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ON THE SMALLEST ENCLOSING BALLS*

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Abstract. In the paper a theoretical analysis is given for the smallest ball that covers a finite number of points $p_1, p_2, \cdots, p_N \in \mathbb{R}^n$. Several fundamental properties of the smallest enclosing ball are described and proved. Particularly, it is proved that the k-circumscribing enclosing ball with smallest k is the smallest enclosing ball, which dramatically reduces a possible large number of computations in the higher dimensional case. General formulas are deduced for calculating circumscribing balls. The difficulty of the closed-form description is discussed. Finally, as an application, the problem of finding a common quadratic Lyapunov function for a set of stable matrices is considered.

Keywords: Smallest enclosing ball, k-dimensional large circle, circumscribing ball

1. Introduction. The problem of the smallest enclosing ball can be described as: Given a set of N points, denoted by

$$P := \{p_i |\ i = 1, \cdots, N\} \subset \mathbb{R}^n,$$

find the smallest ball $B^n(c, r)$, such that

$$P \subseteq B^n(c, r), \tag{1.1}$$

where

$$B^n(c, r) = \{ x \in \mathbb{R}^n |\ \|x - c\| \leq r \}.$$

The boundary of the ball is denoted by

$$\partial B^n(c, r) = \{ x \in \mathbb{R}^n |\ \|x - c\| = r \}.$$

A sphere $B^n(c, r) \subset \mathbb{R}^n$ is called an enclosing ball of P, if and only if (1.1) holds.

The problem is important in many social and engineering problems, such as biological swarms [6], [7] robot communication [8], [9], etc. Numerical algorithms for the construction of the smallest enclosing ball have been developed in [4], [10], [11], and the reference therein.

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The purpose of this paper is to investigate some of the basic theoretical properties of the smallest enclosing ball. It answers the question of where the smallest enclosing ball lies. First, it is proved that the smallest enclosing ball is determined by a k-dimensional "large circle", which is uniquely determined by $k+1$ points of P on its boundary. Then it is proved that the k circumscribing feasible ball with the smallest $k \geq 2$ determines the smallest enclosing ball. The theoretical results in the paper provide a rigorous foundation for various numerical algorithms. Certain formulas in fact can be deduced to the calculation of circumscribing balls.

The paper is organized as follows: Section 2 gives a brief but clear formulation of the problem. Section 3 contains the main results, which provide a rigorous description and method for finding the smallest enclosing ball. Section 4 consists mainly of two set of formulas to calculate circumscribing balls of different dimensions. Section 5 considers the planar and cubic cases. Some examples are given to show the computation process. Section 6 investigates the closed form solution of the problem. As a new application, the common quadratic Lyapunov function of a set of stable matrices is investigated in Section 7. Section 8 is the conclusion.

2. Preliminaries. The smallest enclosing ball problem can be formulated as a min-max optimization problem.

$$\min_{x\in\mathbb{R}^n}\max_{1\le i\le N}\|x-p_i\|^2. \tag{2.1}$$

We begin by establishing some notation. For fixed $x \in \mathbb{R}^n$ we define the smallest radius with respect to a fixed P as

$$r_m(x)=\max_{1\le i\le N}\|x-p_i\|. \tag{2.2}$$

Note that it is easy to see that $r_m : \mathbb{R}^n \to \mathbb{R}_+$ is a well defined continuous and piecewise smooth function. We denote the set of indexes of the points that lie on the boundary of the ball as

$$\mathcal{I}_m(x)=\{i \mid \|p_i-x\|=r_m(x)\}. \tag{2.3}$$

Next we fix i, and write $p_i=(x_1^i,\cdots,x_n^i)^T$. Then

$$\|x-p_i\|^2=\sum_{j=1}^n (x_j-x_j^i)^2.$$

Its (half) gradient is denoted by

$$g_i(x)=\begin{pmatrix} x_1-x_1^i & x_2-x_2^i & \cdots & x_n-x_n^i\end{pmatrix}. \tag{2.4}$$

Let $B^n(c,r)$ be given and let L_k be a k-dimensional affine subspace passing through c. Then a k dimensional ball in L_k with same center (c) and same radius (r), i.e.,

$$B^k(c,r) \subset B^n(c.r), \quad k < n,$$

is called a *k-dimensional large circle of* $B^n(c,r)$. When an arbitrary k-dimensional affine subspace H_k intersects $B^n(c,r)$, the intersection–called a k-dimensional segment –is a k-dimensional ball $B^k(c',r')$ and

$$B^k(c',r') \subset B^n(c,r),$$

where

$$\overline{c',c} \perp H_k, \quad \text{and} \quad r' = \sqrt{r^2 - \overline{c',c}^2}.$$

Through this paper we use $\overline{x,y}$ for both the line segment $[x,y]$ and its length, when there is no possibility of confusion.

If a k-dimensional ball, $B^k(c,r)$, is uniquely determined by $k+1$ points in P and the n-dimensional ball, $B^n(c,r)$, with it as a k-dimensional large circle, is an enclosing ball, i.e., $P \subset B^n(c,r)$, then $B^k(c,r)$ is called a *k circumscribing feasible ball*, and $B^n(c,r)$ is called the *k circumscribing enclosing ball*.

Note that for an enclosing ball $B^n(c,r)$, the corresponding feasible ball may not be unique. We choose the smallest k as its label. Say, in $\mathbb{R}^3$, assume a two dimensional large circle (a disk D) determined by three points $A, B, C \in P$, is a feasible ball (i.e., the ball, $B^3(c,r)$, with D as its large circle is an enclosing ball), if one side, say $\overline{A,B}$, is the diameter of D, then $\overline{A,B}$ is a 1 circumscribing feasible ball and $B^3(c,r)$ is a 1 circumscribing enclosing ball. Otherwise, $B^3(c,r)$ is a 2 circumscribing enclosing ball, and the disk D is the 2 circumscribing feasible ball.

3. Smallest Enclosing Ball. This section discusses the fundamental properties of the smallest enclosing balls. We begin with a critical definition.

DEFINITION 3.1. *x^* is called a best enclosing solution (or $B(x^*, r_m(x^*))$ is the smallest enclosing ball), if for every $x \in \mathbb{R}^n$*

$$r_m(x^*) \le r_m(x). \tag{3.1}$$

PROPOSITION 3.2. *The best enclosing solution is unique. That is, if x^* is a best enclosing solution, then for very $x \neq x^*$*

$$r_m(x^*) < r_m(x). \tag{3.2}$$

Proof. Assume there exist x^* and y^* distinct, such that $r_m(x^*) = r_m(y^*) := r$, we claim that both x^* and y^* are not optimal. To prove the claim we consider $z = \frac{x^*+y^*}{2}$. For any p_i we have

$$\|p_i - z\| = \frac{1}{2}\|(p_i - x^*) + (p_i - y^*)\| \leq \frac{1}{2}\|p_i - x^*\| + \frac{1}{2}\|p_i - y^*\|. \tag{3.3}$$

If $i \notin \mathcal{I}_m(x^*) \cap \mathcal{I}_m(y^*)$, then at least one term on the right hand side of the triangular inequality (3.3) is less than $\frac{1}{2}r$. So $\|p_i - z\| < r$.

