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Classes of mixing stable processes

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Every measurable stationary α-stable process with $0 < \alpha < 2$ can be related to a non-singular flow on a σ-finite measure space. We establish the relationship between properties of the flow and mixing of the stationary stable process. We provide the first example of a mixing stationary stable process corresponding to a conservative flow. We show further the connection between the expected return time of the flow to sets of finite positive measure and the mixing properties of the process.

Keywords: asymptotic singularity; dissipative and conservative flows; ergodicity; expected return time; mixing; non-singular flow; positive and null recurrence; spectral representation; stable processes; stationary processes

1. Introduction

Let $\boldsymbol{X} = \{X_t : t \in \mathbb{T}\}$ be a stationary symmetric α-stable (SαS) process, where $0 < \alpha < 2$ and $\mathbb{T} = \mathbb{Z}$ or $\mathbb{R}$. It is well known that $\boldsymbol{X}$ has a stochastic integral representation

$$\{X_t, t \in \mathbb{T}\} \stackrel{\mathrm{d}}{=} \left\{\int_S f_t \,\mathrm{d}M, t \in \mathbb{T}\right\}, \tag{1.1}$$

where M is an independently scattered SαS random measure on some Lebesgue space $(S, \mathcal{B}, \mu)$ and $\{f_t\}_{t\in\mathbb{T}}$ is the orbit of a one-parameter group $\{U\}_{t\in\mathbb{T}}$ of linear isometries on a subspace of $L^\alpha(S, \mathcal{B}, \mu)$, i.e. $f_t = U^t f$ for some $f \in L^\alpha(S, \mathcal{B}, \mu)$ (see Hardin 1982; or Samorodnitsky and Taqqu 1994). The symbol '$\stackrel{\mathrm{d}}{=}$' means 'equal in distribution'. Furthermore, it was shown in Rosiński (1995) that one can choose f_ts of the form

$$f_t = a_t \left(\frac{\mathrm{d}(\mu \circ \phi_t)}{\mathrm{d}\mu}\right)^{1/\alpha} f \circ \phi_t, t \in \mathbb{T}, \tag{1.2}$$

where $a_t : S \to \{|z| = 1\}$ ($a_t \in \{-1, 1\}$ if $\boldsymbol{X}$ is real) satisfies the equation $a_{t_1+t_2} = a_{t_2} \cdot a_{t_1} \circ \phi_{t_2}$ μ-almost everywhere for every $t_1, t_2 \in \mathbb{T}$, $\phi_t : S \to S$ is a non-singular flow on $(S, \mathcal{B}, \mu)$, and $f \in L^\alpha(S, \mathcal{B}, \mu)$. It is easy to see that in the case $\mathbb{T} = \mathbb{Z}$ one has $\phi_t = V^t$, for some non-singular map $V : S \to S$, and the a_t are determined by a_1 and V.

Since the theory of non-singular flows on measure spaces is so well developed, an obvious direction of research for probabilists with an interest in stable processes is to study the way the properties of the flow determine the properties of the corresponding stable process as well as to identify flows which generate specific classes of stable process. Our goal here is to

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relate the ergodic properties of the stationary stable process to the properties of the underlying flow.

The study of ergodic properties of stationary stable processes (in fact, of general infinitely divisible processes) begins with Maruyama (1970), who showed that the mixing property for such processes is equivalent to weak convergence of the two-dimensional distributions of the process to a limit with independent marginals, when the time-lag increases to infinity. Apparently unaware of this work, Cambanis *et al.* (1987) suggested another, somewhat more complicated, set of necessary and sufficient conditions for mixing (and ergodicity) of stationary stable processes. Treating ergodicity has become easier since the result of Podgórski (1992) showing that ergodicity and weak mixing are equivalent. Finally, Gross (1994) has simplified the necessary and sufficient conditions for mixing of stationary stable processes even further by showing that for those one does not even need to check all of Maruyama's assumptions. It is the work of Gross that served as the starting point for the present research. One should mention that Gross's results for stable processes have been extended to the general stationary infinitely divisible processes by Rosiński and Żak (1995).

Obviously one can assume that the smallest set supporting all f_ts in (1.1) is S, i.e.

$$\operatorname{supp}\{f_t : t \in \mathbb{T}\} = S \qquad \mu - \text{a.e.} \tag{1.3}$$

Otherwise, we can just reduce the space of integration accordingly. Assumption (1.3) will simplify our formulations considerably, so it will be in effect throughout this paper, unless mentioned otherwise.

