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Quadratic covariation and an extension of Itô's formula

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Let X be a standard Brownian motion. We show that for any locally square integrable function f the quadratic covariation $[f(X), X]$ exists as the usual limit of sums converging in probability. For an absolutely continuous function F with derivative f, Itô's formula takes the form $F(X_t) = F(X_0) + \int_0^t f(X_s)\,\mathrm{d}X_s + \frac{1}{2}[f(X), X]_t$. This is extended to the time-dependent case. As an example, we introduce the local time of Brownian motion at a continuous curve.

Keywords: Dirichlet processes; Itô's formula; local time; quadratic covariation; Stratonovich integral

1. Introduction

Let $X = (X_t)_{0 \leq t \leq 1}$ be a standard Brownian motion, and let F be an absolutely continuous function with locally square integrable derivative f. Our purpose is to prove the following extension of Itô's formula:

$$F(X_t) = F(X_0) + \int_0^t f(X_s)\,\mathrm{d}X_s + \tfrac{1}{2}[f(X), X]_t, \tag{1.1}$$

where

$$[f(X), X]_t = \lim_{n \to \infty} \sum_{t_i \in D_n, t_i < t} \{f(X_{t_{i+1}}) - f(X_{t_i})\}(X_{t_{i+1}} - X_{t_i}). \tag{1.2}$$

denotes the *quadratic covariation* of the processes $f(X)$ and X. In particular, we are going to show that the quadratic covariation exists for any locally square integrable function f, as a limit in probability along a sequence of partitions D_n of the time interval $[0, 1]$.

If f is absolutely continuous with derivative f' then the quadratic covariation is given by

$$[f(X), X]_t = \int_0^t f'(X_s)\,\mathrm{d}s, \tag{1.3}$$

and so (1.1) reduces to Itô's formula in its classical form (Itô 1994). If f is locally bounded then we have

$$[f(X), X]_t = -\int f(a) d_a L_t^a \tag{1.4}$$

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where L_t^a denotes the local time of Brownian motion at level a, and so (1.1) reduces to the formula of Bouleau and Yor (1981).

The quadratic covariation in (1.2) admits the representation

$$[f(X), X]_t = \int_0^t f(X_s)\,\mathrm{d}^* X_s - \int_0^t f(X_s)\,\mathrm{d}X_s \tag{1.5}$$

in terms of a backward and a forward stochastic integral, and this is the key to our existence result. In fact, the existence of quadratic covariation will follow from an approximation of these integrals by forward and backward sums. In Section 2 we motivate our approach by looking at the discrete version of (1.1) along a fixed partition. In Section 3 we prove the basic existence result for the quadratic variation $[f(X, \cdot), X]$ in the general time-dependent case where $f(x, t)$ is locally square integrable in x and satisfies a mild continuity condition in t. In general, the quadratic covariation will not be of bounded variation. But we show that it is always a process of *zero energy*, i.e. a process with continuous paths of zero quadratic variation. In Section 4 we derive our Itô formula (1.1). Thus, the process of zero energy appearing in Fukushima's (1980, Chapter 5) decomposition of the Dirichlet process $F(X)$ is identified as a quadratic covariation. Our formula also shows that the Stratonovich integral can be defined on the same level of generality as the Itô integral, without additional restrictions on the function f. In Section 5 we extend these results to the time-dependent case where $F(x, t)$ is absolutely continuous in x and where the derivative $f(x, t)$ satisfies our conditions for the existence of $[f(X, \cdot), X]$. In this case the Itô formula takes the form

$$F(X_t, t) = F(X_0, 0) + \int_0^t f(X_s, s)\,\mathrm{d}X_s + \tfrac{1}{2}[f(X, \cdot), X]_t + \int_0^t F(X_s, \mathrm{d}s), \tag{1.6}$$

where

$$\int_0^t F(X_s, \mathrm{d}s) \equiv \lim_{n \to \infty} \sum_{t_i \in D_n, t_i < t} F(X_{t_{i+1}}, t_{i+1}) - F(X_{t_{i+1}}, t_i) \tag{1.7}$$

exists as a limit in probability. If $F(x, \cdot)$ is absolutely continuous in t with derivative $F_t(x, \cdot)$ then the last term in (1.6) takes the usual form

$$\int_0^t F(X_s, \mathrm{d}s) = \int_0^t F_t(X_s, s)\,\mathrm{d}s. \tag{1.8}$$

In the case where $f(x, t) = I_{[a(t), \infty)}(x)$ for some continuous function $a(\cdot)$, the existence of the quadratic covariation $[f(X, \cdot), X]$ amounts to a construction of the local time of Brownian motion at a continuous curve, and (1.6) may be viewed as a time-dependent version of the Tanaka formula.

The idea of using time reversal and the duality (1.5) for extended versions of the Stratonovich integral and of Itô's formula also appears in recent work by Lyons and Zhang (1994) and by Russo and Vallois (1994), under stronger regularity conditions on the function f. Section 5 shows that, in contrast to the Dirichlet space techniques used in Lyons and Zhang (1994) and Fukushima (1980), our approach admits a straightforward extension to the time-dependent case. Extensions to higher dimensions and applications to the existence of solutions of Stratonovich stochastic differential equations will be discussed elsewhere.

2. Preliminaries

Let F be an absolutely continuous function of the form

$$F(x) = F(0) + \int_0^x f(y)\,\mathrm{d}y \tag{2.1}$$

where f is locally square integrable. In order to motivate our approach, let us first see how the structure of our Itô formula (1.1) appears in discrete time. Let D_n denote a partition of the form $0 = t_0 < \ldots < t_k = 1$ and let us write $|D_n| = \max|t_{i+1} - t_i|$. An increment

$$F(X_{t_{i+1}}) - F(X_{t_i}) = \int_{X_{t_i}}^{X_{t_{i+1}}} f(y)\,\mathrm{d}y \tag{2.2}$$

can be written as

$$f(X_{t_i})(X_{t_{i+1}} - X_{t_i}) + \int_{X_{t_i}}^{X_{t_{i+1}}} \{f(y) - f(X_{t_i})\}\,\mathrm{d}y \tag{2.3}$$

and also as

$$f(X_{t_i})(X_{t_{i+1}} - X_{t_i}) + \{f(X_{t_{i+1}}) - f(X_{t_i})\}(X_{t_{i+1}} - X_{t_i}) + \int_{X_{t_i}}^{X_{t_{i+1}}} \{f(y) - f(X_{t_{i+1}})\}\,\mathrm{d}y. \tag{2.4}$$

Averaging both expressions and summing the increments up to time $t \in D_n$, we obtain

$$F(X_t) - F(X_0) = \sum_{\substack{t_i \in D_n \\ t_i < t}} f(X_{t_i})(X_{t_{i+1}} - X_{t_i}) + \tfrac{1}{2}Q_t^n + R_t^n \tag{2.5}$$

with discrete quadratic covariation

$$Q_t^n \equiv \sum_{\substack{t_i \in D_n \\ t_i < t}} \{f(X_{t_{i+1}}) - f(X_{t_i})\}(X_{t_{i+1}} - X_{t_i}) \tag{2.6}$$

and remainder terms

$$R_t^n \equiv \sum_{\substack{t_i \in D_n \\ t_i < t}} \int_{X_{t_i}}^{X_{t_{i+1}}} [f(y) - \tfrac{1}{2}\{f(X_{t_i}) + f(X_{t_{i+1}})\}]\,\mathrm{d}y. \tag{2.7}$$