Now we assume $i \in \mathcal{I}_m(x^*) \cap \mathcal{I}_m(y^*)$. For the equality of (3.3) to hold, we need

$$p_i - x^* = k(p_i - y^*), \quad k > 0.$$

Note that

$$x^*, y^* \in S^{n-1}(p_i, r).$$

So if we want $p_i - x^*$ and $p_i - y^*$ be linearly dependent, the only possible case is that they are antipodal. That is: $k = -1$. Therefore, the equality in (3.3) can never be true. We conclude that for all i

$$\|p_i - z\| < r,$$

which means both x^* and y^* are suboptimal. □

The following lemma is essential.

PROPOSITION 3.3. *x^* is the best enclosing solution of (2.1), if and only if for all $i \in \mathcal{I}_m(x^*)$*

$$g_i(x^*)Z < 0, \tag{3.4}$$

has no solution.

Proof. (Necessity) Assume there is a solution Z of (2.1), which is obviously nonzero. Without loss of generality, we assume $\|Z\| << 1$. Then for all $i \in \mathcal{I}_m(x^*)$

$$\begin{array}{rcl} \|x^* + Z - p_i\|^2 & = & \|x^* - p_i\|^2 + 2g_i(x^*)Z + O(\|Z\|^2) \\ & < & \|x^* - p_i\|^2 = [r_m(x^*)]^2. \end{array}$$

As for $j \notin \mathcal{I}_m(x^*)$, since

$$\|x^* - p_j\|^2 < [r_m(x^*)]^2,$$

we can choose $\|Z\|$ small enough such that for all $j \notin \mathcal{I}_m(x^*)$

$$\begin{array}{rcl} \|x^* + Z - p_j\|^2 & = & \|x^* - p_j\|^2 + O(\|Z\|) \\ & < & [r_m(x^*)]^2. \end{array}$$

We conclude that $x^* + Z$ is better than x^*, that is,

$$\|x^* + Z - p_i\| < r_m(x^*), \quad \forall i,$$

which is a contradiction.

(Sufficiency) Assume x is not the best enclosing solution.

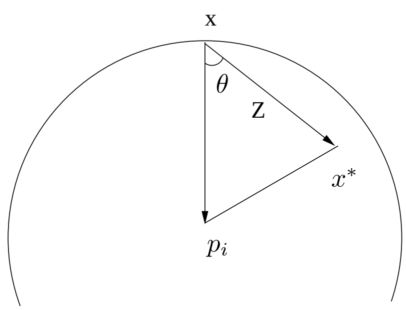

FIG. 1. *The existence of Z.*

Let x^* be the best enclosing solution. Choose $Z = x^* - x$ and assume $i \in \mathcal{I}_m(x)$, i.e.,

$$p_i \in \partial S^{n-1}(x, r_m(x)).$$

Note that x^* is the best enclosing solution, according to Lemma 3.2, we have

$$r_m(x^*) < r_m(x) = \|p_i - x\|.$$

So in triangle $\Delta p_i x x^*$ we have

$$\|p_i - x^*\| \leq r_m(x^*) < \|p_i - x\|$$

which means $\theta := \angle p_i x x^* < 90^o$. Then

$$\langle x - p_i, Z\rangle = -\langle p_i - x, Z\rangle = -\|p_i - x\|\|Z\| \cos(\theta) < 0.$$

Since i is arbitrary, we have for all $i \in \mathcal{I}_m(x)$

$$g_i(x)Z < 0, \quad i \in \mathcal{I}_m(x).$$

□

For a given $x \in \mathbb{R}^n$, let $\mathcal{I}_m(x) = \{i_1, i_2, \cdots, i_s\}$. Then we denote the affine subspace determined by $p_{i_1}, \cdots, p_{i_s}$ by L_x. That is,

$$L_x = Span\left\{p_{i_j} - p_{i_1} : \ j = 2, \cdots, s\right\} + p_{i_1}. \tag{3.5}$$

Note that if $|\mathcal{I}_m(x)| = 1$, we formally define $L_x = \{0\}$, with dimension $\dim(L_x) = 0$.

Using Proposition 3.3, we can prove the following result, which is the key for solving the problem.

PROPOSITION 3.4. *If x^* is the best enclosing solution and assume*

$$\dim(L_{x^*}) = k < n,$$

then

$$x^* \in L_{x^*}. \tag{3.6}$$

Proof. Let $\{\epsilon_1, \cdots, \epsilon_k\}$ be an orthonormal basis of $L_{x^*} - p_{i_1}$. We project $x^* - p_{i_1}$ on $L_{x^*} - p_{i_1}$ as

$$x^* - p_1 = \langle x^* - p_{i_1}, \epsilon_1 \rangle \epsilon_1 + \cdots + \langle x^* - p_{i_1}, \epsilon_k \rangle \epsilon_k + \delta.$$

It follows that

$$\langle x^* - p_{i_1}, \epsilon_i \rangle = \langle x^* - p_{i_1}, \epsilon_i \rangle \|\epsilon_i\|^2 + \langle \delta, \epsilon_i \rangle .$$

Since $\|\epsilon_i\| = 1$, we have

$$\langle \delta, \epsilon_i \rangle = 0, \quad i = 1, \cdots, k. \tag{3.7}$$

That is

$$\delta \perp L_{x^*} - p_{i_1}.$$

We claim that $\delta = 0$. If not, setting $Z = -\delta$ and using (3.7), we have

$$\langle x^* - p_{i_1}, Z \rangle = \langle x^* - p_1, -\delta \rangle = -\|\delta\|^2. \tag{3.8}$$

Next, for $j > 1$ we have

$$\left\langle x^* - p_{i_j}, Z \right\rangle = \left\langle x^* - p_{i_1} + (p_{i_1} - p_{i_j}), -\delta \right\rangle = \langle x^* - p_{i_1}, -\delta \rangle = -\|\delta\|^2.$$

We conclude that for all $i_j \in \mathcal{I}_m(x^*)$

$$\left\langle x^* - p_{i_j}, Z \right\rangle = -\|\delta\|^2, \tag{3.9}$$

which is a contradiction to Proposition 3.3. Hence $\delta = 0$, which implies $x^* - p_{i_1} \in L_{x^*} - p_{i_1}$.

