1 Z - x_1\omega_1\widetilde{Z} \end{pmatrix}$$

where $Z = \mathrm{e}^{-2\eta(t_2-t_1)}$ and $\widetilde{Z} = \frac{b}{a}(1-Z)$ are values of W and $\widetilde{W}$ at $t = t_2$, and

$$S_2X_2(t_2)S_1X_1(t_1)\bar{v} = \frac{v}{\omega_3}\sqrt{\frac{a}{\eta}}\begin{pmatrix} -\frac{\delta^2}{x_1}(\omega_1+\omega_3) - (\frac{\delta^2\omega_1}{x_1} + \frac{\eta\omega_3}{ax_1})Z + \delta\omega_1\widetilde{Z} \\ \delta(\omega_1+\omega_3) + \delta\omega_1 Z - x_1\omega_1\widetilde{Z} \end{pmatrix}$$

Finally

$$X_3(T)S_2X_2(t_2)S_1X_1(t_1)\bar{v} = v\begin{pmatrix} \frac{\delta}{x_1}\frac{\omega_1+\omega_3}{\omega_3} + \frac{\delta}{x_1}\frac{\omega_1+\omega_3}{\omega_3}Z - \frac{\omega_1}{\omega_3}\widetilde{Z} \\ Z \end{pmatrix}$$

Since $Z \leq \exp\left\{-\frac{2\eta}{\omega_3}\pi\right\} < 1$, it is obvious that $\bar{v} = \widetilde{P}_\xi(x_0, 0, \mu, \alpha)\bar{v}$ if and only if $\bar{v} = (0,0)^*$. The verification of condition H3) is finished. $\square$

Because, in general, the formula for $A(t)$ is rather awkward, we move to examples with concrete parameters.

Example 5.4. We take

$$\begin{gathered} a = b = \delta = 1,\ \eta = 2,\ \omega_1 = 1,\ \omega_2 = 5 \\ g(x,y,t,\varepsilon,\mu) = \begin{cases} (\sin\omega t, 0)^* & \text{if } y > 0 \\ (0,0)^* & \text{if } y < 0 \end{cases} \end{gathered} \tag{5.8}$$

Then we have $\omega_3 = 3$, $T = 2\pi$, initial point $(x_0, y_0) = (0, 1+\sqrt{2})$, saltation matrices

$$S_1 = \begin{pmatrix} 1 & -4 \\ 0 & 3 \end{pmatrix} \qquad S_2 = \begin{pmatrix} 1 & -\frac{4}{3} \\ 0 & \frac{1}{3} \end{pmatrix}$$

and

$$\widetilde{P}_\xi(x_0, y_0, 0, \mu, \alpha) = \begin{pmatrix} 1 & 1 + \frac{5}{3}\mathrm{e}^{-2\pi} \\ 0 & \mathrm{e}^{-2\pi} \end{pmatrix}$$

Hence $R_1 = \operatorname{span}\left\{\left(1+\frac{5}{3}\mathrm{e}^{-2\pi}, \mathrm{e}^{-2\pi}-1\right)^*\right\}$ and $\psi = \left(1-\mathrm{e}^{-2\pi}, 1+\frac{5}{3}\mathrm{e}^{-2\pi}\right) \in R_2$. After some algebra we obtain

$$M(\alpha) = \frac{1}{3}\frac{\mathrm{e}^{-2\pi}}{\omega^2-1}\left[(\omega A + B)\sin\omega\alpha + (\omega C + D)\cos\omega\alpha\right]$$