The first sum in (2.5) is the discrete analogue of the stochastic integral in (1.1), and Q_t^n is the discrete version of the quadratic covariation in (1.1). Let us now pass to the limit along a sequence of partitions D_n such that $|D_n|$ converges to 0. Let us say that a sequence of processes $Y^n = (Y_t^n)_{0 \le t \le 1}$ converges *uniformly in probability* (in u.p.) to some process $Y = (Y_t)_{0 \le t \le 1}$ if the supremum norm of the difference converges to 0 in probability. The proof of our Itô formula (1.1) will involve convergence in u.p. of the following terms:

$$\lim_{n \to \infty} \sum_{\substack{t_i \in D_n \\ t_i < t}} f(X_{t_i})(X_{t_{i+1}} - X_{t_i}) = \int_0^t f(X_s)\,\mathrm{d}X_s, \tag{2.8}$$

$$\lim_{n\to\infty} Q_t^n = [f(X), X]_t, \tag{2.9}$$

$$\lim_{n\to\infty} R_t^n = 0. \tag{2.10}$$

Remark 2.1
In each case, convergence in u.p. is known to hold under certain regularity conditions on the function f:

(a) For $f \in C$ the convergence in (2.8) holds in u.p. (Protter 1990, p. 57).
(b) For $f \in C^1$ one can proceed in a strictly pathwise manner (cf. Föllmer 1981). In fact, almost all Brownian paths have the property that the discrete measures

$$\sum_{t_i \in D_n} (X_{t_{i+1}} - X_{t_i})^2 \delta_{t_i}$$

converge weakly to the uniform distribution on $[0, 1]$. But for any such path one can verify the following facts, without any further use of probabilistic arguments. Writing

$$Q_t^n = \sum_{\substack{t_i \in D_n \\ t_i < t}} f'(\xi_i)(X_{t_{i+1}} - X_{t_i})^2 \tag{2.11}$$

for some ξ_i between $X_{t_{i+1}}$ and X_{t_i}, we get convergence of (2.11) to the covariation

$$[f(X), X]_t = \int_0^t f'(X_s)\,\mathrm{d}s. \tag{2.12}$$

Moreover, the stochastic integral in (2.8) can be defined pathwise as the limit of the sums in (2.8), and the remainder terms in (2.10) converge to 0 (for the proofs, see Föllmer 1981).
(c) For a function $f \in C^2$ the convergence of the remainder terms R_t^n to 0 can also be seen from the trapezoidal rule

$$\int_a^b [f(y) - \tfrac{1}{2}\{f(a) + f(b)\}]\,\mathrm{d}y = -\tfrac{1}{12}(b-a)^3 f''(\xi). \tag{2.13}$$

This implies

$$|R_t^n| \le \tfrac{1}{12} \max |f''(X_t)| \sum_{\substack{t_i \in D_n \\ t_i < t}} |X_{t_{i+1}} - X_{t_i}|^3, \tag{2.14}$$

hence convergence to 0 for any continuous path with bounded quadratic variation.

The main point of this paper is to show that no such regularity conditions are needed and that each of the three convergence statements (2.8)–(2.10) holds in u.p. for any locally square integrable function f. Since we can write

$$Q_t^n = \sum_{\substack{t_i \in D_n \\ t_i < t}} f(X_{t_{i+1}})(X_{t_{i+1}} - X_{t_i}) - \sum_{\substack{t_i \in D_n \\ t_i < t}} f(X_{t_i})(X_{t_{i+1}} - X_{t_i}), \tag{2.15}$$

the existence of the quadratic covariation in (2.9) and its representation (1.5) as the difference of a backward and forward stochastic integral will both follow from the discrete approximation (2.8) of the forward integral and from a corresponding approximation of the backward integral by the first sum in (2.15). Let us now recall the definition of the stochastic backward integral.

We assume without loss of generality that our Brownian motion $X = (X_t)_{0 \le t \le 1}$ is defined as the coordinate process on the canonical path space $\Omega = C[0, 1]$ with Wiener measure P. We will have need of the time reversal operator $(RX)_t = X_{1-t}$ on Ω. Note that the law P^* of RX is the law of the Brownian bridge process with initial distribution $N(0, 1)$ and terminal value 0. Thus, the coordinate process X is a semi-martingale under P^* with decomposition

$$X_t = X_0 + W_t + \int_0^t \frac{-X_s}{1-s}\,\mathrm{d}s \tag{2.16}$$

where W is a Brownian motion with respect to P^*. For a measurable function $f(x, t)$ on $\mathbb{R} \times [0, 1]$ such that

$$\begin{aligned} &\mathrm{E}^*\left[\int_0^1 \left\{ f^2(X_s, 1-s) + |f(X_s, 1-s)|\frac{|X_s|}{1-s}\right\} \mathrm{d}s\right] \\ &= \mathrm{E}\left[\int_0^1 \left\{ f^2(X_s, s) + |f(X_s, s)|\frac{|X_s|}{s}\right\} \mathrm{d}s\right] < \infty \end{aligned} \tag{2.17}$$

the stochastic integral

$$\int_0^t f(X_s, 1-s)\,\mathrm{d}X_s = \int_0^t f(X_s, 1-s)\,\mathrm{d}W_s + \int_0^t f(X_s, 1-s)\frac{-X_s}{1-s}\,\mathrm{d}s \tag{2.18}$$

is well defined for any $t \in [0, 1]$ with respect to P^*. Thus, the following *backward integral*

$$\int_0^t f(X_s, s)\,\mathrm{d}^*X_s \equiv -\left(\int_{1-t}^1 f(X_s, 1-s)\,\mathrm{d}X_s\right) \circ \mathrm{R} \tag{2.19}$$

is well defined in terms of Wiener measure P.