That is,

$$x^* \in L_{x^*}.$$

□

REMARK 3.5. *1. Proposition 3.4 implies clearly that the search for the best enclosing solution is finite operation. To see that we may assume* $\dim(L_{x^*}) = k$*,* k *can only be* $1, \cdots, n$*. Then we can search all* $k+1$ *points to see (i) whether they span a* k *dimensional affine plane* L_k*. (ii) construct the smallest inclosing ball* $B^k(c, r)$ *on* L_k *for these* $k+1$ *points and then check whether* $B^n(c, r)$ *is an enclosing ball. In fact, Lemma 3.4 claims that the smallest enclosing ball is one of such enclosing balls.*

2. If x^* *is the optimal solution and* $\dim(L_{x^*}) = k$*, then we have to choose any* $k+1$ *points from* P*, the total number of searches for fixed* k *is*

$$\binom{N}{k+1} = \frac{N!}{(k+1)!(N-k-1)!}.$$

Now for all possible k *the total number of searches is*

$$\sum_{k=1}^{n} \binom{N}{k+1}.$$

Proposition 3.4 proposes a way to search the set of all enclosing balls with finitely many searches and then the smallest enclosing ball can be found by comparing those balls. However, the number of searches is large when N and n become large. Particularly, the search number is polynomial with respect to N but exponential with respect to n.

In the following we will argue that the comparison is necessary. Starting from the lowest dimensional case, where $\dim(L_x) = 1$ and we choose only two points. As long as we find an enclosing ball, we are done! We need some preparation for this.

DEFINITION 3.6. *An enclosing ball* $B^n(c, r)$*, is called a* k *circumscribing enclosing ball if (i) it is uniquely determined by a* k *dimensional large circle, which contains at least* $k+1$ *points of* P *on its boundary; (ii) the* k *is the smallest one which meets (i). The large circle is called a* k *circumscribing feasible ball.*

Note that since the k is the smallest one, the points on the boundary of the large circle determine the large circle uniquely.

Let $B^k(c, r)$ be a k circumscribing feasible ball and denote

$$P_k = P \cap \partial B^k(c, r).$$

Using this notation, we define the concept of irreducibility.

DEFINITION 3.7. *A k circumscribing feasible ball $B^k(c, r)$ is called irreducible if there is no subset $P_s \subset P_k$ such that P_s determines an s circumscribing feasible $B^s(c', r')$, where $s < k$.*

The following two propositions show that once a k circumscribing feasible ball is found its corresponding k circumscribing enclosing ball is the smallest enclosing one.

First, a k circumscribing ball is determined by an irreducible k circumscribing feasible ball, because by definition, k is the smallest one. Next, we prove that such a feasible ball is unique.

PROPOSITION 3.8. *The k circumscribing enclosing ball is unique, which can be determined by any irreducible k circumscribing feasible ball.*

Proof. Assume there are two irreducible k circumscribing feasible balls, namely, $B_1^k(c_1, r_1)$ and $B_2^k(c_2, r_2)$. The k circumscribing enclosing ball determined by $B_1^k(c_1, r_1)$ is $B_1^n(c_1, r_1)$. Now the affine subspace L_k^2 containing $B_2^k(c_2, r_2)$ has a segment $B^k(c', r')$, which contains $B_2^k(c_2, r_2)$, because $B_1^n(c_1, r_2)$ is an enclosing ball. Now it is obvious that

$$r_1 \geq r' \geq r_2.$$

The equality holds, if and only if $B_2^k(c_2, r_2)$ is the large circle. In this case, $B^n(c_2, r_2) = B^n(c_1, r_1)$. The same argument leads to

$$r_2 \geq r_1,$$

and equality holds, if and only if $B^n(c_2, r_2) = B^n(c_1, r_1)$. Hence we have

$$B^n(c_2, r_2) = B^n(c_1, r_1)$$

. □

PROPOSITION 3.9. *The k circumscribing enclosing ball determined by an irreducible k circumscribing feasible ball is the smallest enclosing ball.*

Proof. Let $\{q_1, \cdots, q_s\} \in P$ be the set of points on $\partial B^k(c, r)$, where $B^k(c, r)$ is the k circumscribing feasible ball. Denote by

$$H = \overline{co}\{q_1, \cdots, q_s\}$$

the convex hell of $\{q_1, \cdots, q_s\}$. Refer to Fig. 2, assume $c \notin H$, then there exists a $k-1$ dimensional face on L_{k-1}, an $k-1$ dimensional affine subspace, such that $c \notin L_{k-1}$

and L_{k-1} separate c and $int(H)$, which is the interior of H. Project c perpendicularly on L_{k-1} at c', and let

$$r' = \sqrt{r^2 - \overline{c, c'}^2}$$

Then it is easy to see that $B^n(c', r')$ is an enclosing ball with radius less than r. Note that if $q_i \in L_{k-1} \cap \partial B^k(c, r)$, then $q_i \in Lk - 1 \cap \partial B^n(c', r')$ because of the relationship of the radii. But $L_{k-1} \cap \partial B^n(c', r')$ is a $k-1$ dimensional large circle. This contradicts to the irreducibility.

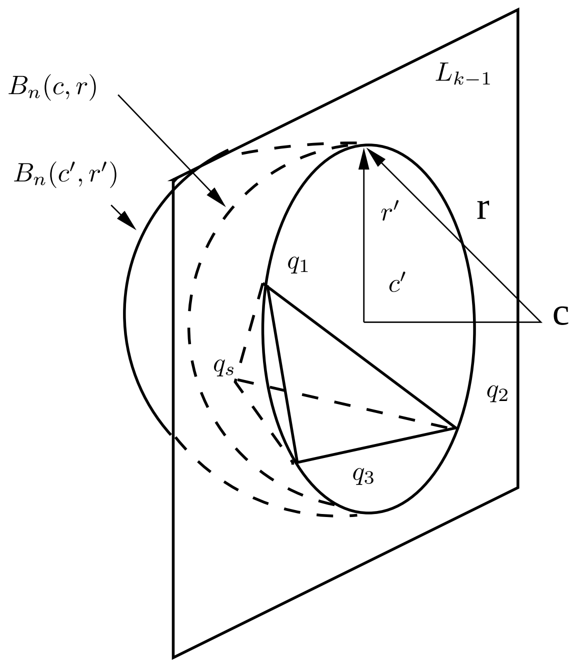

Fig. 2. *Separating Plane.*

Now we can assume $c \in H$. Therefore, for all i

$$c = \sum_{i=1}^{s} \mu_i q_i, \quad \mu_i \geq 0, .$$

We claim that c is the best enclosing solution. Assume there exists Z such that for all $p_i \in P$

$$(c - p_i)Z < 0. \tag{3.10}$$

Particularly, q_i satisfies (3.10), that is,

$$cZ < q_i Z, \quad , i = 1, \cdots, s.$$

Then we have $\sum_{i=1}^{s} \mu_i cZ < \sum_{i=1}^{s} \mu_i q_i Z$, that is, $cZ < cZ$. We have a contradiction. According to Proposition 3.3, c is the best enclosing solution. □

Propositions 3.8 and 3.9 propose the following search procedure:

ALGORITHM 3.10.

Step 1. Choosing any two points $p_i, p_j \in P$, *find the longest one, say* $\overline{p,q}$, *and then check whether* $B^n(\frac{p+q}{2}, \frac{1}{2}[\overline{p,q}])$ *is a* 2*-feasible enclosing ball or not. If "Yes", we are done.*

Step k $(3 \leq k \leq n+1)$*. Choosing any* k *points* $p_{i_1}, \cdots, p_{i_k} \in P$, *find their circumscribing ball. If such ball doesn't exist, ignore it. Check whether the ball is a* k *circumscribing feasible ball or not. If "Yes", we are done. If there is no* k *circumscribing feasible ball, check* $k+1$.