Any stationary SαS process can be decomposed in the form

$$\boldsymbol{X} \stackrel{\mathrm{d}}{=} \boldsymbol{X}^1 + \boldsymbol{X}^2 + \boldsymbol{X}^3 \tag{1.4}$$

(see Rosiński 1995), where $\boldsymbol{X}^1$, $\boldsymbol{X}^2$ and $\boldsymbol{X}^3$ are mutually independent stationary SαS processes such that $\boldsymbol{X}^1$ is a superposition of moving averages (or a mixed moving average process), $\boldsymbol{X}^2$ is harmonizable and $\boldsymbol{X}^3$ does not admit moving average or harmonizable components. The process $\boldsymbol{X}^1$ is generated by a dissipative flow, while the other two processes in the above decomposition are generated by infinitely recurrent flows. Furthermore, the harmonizable process $\boldsymbol{X}^2$ is essentially generated by the identity flow. Process $\boldsymbol{X}^3$ itself has a stochastic integral representation of the form (1.1)–(1.2) and the flow $\{\phi_t\}$ in this case is infinitely recurrent and does not have fixed points (Rosiński 1995). It turns out that $\boldsymbol{X}^1$ is always mixing (Surgailis *et al.* 1993) and $\boldsymbol{X}^2$ is never mixing, provided $\boldsymbol{X}^2 \neq 0$ (Maruyama 1970). In fact, this corresponds perfectly well to our intuition, because the fact that mixed moving averages are generated by dissipative flows implies that the observations of the process $\boldsymbol{X}^1$ at remote instances of time are generated by integration over nearly disjoint sets – and the random measure M in (1.1) is independently scattered. Therefore, one feels that these observations are nearly independent, and so expects mixing. On the other hand, infinitely recurrent flows produce processes for which integration over the same sets plays an important role for time-points far apart, and so the process 'remembers' much from its past. The extreme case is, of course, that of the identity flows, and that is why it is not surprising that harmonizable processes are not even ergodic – the latter is true in every infinitely divisible non-Gaussian case by a result of Maruyama (1970).

By the same token, one would not expect that many of the processes of type $\boldsymbol{X}^3$ were mixing. Indeed, Gross (1994) has shown that a large class of processes of this type consisting of the so-called doubly stationary process with finite control measure μ are not mixing (nor ergodic). This is clear also because such processes share with the harmonizable ones the property of being a non-trivial mixture of stationary processes (via their series representation), and so they cannot be ergodic. However, this implies that the class of mixing SαS processes of type $\boldsymbol{X}^3$ is rather small, and until now it was not even known whether or not this class was not empty, for the only examples of mixing (or ergodic) SαS processes known so far were superpositions of moving averages, i.e. processes of type $\boldsymbol{X}^1$.

In this paper we introduce a new class of mixing SαS processes, consisting of processes of type $\boldsymbol{X}^3$. This class is obtained by considering Markov flows $\{\phi_t\}$ in (1.2) with infinite stationary initial distributions m (see Section 3). A typical example of such a flow is the shift transformation for a symmetric recurrent random walk on $\mathbb{R}$ starting from a point chosen according to the Lebesgue measure. This establishes an interesting connection between Markov and stable processes. In the case of recurrent Markov chains infinite invariant measures correspond to null recurrence, or Markov chains for which expected return time to the initial state is infinite. We will see that, in general, a conservative flow corresponding to an ergodic stable process must take an expected infinite time to return to sets of finite positive measure.

In Section 2 we will show that mixing and ergodicity of SαS processes are entirely determined by properties of the flow $\{\phi_t\}$ in (1.2). We identify these properties and provide some conditions characterizing mixing and ergodicity in terms of ϕ_t and μ, which complement conditions in Gross (1994).

2. Conditions for mixing and ergodicity

Recall that a stationary process $\boldsymbol{X}$ is said to be *ergodic* if

$$\lim_{\tau\to\infty}\frac{1}{\tau}\int_0^\tau P(A\cap S^t B)\,\mathrm{d}t = P(A)P(B),$$

for any $A, B \in \mathcal{F}_{\boldsymbol{X}}$, where $\mathcal{F}_{\boldsymbol{X}}$ is the σ-field generated by $\boldsymbol{X}$ and S^t is the corresponding shift transformation (in discrete time, the integral is replaced by a sum). $\boldsymbol{X}$ is said to be *mixing* if

$$\lim_{t\to\infty} P(A\cap S^t B) = P(A)P(B),$$

for any $A, B \in \mathcal{F}_{\boldsymbol{X}}$. $\boldsymbol{X}$ is said to be *weakly mixing* if the above limit holds with t restricted to a set of density one which may depend on A and B. Alternatively, $\boldsymbol{X}$ is weakly mixing if

$$\lim_{\tau\to\infty}\frac{1}{\tau}\int_0^\tau |P(A\cap S^t B) - P(A)P(B)|\,\mathrm{d}t = 0$$

for any $A, B \in \mathcal{F}_{\boldsymbol{X}}$, which shows that weak mixing is an intermediate property between ergodicity and mixing (see Petersen 1983). However, for SαS processes ergodicity and weak mixing are equivalent by Podgórski (1992). It is also known that there are non-mixing ergodic (and thus weakly mixing) stationary SαS processes (see Gross and Robertson 1993).