3. Existence of quadratic covariation

Our purpose is to show that the quadratic covariation

$$[f(X), X]_t = \lim_{n\to\infty} \sum_{t_i \in D_n,\, t_i \le t} \{f(X_{t_{i+1}}) - f(X_{t_i})\}(X_{t_{i+1}} - X_{t_i}) \tag{3.1}$$

exists as a limit in u.p., beyond the usual cases where $f(X)$ is a semi-martingale. In order to illustrate the basic idea we begin with the following preliminary result:

Proposition 3.1 Let f be a continuous function. Then the quadratic covariation $[f(X), X]$ exists as a continuous process, and satisfies

$$[f(X), X]_t = \int_0^t f(X_s)\,\mathrm{d}^*X_s - \int_0^t f(X_s)\,\mathrm{d}X_s. \tag{3.2}$$

Proof

Since X is a semi-martingale and $f(X)$ is an adapted continuous process, we know that

$$\lim_{n\to\infty} \sum_{t_i \in D_n, t_i \leq t} f(X_{t_i})(X_{t_{i+1}} - X_{t_i}) = \int_0^t f(X_s)\,\mathrm{d}X_s \tag{3.3}$$

with convergence in u.p. (cf. Protter 1990, p. 57). On the other hand we have

$$\lim_{n\to\infty} \sum_{\substack{t_i \in D_n \\ t_i \leq t}} f(X_{t_{i+1}})(X_{t_{i+1}} - X_{t_i}) = \lim_{n\to\infty} \left\{ \sum_{\substack{t_i \in D_n \\ t_i \leq t}} f(X_{1-t_{i+1}})(X_{1-t_{i+1}} - X_{1-t_i}) \right\} \circ \mathrm{R}$$

$$= \lim_{n\to\infty} -\left\{ \sum_{\substack{s_i \in D_n \\ s_{i+1} \geq 1-t}} f(X_{s_i})(X_{s_{i+1}} - X_{s_i}) \right\} \circ \mathrm{R}.$$

Since the time-reversed process $X \circ \mathrm{R}$ is also a semi-martingale (as shown by (2.16)), we have convergence in u.p. of the above sums (again by Protter 1990, p. 57) to

$$-\left\{ \int_{1-t}^1 f(X_s)\,\mathrm{d}X_s \right\} \circ \mathrm{R} = \int_0^t f(X_s)\,\mathrm{d}^* X_s,$$

hence

$$\lim_{n\to\infty} \sum_{t_i \in D_n, t_i \leq t} f(X_{t_{i+1}})(X_{t_{i+1}} - X_{t_i}) = \int_0^t f(X_s)\,\mathrm{d}^* X_s. \tag{3.4}$$

If we now subtract (3.3) from (3.4) we obtain the existence of $[f(X), X]$ as a limit of sums in u.p., as well as its identification as the difference of two stochastic integrals. Since both stochastic integral processes are continuous, so also is $[f(X), X]$.

Remark 3.1

The process $[f(X), X]$, while continuous a.s., need not be of bounded variation as it is in the usual theory when f is taken to be $\mathcal{C}^1$. Indeed, if f is continuous but not of bounded variation, then its primitive F is not the difference of two convex functions, and therefore $F(X)$ is *not* a semi-martingale (cf. Protter 1990, p. 162). Theorem 4.1 will show that $F(X_t) - \frac{1}{2}[f(X), X]_t$ is a local martingale. Thus, $[f(X), X]$ cannot have paths of bounded variation if f is not a function of bounded variation. However, Theorem 3.5 will show that the quadratic covariation is always a process of 'zero energy'.

Let us now show that the quadratic covariation $[f(X), X]$ exists for any f which is locally square integrable. The idea is quite simple. As in the proof of (3.2), it is enough to show that the forward and backward stochastic integral can each be approximated by the simple sums in (3.3) and (3.4), without any additional smoothing. The existence of quadratic covariation then follows by subtracting these sums.

Proposition 3.2 (a) Let f be a square integrable function on $\mathbb{R}^1$. Then

$$\lim_{n\to\infty} \sum_{0<t_i\in D_n,\, t_i\le t} f(X_{t_i})(X_{t_{i+1}} - X_{t_i}) = \int_0^t f(X_s)\,\mathrm{d}X_s \tag{3.5}$$

in $L^2(P)$ and

$$\lim_{n\to\infty} \sum_{t_i\in D_n,\, t_i\le t} f(X_{t_{i+1}})(X_{t_{i+1}} - X_{t_i}) = \int_0^t f(X_s)\,\mathrm{d}^* X_s \tag{3.6}$$

in $L^1(P)$.

(b) If f is locally square integrable then convergence in (3.5) and (3.6) holds in u.p., and so the quadratic covariation exists in u.p. and satisfies

$$[f(X), X]_t = \int_0^t f(X_s)\,\mathrm{d}^* X_s - \int_0^t f(X_s)\,\mathrm{d}X_s. \tag{3.7}$$

Instead of proving Proposition 3.2, we pass directly to the *time-dependent* case and prove the corresponding general version. Let $f(x,t)$ be a measurable function on $\mathbb{R}^1 \times [0,1]$. The following proof shows that the analogue of part (a) holds if $f(\cdot,t)$ is square integrable and continuous in t as a map from $[0,1]$ to $L^2(\mathbb{R}^1)$. The analogue of part (b) involves continuity of $f(\cdot,t)$ as a map from $[0,1]$ to $L^2_{\mathrm{loc}}(\mathbb{R}^1)$ in the sense that $f(\cdot,t)I_K(\cdot) \in L^2(\mathbb{R}^1)$ is norm-continuous in t for any compact set K.

Theorem 3.3 Suppose that $f(\cdot,t)$ is locally square integrable and that

$$f(\cdot,t) \text{ is continuous in } t \tag{3.8}$$

as a map from $[0,1]$ to $L^2_{\mathrm{loc}}(\mathbb{R}^1)$. Then the quadratic covariation

$$[f(X,\cdot), X]_t = \lim_{n\to\infty} \sum_{t_i\in D_n,\, 0<t_i\le t} \{f(X_{t_{i+1}}, t_{i+1}) - f(X_{t_i}, t_i)\}(X_{t_{i+1}} - X_{t_i}) \tag{3.9}$$

exists as a limit in u.p., and

$$[f(X,\cdot), X]_t = \int_0^t f(X_s, s)\,\mathrm{d}^* X_s - \int_0^t f(X_s, s)\,\mathrm{d}X_s. \tag{3.10}$$

Proof

By the usual localization argument, it is enough to consider the case where $f(\cdot,t)$ is square integrable and continuous in t as a map from $[0,1]$ to $L^2(\mathbb{R}^1)$. It is also enough to consider the case $t = 1$.

First, consider the processes ϕ and ϕ_n defined by

$$\phi(\omega,t) = f(X_t(\omega), t) \tag{3.11}$$

and

$$\phi_n(\omega,t) = \sum_{0<t_i\in D_n} f(X_{t_i}(\omega), t_i) I_{(t_i, t_{i+1}]}(t). \tag{3.12}$$

In order to prove

$$\lim_{n\to\infty}\sum_{0<t_i\in D_n} f(X_{t_i},t_i)(X_{t_{i+1}}-X_{t_i})=\int_0^1 f(X_s,s)\,\mathrm{d}X_s \tag{3.13}$$

in $L^2(P)$, we have to show that

$$\|\phi-\phi_n\|_2\equiv \mathrm{E}\left[\int_0^1(\phi(\omega,t)-\phi_n(\omega,t))^2\,\mathrm{d}t\right]^{1/2}$$

converges to 0. Let us use the notation

$$p_t(x)\equiv\frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{x^2}{2t}\right).$$