THEOREM 3.11. *The first* k *circumscribing feasible ball found in Algorithm 3.10 is the optimal enclosing solution.*

Proof. According to Proposition 3.8, the k circumscribing feasible ball is unique, so as long as we found one, we don't need the search any other k circumscribing feasible ball. Now it is obvious that the first k circumscribing feasible ball is irreducible. Otherwise, the reducible one should be found in at most $n+1$ steps. Now Proposition 3.9 assures that it is the optimal enclosing ball. □

4. Searching k Circumscribing Feasible Balls. This section provides precise formulas to calculate the center and radius of the k circumscribing feasible balls. Note that in our searching process, the points used to construct the ball are known. They are on the boundary of the ball. So as long as the center of the candidate is known, the radius is also known.

First, we construct a ball B^k in $\mathbb{R}^k$ by using $k+1$ points, such that ∂B^k circumscribing the $k+1$ points.

PROPOSITION 4.1. *Let* $q_1, \cdots, q_{k+1}$ *be* $k+1$ *points in* $\mathbb{R}^k$ *with* $q_i = (x_1^i, \cdots, x_k^i)^T$. *They lie on the boundary of a ball* $B^k(c,r)$, *if and only if the following* A *is invertible:*

$$A = \begin{pmatrix} x_1^2 - x_1^1 & \cdots & x_k^2 - x_k^1 \\ x_1^3 - x_1^2 & \cdots & x_k^3 - x_k^2 \\ \vdots & & \\ x_1^{k+1} - x_1^k & \cdots & x_k^{k+1} - x_k^k \end{pmatrix} \tag{4.1}$$

Moreover, as A *is invertible the center is*

$$c = A^{-1}B,$$

where

$$B = \frac{1}{2}\begin{pmatrix} \sum_{j=1}^{k}[(x_j^2)^2 - (x_j^1)^2] \\ \sum_{j=1}^{k}[(x_j^3)^2 - (x_j^2)^2] \\ \vdots \\ \sum_{j=1}^{k}[(x_j^{k+1})^2 - (x_j^k)^2] \end{pmatrix}. \tag{4.2}$$

Proof. Let $(X_1, \cdots, X_k)$ be the points on the perpendicular dividing affine subspace of the segment $\overline{q_i q_{i+1}}$, then it satisfies

$$\sum_{j=1}^{k}\left(x_j^{i+1} - x_j^i\right)\left(X_j - \frac{x_j^{i+1} + x_j^i}{2}\right) = 0, \quad i = 1, \cdots, k. \tag{4.3}$$

Equivalently, we have

$$\sum_{j=1}^{k}\left(x_j^{i+1} - x_j^i\right)X_j = \frac{1}{2}\sum_{j=1}^{k}[(x_j^{i+1})^2 - (x_j^i)^2], \quad i = 1, \cdots, k. \tag{4.4}$$

which is equivalent to the aforementioned equation $AX = B$. We know that if c exists, then it must lie on the intersection of these k affine subspaces.

Moreover, it is easy to verify that $q_1, \cdots, q_{k+1}$ lie on a $k-1$ dimensional affine plane, if and only if A is singular. $\square$

Next, we consider a more general case, where the $k+1$ points are on $\mathbb{R}^n$. Assume

$$L_k = Span\,\{q_{i+1} - q_i | i = 1, \cdots, k\}$$

is a k dimensional hyperplane. (Otherwise, it becomes degenerate and the sphere circumscribing the $k+1$ points doesn't exist.) First, we determine $L_k^\perp$. Let

$$L_k^\perp = Span\{h_1, \cdots, h_{n-k}\}.$$

Then we have

$$\langle h_j, q_{i+1} - q_i\rangle = 0, \quad j = 1, \cdots, n-k;\ i = 1, \cdots, k. \tag{4.5}$$

Denote

$$Q = \begin{pmatrix} q_2 - q_1 \\ \vdots \\ q_{k+1} - q_k \end{pmatrix} = \begin{pmatrix} x_1^2 - x_1^1 & x_2^2 - x_2^1 & \cdots & x_n^2 - x_n^1 \\ x_1^3 - x_1^2 & x_2^3 - x_2^2 & \cdots & x_k^3 - x_k^2 \\ \vdots & & & \\ x_1^{k+1} - x_1^k & x_2^{k+1} - x_2^k & \cdots & x_n^{k+1} - x_n^k \end{pmatrix}.$$

without loss of generality, we assume the first $k \times k$ block of Q is nonsingular. Then we can set

$$h_j = \left(\mu_1^j, \cdots, \mu_k^j, -\delta_j^{n-k}\mu_0\right)^T, \quad j = 1, \cdots, n-k, \tag{4.6}$$

where the delta function is defined as

$$\delta_j^{n-k} = (0, 0, \cdots, \underbrace{1}_{j-th}, 0, \cdots, 0) \in \mathbb{R}^{n-k}.$$

Now (4.5) becomes

$$Qh_j = 0, \quad j = 1, \cdots, n-k. \tag{4.7}$$

It is easy to see that μ_i^j and μ_0 in h_j can be solved as (up to a non-zero constant coefficient)

$$\mu_0 = \det \begin{pmatrix} x_1^2 - x_1^1 & x_2^2 - x_2^1 & \cdots & x_k^2 - x_k^1 \\ x_1^3 - x_1^2 & x_2^3 - x_2^2 & \cdots & x_k^3 - x_k^2 \\ \vdots & & & \\ x_1^{k+1} - x_1^k & x_2^{k+1} - x_2^k & \cdots & x_k^{k+1} - x_k^k \end{pmatrix}$$

$$\mu_i^j = \det \begin{pmatrix} x_1^2 - x_1^1 & \cdots & x_{k+j}^2 - x_{k+j}^1 & \cdots & x_k^2 - x_k^1 \\ x_1^3 - x_1^2 & \cdots & x_{k+j}^3 - x_{k+j}^2 & \cdots & x_k^3 - x_k^2 \\ \vdots & & & & \\ x_1^{k+1} - x_1^k & \cdots & \underbrace{x_{k+j}^{k+1} - x_{k+j}^k}_{i-th} & \cdots & x_k^{k+1} - x_k^k \end{pmatrix}, \tag{4.8}$$
$$j = 1, \cdots, n-k, \; i = 1, \cdots, k.$$

Note that μ_i^j is obtained from μ_0 by replacing its i-th column by

$$\begin{pmatrix} x_{k+j}^2 - x_{k+j}^1 \\ x_{k+j}^3 - x_{k+j}^2 \\ \vdots \\ x_{k+j}^{k+1} - x_{k+j}^k \end{pmatrix}.$$

Now we are ready to solve for the center, c, of the circumscribing sphere.

