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Lower tails of self-similar stable processes

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For a self-similar α-stable process with stationary increments $\{X(t), 0 \leqslant t \leqslant 1\}$ we study the asymptotic behaviour of the probability that the process stays within the interval $[-\varepsilon, \varepsilon]$, as ε becomes small. This behaviour turns out to be only partially determined by the index of stability α and parameter of self-similarity H.

Keywords: self-similar processes; small tails; stable processes; stationary increments

1. Introduction

Let $\{X(t), t \geqslant 0\}$ be an H-self-similar stochastic process with stationary increments. This means that for any $c > 0$,

$$\{X(ct), t \geqslant 0\} \overset{\mathrm{d}}{=} \{c^H X(t), t \geqslant 0\}$$

(property of self-similarity) and

$$\{X(t+c) - X(c), t \geqslant 0\} \overset{\mathrm{d}}{=} \{X(t) - X(0), t \geqslant 0\}$$

(property of stationarity of the increments). These processes arise naturally as the only possible limits when an arbitrary process with stationary increments undergoes a time rescaling, $X(ct)$ with $c \to \infty$, and a simultaneous space rescaling, as is known from various sources, beginning with Lamperti (1962); see also Vervaat (1987). A large number of papers have appeared on the subject in the last two decades, the interest in self-similar processes being generated by their fractal-type behaviour and by their common usage as stochastic models with long range dependence, beginning with Mandelbrot and Van Ness (1968). An extensive survey is presented in Taqqu (1986), and for more recent information the reader can consult Chapters 7 and 8 in Samorodnitsky and Taqqu (1994).

A very important class of heavy-tailed H-self-similar processes with stationary increments (H-sssi processes) is that of symmetric α-stable (SαS) H-sssi processes. These are processes as above, for which every linear combination $Y = \sum_{j=1}^{k} a_j X(t_j)$ has an SαS distribution, $0 < \alpha \leqslant 2$. That is, $\mathrm{E}\mathrm{e}^{\mathrm{i}\theta Y} = \mathrm{e}^{-\sigma^\alpha |\theta|^\alpha}$ for some $\sigma \geqslant 0$. We refer the reader to the recent books by Samorodnitsky and Taqqu (1994) and Janicki and Weron (1994) for more information on SαS random variables and processes. Note that an SαS process with $\alpha = 2$ is simply a zero-mean Gaussian process.

Once one gets away from the assumption of Gaussianity, the family of H-sssi processes becomes rich indeed, and it is therefore important to establish to what extent the properties of H-sssi processes are determined by H alone, and to which extent SαS H-sssi processes are determined by H and α alone. Much of what we know today on this subject is due to the work of Vervaat and O'Brien – see, for instance, O'Brien and Vervaat (1983) and Vervaat (1985; 1987).

In this paper we address the question of the so-called *lower tails* of SαS H-sssi processes. That is, we are interested in the behaviour of the 'small ball' probability

$$P(\sup_{0\leqslant t\leqslant 1}|X(t)|\leqslant\varepsilon),\quad \varepsilon>0, \tag{1.1}$$

as $\varepsilon\to 0$. (We always take a separable version of the process to ensure that its supremum is a well-defined random variable.) This question is important for many reasons. We mention problems related to the law of the iterated logarithm for which lower tails are of crucial importance, and problems related to the geometry of Banach spaces for which both 'small ball' and 'large ball' behaviour of stable measures is very informative as well.

The present work started when the author saw in Monrad and Rootzén (1992) the following elegant result describing the logarithmic behaviour of the lower tails of H-sssi Gaussian processes, or of fractional Brownian motions. For a fractional Brownian motion $\{X(t),\, 0\leqslant t\leqslant 1\}$ with parameter of self-similarity H one has

$$C^{-1}\varepsilon^{-1/H}\leqslant -\log P(\sup_{0\leqslant t\leqslant 1}|X(t)|\leqslant\varepsilon)\leqslant C\varepsilon^{-1/H},\qquad 0<\varepsilon<1, \tag{1.2}$$

for some $C>0$. The above paper was published as Monrad and Rootzén (1995), and (1.2) is also given in Talagrand (1995). It actually turned out that both bounds in (1.2) had been known earlier. Its left-hand side follows from Pitt (1978), while its right-hand side is a particular case of a bound given in Talagrand (1993) (historical information provided by the referee).