Then

$$\|\phi\|_2^2=\mathrm{E}\left\{\int_0^1 f^2(X_t,t)\,\mathrm{d}t\right\}=\int_0^1\int f^2(x,t)\,p_t(x)\,\mathrm{d}x\,\mathrm{d}t$$

$$\le\frac{1}{\sqrt{2\pi}}\int_0^1\int f^2(x,t)\,\mathrm{d}x\frac{1}{\sqrt{t}}\,\mathrm{d}t\equiv\|f\|_2^2 \tag{3.14}$$

and

$$\|\phi_n\|_2^2=\sum_{0<t_i\in D_n}\int f^2(x,t_i)\,p_{t_i}(x)\,\mathrm{d}x(t_{i+1}-t_i)$$

$$\le\frac{1}{\sqrt{2\pi}}\sum_{0<t_i\in D_n}\int f^2(x,t_i)\,\mathrm{d}x\frac{1}{\sqrt{t_i}}(t_{i+1}-t_i),$$

hence

$$\overline{\lim_n}\,\|\phi_n\|_2^2\le\|f\|_2^2 \tag{3.15}$$

due to the continuity of $\int f(x,t)^2\,\mathrm{d}x$ in t. Now take a continuous function g on $\mathbb{R}\times[0,1]$ with compact support such that $\|g-f\|_2\le\epsilon$ and define the corresponding processes ψ and ψ_n as in (3.11) and (3.12). Then

$$\|\phi-\phi_n\|_2\le\|\phi-\psi\|_2+\|\psi-\psi_n\|_2+\|\psi_n-\phi_n\|_2.$$

Since $\lim\|\psi-\psi_n\|_2=0$ by continuity of g and by Lebesgue's theorem, the two preceding estimates (3.14) and (3.15) applied to $f-g$ imply

$$\overline{\lim_n}\,\|\phi-\phi_n\|_2\le 2\|f-g\|_2\le 2\epsilon,$$

hence

$$\lim_n\|\phi-\phi_n\|_2=0. \tag{3.16}$$

Second, the convergence

$$\lim_{n\to\infty}\sum_{t_i\in D_n,\, t_i\le t} f(X_{t_{i+1}}, t_{i+1})(X_{t_{i+1}} - X_{t_i}) = \int_0^t f(X_s, s)\,\mathrm{d}^* X_s \tag{3.17}$$

in $L^1(P)$ for $t = 1$ is equivalent to the convergence

$$\lim_n \sum_{0<t_i\in D_n} f(X_{t_i}, 1 - t_i)(X_{t_{i+1}} - X_{t_i}) = \int_0^1 f(X_t, 1 - t)\,\mathrm{d}X_t \tag{3.18}$$

in $L^1(P^*)$, where P^* is the distribution of the time-reversed process $X \circ \mathrm{R}$. Recall the decomposition (2.16) of X under P^*, and let us first check that

$$\lim_n \sum_{0<t_i\in D_n} f(X_{t_i}, 1 - t_i)(W_{t_{i+1}} - W_{t_i}) = \int_0^1 f(X_t, 1 - t)\,\mathrm{d}W_t \tag{3.19}$$

in $L^2(P^*)$. But (3.19) is equivalent to the convergence of

$$\mathrm{E}^*\left[\int_0^1\left\{ f(X_t, 1-t) - \sum_{0<t_i\in D_n} f(X_{t_i}, 1-t_i) I_{(t_i, t_{i+1}]}(t)\right\}^2 \mathrm{d}t\right]$$

$$= \mathrm{E}\left[\int_0^1\left\{ f(X_s, s) - \sum_{1>s_{i+1}\in D_n} f(X_{s_{i+1}}, s_{i+1}) I_{[s_i, s_{i+1})}(s)\right\}^2 \mathrm{d}s\right]$$

to 0, and this follows as in the first part of this proof. In order to prove (3.18) it remains to show, due to (2.16), that

$$\lim_n \sum_{0<t_i\in D_n} f(X_{t_i}, 1 - t_i)\int_{t_i}^{t_{i+1}} \frac{X_s}{1-s}\,\mathrm{d}s = \int_0^1 f(X_t, 1-t)\frac{X_t}{1-t}\,\mathrm{d}t$$

in $L^1(P^*)$, or equivalently,

$$\lim_n \sum_{1>s_{i+1}\in D_n} f(X_{s_{i+1}}, s_{i+1})\int_{s_i}^{s_{i+1}} \frac{X_s}{s}\,\mathrm{d}s = \int_0^1 f(X_s, s)\frac{X_s}{s}\,\mathrm{d}s \tag{3.20}$$

in $L^1(P)$. For a measurable function $\psi(\omega, t)$ on $\Omega \times [0, 1]$ we introduce the norm

$$\|\psi\|_1 \equiv \mathrm{E}\left(\int_0^1 |\psi(\omega, s)|\frac{|X_s|}{s}\,\mathrm{d}s\right). \tag{3.21}$$

For the function ϕ defined in (3.11) we get

$$\begin{aligned}
\|\phi\|_1 &\equiv \mathrm{E}\left(\int_0^1 |\phi(\omega, s)|\frac{|X_s|}{s}\,\mathrm{d}s\right)\\
&\le \int_0^1 \frac{1}{s}\mathrm{E}\{f^2(X_s, s)\}^{1/2}\mathrm{E}(X_s^2)^{1/2}\,\mathrm{d}s\\
&= \int_0^1 \frac{1}{s}\left(\int f^2(x, s)\,p_s(x)\,\mathrm{d}x\right)^{1/2} s^{1/2}\,\mathrm{d}s\\
&\le \left(\frac{1}{2\pi}\right)^{1/4}\int_0^1\left(\int f^2(x, s)\,\mathrm{d}x\right)^{1/2} s^{-3/4}\,\mathrm{d}s \equiv \|f\|_1.
\end{aligned}$$

Similarly, we see that

$$\phi_n^*(\omega, s) \equiv \sum_{1 > s_{i+1} \in D_n} f(X_{s_{i+1}}, s_{i+1}) I_{[s_i, s_{i+1})}(s)$$

satisfies

$$\begin{aligned} ||\phi_n^*||_1 &= \sum_{1 > s_{i+1} \in D_n} \mathrm{E}\left\{ |f(X_{s_{i+1}}, s_{i+1})| \int_{s_i}^{s_{i+1}} \frac{|X_s|}{s} \mathrm{d}s \right\} \\ &\leq \left(\frac{1}{2\pi}\right)^{1/4} \sum_{1 > s_i \in D_n} \left(\int f^2(x, s_{i+1}) \, \mathrm{d}x \right)^{1/2} s_{i+1}^{-1/4} \int_{s_i}^{s_{i+1}} s^{-1/2} \, \mathrm{d}s \\ &\leq \left(\frac{1}{2\pi}\right)^{1/4} \sum_{1 > s_i \in D_n} \left(\int f^2(x, s_{i+1}) \, \mathrm{d}x \right)^{1/2} s_{i+1}^{-1/4} s_i^{-1/2} (s_{i+1} - s_i), \end{aligned}$$

hence

$$\overline{\lim} \, ||\phi_n^*||_1 \leq \left(\frac{1}{2\pi}\right)^{1/4} \int_0^1 \left(\int f^2(x, s) \, \mathrm{d}x \right)^{1/2} s^{-3/4} \, \mathrm{d}s = ||f||_1. \tag{3.22}$$

Now we choose a function $g \in C(\mathbb{R}^1 \times [0, 1])$ with compact support such that $||g - f||_1 \leq \epsilon$ and conclude as in the first part of this proof that

$$\lim_n ||\phi - \phi_n^*||_1 = 0. \tag{3.23}$$

This implies (3.20), and so we have shown (3.17).