where

$$\begin{aligned} A &= 4\sqrt{2}\sin\left(\frac{3}{4}\pi\omega\right) + \left(3\mathrm{e}^{2\pi}\sqrt{2}+\sqrt{2}\right)\sin\left(\frac{5}{4}\pi\omega\right) + \left(3\mathrm{e}^{2\pi}-3\right)\sin(2\pi\omega) \\ B &= -5 - 3\mathrm{e}^{2\pi} - \left(\sqrt{2}+3\sqrt{2}\mathrm{e}^{2\pi}\right)\cos\left(\frac{3}{4}\pi\omega\right) + 4\sqrt{2}\cos\left(\frac{5}{4}\pi\omega\right) + \left(5+3\mathrm{e}^{2\pi}\right)\cos(2\pi\omega) \\ C &= 3\mathrm{e}^{2\pi} - 3 - 4\sqrt{2}\cos\left(\frac{3}{4}\pi\omega\right) - \left(\sqrt{2}+3\sqrt{2}\mathrm{e}^{2\pi}\right)\cos\left(\frac{5}{4}\pi\omega\right) + \left(3-3\mathrm{e}^{2\pi}\right)\cos(2\pi\omega) \\ D &= -\left(\sqrt{2}+3\sqrt{2}\mathrm{e}^{2\pi}\right)\sin\left(\frac{3}{4}\pi\omega\right) + 4\sqrt{2}\sin\left(\frac{5}{4}\pi\omega\right) + \left(5+3\mathrm{e}^{2\pi}\right)\sin(2\pi\omega) \end{aligned} \tag{5.9}$$

For $\omega > 0$, $\omega \neq 1$, $M(\alpha)$ has a simple root if and only if $(\omega A + B)^2 + (\omega C + D)^2 > 0$. Since

$$\sqrt{B^2+D^2} \leq \left(\left(5+3e^{2\pi}+\sqrt{2}+3\sqrt{2}e^{2\pi}+4\sqrt{2}+5+3e^{2\pi}\right)^2 + \left(5+3e^{2\pi}+\sqrt{2}+3\sqrt{2}e^{2\pi}+4\sqrt{2}\right)^2\right)^{\frac{1}{2}} = \sqrt{3}\left(1+\sqrt{2}\right)\sqrt{9e^{4\pi}+30e^{2\pi}+25} \leq 6739$$

that A and C are 8–periodic functions and according to Figure 2 we have

$$\sqrt{(\omega A+B)^2+(\omega C+D)^2} \geq \omega\sqrt{A^2+C^2} - \sqrt{B^2+D^2} \geq 420\omega - 6739$$

and one can see that for $\omega \geq 16$, T–periodic orbit in perturbed system (5.1) persists for $\varepsilon \neq 0$ small. It can be proved numerically (see Figure 2) that

$$\frac{1}{|\omega^2-1|}\sqrt{(\omega A+B)^2+(\omega C+D)^2} > 0 \tag{5.10}$$

for $\omega \in (0,16)$. We conclude

Corollary 5.5. *Consider* (5.1) *with parameters* (5.8)*. Then* 2π*–periodic orbit persists for all* $\omega > 0$ *and* $\varepsilon \neq 0$ *small.*

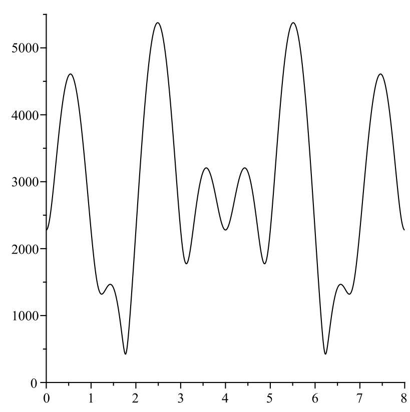

Figure 2. Graphs of the functions $\sqrt{A^2+C^2}$ and the left-hand side of (5.10)

Example 5.6. Now let

$$a = b = \delta = 1,\ \eta = 2,\ \omega_1 = 1,\ \omega_2 = 5$$
$$g(x,y,t,\varepsilon,\mu) = \begin{cases} \mu_1(\sin\omega t, 0)^* & \text{if } y > 0 \\ \mu_2(x+y, 0)^* & \text{if } y < 0 \end{cases} \tag{5.11}$$

Consequently, Melnikov function is

$$M(\alpha) = \mu_1 \frac{1}{3} \frac{\mathrm{e}^{-2\pi}}{\omega^2 - 1} \left[(\omega A + B) \sin \omega\alpha + (\omega C + D) \cos \omega\alpha \right] + \mu_2 E$$

where A, B, C, D are given by (5.9) and

$$E = \frac{\sqrt{2}}{975} \left(739 - 223\mathrm{e}^{-2\pi} \right)$$

$M(\alpha)$ possesses a simple root if and only if

$$|\mu_2| < \frac{1}{3} \frac{\mathrm{e}^{-2\pi}}{|\omega^2 - 1|} \frac{\sqrt{(\omega A + B)^2 + (\omega C + D)^2}}{E} |\mu_1| \tag{5.12}$$