Suppose now that $\{X(t),\, 0\leqslant t\leqslant 1\}$ is an SαS H-sssi process, $0<\alpha<2$. To what extent does (1.2) extend to this case? Does it hold as it is? Or is there a $d>0$ such that

$$C^{-1}\varepsilon^{-d}\leqslant -\log P(\sup_{0\leqslant t\leqslant 1}|X(t)|\leqslant\varepsilon)\leqslant C\varepsilon^{-d},\qquad \varepsilon>0, \tag{1.3}$$

for some $C>0$? If there is, is the d in (1.3) determined by α and H?

Before embarking on an analysis of the above questions, let us place our problem in the general context of 'small ball' problems. Let X be an SαS random vector in a measurable vector space E, and let q be a measurable seminorm on E. The 'small ball' problem concerns the behaviour of $P(q(X)\leqslant\varepsilon)$ as $\varepsilon\to 0$, and has been discussed in many papers. When $(E,\|\cdot\|)$ is a separable Banach space, and $q(x)=\|x\|$, the best possible general upper bound is

$$P(\|X\|\leqslant\varepsilon)\leqslant K\varepsilon,\qquad \varepsilon>0,$$

where K is a finite constant that depends only on $P(\|X\|\leqslant 1)$; see Fernique (1985) for the case $\alpha=2$ and Lewandowski *et al.* (1992) for the case $0<\alpha<2$. Clearly, general bounds of this type are too crude to be of help in our analysis of lower tails of SαS H-sssi processes.

Finer estimates are available for certain particular seminorms in the Gaussian case, especially for Hilbertian norms (Sytaya 1974; Hoffmann-Jørgensen *et al.* 1979; Ibragimov 1982; Zolotarev 1986; Mayer-Wolf and Zeitouni 1993; Dembo *et al.* 1995). Moreover, one is sometimes able to relate small ball probabilities for Gaussian measures on separable Banach spaces to the metric entropy of particular sets (see Kuelbs and Li 1993; Talagrand 1993).

In the proper α-stable case $0<\alpha<2$ an important contribution is due to Ryznar (1986), where (in addition to considering specific measurable seminorms and relating small ball probability behaviour in a separable Banach space to the geometry of the space) it has been proved that

$$-\log P(q(X) \leqslant \varepsilon) \leqslant C\varepsilon^{-\alpha/(1-\alpha)}, \qquad 0<\varepsilon<1, \tag{1.4}$$

for every measurable seminorm q in a measurable vector space E and any strictly α-stable random vector X in E with $0<\alpha<1$. Further, Ryznar showed that the exponent $d = \alpha/(1-\alpha)$ in the upper bound (1.4) cannot be improved in the strictly α-stable case. The following example shows that it cannot be improved in the SαS case either.

Example 1.1. Consider an SαS random vector $\mathbf{X}$ in l^∞ given by

$$X_i = \int_0^1 f_i(x)M(\mathrm{d}x), \qquad i \geqslant 1,$$

where f_i: $(0, 1) \to \{-1, 1\}$ is, for any $i \geqslant 1$, a measurable non-random function defined below, and M is an (independently scattered) SαS random measure on (0, 1) with Lebesgue control measure. See Samorodnitsky and Taqqu (1994) for detailed information on stable random measures and stochastic integrals with respect to these measures. The functions f_i are defined as follows. For $n \geqslant 1$ let $I_j^{(n)}$ be the jth binary interval of order n, $j = 1, 2, \ldots, 2^n$, and $A_1, A_2, \ldots, A_{2^{2^n}}$ be an enumeration of $2^{\{1,2\ldots,2^n\}}$. Let

$$h_j^{(n)}(x) = \begin{cases} 1 & \text{if } x \in I_k^{(n)}, \text{ for some } k \in A_j, \\ -1 & \text{if } x \in I_k^{(n)}, \text{ for some } k \notin A_j, \end{cases}$$