Finally, subtracting (3.13) from (3.17), we obtain (3.9).

Even though $f(X, \cdot)$ is in general not a semi-martingale, the existence of quadratic covariation $[f(X, \cdot), X]$ implies the *existence* of the stochastic integral

$$\int_0^t X_s \, \mathrm{d}f(X_s, s) = \lim_{n \to \infty} \sum_{t_i \in D_n, 0 < t_i \leq t} X_{t_i} \{ f(X_{t_{i+1}}, t_{i+1}) - f(X_{t_i}, t_i) \} \tag{3.24}$$

and the following integration by parts formula:

Corollary 3.4 Under the assumptions of Theorem 3.3 we have

$$f(X_t, t) X_t = f(X_0, 0) X_0 + \int_0^t X_s \, \mathrm{d}f(X_s, s) + \int_0^t f(X_s, s) \, \mathrm{d}X_s + [f(X, \cdot), X]_t. \tag{3.25}$$

Proof

This follows from the relation

$$\begin{aligned} f(X_{t_{i+1}}, t_{i+1}) X_{t_{i+1}} - f(Xt_i, t_i) X_{t_i} = X_{t_i} \{ & f(X_{t_{i+1}}, t_{i+1}) - f(X_{t_i}, t_i) \} \\ & + f(X_{t_i}, t_i)(X_{t_{i+1}} - X_{t_i}) \\ & + \{ f(X_{t_{i+1}}, t_{i+1}) - f(X_{t_i}, t_i) \} (X_{t_{i+1}} - X_{t_i}) \end{aligned}$$

and the convergence in (3.9) and (3.13). □

Remark 3.2

(a) If we average (3.13) and (3.17) and use (3.10) then we obtain

$$\lim_{n\to\infty} \sum_{t_i\in D_n, t_i\le t} \tfrac{1}{2}\{f(X_{t_{i+1}}, t_{i+1}) + f(X_{t_i}, t_i)\}(X_{t_{i+1}} - X_{t_i})$$

$$= \int_0^t f(X_s, s)\,\mathrm{d}X_s + \tfrac{1}{2}[f(X,\cdot), X]_t \tag{3.26}$$

in u.p. This will be used in our definition of the Stratonovich integral (Definition 4.1 and equation (5.9)).

(b) Suppose that f_n converges to f in the norms $\|\cdot\|_i$ $(i = 1, 2)$ used in the proof of Theorem 3.3. The proof shows that the corresponding forward and backward stochastic integrals converge, and so (3.10) implies

$$[f(X,\cdot), X]_t = \lim_{n\to\infty} [f_n(X,\cdot), X]_t. \tag{3.27}$$

(c) If $f(\cdot, t)$ is absolutely continuous with derivative $f_x(\cdot, t)$ then

$$[f(X,\cdot), X] = \int_0^t f_x(X_s, s)\,\mathrm{d}s. \tag{3.28}$$

For a smooth function $f \in C^1(\mathbb{R}^1 \times [0,1])$ this is well known (cf. Protter 1990, p. 75) and can be checked directly from (3.9). The extension of (3.28) to the absolutely continuous case follows by approximating f with smooth functions f_n in the norms $\|\cdot\|_i$ $(i = 1, 2)$ and applying (3.27); see the proof of Theorem 4.1.

Example 3.1

Let $a(t)$ denote a continuous function on $[0,1]$. The function

$$f(x,t) = I_{[a(t),\infty)}(x) \tag{3.29}$$

satisfies the continuity condition of Theorem 3.3, and so the quadratic covariation in (3.9) exists. Let us define the *local time of Brownian motion at the continuous curve* $a(\cdot)$ as the continuous process $L^{a(\cdot)}$ given by

$$L_t^{a(\cdot)} = [f(X,\cdot), X]_t. \tag{3.30}$$

In order to motivate this definition, we take a sequence (ϵ_n) decreasing to 0. By (3.28) the function

$$f_n(x,t) = \frac{1}{2\epsilon_n}\int_{-\infty}^{x} I_{(a(t)-\epsilon_n,\, a(t)+\epsilon_n)}(y)\,\mathrm{d}y$$

has quadratic covariation

$$[f_n(X,\cdot), X]_t = \frac{1}{2\epsilon_n}\int_0^t I_{(a(s)-\epsilon_n,\, a(s)+\epsilon_n)}(X_s)\,\mathrm{d}s.$$

But f_n converges to f in the norms $\|\cdot\|_i$ $(i = 1, 2)$ used in the proof of Theorem 3.3, and so (3.27) implies

$$L_t^{a(\cdot)} = \lim_{n\to\infty} \frac{1}{2\epsilon_n}\int_0^t I_{(a(s)-\epsilon_n,\, a(s)+\epsilon_n)}(X_s)\,\mathrm{d}s \tag{3.31}$$

in u.p. In particular, we can conclude that the continuous process $L^{a(\cdot)}$ has increasing paths, and that the corresponding random measure on $[0, 1]$ is a.s. concentrated on times where the Brownian path intersects the given function $a(\cdot)$:

$$\int_0^1 I_{\{X_t \neq a(t)\}} \, \mathrm{d}_t L_t^{a(\cdot)} = 0. \tag{3.32}$$

If the function $a(\cdot)$ assumes a constant level a then

$$\{f(X_{t_{i+1}}, t_{i+1}) - f(X_{t_i}, t_i)\}(X_{t_{i+1}} - X_{t_i}) = |X_{t_{i+1}} - X_{t_i}| I_{C_{n,i}}$$

where

$$C_{n,i} = \{\mathrm{sign}(X_{t_i} - a) \neq \mathrm{sign}(X_{t_{i+1}} - a)\},$$

with $t_i \in D_n$, denotes the set of paths which exhibit a crossing of the level a if checked at times t_i and t_{i+1}. Thus the existence of the quadratic covariation (3.9) implies the identity

$$L_t^a = \lim_{n \to \infty} \sum_{t_i \in D_n, t_i \leq t} |X_{t_{i+1}} - X_{t_i}| I_{C_{n,i}} \tag{3.33}$$

for the local time of Brownian motion at a constant level a.

Let us now return to the general situation of Theorem 3.3 and show that quadratic covariation is a continuous process of zero energy.

Definition 3.1
For a process $Y = (Y_t)_{0 \leq t \leq 1}$ with continuous paths we define the *quadratic variation*

$$[Y]_t = \lim_{n \to \infty} \sum_{t_i \in D_n, t_i \leq t} (Y_{t_{i+1}} - Y_{t_i})^2 \tag{3.34}$$

whenever this limit exists uniformly in probability. If $[Y]_1 = 0$ a.s. with respect to P, then Y is called a process of *zero energy*.

Remark 3.3
(a) Any process with continuous paths of bounded variation has zero energy.

(b) A local martingale of the form

$$Y_t = \int_0^t f(X_s, s) \, \mathrm{d}X_s$$

has quadratic variation

$$[Y]_t = \int_0^t f^2(X_s, s) \, \mathrm{d}s. \tag{3.35}$$

Theorem 3.5 The quadratic covariation $[f(X, \cdot), X]$ is a continuous process of zero energy.