In this section we will establish a one-to-one correspondence between ergodic properties of a stochastic process $\boldsymbol{X}$ and certain asymptotic properties of the flow $\{\phi_t\}$ in (1.2). We will need the following definitions. A measurable flow $\{\phi_t\}_{t\in\mathbb{T}}$ on a *finite* measure space $(S, \mathcal{B}, \mu)$ is said to be *asymptotically singular* if, for all $\epsilon > 0$, there is a value t_0 such that for all $|t| > t_0$, there exists an $A = A(t, \epsilon) \in \mathcal{B}$ such that

$$\mu(A) < \epsilon \qquad \text{and} \qquad \mu(\phi_t A) > \mu(S) - \epsilon. \tag{2.1}$$

It is trivial that an asymptotically singular flow is not μ-preserving; moreover, there cannot be any *finite* ϕ_t-preserving measure ν equivalent to μ. The usual tool in a study of singularity/regularity of measures is the Hellinger integral. This is defined for finite measures μ and ν by

$$H(\mu, \nu) = \int_S \sqrt{\frac{\mathrm{d}\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}\nu}{\mathrm{d}\lambda}}\,\mathrm{d}\lambda, \tag{2.2}$$

where λ is a finite measure such that $\mu \ll \lambda$ and $\nu \ll \lambda$. A convenient (but somewhat loose) way of writing (2.2) is

$$H(\mu, \nu) = \int_S \sqrt{\mathrm{d}\mu\,\mathrm{d}\nu}.$$

It is easy to show that $\{\phi_t\}_{t\in\mathbb{T}}$ is asymptotically singular with respect to μ if and only if

$$\lim_{|t|\to\infty} H(\mu, \mu\circ\phi_t) = 0. \tag{2.3}$$

We will need the following characterization.

Proposition 2.1. *Let $\boldsymbol{X} = \{X_t\}_{t\in\mathbb{T}}$ be a stationary SαS process with representation $\{f_t\}_{t\in\mathbb{T}}$ satisfying* (1.1). *$\boldsymbol{X}$ is mixing if and only if for some (equivalently, any) $\delta \in (0, 1)$,*

$$\lim_{t\to\infty} \int_S |f_0|^{\alpha\delta}|f_t|^{\alpha(1-\delta)}\,\mathrm{d}\mu = 0. \tag{2.4}$$

Proof. We start by recalling that $\boldsymbol{X}$ is mixing if and only if

$$\mu\{f_0 \in K, |f_t| > \epsilon\} \to 0, \tag{2.5}$$

for every compact set $K \subset \mathbb{R}$ bounded away from 0 and $\epsilon > 0$. This follows from the general description in Theorem 5 of Rosiński and Żak (1995), and in the stable case it has been known since a result of Gross (1994, Theorem 2.7).

Suppose that $\boldsymbol{X}$ is mixing. We have for any $\eta \in (0, 1)$ and $\epsilon > 0$ (to be chosen later)

$$\begin{aligned}
\int_S |f_0|^{\alpha\delta}|f_t|^{\alpha(1-\delta)}\,\mathrm{d}\mu &= \int_{|f_0|\in[\eta,\eta^{-1}],\,|f_t|>\epsilon} |f_0|^{\alpha\delta}|f_t|^{\alpha(1-\delta)}\,\mathrm{d}\mu + \int_{|f_0|\notin[\eta,\eta^{-1}]} |f_0|^{\alpha\delta}|f_t|^{\alpha(1-\delta)}\,\mathrm{d}\mu \\
&\quad + \int_{|f_0|\in[\eta,\eta^{-1}],\,|f_t|\le\epsilon} |f_0|^{\alpha\delta}|f_t|^{\alpha(1-\delta)}\,\mathrm{d}\mu \\
&= I_1 + I_2 + I_3.
\end{aligned}$$

By Hölder's inequality,

$$I_2 \le \left(\int_{|f_0| \notin [\eta, \eta^{-1}]} |f_0|^\alpha \, \mathrm{d}\mu\right)^\delta \left(\int_S |f_t|^\alpha \, \mathrm{d}\mu\right)^{1-\delta},$$

hence, by choosing η sufficiently small, we make I_2 small uniformly in t. Since

$$I_3 \le (\epsilon\eta^{-1})^{\alpha(1-\delta)} \int_S |f_0|^\alpha \, \mathrm{d}\mu,$$

we can choose ϵ small to make I_3 small uniformly in t. Having fixed ϵ and η, we obtain by Hölder's inequality and (2.5),

$$\limsup_{t\to\infty} I_1 \le \lim_{t\to\infty} \left(\int_{|f_0| \in [\eta, \eta^{-1}],\, |f_t| > \epsilon} |f_0|^\alpha \, \mathrm{d}\mu\right)^\delta \left(\int_S |f_t|^\alpha \, \mathrm{d}\mu\right)^{1-\delta} = 0.$$