$n \geqslant 1$, $j = 1, 2, \ldots, 2^{2^n}$. Finally, define $f_1 = h_1^{(1)}$, $f_2 = h_2^{(1)}$, $f_3 = h_3^{(1)}$, $f_4 = h_4^{(1)}$, $f_5 = h_1^{(2)}$, $\ldots$.

Note that an alternative way of representing $\{X_i, i \geqslant 1\}$ is by

$$X_i = c_\alpha \sum_{j=1}^{\infty} \varepsilon_j \Gamma_j^{-1/\alpha} f_i(U_j), \qquad i \geqslant 1,$$

where c_α is a finite positive constant, and $(\varepsilon_j, j \geqslant 1)$, $(U_j, j \geqslant 1)$ and $(\Gamma_j, j \geqslant 1)$ are three independent sequences of random variables. Here $(\varepsilon_j, j \geqslant 1)$ are i.i.d. Rademacher random variables $(P(\varepsilon_j = 1) = 1 - P(\varepsilon_j = -1) = 1/2)$, $(U_j, j \geqslant 1)$ are i.i.d. random variables with uniform distribution on (0, 1), and $(\Gamma_j, j \geqslant 1)$ are the arrival times of a unit-rate Poisson process (Samorodnitsky and Taqqu 1994). We claim that X belongs to l^∞, with

$$\sup_{j\geqslant 1} |X_j| = c_\alpha \sum_{j=1}^{\infty} \Gamma_j^{-1/\alpha} \text{ a.s.} \tag{1.5}$$

(Note that the right-hand side of (1.5) is finite with probability 1, because $\alpha<1$.)

Clearly, it is enough to prove that, for almost every ω,

$$\sup_{j\geq 1} X_j \geq c_\alpha \sum_{j=1}^\infty \Gamma_j^{-1/\alpha},$$

the inequality in the opposite direction being trivial. Fix an ω such that $\sum_{j=1}^\infty \Gamma_j^{-1/\alpha} < \infty$ and such that the U_j are all different. Fix a $\delta \in (0, 1/2)$, and let $N = N(\omega)$ be such that

$$\sum_{j=N+1}^\infty \Gamma_j^{-1/\alpha} \leq \delta \sum_{j=1}^\infty \Gamma_j^{-1/\alpha}.$$

There is an n such that $U_1, U_2, \ldots, U_N$ all belong to different binary intervals of order n. Suppose that $U_k \in I_{j_k}^{(n)}$, $k = 1, 2, \ldots, N$. By construction, there is an i such that

$$f_i(x) = \varepsilon_k \text{ on } I_{j_k}^{(n)}, \qquad k = 1, \ldots, N.$$

Therefore,

$$\begin{aligned} X_i &= c_\alpha \sum_{j=1}^N \Gamma_j^{-1/\alpha} + c_\alpha \sum_{j=N+1}^\infty \varepsilon_j \Gamma_j^{-1/\alpha} f_i(U_j) \\ &\geq c_\alpha \sum_{j=1}^N \Gamma_j^{-1/\alpha} - \delta c_\alpha \sum_{j=1}^\infty \Gamma_j^{-1/\alpha} \geq (1-2\delta) c_\alpha \sum_{j=1}^\infty \Gamma_j^{-1/\alpha}, \end{aligned}$$

which proves (1.5) because δ can be taken as close to 0 as we wish.

Since the random variable on the right-hand side of (1.5) is a positive strictly α-stable random variable, we conclude that, in this case,

$$-\log P(\sup_{i\geq 1} |X_i| \leq \varepsilon) \sim C\varepsilon^{-\alpha/(1-\alpha)}$$

as $\varepsilon \to 0$, and so the exponent $d = \alpha/(1-\alpha)$ in (1.4) cannot be improved even in the SαS case.