Proof
The quadratic covariation $Y_t = [f(X, \cdot), X]_t$ is of the form

$$Y_t = Y_t^{(1)} + Y_t^{(2)} \tag{3.36}$$

with

$$Y_t^{(1)} = \int_0^t f(X_s, s)\,\mathrm{d}^* X_s, \qquad Y_t^{(2)} = -\int_0^t f(X_s, s)\,\mathrm{d}X_s.$$

The process $Y^{(2)}$ has quadratic variation

$$[Y^{(2)}]_t = \int_0^t f^2(X_s, s)\,\mathrm{d}s. \tag{3.37}$$

Since

$$Y_t^* = \int_{1-t}^1 f(X_s, 1-s)\,\mathrm{d}X_s$$

has quadratic variation

$$[Y^*]_t = \int_{1-t}^1 f^2(X_s, 1-s)\,\mathrm{d}s$$

under P^*, the process $Y^{(1)} = Y^* \circ \mathrm{R}$ has quadratic variation

$$[Y^{(1)}]_t = \left\{ \int_{1-t}^1 f^2(X_s, 1-s)\,\mathrm{d}s \right\} \circ \mathrm{R} = \int_0^t f^2(X_s, s)\,\mathrm{d}s \tag{3.38}$$

under P. Thus, the decomposition (3.36) implies the a priori estimate

$$[Y]_t \leq 2([Y^{(1)}]_t + [Y^{(2)}]_t) \leq 4 \int_0^t f^2(X_s, s)\,\mathrm{d}s \tag{3.39}$$

if, for the moment, we define $[Y]_t$ as the supremum limit of the sums in (3.34).

Assume that f has compact support and take a sequence of functions $f_n \in C^1(\mathbb{R}^1 \times [0,1])$ such that f_n converges to f in the norms $\|\cdot\|_i$ $(i = 1, 2)$ used in the proof of Theorem 3.3. The process

$$Y_t^{(n)} = [f_n(X, \cdot), X]_t = \int_0^t (f_n)_x(X_s, s)\,\mathrm{d}s$$

has continuous paths of bounded variation and therefore has zero energy. Evaluating the squares in (3.34) for $[Y - Y^{(n)}]_1$ and using Cauchy–Schwarz, we see that

$$[Y - Y^{(n)}]_1 = [Y]_1. \tag{3.40}$$

Applying our a priori estimate (3.39) to the difference $f - f_n$, we get

$$[Y]_1 = [Y - Y^{(n)}]_1 \leq 4 \int_0^t (f - f_n)^2(X_s, s)\,\mathrm{d}s. \tag{3.41}$$

But the expectation of the last term converges to 0 due to our estimate (3.14). Thus we have $\mathrm{E}[[Y]_1] = 0$, hence $[Y]_1 = 0$ almost surely with respect to P.

4. An extension of Itô's formula

In this section we prove the time-independent extension (1.1) of Itô's formula. For $f \in \mathcal{C}^1$ the quadratic covariation takes the well-known form

$$[f(X), X]_t = \int_0^t f'(X_s)\,\mathrm{d}s \tag{4.1}$$

which appears in Itô's formula

$$F(X_t) = F(X_0) + \int_0^t f(X_s)\,\mathrm{d}X_s + \frac{1}{2}\int_0^t f'(X_s)\,\mathrm{d}s \tag{4.2}$$

for $F \in \mathcal{C}^2$ and $f = F'$. Thus, the following formula is an extension of Itô's formula to the case where f is locally square integrable:

Theorem 4.1 Let F be absolutely continuous with locally square integrable derivative f. Then

$$F(X_t) = F(X_0) + \int_0^t f(X_s)\,\mathrm{d}X_s + \tfrac{1}{2}[f(X), X]_t. \tag{4.3}$$

Proof
The quadratic covariation process $[f(X), X]$ exists as shown in Proposition 3.2. By the usual localization argument we may assume that f has compact support. Take a sequence $f_n \in \mathcal{C}^1$ such that f_n converges to f in L^2. The estimates in the proof of Theorem 3.3 show that

$$\lim_{n\to\infty} \int_0^t f_n(X_s)\,\mathrm{d}X_s = \int_0^t f(X_s)\,\mathrm{d}X_s \tag{4.4}$$

in $L^2(P)$ and

$$\lim_{n\to\infty} \int_0^t f_n(X_s)\,\mathrm{d}^*X_s = \int_0^t f(X_s)\,\mathrm{d}^*X_s \tag{4.5}$$

in $L^1(P)$. By (3.7) and the usual Itô formula (4.2) applied to the functions

$$F_n(x) \equiv F(0) + \int_0^x f_n(y)\,\mathrm{d}y, \tag{4.6}$$

we obtain

$$\begin{aligned}\tfrac{1}{2}[f(X), X]_t &= \lim_{n\to\infty} \tfrac{1}{2}[f^n(X), X]_t \\ &= \lim_{n\to\infty}\left\{F_n(X_t) - F_n(X_0) - \int_0^t f_n(X_s)\,\mathrm{d}X_s\right\},\end{aligned}$$

and this is equal to

$$F(X_t) - F(X_0) - \int_0^t f(X_s)\,\mathrm{d}X_s,$$

again by (4.4).

Remark 4.1

For $t \in D_n$ we can write

$$F(X_t) - F(X_0) = \sum_{\substack{t_i \in D_n \\ t_i \le t}} \left\{ f(X_{t_i})(X_{t_{i+1}} - X_{t_i}) + \int_{X_{t_i}}^{X_{t_{i+1}}} (f(y) - f(X_{t_i}))\,\mathrm{d}y \right\} \tag{4.7}$$

and also

$$F(X_t) - F(X_0) = \sum_{\substack{t_i \in D_n \\ t_i \le t}} \left\{ f(X_{t_{i+1}})(X_{t_{i+1}} - X_{t_i}) + \int_{X_{t_i}}^{X_{t_{i+1}}} (f(y) - f(X_{t_{i+1}}))\,\mathrm{d}y \right\}. \tag{4.8}$$

Equation (4.7), together with (4.3) and (3.5), implies

$$\tfrac{1}{2}[f(X), X]_t = \lim_{n\to\infty} \sum_{\substack{t_i \in D_n \\ t_i \le t}} \int_{X_{t_i}}^{X_{t_{i+1}}} (f(y) - f(X_{t_i}))\,\mathrm{d}y. \tag{4.9}$$

On the other hand, equation (4.8), together with (4.3), (3.6) and (3.7), implies

$$\tfrac{1}{2}[f(X), X]_t = \lim_{n\to\infty} \sum_{\substack{t_i \in D_n \\ t_i \le t}} \int_{X_{t_i}}^{X_{t_{i+1}}} (f(X_{t_{i+1}}) - f(y))\,\mathrm{d}y. \tag{4.10}$$

If we subtract (4.10) from (4.9) we obtain the remarkable relation

$$\lim_{n\to\infty} \sum_{\substack{t_i \in D_n \\ t_i \le t}} \int_{X_{t_i}}^{X_{t_{i+1}}} \{ f(y) - \tfrac{1}{2}(f(X_{t_i}) + f(X_{t_{i+1}}))\}\,\mathrm{d}y = 0 \tag{4.11}$$

uniformly in probability for any locally square integrable function f.