This completes the proof of (2.4). To prove the converse, we notice that Markov's inequality and (2.4) imply (2.5) for $t \to \infty$. Since

$$\int_S |f_0|^{\alpha(1-\delta)} |f_{-t}|^{\alpha\delta} \, \mathrm{d}\mu = \int_S |f_0|^{\alpha\delta} |f_t|^{\alpha(1-\delta)} \, \mathrm{d}\mu,$$

the same argument gives (2.5) when $t \to -\infty$. The proof is complete. □

Remark 2.1. Alternatively, one can derive Proposition 2.1 directly from Theorem 2 of Cambanis *et al.* (1987). However, that way seems to be more tedious than a simple use of (2.5) above.

The following theorem characterizes the flows on *finite* measure spaces generating mixing SαS processes.

Theorem 2.1. *Let $\mathbf{X} = \{X_t\}_{t\in\mathbb{T}}$ be a stationary $S\alpha S$ process with representation* (1.1)–(1.3). *Assume $\mu(S) < \infty$. Then the following conditions are equivalent:*

(a) $\mathbf{X}$ is mixing;
(b) $\{\phi_t\}_{t\in\mathbb{T}}$ is asymptotically singular.

Proof. Since $\mu \circ \phi_t$ and μ are mutually absolutely continuous, we can take $\lambda = \mu$ in (2.2), thus

$$H(\mu, \mu \circ \phi_t) = \int_S \left(\frac{\mathrm{d}(\mu \circ \phi_t)}{\mathrm{d}\mu}\right)^{1/2} \mathrm{d}\mu.$$

We will use Proposition 2.1 with $\delta = \frac{1}{2}$. Since for every r, t

$$\int_S |f_r f_{r+t}|^{\alpha/2} \, \mathrm{d}\mu = \int_S |f_0 f_t|^{\alpha/2} \, \mathrm{d}\mu,$$

we obtain from (2.4)

$$\lim_{t\to\infty} \int_S |f_r f_t|^{\alpha/2} \, \mathrm{d}\mu = 0 \qquad \text{for each } r. \tag{2.6}$$

By (1.3) and the finiteness of μ, for every $\epsilon > 0$ there exists a finite set $t_1, \ldots, t_n \in \mathbb{T}$ and $\delta > 0$ such that

$$\mu(S \setminus \cup \{|f_{t_i}| > \delta\}) < \epsilon. \tag{2.7}$$

Consider

$$g = \sum |f_{t_i}|,$$

and

$$g_t = \sum |f_{t_i+t}| = \left(\frac{\mathrm{d}(\mu \circ \phi_t)}{\mathrm{d}\mu}\right)^{1/\alpha} g \circ \phi_t.$$

From (2.6) we obtain

$$\limsup_{t \to \infty} \int_S |gg_t|^{\alpha/2} \, \mathrm{d}\mu \leq \sum_{i,j} \lim_{t \to \infty} \int_S |f_{t_i} f_{t_j+t}|^{\alpha/2} \, \mathrm{d}\mu = 0.$$

Let $A = \{|g| > \delta\}$; by (2.7), $\mu(S - A) < \epsilon$. We obtain

$$\begin{aligned} H(\mu, \mu \circ \phi_t) &\leq \int_{A \cap \phi_t^{-1} A} \left(\frac{\mathrm{d}(\mu \circ \phi_t)}{\mathrm{d}\mu}\right)^{1/2} \mathrm{d}\mu + \int_{A^{\mathrm{c}}} \left(\frac{\mathrm{d}(\mu \circ \phi_t)}{\mathrm{d}\mu}\right)^{1/2} \mathrm{d}\mu + \int_{\phi_t^{-1} A^{\mathrm{c}}} \left(\frac{\mathrm{d}(\mu \circ \phi_t)}{\mathrm{d}\mu}\right)^{1/2} \mathrm{d}\mu \\ &\leq \delta^{-\alpha} \int_S |gg_t|^{\alpha/2} \, \mathrm{d}\mu + 2\mu(A^{\mathrm{c}})^{1/2} \mu(S)^{1/2}. \end{aligned}$$

Thus

$$\limsup_{t \to \infty} H(\mu, \mu \circ \phi_t) \leq 2\mu(S)^{1/2} \epsilon^{1/2},$$

proving (b) after letting ϵ go to zero.