Denote the middle point of $\overline{q_i q_{i+1}}$ by m_i, then the center, c, should satisfy the conditions

$$\overline{cm_i} \perp \overline{q_i q_{i+1}}, \quad i = 1, \cdots, k, \tag{4.9}$$

and

$$\overline{cm_1} \perp h_j, \quad j = 1, \cdots, n-k. \tag{4.10}$$

(4.9)-(4.10) are enough to determine c. Note that as long as (4.9)-(4.10) hold we have $c - m_1 \in L_k$, which implies that

$$\overline{cm_i} \perp h_j, \quad i = 1, \cdots, k;\ j = 1, \cdots, n-k.$$

(4.9) can be written as

$$\sum_{s=1}^{n} \left(x_s^{j+1} - x_s^j\right) \left(c_s - \frac{x_s^{j+1} + x_s^j}{2}\right) = 0, \quad j = 1, \cdots, k. \tag{4.11}$$

In matrix form it becomes

$$Qc = \frac{1}{2} \begin{pmatrix} \sum\limits_{s=1}^{n} (x_s^2)^2 - (x_s^1)^2 \\ \vdots \\ \sum\limits_{s=1}^{n} (x_s^{k+1})^2 - (x_s^k)^2 \end{pmatrix}. \tag{4.12}$$

(4.10) can be written as

$$h_j \begin{pmatrix} c_1 - \frac{x_1^2 + x_1^1}{2} \\ \vdots \\ c_n - \frac{x_n^2 + x_n^1}{2} \end{pmatrix}, \quad j = 1, \cdots, n-k. \tag{4.13}$$

Denote

$$H = \begin{pmatrix} h_1 \\ \vdots \\ h_{n-k} \end{pmatrix}; \quad h_j = (h_1^j, \cdots, h_n^j),$$

then (4.13) can be written as

$$Hc = \frac{1}{2} \begin{pmatrix} \sum\limits_{s=1}^{n} h_s^1 (x_s^2 + x_s^1) \\ \vdots \\ \sum\limits_{s=1}^{n} h_s^{n-k} (x_s^2 + x_s^1) \end{pmatrix}. \tag{4.14}$$

Summarizing the above argument, we have the following result:

PROPOSITION 4.2. *Let $q_1, \cdots, q_{k+1}$ be $k+1$ points in $\mathbb{R}^n$ with $q_i = (x_1^i, \cdots, x_n^i)^T$. They lie on the boundary of a ball, $\partial B^k(c, r)$, if and only if the following matrix A is* *invertible.*

$$A = \begin{pmatrix} Q \\ H \end{pmatrix} = \begin{pmatrix} x_1^2 - x_1^1 & x_2^2 - x_2^1 & \cdots & x_n^2 - x_n^1 \\ x_1^3 - x_1^2 & x_2^3 - x_2^2 & \cdots & x_n^3 - x_n^2 \\ \vdots & & & \\ x_1^{k+1} - x_1^k & x_2^{k+1} - x_2^k & \cdots & x_n^{k+1} - x_n^k \\ h_1^1 & h_2^1 & \cdots & h_n^1 \\ \vdots & & & \\ h_1^{n-k} & h_2^{n-k} & \cdots & h_n^{n-k} \end{pmatrix} \tag{4.15}$$

Moreover, as A is invertible, the center c is

$$c = A^{-1}B,$$

where B is

$$B = \frac{1}{2} \begin{pmatrix} \sum_{s=1}^{n} (x_s^2)^2 - (x_s^1)^2 \\ \vdots \\ \sum_{s=1}^{n} (x_s^{k+1})^2 - (x_s^k)^2 \\ \sum_{s=1}^{n} h_s^1 (x_s^2 + x_s^1) \\ \vdots \\ \sum_{s=1}^{n} h_s^{n-k} (x_s^2 + x_s^1). \end{pmatrix} \tag{4.16}$$

We use a numerical example to demonstrate the formulas.

EXAMPLE 4.3. Given 5 points in $\mathbb{R}^4$ as

$$x_1 = \begin{pmatrix} 2 \\ 1 \\ 3 \\ -1 \end{pmatrix}; \quad x_2 = \begin{pmatrix} -2 \\ 3 \\ -1 \\ 0 \end{pmatrix}; \quad x_3 = \begin{pmatrix} 1 \\ 3 \\ -1 \\ -2 \end{pmatrix}; \quad x_4 = \begin{pmatrix} 0 \\ 2 \\ 3 \\ -3 \end{pmatrix}; \quad x_5 = \begin{pmatrix} 2 \\ -1 \\ 3 \\ 5 \end{pmatrix}.$$

1. Find a sphere $\partial B^4(c, r)$, circumscribing them.

Using (4.1) and (4.2), it is easy to calculate A and B as

$$A = \begin{pmatrix} -4 & 2 & -4 & 1 \\ 3 & 0 & 0 & -2 \\ -1 & -1 & 4 & -1 \\ 2 & -3 & 0 & 8 \end{pmatrix}$$

$$B = \begin{pmatrix} -0.5 \\ 0.5 \\ 3.5 \\ 8.5 \end{pmatrix}$$

Then the center is

$$c = A^{-1}B = \begin{pmatrix} -19.5 \\ -94.5 \\ -35 \\ -29 \end{pmatrix}.$$

Finally it is easy to find the radius

$$r = 108.8060.$$

2. Find a circle $\partial B^3(c, r)$, circumscribing x_1, x_2, x_3.

Using (4.8), we have

$$\mu_0 = \det \begin{pmatrix} -4 & 2 \\ 3 & 0 \end{pmatrix} = -6;$$

$$\mu_1^1 = \det \begin{pmatrix} -4 & 2 \\ 0 & 0 \end{pmatrix} = 0; \quad \mu_2^1 = \det \begin{pmatrix} -4 & -4 \\ 3 & 0 \end{pmatrix} = 12;$$

$$\mu_1^2 = \det \begin{pmatrix} 1 & 2 \\ -2 & 0 \end{pmatrix} = 4; \quad \mu_2^2 = \det \begin{pmatrix} -4 & 1 \\ 3 & -2 \end{pmatrix} = 5.$$

Then (4.6) yields

$$h_1 = (\mu_1^1, \mu_2^1, -\mu_0, 0) = (0, 12, 6, 0);$$

$$h_2 = (\mu_1^2, \mu_2^2, 0, -\mu_0) = (4, 5, 0, 6).$$

Using (4.15) and (4.16), we have

$$A = \begin{pmatrix} -4 & 2 & -4 & 1 \\ 3 & 0 & 0 & -2 \\ 0 & 12 & 6 & 0 \\ 4 & 5 & 0 & 6 \end{pmatrix}, \quad B = \begin{pmatrix} -0.5 \\ 0.5 \\ 30 \\ 7 \end{pmatrix}.$$

Then c is solved as

$$c = A^{-1}B = \begin{pmatrix} -0.0965 \\ 1.9509 \\ 1.0982 \\ -0.3947 \end{pmatrix}.$$