Example 4.1

Let us return to Example 3.1 in the case where the function $a(\cdot)$ assumes a constant level a. The function $f(x) = I_{[a,\infty)}(x)$ is the derivative of the absolutely continuous function $F(x) = (x-a)^+$. Due to (3.33), our Itô formula (4.3) takes the form

$$(X_t - a)^+ = (X_0 - a)^+ + \int_0^t I_{[a,\infty)}(X_s)\,\mathrm{d}X_s + \tfrac{1}{2}L_t^a \tag{4.12}$$

where

$$L_t^a = \lim_{n\to\infty} \sum_{t_i \in D_n, t_i \le t} |X_{t_{i+1}} - X_{t_i}| I_{C_{n,i}}. \tag{4.13}$$

Comparing (4.12) with the well-known Tanaka formula (see, for example, Protter 1990, p. 169), we see that L_t^a coincides with Lévy's local time at level a. Note that (4.9) and (4.10) imply the alternative descriptions

$$\begin{aligned}\tfrac{1}{2}L_t^a &= \lim_{n\to\infty} \sum_{t_i\in D_n,\, t_i\le t} |X_{t_i} - a| I_{C_{n,i}} \\ &= \lim_{n\to\infty} \sum_{t_i\in D_n,\, t_i\le t} |X_{t_{i+1}} - a| I_{C_{n,i}}. \end{aligned} \tag{4.14}$$

Itô (1974; 1975) proposed defining the Stratonovich integral by the following formula (4.15) when the quadratic covariation $[f(X), X]$ exists (see also Protter 1990 p. 216). Proposition 3.2 shows that this definition makes sense for any locally square integrable function f, and Remark 3.2 shows that the Stratonovich integral can be described as well as the following limit of sums:

Definition 4.1
For any locally square integrable function f the Stratonovich integral is well defined by

$$\int_0^t f(X_s) \circ \mathrm{d}X_s \equiv \int_0^t f(X_s)\,\mathrm{d}X_s + \tfrac{1}{2}[f(X), X]_t, \tag{4.15}$$

and can be computed as

$$\int_0^t f(X_s) \circ \mathrm{d}X_s = \lim_{n\to\infty} \sum_{\substack{t_i\in D_n \\ t_i\le t}} \tfrac{1}{2}\{f(X_{t_i}) + f(X_{t_{i+1}})\}(X_{t_{i+1}} - X_{t_i}). \tag{4.16}$$

With this definition, the Itô formula (4.3) can be reformulated as follows:

Corollary 4.2 For an absolutely continuous function F with locally square integrable derivative $F' = f$,

$$F(X_t) = F(X_0) + \int_0^t f(X_s) \circ \mathrm{d}X_s. \tag{4.17}$$

□

Note that this is an improvement upon Itô (1974) and Protter (1990, pp. 222–224), where F is required to be C^2. Note also that $\int_0^t f(X_s) \circ \mathrm{d}X_s$ need not be a semi-martingale, but that it is a *Dirichlet process* in the sense of the following remark.

Remark 4.2
(a) In order to compare Theorem 4.1 with the extension of Itô's formula obtained by Bouleau and Yor (see Bouleau and Yor 1981, or alternatively Protter 1990, p. 179), let L_t^a denote the local time of our standard Brownian motion X at level a. Let F be absolutely continuous with a locally bounded Borel measurable derivative f. Using a vector-valued measure approach Bouleau and Yor show

$$F(X_t) = F(X_0) + \int_0^t f(X_s)\,\mathrm{d}X_s - \frac{1}{2}\int_{\mathbb{R}} f(a)\,\mathrm{d}_a L_t^a. \tag{4.18}$$

Comparing the Bouleau–Yor formula (4.18) with our formula (4.3), we see that if f is Borel and locally bounded, then

$$[f(X), X]_t = -\int_{\mathbb{R}} f(a)\, \mathrm{d}_a L_t^a. \tag{4.19}$$

(b) A general result of Fukushima in the context of Dirichlet spaces implies that, for an absolutely continuous function F on $\mathbb{R}^1$ with locally square integrable derivative f, the process $F(X)$ is a *Dirichlet process*, i.e. $F(X)$ can be represented as the sum

$$F(X_t) = F(X_0) + \int_0^t f(X_s)\, \mathrm{d}X_s + A_t \tag{4.20}$$

of a local martingale and a continuous process A of zero energy (cf. Fukushima 1980, Chapter 5). Thus, our Itô formula (4.3), together with Theorem 3.5, identifies the process of zero energy A in the Fukushima decomposition (4.20) as a quadratic covariation:

$$A_t = \tfrac{1}{2}[f(X), X]_t. \tag{4.21}$$

Note that in our approach the process A is computed directly on the paths of Brownian motion, without any smoothing of the function f as is usually done in the theory of Dirichlet spaces.

The following section shows that, in contrast to the Bouleau–Yor formula (4.18) and to the Fukushima formula (4.20), our version (4.3) of the Itô formula for locally square integrable functions f admits a straightforward extension to the time-dependent case.

5. The time-dependent case

Let $f(x, t)$ be a measurable function on $\mathbb{R}^1 \times [0, 1]$ such that $f(\cdot, t)$ is locally square integrable and assume that

$$f(\cdot, t) \text{ is continuous in } t \tag{5.1}$$

as a map from $[0, 1]$ to L^2_{loc}. Under this assumption, the existence of quadratic covariation

$$[f(X, \cdot), X]_t = \lim_{n\to\infty} \sum_{t_i \in D_n, t_i \le t} \{f(X_{t_{i+1}}, t_{i+1}) - f(X_{t_i}, t_i)\}(X_{t_{i+1}} - X_{t_i}) \tag{5.2}$$

was already shown in the proof of Theorem 3.3. Let us now establish the corresponding time-dependent version of Itô's formula:

Theorem 5.1 Suppose that $F(x, t)$ is absolutely continuous in x and that the partial derivative $f(\cdot, t) \equiv F_x(\cdot, t)$ satisfies the preceding assumptions. Then Itô's formula holds in the form

$$F(X_t, t) = F(X_0, 0) + \int_0^t f(X_s, s)\, \mathrm{d}X_s + \tfrac{1}{2}[f(X, \cdot), X]_t + \int_0^t F(X_s, \mathrm{d}s) \tag{5.3}$$

where

$$\int_0^t F(X_s, \mathrm{d}s) \equiv \lim_{n\to\infty} \sum_{t_i \in D_n, t_i \le t} F(X_{t_{i+1}}, t_{i+1}) - F(X_{t_{i+1}}, t_i) \tag{5.4}$$

exists uniformly in probability.