In Section 3 of this paper we will discuss the possible values of the exponent d in (1.3) for SαS H-sssi processes. We will see that, even though its value may differ between different classes of such processes, certain information on d can be obtained through H and α only. In particular, (1.2) holds for some of these processes and fails for others. The next section collects together some preliminary results.

2. Preliminary results

We start with the following lemma that extends Šidák's inequality for Gaussian random vectors (Šidák 1968) to SαS random vectors.

Lemma 2.1. *Let $\boldsymbol{X} = (X_1, \ldots, X_n)$ be an $S\alpha S$ random vector, $0 < \alpha < 2$. Then for every $x_j > 0$, $j = 1, \ldots, n$,*

$$P(|X_j| \leqslant x_j,\, j = 1, \ldots, n) \geqslant \prod_{j=1}^{n} P(|X_j| \leqslant x_j). \tag{2.1}$$

Proof. Since every SαS random vector is a weak limit of SαS random vectors with 'discrete spectral measures', that is, SαS random vectors of the form

$$X_j = \sum_{i=1}^{m} a_{ij} Y_i, \qquad j = 1, \ldots, n, \tag{2.2}$$

where $Y_1, \ldots, Y_m$ are i.i.d. $S_\alpha(1, 0, 0)$ random variables, it is enough to prove the lemma for SαS random vectors of the form (2.2). Observe that we can write

$$Y_i = G_i A_i^{1/2}, \qquad i = 1, \ldots, m,$$

where $G_1, \ldots, G_m$ are i.i.d. centred normal random variables living on, say, the probability space $(\Omega_1, \mathscr{F}_1, P_1)$, and $A_1, \ldots, A_m$ are i.i.d. $S_{\alpha/2}(1, 1, 0)$ random variables living, say, on another probability space $(\Omega_2, \mathscr{F}_2, P_2)$. We denote by E_i and P_i the expectation and probability operators taken with respect to the ith probability space, $i = 1, 2$. By Šidák's inequality for Gaussian random vectors,

$$\begin{aligned} P(|X_j| \leqslant x_j,\, j = 1, \ldots, n) &= \mathrm{E}_2\left[P_1\left(\left|\sum_{i=1}^{m} a_{ij} G_i A_i^{1/2}\right| \leqslant x_j,\, j = 1, \ldots, n,\right)\right] \\ &\geqslant \mathrm{E}_2\left[\prod_{j=1}^{n} P_1\left(\left|\sum_{i=1}^{m} a_{ij} G_i A_i^{1/2}\right| \leqslant x_j\right)\right]. \end{aligned} \tag{2.3}$$

Observe that the random variables $A_1, \ldots, A_m$ are independent, thus associated. Furthermore, each Ω_2 random variable

$$Z_j = P_1\left(\left|\sum_{i=1}^{m} a_{ij} G_i A_i^{1/2}\right| \leqslant x_j\right)$$

is a non-increasing function of each of $A_1, \ldots, A_m$, $j = 1, \ldots, n$, and so $Z_1, \ldots, Z_n$ are associated as well. Since they are also non-negative, it follows that Z_j and $\prod_{k=1}^{j-1} Z_k$ are associated for each $j = 2, \ldots, n$, and so

$$\mathrm{E}_2\left(\prod_{k=1}^{n} Z_k\right) \geqslant \mathrm{E}_2 Z_n \mathrm{E}_2\left(\prod_{k=1}^{n-1} Z_k\right) \geqslant \cdots \geqslant \prod_{k=1}^{n} \mathrm{E}_2 Z_k,$$

which, together with (2.3), proves (2.1). □

We will need the following simple estimate for conditional probabilities of hypercubes for sums of symmetric random variables.