Finally we can calculate the radius

$$r = 3.0467.$$

It is easy to check that $\overline{cx_1} = \overline{cx_2} = \overline{cx_3}$. To prove that it is a circle, we have to prove that c is on the plane of $\Delta x_1x_2x_3$. This is done by verifying that

$$\langle h_j, c - x_i\rangle = 0, \quad j = 1,2;\ i = 1,2,3.$$

5. For 2D and 3D Cases. In practical applications $\mathbb{R}^2$ and $\mathbb{R}^3$ are particularly important. In plane, as a special case of Proposition 4.1, we have

COROLLARY 5.1. *Three points* (x_1, y_1), (x_2, y_2), (x_3, y_3) *lie on a circle if and only if*

$$d = (x_2 - x_1)(y_3 - y_2) - (x_3 - x_2)(y_2 - y_1) \neq 0, \tag{5.1}$$

in other words, the points do not lie on a line. Then the center of the circle is

$$\begin{aligned} c_x &= \tfrac{1}{2d}[(y_3 - y_2)(x_2^2 - x_1^2 + y_2^2 - y_1^2) + (y_1 - y_2)(x_3^2 - x_2^2 + y_3^2 - y_2^2)],\\ c_y &= \tfrac{1}{2d}[(x_2 - x_3)(x_2^2 - x_1^2 + y_2^2 - y_1^2) + (x_2 - x_1)(x_3^2 - x_2^2 + y_3^2 - y_2^2)]. \end{aligned} \tag{5.2}$$

We give an example to show this:

EXAMPLE 5.2. Consider $N = 100$ points. To create initial data, let

$$x_0 = (3, -2, 4, 4, -2, 2, 0, -1, 5, 1); \quad y_0 = (-2, 3, -1, 0, 1, 3, -1, -2, 3, 5),$$

and

$$\begin{aligned} x_{10\times(i-1)+j} &:= x_0(i) * x_0(j), \quad 1 \le i, j \le 10\\ y_{10\times(i-1)+j} &:= y_0(i) * y_0(j), \quad 1 \le i, j \le 10 \end{aligned}$$

Then the 100 points are

$$p_1 = (x_1, y_1),\ p_2 = (x_2, y_2),\ \cdots,\ p_{100} = (x_{100}, y_{100}).$$

Coding in MatLab, the following result is reported. For $k = 2$, the longest segment is $\overline{p_{49}p_{89}}$, with length 17.7553. The circle using $\overline{p_{49}p_{89}}$ as diameter is not an enclosing circle.

For $k = 3$, $\mathcal{I}_m(x^*) = \{89, 80, 100\}$, the center is $c = (6.4220, 7.1330)$. The radius is $r = 18.6716$.

Total computing time in PC is 2 sec.

As a particular case, if there are only three points, it is easy to see that if the triangle is right or oblique, the optimal circle with the longest side as its diameter. Otherwise, it is the circle circumscribing the triangle.

Next, we consider the three dimensional case. As an application of Propositions 4.1 and 4.2, we have

COROLLARY 5.3. *Let $p_1, p_2, p_3, p_4 \in \mathbb{R}^3$ be given as*

$$p_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}; \quad p_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}; \quad p_3 = \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix}; \quad p_4 = \begin{pmatrix} x_4 \\ y_4 \\ z_4 \end{pmatrix}.$$

1. There is a sphere circumscribing them, if and only if the following formula for the center is executable:

$$c = \frac{1}{2} \begin{pmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_2 & y_3 - y_2 & z_3 - z_2 \\ x_4 - x_3 & y_4 - y_3 & z_4 - z_3 \end{pmatrix}^{-1} \begin{pmatrix} \|p_2\|^2 - \|p_1\|^2 \\ \|p_3\|^2 - \|p_2\|^2 \\ \|p_4\|^2 - \|p_3\|^2 \end{pmatrix} \tag{5.3}$$

2.There is a circle circumscribing p_1, p_2, p_3, if and only if the following formula for the center is executable:

(5.4)

$$c = \frac{1}{2} \begin{pmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_2 & y_3 - y_2 & z_3 - z_2 \\ h_1 & h_2 & h_3 \end{pmatrix}^{-1} \begin{pmatrix} \|p_2\|^2 - \|p_1\|^2 \\ \|p_3\|^2 - \|p_2\|^2 \\ h_1(x_2 + x_1) + h_2(y_2 + y_1) + h_3(z_2 + z_1) \end{pmatrix},$$

where

$$\begin{aligned} h_1 &= (x_2 - x_1)(z_3 - z_2) + (x_2 - x_3)(z_2 - z_1); \\ h_2 &= (z_2 - z_1)(y_3 - y_2) + (z_2 - z_3)(y_2 - y_1); \\ h_3 &= (y_2 - y_1)(x_3 - x_2) + (y_2 - y_3)(x_2 - x_1). \end{aligned}$$

EXAMPLE 5.4. Consider 10 points in $\mathbb{R}^3$: $p_1 = (1, 3, 2)$, $p_2 = (-2, -3, 1)$, $p_3 = (4, -1, -3)$, $p_4 = (4, 4, 2)$, $p_5 = (-2, 1, -2)$, $p_6 = (2, 3, 3)$, $p_7 = (1, -1, 1)$, $p_8 = (-1, -2, 2)$, $p_9 = (5, 3, -1)$, $p_{10} = (1, 2, -2)$.

$k = 2$: The largest distance is between p_2 and p_9. The radius determined by it is $r = 4.7648$. It doesn't work.

$k = 3$. The first feasible solution is the circle circumscribing p_9, p_4, and p_2. The center is $c = (1.2394, 0.4475, 0.4306)$, the radius is $r = 4.7648$.

$k = 4$. The first feasible solution is the sphere circumscribing p_2, p_4, p_5, p_9. The center is $c = (1.7609, -0.0478, 0.7696)$, the radius is $r = 4.7867$.

We conclude that the optimal covering sphere is:

$$S^2((1.2394, 0.4475, 0.4306), 4.7648).$$

In fact, according to Theorem 3.11, calculation for $k = 4$ is redundant, because as long as we find a 3 circumscribing feasible ball, it is unique and its corresponding 3 circumscribing enclosing ball is the optimal solution.

6. Closed-form Expression. This section considers the closed-form expression. We started from three points $p_1 = (x_1, y_1)$, $p_2 = (x_2, y_2)$, and $p_3 = (x_3, y_3)$ in $\mathbb{R}^2$.