We have the next result

Corollary 5.7. *Consider* (5.1) *with parameters* (5.8)*. If* μ_1*,* μ_2 *and* ω *satisfy* (5.12) *then* 2π*–periodic orbit persists for* $\varepsilon \neq 0$ *small.*

Remark 5.8. Inequality (5.12) means that if the periodic perturbation is sufficiently large (with respect to non-periodic part of perturbation) then the T–periodic trajectory persists. Note that the right-hand side of (5.12) can be estimated from above by

$$\frac{\sqrt{c_1 \omega^2 + c_2 \omega + c_3}}{|\omega^2 - 1|}$$

for appropriate constants c_1, c_2, c_3, which tends to 0, if ω tends to $+\infty$. Hence the bigger frequency ω, the bigger $|\mu_1|$ for $\mu_2 \neq 0$ fixed, for persistence of the T–periodic orbit.

5.2 Piecewise linear problems

Here we consider the system

$$\begin{aligned} \dot{x} &= b_1 + \varepsilon\mu_1 \sin \omega t \\ \dot{y} &= -2a_1 b_1 x + \varepsilon\mu_2 \cos \omega t \end{aligned} \qquad \text{for} \quad y > 0$$
$$\begin{aligned} \dot{x} &= -b_2 + \varepsilon\mu_1 \sin \omega t \\ \dot{y} &= -2a_2 b_2 x + \varepsilon\mu_2 \cos \omega t \end{aligned} \qquad \text{for} \quad y < 0 \tag{5.13}$$

where all constants a_i, b_i for $i = 1, 2$ are assumed to be positive and $(\mu_1, \mu_2) \neq (0, 0)$, $\omega > 0$.

The starting point can be chosen in the form $(x_0, y_0) = (0, y_0)$ with $y_0 > 0$. Then with $h(x, y) = y$ we obtain results similar to those by the previous case.

Firstly, it is not difficult to find the trajectory starting from $(0, y_0)$ in $\Omega_+ = \{(x, y) \in \mathbb{R}^2 | y > 0\}$ to $\Omega_- = \{(x, y) \in \mathbb{R}^2 | y < 0\}$ and then back.

Lemma 5.9. *For any* $y_0 > 0$*, unperturbed system*

$$\begin{aligned} \dot{x} &= b_1 \\ \dot{y} &= -2a_1 b_1 x \end{aligned} \quad \textit{for} \;\; y > 0 \qquad \begin{aligned} \dot{x} &= -b_2 \\ \dot{y} &= -2a_2 b_2 x \end{aligned} \quad \textit{for} \;\; y < 0 \tag{5.14}$$

possesses a unique periodic trajectory containing point $(0, y_0)$ *given by*

$$\gamma(t) = \begin{cases} \gamma_1(t) = \left(b_1 t, -a_1 b_1^2 t^2 + y_0\right) & \text{if } t \in [0, t_1] \\ \gamma_2(t) = \left(x_1 - b_2(t - t_1), a_2(x_1 - b_2(t - t_1))^2 - a_2 x_1^2\right) & \text{if } t \in [t_1, t_2] \\ \gamma_3(t) = \left(x_2 + b_1(t - t_2), -a_1(x_2 + b_1(t - t_2))^2 + a_1 x_2^2\right) & \text{if } t \in [t_2, T] \end{cases}$$

where

$$t_1 = \frac{1}{b_1}\sqrt{\frac{y_0}{a_1}} \qquad (x_1, y_1) = \left(\sqrt{\frac{y_0}{a_1}}, 0\right) \qquad t_2 = \frac{2}{b_2}\sqrt{\frac{y_0}{a_1}} + t_1$$
$$(x_2, y_2) = (-x_1, 0) \qquad T = \frac{1}{b_1}\sqrt{\frac{y_0}{a_1}} + t_2$$