Lemma 2.2. *Let* $Y_1, \ldots, Y_n, X_{11}, X_{12}, \ldots, X_{nm}$ *be independent symmetric random variables. Then, for all* $0 < x \leqslant y$, *we have*

$$P\left(\left|\sum_{i=1}^{k}\left(Y_i + \sum_{j=1}^{m} X_{ij}\right)\right| < y,\ k = 1, \ldots, n \,\middle|\, \left|\sum_{j=1}^{l} X_{ij}\right| < x,\ l = 1, \ldots, m,\ i = 1, \ldots, n\right) \geqslant 2^{-n}\prod_{i=1}^{n} P(0 < Y_i < y). \tag{2.4}$$

Proof. The proof is by induction on n. For $n = 1$ we need to consider

$$P\left(\left|Y_1 + \sum_{j=1}^{m} X_{1j}\right| < y \,\middle|\, \left|\sum_{j=1}^{l} X_{1j}\right| < x,\ l = 1, \ldots, m\right). \tag{2.5}$$

Let Q^* be the conditional distribution of $\sum_{j=1}^{m} X_{1j}$ given the condition in (2.5). Then the probability in (2.5) is equal to

$$\begin{aligned}\int_{-x}^{x} P(|Y_1 + z| < y)Q^*(\mathrm{d}z) &\geqslant \int_{0}^{x} P(Y_1 \in (-y, 0))Q^*(\mathrm{d}z) + \int_{-x}^{0} P(Y_1 \in (0, y))Q^*(\mathrm{d}z)\\ &= P(Y_1 \in (0, y)) \geqslant \tfrac{1}{2}P(0 < Y_1 < y)\end{aligned}$$

by the symmetry of Y_1, thus establishing the basis of the induction.

Suppose our claim is true for $n - 1 \geqslant 1$. Rewrite the probability on the left-hand side of (2.4) as

$$\begin{aligned}&P\left(\left|Y_n + \sum_{j=1}^{m} X_{nj}\right| < y \,\middle|\, \left|\sum_{i=1}^{k}\left(Y_i + \sum_{j=1}^{m} X_{ij}\right)\right| < y,\ k = 1, \ldots, n-1,\right.\\ &\qquad\qquad\left.\left|\sum_{j=1}^{l} X_{ij}\right| < x,\ l = 1, \ldots, m,\ i = 1, \ldots, n\right) \qquad (2.6)\\ &\times P\left(\left|\sum_{i=1}^{k}\left(Y_i + \sum_{j=1}^{m} X_{ij}\right)\right| < y,\ k = 1, \ldots, n-1 \,\middle|\, \left|\sum_{j=1}^{l} X_{ij}\right| < x,\ l = 1, \ldots, m,\right.\\ &\qquad\qquad\left. i = 1, \ldots, n-1\right).\end{aligned}$$

By the assumption of the induction, the second term in (2.6) is greater than or equal to

$$2^{-(n-1)}\prod_{i=1}^{n-1} P(0 < Y_i < y).$$

We consider now the first term in (2.6). Let Q^{**} be the conditional distribution of $\sum_{i=1}^{n-1}(Y_i + \sum_{j=1}^m X_{ij})$ given $|\sum_{i=1}^k (Y_i + \sum_{j=1}^m X_{ij})| < y$, $k = 1, \ldots, n-1$, $|\sum_{j=1}^l X_{ij}| < x$, $l = 1, \ldots, m$, $i = 1, \ldots, n-1$, and let Q^{***} be the conditional distribution of $\sum_{j=1}^m X_{nj}$ given $|\sum_{j=1}^l X_{nj}| < x$, $l = 1, \ldots, m$. Note that both Q^{**} and Q^{***} are symmetric about the origin. Then the first term in (2.6) is

$$= \int_{-y}^{y} Q^{**}(\mathrm{d}z_1) \int_{-x}^{x} Q^{***}(\mathrm{d}z_2) P(|Y_n + z_1 + z_2| < y)$$

$$\geqslant 2\int_{0}^{y} Q^{**}(\mathrm{d}z_1) \int_{-x}^{0} Q^{***}(\mathrm{d}z_2) P(|Y_n + z_1 + z_2| < y).$$