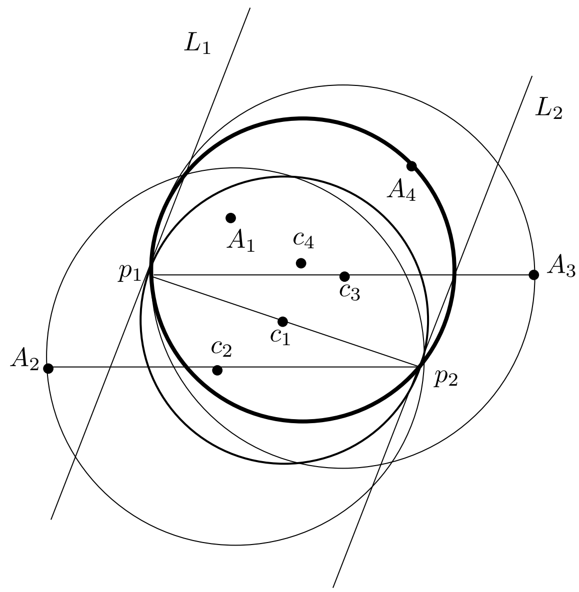

FIG. 3. *Three Points.*

We can choose any two points, say p_1 and p_2. 1. Draw a circle with $\overline{p_1p_2}$ as its diameter. 2. Draw two lines L_1, and L_2, perpendicular to $\overline{p_1p_2}$, and passing p_1 and p_2 respectively. Then $\mathbb{R}^2$ is divided into four parts: 1. inside the circle (say, $p_3 = A_1$); 2. outside of L_1 (say, $p_3 = A_2$); 3. outside of L_2 (say, $p_3 = A_3$); 4. inside the strip bounded by L_1, L_2 (say, $p_3 = A_4$);. Precisely, we define four regions as

$$R_1 = \{(x,y) \Big| (x - \frac{x_1+x_2}{2})^2 + (y - \frac{y_1+y_2}{2})^2 \le (x_2-x_1)^2 + (y_2-y_1)^2\};$$

$$\begin{aligned} R_2 \quad = \quad & \{(x,y)\,|(x_2-x_1)(x-x_1) + (y_2-y_1)(y-y_1), \le 0 \\ & (x_1-x_2)(x-x_2) + (y_1-y_2)(y-y_2) > 0; \}; \end{aligned}$$

$$\begin{aligned} R_3 \quad = \quad & \{(x,y)\,|(x_2-x_1)(x-x_1) + (y_2-y_1)(y-y_1) > 0, \\ & (x_1-x_2)(x-x_2) + (y_1-y_2)(y-y_2) \le 0; \}; \end{aligned}$$

$$\begin{aligned} R_4 \quad = \quad & \big\{(x,y)\,\big|(x - \tfrac{x_1+x_2}{2})^2 + (y - \tfrac{y_1+y_2}{2})^2 < (x_2-x_1)^2 + (y_2-y_1)^2, \\ & (x_2-x_1)(x-x_1) + (y_2-y_1)(y-y_1) > 0, \\ & (x_1-x_2)(x-x_2) + (y_1-y_2)(y-y_2) > 0; \}; \end{aligned}$$

Then the closed form for the center of circumscribing circle is

$$c_x = \begin{cases} \frac{x_1+x_2}{2}, & p_3 \in R_1, \\ \frac{x_3+x_2}{2}, & p_3 \in R_2, \\ \frac{x_1+x_3}{2}, & p_3 \in R_3, \\ \frac{1}{2d}[(y_3-y_2)(x_2^2-x_1^2+y_2^2-y_1^2)+(y_1-y_2)(x_3^2-x_2^2+y_3^2-y_2^2)]; & p_3 \in R_4; \end{cases}$$

$$c_y = \begin{cases} \frac{y_1+y_2}{2}, & p_3 \in R_1, \\ \frac{y_3+y_2}{2}, & p_3 \in R_2, \\ \frac{y_1+y_3}{2}, & p_3 \in R_3, \\ \frac{1}{2d}[(x_2-x_3)(x_2^2-x_1^2+y_2^2-y_1^2)+(x_2-x_1)(x_3^2-x_2^2+y_3^2-y_2^2)], p_3 \in R_4. \end{cases}$$

Here d is defined in (5.1).

In fact, we defined four circles, we denote them by

$$D_1^{123}, \quad D_2^{123}, \quad D_3^{123}, \quad D_4^{123}.$$

The first three circles use three sides of the triangle as their diameters respectively, and the last one is the circle circumscribing the triangle. The analytic form for the centers and radii of these four circles are given as in above. Then according to different case, one is available. Since each case may happen, we are not able to improve them.

Now if we consider four points. We have to construct 16 pairs of analytic forms for the centers and radii of 16 circles: D_1^{123}, $\cdots$, D_4^{123}, D_1^{124}, $\cdots$, D_4^{124}, D_1^{134}, $\cdots$, D_4^{134}, D_1^{234}, $\cdots$, D_4^{234}. Then according to one of the following 16 cases to choose one expression:

$$x_4 \in D_i^{123}; \text{ or } x_3 \in D_i^{124}; \text{ or } x_2 \in D_i^{134}; \text{ or } x_1 \in D_i^{234}; \quad 1 \le i \le 4.$$

In general, we need

$$D_N = \frac{4 \times N!}{3!(n_3)!}$$

different analytic expressions. Even though it is not difficult to write them down, they are difficult to be used.

Similarly, using the formulas in Section 5, it is not difficult to write all possible spheres for four points in $\mathbb{R}^3$. Since there are 6 sides, 4 faces and 1 body of a pyramid, there are 11 possible feasible spheres. In general, we need

$$S_N = \frac{11 \times N!}{4!(n-4)!}$$

different analytic expression. In $\mathbb{R}^n$ it is also easy to write all the analytic expressions, but the total number of expressions is

$$B_N^n = \frac{[2^{n+1} - (n+2)] \times N!}{(n+1)!(N-n-1)!}.$$

When the smallest enclosing ball for N moving bodies is considered, the closed-form expression may be useful. In this case the system could be considered as a switched system. the switching rule is based on adjacency matrix of graph. We refer to [6] for such kind of approaches.

7. Common Quadratic Lyapunov Functions. As an application, we consider the problem of finding a common quadratic Lyapunov function (QLF) for a set of stable matrices. The problem has been extensively studied in the literature, for example, [1, 2, 3].

The problem is described as follows. We say a matrix is stable if its eigenvalues lie in the open left hand complex plane, i.e. $\lambda \in \sigma(A)$ implies that $Real(\lambda) < 0$, where $\sigma(A)$ is the set of eigenvalues of A. Given a set of stable matrices A_i, $i = 1, \cdots, N$, we ask if there is a a positive definite matrix $P > 0$ such that

$$PA_i + A_i^T P < 0, \quad \forall i. \tag{7.1}$$

If so then this P acts as a Lyapunov function for each system $\dot{x} = A_i x$ and P is referred to as a common quadratic Lyapunov function for the set of matrices A_i.

Let C be the center of the points in $\mathbb{R}^{n^2}$ represented by the matrices A_i. Assume all A_i, $i = 1, \cdots, N$ share a common QLF, P. Then we claim that C is stable. In fact, from the discussion in Section 3 one sees easily that C is in the convex hell of A_i. That is,

$$C = \sum_{i=1}^N \lambda_i A_i, \quad \lambda_i \geq 0 \text{ and } \sum_{i=1}^N \lambda_i = 1.$$

Since

$$PA_i + A_i^T P < 0, \quad i = 1, \cdots, N,$$

multiplying each by non-negative λ_i and sum up, we have

$$\sum_{i=1}^N \lambda_i [PA_i + A_i^T P] = PC + C^T P < 0.$$

We propose the following test for existence of common QLF: Find the center of the smallest inclosing ball, Check if C is stable. If “Not”, the set doesn't have common

QLF. Otherwise we can use C to find a positive P by solving (following Gantmacher, [5])

$$PC + C^T P = -I. \tag{7.2}$$

Finally, check whether this P is a common QLF.