Remark 5.1
If $F(x, \cdot)$ is absolutely continuous in t with derivative $F_t(x, \cdot)$ and if

$$F_t(\cdot, t) \text{ is continuous in } t \tag{5.5}$$

as a map from $[0, 1]$ to L^1_{loc}, then the last term in our Itô formula (5.3) takes the usual form

$$\int_0^t F(X_s, \mathrm{d}s) = \int_0^t F_t(X_s, s)\,\mathrm{d}s. \tag{5.6}$$

Proof
Let us write

$$F(X_t, t) - F(X_0, 0) = A_t^n + B_t^n$$

where

$$A_t^n = \sum_{t_i \in D_n, t_i \le t} F(X_{t_{i+1}}, t_{i+1}) - F(X_{t_{i+1}}, t_i)$$

and

$$B_t^n = \sum_{t_i \in D_n, t_i \le t} F(X_{t_{i+1}}, t_i) - F(X_{t_i}, t_i).$$

By the time-homogeneous Itô formula (4.3),

$$\begin{aligned} F(X_{t_{i+1}}, t_i) - F(X_{t_i}, t_i) &= \int_{t_i}^{t_{i+1}} f(X_t, t_i)\,\mathrm{d}X_t + \tfrac{1}{2}([f(X, t_i), X]_{t_{i+1}} - [f(X, t_i), X]_{t_i}) \\ &= \frac{1}{2}\left\{ \int_{t_i}^{t_{i+1}} f(X_t, t_i)\,\mathrm{d}^* X_t + \int_{t_i}^{t_{i+1}} f(X_t, t_i)\,\mathrm{d}X_t \right\} \end{aligned}$$

due to (3.7). But under our continuity assumption (5.1), the arguments in the proof of Theorem 3.3 show that the sum B_t^n of these terms converges to

$$\begin{aligned} \lim_{n\to\infty} B_t^n &= \frac{1}{2}\left\{ \int_0^t f(X_s, s)\,\mathrm{d}^* X_s + \int_0^t f(X_s, s)\,\mathrm{d}X_s \right\} \\ &= \int_0^t f(X_s, s)\,\mathrm{d}X_s + \tfrac{1}{2}[f(X, \cdot), X]_t \end{aligned}$$

uniformly in probability. Thus, the sum A_t^n also converges uniformly in probability, and

$$\int_0^t F(X_s, \mathrm{d}s) \equiv \lim_{n\to\infty} A_t^n$$
$$= F(X_t, 0) - F(X_0, 0) - \int_0^t f(X_s, s)\,\mathrm{d}X_s - \tfrac{1}{2}[f(X, \cdot), X]_t.$$

This shows that the limit in (5.4) exists and that (5.3) holds.

In view of (5.6) suppose that $F(x, \cdot)$ is absolutely continuous in t with derivative $F_t(x, \cdot)$. For $t \in D_n$ we have

$$A_t^n = \sum_{t_i \in D_n, t_i \le t} \int_{t_i}^{t_{i+1}} F_t(X_{t_{i+1}}, s)\,\mathrm{d}s = \int_0^t F_t(X_s^{(n)}, s)\,\mathrm{d}s$$

where $X_s^{(n)} \equiv X_{t_{i+1}}$ for $s \in (t_i, t_{i+1}]$. Under an additional continuity assumption on F_t we see that

$$\lim_{n\to\infty} A_t^n = \int_0^t F_t(X_s, s)\,\mathrm{d}s$$

uniformly in probability, and this implies (5.6). For example, the arguments in the proof of Theorem 3.3 show that it is enough to assume condition (5.5).

Remark 5.2

(a) Our Itô formula (5.3) shows that the process

$$F(X_t, t) - \int_0^t F(X_s, \mathrm{d}s) \qquad 0 \le t \le 1 \tag{5.7}$$

is a Dirichlet process in the sense of (4.20). Under the additional assumptions in Remark 5.1 the second term has continuous paths of bounded variation. In this case the process $F(X_t, t)$ is itself a Dirichlet process.

(b) In analogy to the previous section we see that, under the assumptions on f in Theorem 5.1, the Stratonovich integral is well defined by

$$\int_0^t f(X_s, s) \circ \mathrm{d}X_s \equiv \int_0^t f(X_s, s)\,\mathrm{d}X_s + \tfrac{1}{2}[f(X, \cdot), X]_t, \tag{5.8}$$

and that it can be computed as

$$\int_0^t f(X_s, s) \circ \mathrm{d}X_s = \lim_{n\to\infty} \sum_{\substack{t_i \in D_n \\ t_i \le t}} \tfrac{1}{2}\{f(X_{t_i}, t_i) + f(X_{t_{i+1}}, t_{i+1})\}(X_{t_{i+1}} - X_{t_i}). \tag{5.9}$$

Thus, the Itô formula (5.3) takes the form

$$F(X_t, t) = F(X_0, 0) + \int_0^t f(X_s, s) \circ \mathrm{d}X_s + \int_0^t F(X_s, \mathrm{d}s). \tag{5.10}$$

(c) The time-dependent version of the argument for (4.9) shows that

$$\tfrac{1}{2}[f(X,\cdot),X]_t = \lim_{n\to\infty}\sum_{\substack{t_i\in D_n\\ t_i\le t}}\int_{X_{t_i}}^{X_{t_{i+1}}}(f(y,t_i)-f(X_{t_i},t_i))\,dy. \tag{5.11}$$

Example 5.1

Let us return to the situation in Example 3.1 where $f(x,t) = I_{[a(t),\infty)}(x)$ is the partial derivative F_x of the function $F(x,t) = (x-a(t))^+$. Itô's formula takes the form

$$(X_t - a(t))^+ = (X_0 - a(0))^+ + \int_0^t I_{[a(s)\,\infty)}(X_s)\,dX_s + \tfrac{1}{2}L_t^{a(\cdot)} + \int_0^t F(X_s, ds). \tag{5.12}$$

In the special case where $a(\cdot)$ is a function of bounded variation we get

$$\int_0^t F(X_s, ds) = -\int_0^t I_{[a(s),\infty)}(X_s)\,da(t), \tag{5.13}$$

and so (5.12) reduces to

$$(X_t - a(t))^+ = (X_0 - a(0))^+ + \int_0^t I_{[a(s),\infty)}(X_s)d(X - a(\cdot)) + \tfrac{1}{2}L_t^{a(\cdot)}. \tag{5.14}$$

In this special case, the process $X - a(\cdot)$ is a semi-martingale, $L^{a(\cdot)}$ may be viewed as the local time of this semi-martingale at level 0, and (5.14) is the corresponding Tanaka formula. Thus (5.12) is an extension of the Tanaka formula to the general case of a continuous function $a(\cdot)$.

As a special case of (5.11) we obtain the following identity for the local time at a continuous curve, in analogy to (4.14):

$$\tfrac{1}{2}L_t^{a(\cdot)} = \lim_{n\to\infty}\sum_{t_i\in D_n,\, t_i\le t}|X_{t_{i+1}} - a(t_i)|I_{C_{n,i}} \tag{5.15}$$

where

$$C_{n,i} = \{\mathrm{sign}(X_{t_i} - a(t_i)) \neq \mathrm{sign}(X_{t_{i+1}} - a(t_i))\}$$

denotes the set of paths which exhibit a crossing of the level $a(t_i)$ if checked at times t_i and t_{i+1}.

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