Clearly, $|z_1 + z_2| \leqslant y$ under the above integral, implying, by the symmetry of Y_n, that

$$P(|Y_n + z_1 + z_2| < y) \geqslant P(0 < Y_n < y).$$

Therefore, the first term in (2.6) is

$$\geqslant 2P(0 < Y_n < y) \int_0^y Q^{**}(\mathrm{d}z_1) \int_{-x}^0 Q^{***}(\mathrm{d}z_2) = \tfrac{1}{2} P(0 < Y_n < y),$$

completing the proof of the lemma. □

3. Small ball probabilities for self-similar SαS processes

Let $\{X(t), t \geqslant 0\}$ be an SαS H-sssi process, $0 < \alpha < 2$, given in the form

$$X(t) = \int_S f(t, x) M(\mathrm{d}x), \qquad t \geqslant 0, \tag{3.1}$$

where M is an SαS random measure on a measurable space $(S, \mathscr{S})$ with a σ-finite control measure m, and $f(t, \cdot) \in L^\alpha(S, \mathscr{S}, m)$ for all $t \geqslant 0$. We will always assume that the process is continuous in probability, in which case we must have $H > 0$, unless the process is constant with probability 1 – a not very interesting case which we do not consider (Vervaat 1987). Furthermore, moment considerations (see Maejima 1986) imply that the feasible range for the pair (H, α) is

$$0 < H \leqslant 1/\alpha \qquad \text{if } \alpha < 1$$

and

$$0 < H \leqslant 1 \qquad \text{if } 1 \leqslant \alpha < 2.$$

Our first observation is that one can obtain, in certain cases, a general lower bound in (1.3).

Theorem 3.1. *Let $\{X(t),\ t \geqslant 0\}$ be an SαS H-sssi process which is continuous in probability. Suppose that there is a $b > 0$ such that*

$$\int_{\mathbb{R}^n} \exp\left(-\int_S \left|\sum_{j=1}^n y_j f(j, x)\right|^\alpha m(\mathrm{d}x)\right) \mathrm{d}\mathbf{y} \leqslant b^n \tag{3.2}$$

for all $n \geqslant 1$. Then there is a $C > 0$ such that

$$-\log P(\sup_{0 \leqslant t \leqslant 1} |X(t)| \leqslant \varepsilon) \geqslant C\varepsilon^{-1/H} \tag{3.3}$$

for all $0 < \varepsilon < 1$.

Proof. We are using the obvious fact that the density $f_{\mathbf{X}}$ of any non-degenerate SαS random vector $\mathbf{X} = (X_1, \ldots, X_n)$ satisfies

$$\begin{aligned} f_{\mathbf{X}}(x_1, \ldots, x_n) &\leqslant f_{\mathbf{X}}(0, \ldots, 0) \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \mathrm{E} \exp\left(\mathrm{i} \sum_{j=1}^n y_j X_j\right) \mathrm{d}\mathbf{y}. \end{aligned}$$

Therefore, for every $\varepsilon > 0$ and $n \geqslant 1$ we have, using the self-similarity property,

$$\begin{aligned} P(\sup_{0 \leqslant t \leqslant 1} |X(t)| \leqslant \varepsilon) &= P(\sup_{0 \leqslant t \leqslant n} |X(t)| \leqslant n^H \varepsilon) \\ &\leqslant P(|X(j)| \leqslant n^H \varepsilon,\ j = 1, \ldots, n) \\ &\leqslant \frac{1}{\pi^n} \varepsilon^n n^{nH} \int_{\mathbb{R}^n} \exp\left(-\int_S \left|\sum_{j=1}^n y_j f(j, x)\right|^\alpha m(\mathrm{d}x)\right) \mathrm{d}\mathbf{y} \\ &\leqslant \frac{1}{\pi^n} \varepsilon^n n^{nH} b^n \end{aligned} \tag{3.4}$$