Fundamental and saltation matrices are described in the next lemma

Lemma 5.10. *Unperturbed system* (5.14) *has the corresponding fundamental matrices*

$$X_1(t) = \begin{pmatrix} 1 & 0 \\ -2a_1 b_1 t & 1 \end{pmatrix} \qquad X_2(t) = \begin{pmatrix} 1 & 0 \\ -2a_2 b_2 (t - t_1) & 1 \end{pmatrix}$$
$$X_3(t) = \begin{pmatrix} 1 & 0 \\ -2a_1 b_1 (t - t_2) & 1 \end{pmatrix}$$

and saltation matrices

$$S_1 = \begin{pmatrix} 1 & \frac{\sqrt{a_1}(b_1 + b_2)}{2a_1 b_1 \sqrt{y_0}} \\ 0 & \frac{a_2 b_2}{a_1 b_1} \end{pmatrix} \qquad S_2 = \begin{pmatrix} 1 & \frac{\sqrt{a_1}(b_1 + b_2)}{2a_2 b_2 \sqrt{y_0}} \\ 0 & \frac{a_1 b_1}{a_2 b_2} \end{pmatrix}$$

Proof. Because of the linearity of this case, fundamental matrices are obtained via equations

$$X_1(t) = e^{At} \qquad X_2(t) = e^{B(t - t_1)} \qquad X_3(t) = e^{A(t - t_2)}$$

where

$$A = \begin{pmatrix} 0 & 0 \\ -2a_1 b_1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 0 \\ -2a_2 b_2 & 0 \end{pmatrix}$$

are Jacobi matrices of the functions $f_+(x, y)$ and $f_-(x, y)$, respectively. Saltation matrices are given by their definitions in (3.8) and (3.12) where $\nabla h(x, y) = (0, 1)$ in $\overline{\Omega}$ and

$$\begin{aligned} f_+(x_1, y_1) &= \begin{pmatrix} b_1 \\ -2a_1 b_1 \sqrt{\frac{y_0}{a_1}} \end{pmatrix} \quad f_-(x_1, y_1) = \begin{pmatrix} -b_2 \\ -2a_2 b_2 \sqrt{\frac{y_0}{a_1}} \end{pmatrix} \\ f_+(x_2, y_2) &= \begin{pmatrix} b_1 \\ 2a_1 b_1 \sqrt{\frac{y_0}{a_1}} \end{pmatrix} \quad f_-(x_2, y_2) = \begin{pmatrix} -b_2 \\ 2a_2 b_2 \sqrt{\frac{y_0}{a_1}} \end{pmatrix} \end{aligned} \tag{5.15}$$

□

In this case, the corresponding matrices can be easily multiplied to derive the following result

Lemma 5.11. *Function $A(t)$ of* (3.13) *for the system* (5.13) *possesses the form*

$$A(t)=\begin{cases}\begin{pmatrix}1-\frac{2\sqrt{a_1}b_1t(b_1+b_2)}{b_2\sqrt{y_0}} & -\frac{b_1+b_2}{b_2\sqrt{a_1y_0}}\\ 2a_1b_1t & 1\end{pmatrix} & \text{if } t\in[0,t_1)\\ \begin{pmatrix}-1-\frac{2b_1}{b_2}+\frac{\sqrt{a_1}(t-t_1)(b_1+b_2)}{\sqrt{y_0}} & \frac{\sqrt{a_1}(b_1+b_2)}{2a_2b_2\sqrt{y_0}}\\ 2(\sqrt{a_1y_0}-a_1b_2(t-t_1)) & -\frac{a_1}{a_2}\end{pmatrix} & \text{if } t\in[t_1,t_2)\\ \begin{pmatrix}1 & 0\\ 2(a_1b_1(t-t_2)-\sqrt{a_1y_0}) & 1\end{pmatrix} & \text{if } t\in[t_2,T]\end{cases} \tag{5.16}$$

where all constants are assumed to be positive.

Proposition 5.12. *Conditions* H1)*,* H2) *and* H3) *are satisfied.*

Proof. Since we already have the periodic orbit, $\nabla h(x,y)=(0,1)$ in $\overline{\Omega}$ and (5.15), conditions H1) and H2) are immediately satisfied.