The converse is easy. For any $M > 0$ we have

$$\begin{aligned} \int_S |f_0 f_t|^{\alpha/2} \, \mathrm{d}\mu &\leq \int_{|f_0| > M} |f_0 f_t|^{\alpha/2} \, \mathrm{d}\mu + \int_{|f_0 \circ \phi_t| > M} |f_0 f_t|^{\alpha/2} \, \mathrm{d}\mu + M^{\alpha} \int_S \left(\frac{\mathrm{d}(\mu \circ \phi_t)}{\mathrm{d}\mu}\right)^{1/2} \mathrm{d}\mu \\ &\leq 2\left(\int_{|f_0| > M} |f_0|^{\alpha} \, \mathrm{d}\mu\right)^{1/2} \left(\int_S |f_0|^{\alpha} \, \mathrm{d}\mu\right)^{1/2} + M^{\alpha} \int_S \left(\frac{\mathrm{d}(\mu \circ \phi_t)}{\mathrm{d}\mu}\right)^{1/2} \mathrm{d}\mu. \end{aligned}$$

Taking M sufficiently large and then letting $t \to \infty$ we prove (2.4), and so (a). □

Remark 2.2. Assumption (1.3) is not needed for the implication (b) ⇒ (a): asymptotic singularity gives mixing.

Despite the fact that one can always choose a representation (1.1) with finite measure μ, in certain cases the natural representation for an SαS process involves an infinite measure, and so it is useful to have conditions for mixing when μ is arbitrary.

Theorem 2.2. *Let $\boldsymbol{X} = \{X_t\}_{t \in \mathbb{T}}$ be a stationary $S\alpha S$ process with representation* (1.1)–(1.3). *Then the following conditions are equivalent:*

(a) $\boldsymbol{X}$ *is mixing;*
(b) for every $A \subset S$ *of finite measure* μ,

$$\lim_{t\to\infty} \int_{A\cap\phi_t A} \left(\frac{\mathrm{d}(\mu\circ\phi_t)}{\mathrm{d}\mu}\right)^{1/2} \mathrm{d}\mu = 0;$$

(c) there exist $A_k \nearrow S$ *such that for each* $k \geq 1$,

$$\lim_{t\to\infty} \int_{A_k\cap\phi_t A_k} \left(\frac{\mathrm{d}(\mu\circ\phi_t)}{\mathrm{d}\mu}\right)^{1/2} \mathrm{d}\mu = 0.$$

Moreover, the implication $(c) \Rightarrow (a)$ *holds without assumption* (1.3).

Proof. Choose $h > 0$ such that $\int_S h^\alpha \,\mathrm{d}\mu = 1$ and define

$$f_t^* = \frac{f_t}{h}.$$

$\{f_t^*\}$ is another representation of $\boldsymbol{X}$ with respect to a random measure M^* with control measure μ^* such that $\mathrm{d}\mu^* = h^\alpha\,\mathrm{d}\mu$. Since f_t is given by (1.2),

$$f_t^* = a_t\left(\frac{\mathrm{d}(\mu^*\circ\phi_t)}{\mathrm{d}\mu^*}\right)^{1/\alpha} f^*\circ\phi_t.$$

Thus, by Theorem 2.1, $\boldsymbol{X}$ is mixing if and only if $\{\phi_t\}$ is asymptotically singular with respect to the probability measure μ^*. Assume (a) and let A be such that $\mu(A) < \infty$. We have

$$\begin{aligned}
\int_{A\cap\phi_t A} \left(\frac{\mathrm{d}(\mu\circ\phi_t)}{\mathrm{d}\mu}\right)^{1/2} \mathrm{d}\mu &= \int_S (\mathbf{1}_A h^{-\alpha/2})[(\mathbf{1}_A h^{-\alpha/2})\circ\phi_t]\left(\frac{\mathrm{d}(\mu^*\circ\phi_t)}{\mathrm{d}\mu^*}\right)^{1/2} \mathrm{d}\mu^* \\
&\leq \int_{h^{-\alpha/2}>M} (\mathbf{1}_A h^{-\alpha/2})[(\mathbf{1}_A h^{-\alpha/2})\circ\phi_t]\left(\frac{\mathrm{d}(\mu^*\circ\phi_t)}{\mathrm{d}\mu^*}\right)^{1/2} \mathrm{d}\mu^* \\
&\quad + \int_{h^{-\alpha/2}\circ\phi_t>M} (\mathbf{1}_A h^{-\alpha/2})[(\mathbf{1}_A h^{-\alpha/2})\circ\phi_t]\left(\frac{\mathrm{d}(\mu^*\circ\phi_t)}{\mathrm{d}\mu^*}\right)^{1/2} \mathrm{d}\mu^* \\
&\quad + M^2\int_S \left(\frac{\mathrm{d}(\mu^*\circ\phi_t)}{\mathrm{d}\mu^*}\right)^{1/2} \mathrm{d}\mu^* \\
&\leq 2\mu(A\cap\{h^{-\alpha/2}>M\})^{1/2}\mu(A)^{1/2} + M^2\int_S \left(\frac{\mathrm{d}(\mu^*\circ\phi_t)}{\mathrm{d}\mu^*}\right)^{1/2} \mathrm{d}\mu^*.
\end{aligned}$$

Letting $t\to\infty$ and then $M\to\infty$, we complete the proof of (b).