by (3.2). Since (3.4) holds for every n, we may choose

$$n = [\mathrm{e}^{-1} \varepsilon^{-1/H} (b/\pi)^{-1/H}].$$

We then have

$$\begin{aligned} P(\sup_{0 \leqslant t \leqslant 1} |X(t)| \leqslant \varepsilon) &\leqslant \exp\left(n \log\left(\frac{b}{\pi} \varepsilon n^H\right)\right) \\ &\leqslant \mathrm{e}^{-nH} \leqslant \exp\left(-H\left(\mathrm{e}^{-1} \varepsilon^{-1/H} \left(\frac{b}{\pi}\right)^{-1/H}\right) - 1\right), \end{aligned}$$

thus completing the proof of the theorem. □

This result applies immediately to the following situation.

Example 3.1 *Lévy SαS motion*. This is the simplest example of SαS H-sssi processes: a process with stationary and independent SαS-distributed increments. The process can be written formally in the form

$$X(t) = \int_0^\infty \mathbf{1}(x \leqslant t) M(\mathrm{d}x), \qquad t \geqslant 0, \tag{3.5}$$

where M is an SαS random measure with Lebesgue control measure. The exponent of self-similarity is, in this case, $H = 1/\alpha$. It is the unique SαS H-sssi process with such an H if $0 < \alpha < 1$, but not so in the case $1 \leqslant \alpha < 2$; see Kasahara *et al.* (1988) and Samorodnitsky and Taqqu (1990). We will see that (1.2) extends directly to the case of SαS Lévy motion. That is, there is a finite positive constant $C = C(\alpha)$ such that, for every $0 < \varepsilon < 1$,

$$C^{-1}\varepsilon^{-\alpha} \leqslant -\log P(\sup_{0 \leqslant t \leqslant 1} |X(t)| \leqslant \varepsilon) \leqslant C\varepsilon^{-\alpha}. \tag{3.6}$$

The lower bound in (3.6) is a simple application of Theorem 3.1. We need to check condition (3.2). Since in the present case

$$f(t, x) = \mathbf{1}(x \leqslant t), \qquad x \geqslant 0,\ t \geqslant 0,$$

we conclude that

$$\begin{aligned} \int_{\mathbb{R}^n} \exp\left(-\int_0^\infty \left| \sum_{j=1}^n y_j f(j, x) \right|^\alpha m(\mathrm{d}x) \right) \mathrm{d}\mathbf{y} &= \int_{\mathbb{R}^n} \exp\left(-\sum_{j=1}^n \sum_{k=j}^n |y_k|^\alpha \right) \mathrm{d}\mathbf{y} \\ &= \left(\int_{-\infty}^\infty \mathrm{e}^{-|y|^\alpha}\,\mathrm{d}y \right)^n := b^n, \end{aligned}$$

thus verifying (3.2), and so proving the lower bound in (3.6).

It is easy to get a matching upper bound. By separability,

$$P(\sup_{0 \leqslant t \leqslant 1} |X(t)| \leqslant \varepsilon) = \lim_{n\to\infty} P\left(\max_{i=1,\ldots,2^n} \left| X\left(\frac{i}{2^n}\right) \right| < \varepsilon \right). \tag{3.7}$$

Let $Y_1, Y_2, \ldots$ be i.i.d. $S_\alpha(1, 0, 0)$ random variables. By the $1/\alpha$-self-similarity of the process and independence of the increments we have, for every $n > \varepsilon^{-2\alpha}$,