Now let dim $\mathcal{N}(\mathbb{I}-\widetilde{P}_\xi(x_0,y_0,0,\mu,\alpha))>1$. Then there exists

$$\bar v\in\mathcal{N}(\mathbb{I}-\widetilde{P}_\xi(x_0,y_0,0,\mu,\alpha))$$

such that $\langle\bar v,f_+(x_0,y_0)\rangle=0$ and we can write $\bar v=(0,v)^*$. Since

$$\widetilde{P}_\xi(x_0,y_0,0,\mu,\alpha)\bar v=A(0)\bar v=\begin{pmatrix}-\frac{v(b_1+b_2)}{b_2\sqrt{a_1y_0}}\\ v\end{pmatrix}$$

then $v=0$, dim $\mathcal{N}(\mathbb{I}-\widetilde{P}_\xi(x_0,y_0,0,\mu,\alpha))=1$ and the condition H3) is verified. □

Note that there is a lot of periodic trajectories in the neigbourhood of $\gamma(t)$ but none of them has the same period, because $T=2\sqrt{\frac{y_0}{a_1}}\left(\frac{1}{b_1}+\frac{1}{b_2}\right)$ depends on the initial point (x_0,y_0).

We have

$$\mathcal{R}(\mathbb{I}-\widetilde{P}_\xi(x_0,y_0,0,\mu,\alpha))=\mathcal{R}(\mathbb{I}-A(0))=\mathbb{R}\times\{0\}$$

Then $\psi=(0,1)^*$ and $A(t)^*\psi=\begin{pmatrix}a_{21}(t)\\ a_{22}(t)\end{pmatrix}$. The assumptions of the Theorem 3.5 are equivalent to say that

$$\begin{aligned}M(\alpha)=\Bigg[&\sin\omega\alpha\left(\mu_1\int_0^T a_{21}(t)\cos\omega t\,dt-\mu_2\int_0^T a_{22}(t)\sin\omega t\,dt\right)\\ &+\cos\omega\alpha\left(\mu_1\int_0^T a_{21}(t)\sin\omega t\,dt+\mu_2\int_0^T a_{22}(t)\cos\omega t\,dt\right)\Bigg]\end{aligned}$$

has a simple root. It is easy to see, that this happens if and only if

$$\Phi(\omega)=\int_0^T e^{-\imath\omega t}(\mu_1a_{21}(t)-\imath\mu_2a_{22}(t))dt\neq 0 \tag{5.17}$$

Similarly to [1], function $\Phi(\omega)$ is analytic for $\omega>0$ and hence the following theorem holds (see Theorem 4.2 in [1] and [15])

Theorem 5.13. *When $\Phi(\omega)$ is not identically equal to zero, then there is at most a countable set $\{\omega_j\} \subset (0,\infty)$ with possible accumulating point at $+\infty$ such that for any $\omega \in (0,\infty)\backslash\{\omega_j\}$, the T–periodic orbit $\gamma(t)$ persists for* (5.13) *under perturbations for $\varepsilon \neq 0$ small.*

Because for general parameters, conditions on μ_1 and μ_2, so we could decide when $\Phi(\omega)$ is identically zero, or the set of roots is finite or countable, are too complicated, we rather provide an example with concrete numerical values of parameters.

Example 5.14. We take

$$a_1 = a_2 = b_1 = b_2 = y_0 = 1 \tag{5.18}$$

Then from (5.17) we have

$$\Phi(\omega) = -4\imath\frac{e^{-2\imath\omega}}{\omega^2}(2\mu_1 + \omega\mu_2)\sin\omega(\cos\omega - 1)$$

Thence for $\omega \in (0,\infty)$ it holds: if $\omega = k\pi$ for some $k \in \mathbb{N}$ or $\omega = -\frac{2\mu_1}{\mu_2}$ for $-\frac{\mu_1}{\mu_2} > 0$ then $\Phi(\omega) = 0$.

We conclude

Corollary 5.15. *Consider* (5.13) *with parameters* (5.18). *If $\omega > 0$ is such that $\omega \neq k\pi$ for all $k \in \mathbb{N}$ and $\omega \neq -\frac{2\mu_1}{\mu_2}$ with $\mu_2 \neq 0$ then T–periodic orbit $\gamma(t)$ persists under perturbations for $\varepsilon \neq 0$ small.*

Finally, if $\Phi(\omega)$ is identically zero then higher order Melnikov function must be derived [2]. We omit those computations in our case, because they are very awkward.

Stability of persisting periodic trajectory will be investigated in our forthcoming paper.

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