Obviously (b) $\Rightarrow$ (c). Assume (c) (and (1.1)–(1.2), but not (1.3)). Using the fact that μ^* is a probability measure, we obtain for each $k \geq 1$,

$$\int_S \left(\frac{\mathrm{d}(\mu^* \circ \phi_t)}{\mathrm{d}\mu^*}\right)^{1/2} \mathrm{d}\mu^* \leq \int_{A_k \cap \phi_t A_k} \left(\frac{\mathrm{d}(\mu^* \circ \phi_t)}{\mathrm{d}\mu^*}\right)^{1/2} \mathrm{d}\mu^*$$

$$+ \int_{A_k^c} \left(\frac{\mathrm{d}(\mu^* \circ \phi_t)}{\mathrm{d}\mu^*}\right)^{1/2} \mathrm{d}\mu^* + \int_{\phi_t A_k^c} \left(\frac{\mathrm{d}(\mu^* \circ \phi_t)}{\mathrm{d}\mu^*}\right)^{1/2} \mathrm{d}\mu^*$$

$$\leq \int_{A_k \cap \phi_t A_k} \left(\frac{\mathrm{d}(\mu^* \circ \phi_t)}{\mathrm{d}\mu^*}\right)^{1/2} \mathrm{d}\mu^* + 2\mu^*(A_k^c)^{1/2}.$$

Letting $t \to \infty$ and then $k \to \infty$ gives (b) of Theorem 2.1. This implies (a) by Remark 2.2 and ends the proof. □

Probably the most interesting case of Theorem 2.2 is when μ is an infinite measure invariant under $\{\phi_t\}$. Since the Radon–Nikodym derivative equals 1, we obtain the following corollary, equivalent to Theorem 4.1 of Gross (1994).

Corollary 2.1. *Under the assumptions of Theorem* 2.2, *suppose that μ is invariant under $\{\phi_t\}_{t \in \mathbb{T}}$. Then the following conditions are equivalent:*

*(a) **X** is mixing;*

(b) for every $A \subset S$ of finite measure μ,

$$\lim_{t \to \infty} \mu(A \cap \phi_t A) = 0;$$

(c) there exist $A_k \nearrow S$ such that for each $k \geq 1$,

$$\lim_{t \to \infty} \mu(A_k \cap \phi_t A_k) = 0.$$

Moreover, the implication $(c) \Rightarrow (a)$ *holds without assumption* (1.3).

Remark 2.3. As observed by Gross (1994), restricting t in (2.5) to a set of density one (which may depend on K and ϵ) gives weak mixing of the process instead of mixing. Since our computations do not depend on whether t runs over the entire set $\mathbb{T}$ or only over its infinite subset, we conclude that by adding the phrase 'there exists a set $D \subset \mathbb{T}$ of density one' to the statements characterizing mixing in this section we obtain corresponding results on ergodicity (and weak mixing) of SαS processes.

The last part of this section (as well as the next) is devoted to SαS processes of type $\boldsymbol{X}^3$ (recall the decomposition (1.4)). As mentioned in Section 1, for such processes to be mixing (or even ergodic) it is necessary to be such in spite of the fact that the flow $\{\phi_t\}$ is infinitely recurrent, and so generates 'memory' within the process. It turns out that the flow should not return 'too often' to the starting point, and we now start making this statement precise. The most logical way of doing so is, of course, through the *expected return time*.

Suppose that μ is invariant under $\{\phi_t\}$. The *return time* to a set $A \in \mathcal{B}$ is defined by

$\tau_A(s) = \inf\{t > 0 : \phi_t(s) \in A\}$ $(s \in A)$ and, if $\mu(A) < \infty$,

$$\frac{1}{\mu(A)} \int_A \tau_A \, \mathrm{d}\mu$$

is the *expected return time*. To avoid measurability problems, we consider the case $\mathbb{T} = \mathbb{Z}$. Under these assumptions we obtain the following proposition.

Proposition 2.2. *Suppose that $\boldsymbol{X}^3$ is ergodic. Then the expected return time is infinite for any set A of a finite positive measure.*

Proof. Kac's theorem (see Krengel 1985, p. 19) gives

$$\int_A \tau_A \, \mathrm{d}\mu = \mu\left(\bigcup_{n=0}^{\infty} \phi_{-n}(A)\right). \tag{2.8}$$