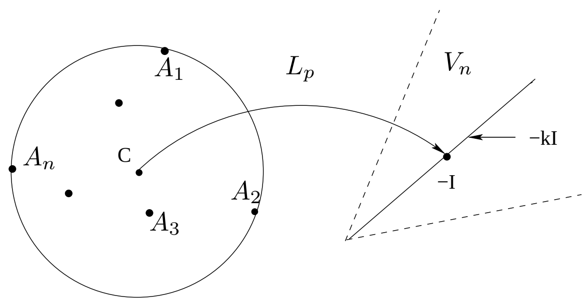

FIG. 4. *Lyapunov mapping L_p.*

Of course, this is a sufficient condition for finding a common QLF. If the last test fails, we can not say anything about the problem. But we would like to explain that geometrically it is reasonable. It is well known that the set of negative definite matrices forms an open convex cone $V_n \subset \mathbb{R}^{n^2}$ (precisely, in the subset of symmetric matrices of $\mathbb{R}^{n^2}$ with subset topology.) We claim that $-kI$, $k > 0$ is the center of the cone V_n in the following sense: Refer to Fig. 4, assume $Q < 0$, the "robust radius", $r(Q)$, for Q to remain negative is defined as

$$r(Q) = \max_r \{Q + Z < 0 \mid \|Z\| < r\}.$$

Now it is easy to show that when $\|Q\|$ is constant, then $-kI$ has largest robust radius. That is,

$$max_{\|Q\|=cons.} r(Q) = r(-kI).$$

Now we are looking for a $P > 0$, such that the Lyapunov mapping of P, $L_P : \mathbb{R}^{n^2} \to \mathbb{R}^{n^2}$, defined as $L_P : X \mapsto PX + X^T P$, can map all the points A_i, $i = 1, \cdots, N$ into the cone V_n. It is very reasonable to cover all points by a smallest ball and maps the center of the ball, C, into the center of the cone V_n, $-I$. (It is obvious that a constant $k > 0$ doesn't affect P, because kP plays same role as P.)

The following example demonstrate this procedure.

EXAMPLE 7.1. Consider the following three matrices

$$M_1 = \begin{pmatrix} -4 & -1 & -1 & 1 \\ 1 & -4 & 2 & 3 \\ 2 & 1 & -3 & 1 \\ -1 & 0 & 2 & -3 \end{pmatrix}; \quad M_2 = \begin{pmatrix} -3 & -2 & 1 & -1 \\ 2 & -3 & 1 & 0 \\ 2 & 2 & -5 & 2 \\ 2 & 1 & -1 & -2 \end{pmatrix};$$

$$M_3 = \begin{pmatrix} -3 & -1 & -1 & 2 \\ 1 & -3 & 1 & 1 \\ 2 & 2 & -4 & 2 \\ 1 & 1 & -2 & -2 \end{pmatrix}.$$

First, the longest distance is $\overline{M_1, M_2}$, and the ball in $\mathbb{R}^{16}$ with $\overline{M_1, M_2}$ as its diameter has radius $r = 3.4641$. But the distance between the center $C = (M_1 + M_2)/2$ and M_2 is $\overline{M_3 C} = 3.7417$, so there is no $k = 1$ feasible circumscribing ball.

Using formulas (4.15) and (4.16), we have A as

$$A = \begin{pmatrix}
1 & -1 & 2 & -2 & 1 & 1 & -1 & -3 & 0 & 1 & -2 & 1 & 3 & 1 & -3 & 1 \\
0 & 1 & -2 & 3 & -1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & -1 & 0 & -1 & 0 \\
-1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
7 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-6 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-4 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
-5 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
5 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
3 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
-5 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
3 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix}$$

$$B = (-1, -5.5, 5, -2, 7, -19.5, 21, 14, 6.5, -7, -8.5, 17.5, -16.5, -9.5, 18.5, -6.5)^T$$

Then we can solve the center C by $C = A^{-1}B$ as

$$C = \begin{pmatrix} -3.4808 & -0.6346 & -0.8846 & -0.8462 \\ 0.5962 & -4.2308 & 0.5192 & 0.5577 \\ 1.0962 & 0.6731 & -4.7885 & 0.5385 \\ -0.2692 & -0.3077 & -0.4615 & -3.3077 \end{pmatrix}.$$

Using this C, P can be found by solving equation (7.1) following [5]

$$P = \begin{pmatrix} 0.1489 & -0.0020 & 0.0052 & -0.0256 \\ -0.0020 & 0.1203 & 0.0155 & 0.0035 \\ 0.0052 & 0.0155 & 0.1078 & 0.0001 \\ -0.0256 & 0.0035 & 0.0001 & 0.1529 \end{pmatrix}.$$

Note that a necessary condition for the existence of common QLF is C is stable. Then it is well known that (7.1) has unique solution.

Finally, we can check that

$$PM_1 + M_1^T P = \begin{pmatrix} -1.1236 & 0.0204 & -0.0094 & 0.1777 \\ 0.0204 & -0.9270 & 0.2438 & 0.3753 \\ -0.0094 & 0.2438 & -0.5948 & 0.4973 \\ 0.1777 & 0.3753 & 0.4973 & -0.9473 \end{pmatrix} < 0.$$

$$PM_2 + M_2^T P = \begin{pmatrix} -0.9832 & -0.0222 & 0.3777 & 0.3023 \\ -0.0222 & -0.6444 & 0.1961 & 0.2197 \\ 0.3777 & 0.1961 & -1.0368 & 0.0348 \\ 0.3023 & 0.2197 & 0.0348 & -0.5601 \end{pmatrix} < 0.$$

$$PM_3 + M_3^T P = \begin{pmatrix} -1.0140 & 0.0204 & -0.0365 & 0.0102 \\ 0.0204 & -0.9963 & -0.0078 & 0.0199 \\ -0.0365 & -0.0078 & -1.0255 & 0.0154 \\ 0.0102 & 0.0199 & 0.0154 & -0.9642 \end{pmatrix} < 0.$$

Hence, P is shown to be a common QLF for M_1, M_2, M_3, following [5].

This example shows that constructing the higher dimensional smallest enclosing ball is sometimes useful. In the aforementioned example the dimension n is 16.

8. Conclusion. The problem of smallest enclosing balls for a finite number points was considered. Theoretically, it was proved that the feasible circumscribing ball with smallest dimension of the affine plane, spanned by the points on its boundary, determines the smallest enclosing ball. It characterized the smallest enclosing ball and is more meaningful in searching higher dimensional smallest enclosing balls. Several formulas were deduced to calculate the circumscribing sphere for given points. Some numerical examples were given to substantiate the theoretical results and the formulas. Finally, as an application for searching smallest enclosing ball in higher dimensional space, the common QLF of a set of stable matrices was investigated. What remains to be studied is how the dimension of the space can be reduced in such a symmetric case.

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