Let $\delta = \mu(A) > 0$. Since, by Corollary 2.1,

$$\mu(A \cap \phi_{-n}(A)) = \mu(A \cap \phi_n(A)) \to 0,$$

as $n \to \infty$, for every $\theta > 0$ there is an N such that, for every $n \geq N$,

$$\mu(A \cap \phi_{-n}(A)) < \theta.$$

For every $k \geq 1$ we have

$$\begin{aligned}
\mu\left(\bigcup_{n=0}^{\infty} \phi_{-n}(A)\right) &\geq \mu\left(\bigcup_{n=0}^{k} \phi_{-nN}(A)\right) \\
&\geq \sum_{n=0}^{k} \mu(\phi_{-nN}(A)) - \sum_{n=0}^{k} \sum_{m=n+1}^{k} \mu(\phi_{-nN}(A) \cap \phi_{-mN}(A)) \\
&= (k+1)\mu(A) - \sum_{n=0}^{k} \sum_{m=n+1}^{k} \mu(A \cap \phi_{-(m-n)N}(A)) \\
&\geq (k+1)\delta - 2^{-1}k(k+1)\theta.
\end{aligned}$$

Letting first $\theta \to 0$ and then $k \to \infty$, we see immediately that the right-hand side of (2.8) is infinite. □

3. Mixing SαS processes generated by conservative flows

In this section we will show that mixing stationary SαS processes of type $\boldsymbol{X}^3$ really exist. For simplicity we assume $\mathbb{T} = \mathbb{Z}$, but see the construction at the end of the section.

Consider a bilateral real-valued Markov chain $\{S_n\}_{n \in \mathbb{Z}}$ defined on the canonical coordinate space $S = \mathbb{R}^{\mathbb{Z}}$ with a stationary transition probability function

$$Q(x, B) = P(S_{n+1} \in B \mid S_n = x).$$

Suppose that m is an invariant measure for $\{S_n\}_{n\in\mathbb{Z}}$. That is, m is a Radon measure on $\mathbb{R}$ such that

$$m(B) = \int_{\mathbb{R}} Q(x, B)m(\mathrm{d}x), \qquad \forall\, B \in \mathcal{B}_{\mathbb{R}}.$$

Denoting by Q^x the distribution of $\{S_n\}_{n\in\mathbb{Z}}$ as a random vector in $(S, \mathcal{B}_S)$ starting from $x \in \mathbb{R}$ (i.e. $Q^x\{S_0 = x\} = 1$), we define a measure μ on $(S, \mathcal{B}_S)$ by

$$\mu(A) = \int_{\mathbb{R}} Q^x(A)m(\mathrm{d}x), \qquad A \in \mathcal{B}_S. \tag{3.1}$$

Then, under (possibly infinite) law μ, $\{S_n\}$ can be viewed as a stationary Markov chain with the initial 'distribution' m. We suppose that $\{S_n\}$ is m-recurrent. That is, for all $B \in \mathcal{B}_{\mathbb{R}}$ with $m(B) > 0$,

$$\Pr(S_n \in B \text{ infinitely often} \mid S_0 = x) = 1 \qquad \text{for } m\text{-almost all } x \in \mathbb{R} \tag{3.2}$$

(see, for example, Orey 1971, p. 4). We have the following proposition.

Proposition 3.1. *The shift transformation θ on S is measure-preserving, conservative, and the following are equivalent:*

(a) the measure μ is infinite;
(b) the invariant measure m is infinite;
(c) for all bounded sets $B \in \mathcal{B}_{\mathbb{R}}$

$$\lim_{n\to\infty} \Pr(S_n \in B \mid S_0 = x) = 0 \qquad \textit{for } m\text{-almost all } x \in \mathbb{R}. \tag{3.3}$$

Furthermore, under any of the above, the flow $\phi_n := \theta^n$, $n \in \mathbb{Z}$ satisfies (c) of Corollary 2.1.

Proof. The fact that θ preserves measure μ follows simply from invariance of the measure m, while the fact that it is conservative follows from Harris and Robbins (1953).

The equivalence of (a) and (b) is trivial, as is the fact that (c) implies (b). The implication (b) $\Rightarrow$ (c) follows from Theorem 7.3 of Orey (1971).

It remains to show that (3.3) implies (c) of Corollary 2.1. Recall that S_n is the projection onto the nth coordinate of S and let

$$A_k = \{|S_0| \le k\}.$$

Then

$$\begin{aligned}\mu(A_k \cap \phi_n A_k) &= \mu(\theta^{-n} A_k \cap A_k)\\ &= \int_{[-k,k]} \Pr(S_n \in [-k,k] \mid S_0 = x)m(\mathrm{d}x) \to 0\end{aligned}$$

as $n \to \infty$ by (3.3), completing the proof. □

As an immediate consequence of Proposition 3.1, we obtain the following class of mixing stationary SαS processes generated by conservative flows.

Corollary 3.1. *Under the above assumptions, suppose that the invariant measure* m *is infinite. Let* $f \in L^{\alpha}(S, \mu)$ *and let* $a_n : S \to \{-1, 1\}$ *satisfy* $a_{n+m} = a_n \cdot a_m \circ \theta^n \mu$-a.e. *Then the stationary* $S\alpha S$ *process*

$$X_n = \int_S a_n f \circ \theta^n \, \mathrm{d}M, \tag{3.4}$$

is of type $\boldsymbol{X}^3$*, and is mixing.*

Remarks 3.1

(i) The simplest choice of parameters in Corollary 3.1 that produces a rich class of processes is to take $a_n \equiv 1$, and $f(\ldots, x_{-1}, x_0, x_1, \ldots) = \mathbf{1}(x_0 \in A)$ for a Borel subset A of $\mathbb{R}$.

(ii) Processes of this form belong to the class of doubly stationary processes (see Cambanis *et al.* 1987; Gross and Weron 1994).

(iii) The connection described above between the recurrence properties of a Markov chain $\{S_n\}$ and ergodic properties of the stationary $\mathrm{S}\alpha\mathrm{S}$ process $\{X_n\}$ becomes especially transparent if the Markov chain is, say, integer-valued. If it is null recurrent, then (3.3) holds by Theorem 69(b) in Freedman (1983, p. 25), and so the stationary $\mathrm{S}\alpha\mathrm{S}$ process is mixing. On the other hand, if the Markov chain is positive recurrent, then the invariant measure m is finite, and then so is the measure μ, implying that the $\mathrm{S}\alpha\mathrm{S}$ process is not even ergodic, as was observed above. This is another demonstration of the phenomenon presented in Proposition 2.2: mixing (or even ergodicity) of an $\mathrm{S}\alpha\mathrm{S}$ process requires the underlying flows to have infinite expected return times.

(iv) The simplest case when the conditions for mixing of (3.4) are easy to check is the case of a random walk $\{S_n\}$. Let F be the common distribution of $S_{n+1} - S_n$. It is then well known that $\{S_n\}$ is m-recurrent (with m being the Lebesgue measure if F is not concentrated on any lattice $L_d = \{nd : n \in \mathbb{Z}\}$, or the counting measure on L_d if F is concentrated on it) if and only if the characteristic function $\hat{F}$ of F satisfies

$$\lim_{s \to 1^+} \int_{-1}^{1} \Re \frac{1}{1 - s\hat{F}(u)} \, \mathrm{d}u = \infty$$

(see Chung and Fuchs 1951, or Feller 1971, Section XVII.6). If the random walk is actually concentrated on L_d, then an equivalent condition is

$$\int_{-1}^{1} \Re \frac{1}{1 - \hat{F}(u)} \, \mathrm{d}u = \infty$$

(Spitzer 1964). If $\int_{\mathbb{R}} |x| F(\mathrm{d}x) < \infty$, then a more easily verifiable necessary and sufficient condition for recurrence is $\int_{\mathbb{R}} x F(\mathrm{d}x) = 0$ (see Feller 1971, Section VI.10). Of course, for a recurrent random walk condition (3.3) follows by the concentration inequality for sums of independent random variables. Therefore, Markov shifts corresponding to recurrent random walks generate mixing $\mathrm{S}\alpha\mathrm{S}$ processes of type $\boldsymbol{X}^3$.

Remark 3.1(iv) has an obvious extension to continuous time, $\mathbb{T} = \mathbb{R}$. We replace the random walk by a bilateral Lévy process $\{S_t\}_{t \in \mathbb{R}}$ with stationary independent increments (for example, a Wiener process) having sample paths in $S := D(\mathbb{R})$. In this case the invariant initial measure m is either the counting measure on a lattice L_d or the Lebesgue measure on $\mathbb{R}$, depending on whether $F_t := \mathcal{L}\{S_{u+t} - S_u\}$, $t, u \in \mathbb{R}$ are all concentrated on the lattice L_d, or whether the latter condition does not hold. The usual notion of recurrence in this context means that $P\{\overline{\lim}_{t\to\infty} \mathbf{1}_G(S_t) = 1\} = 1$ for every open neighbourhood G of the origin and transience means that $P\{\lim_{t\to\infty} |S_t| = \infty\} = 1$. It follows from Kingman (1964) that $\{S_t\}_{t\geq 0}$ is recurrent (transient) if and only if the discrete random walk $\{S_n\}_{n\in\mathbb{Z}^+}$ is recurrent (transient).

Defining a measure μ on $S = D(\mathbb{R})$ by (3.1), where as before, Q^x is the distribution on S of $\{S_t\}_{t\in\mathbb{R}}$ starting from x, one observes, as above, that μ satisfies (c) of Corollary 2.1 with ϕ_t defined as the shift on S, $(\phi_t(s))_u = s_{u+t}$. Therefore, any SαS process given by

$$X_t = \int_S a_t f \circ \phi_t \, \mathrm{d}M, \qquad t \in \mathbb{R},$$

is mixing and of type $\boldsymbol{X}^3$. Here $f \in L^\alpha(S, \mu)$ and $a : S \to \{-1, 1\}$ satisfies $a_{t_1+t_2} = a_{t_2} a_{t_1} \circ \phi_{t_2}$ μ-a.e., for every $t_1, t_2 \in \mathbb{R}$.

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