$$\begin{aligned} P\left(\max_{i=1,\ldots,2^n} \left| X\left(\frac{i}{2^n}\right) \right| < \varepsilon \right) &= P(\max_{i=1,\ldots,2^n} |Y_1 + \cdots + Y_i| < \varepsilon 2^{n/\alpha}) \\ &\geqslant P\bigg(|Y_1 + \cdots + Y_{j2^{n-n_0}}| < \frac{\varepsilon}{2} 2^{n/\alpha},\ j = 1, 2, 3, \ldots, 2^{n_0}, \\ &\quad |Y_{(j-1)2^{n-n_0}+1} + \cdots + Y_{(j-1)2^{n-n_0}+k}| < \frac{\varepsilon}{2} 2^{n/\alpha},\ j = 1, 2, 3, \ldots, 2^{n_0},\ k = 1, \ldots, 2^{n-n_0} - 1 \bigg) \\ &= P\bigg(|Y_1 + \cdots + Y_{j2^{n-n_0}}| < \frac{\varepsilon}{2} 2^{n/\alpha},\ j = 1, 2, 3, \ldots, 2^{n_0} \bigg| |Y_{(j-1)2^{n-n_0}+1} + \cdots \end{aligned}$$

$$+ \left. Y_{(j-1)2^{n-n_0}+k} \right| < \frac{\varepsilon}{2} 2^{n/\alpha},\ j = 1, 2, 3, \ldots, 2^{n_0},\ k = 1, \ldots, 2^{n-n_0} - 1 \Bigg)$$

$$P\left(|Y_{(j-1)2^{n-n_0}+1} + \cdots + Y_{(j-1)2^{n-n_0}+k}| < \frac{\varepsilon}{2} 2^{n/\alpha},\ j = 1, \ldots, 2^{n_0},\ k = 1, \ldots, 2^{n-n_0} - 1 \right). \tag{3.8}$$

Here n_0 is given by

$$2^{n_0} = [M\varepsilon^{\alpha}], \tag{3.9}$$

with M satisfying

$$M > \max(2,\ x_0 c_0^{-1} 2^{-(1+\alpha)}),$$

where c_0 and x_0 are as in (3.13) and (3.14) below. By Lemma 2.2 the first term on the right-hand side of (3.8) is greater than or equal to

$$2^{-2^{n_0}} (P(0 < Y_1 < \varepsilon_2 2^{n/\alpha}))^{2^{n_0}}$$

which converges, as $n \to \infty$, to

$$2^{-2(2^{n_0})} \geqslant \mathrm{e}^{-2\log(2M\varepsilon^{-\alpha})}.$$

Therefore, we only need to get a matching lower bound on the second term on the right-hand side of (3.8). Observe that it can be written as

$$\begin{aligned}
(P(\max_{i=1,\ldots,2^{n-n_0}-1} |Y_1 + \cdots + Y_i| < \varepsilon 2^{n/\alpha}))^{2^{n_0}} \\
&\geqslant (1 - 2P(\max_{i=1,\ldots,2^{n-n_0}-1} (Y_1 + \cdots + Y_i) \geqslant \varepsilon 2^{n/\alpha}))^{2^{n_0}} \\
&\geqslant (1 - 4P(Y_1 + \cdots + Y_{2^{n-n_0}} \geqslant \varepsilon 2^{n/\alpha}))^{2^{n_0}} \\
&= (1 - 4P(Y_1 \geqslant \varepsilon 2^{n_0/\alpha}))^{2^{n_0}} \\
&\geqslant (1 - 4(2^{\alpha}) c_0 \varepsilon^{-\alpha} 2^{-n_0})^{2^{n_0}} \geqslant \exp(-x_0 c_0^{-1} 2^{-(1+\alpha)} \varepsilon^{-\alpha})
\end{aligned}$$

by Lévy's inequality, (3.13), (3.14) and the choice of M. This establishes (3.6) completely.

In general, to obtain upper bounds in (1.3) one can use the following theorem.

Theorem 3.2. *Let $\{X(t),\ t \geqslant 0\}$ be an SαS H-sssi process which is continuous in probability and bounded with probability 1 on compact intervals.*

(i) If $0 < \alpha < 1$, then there is a $C > 0$ such that

$$-\log P(\sup_{0 \leqslant t \leqslant 1} |X(t)| \leqslant \varepsilon) \leqslant C\varepsilon^{-\alpha/(1-\alpha)} \tag{3.10}$$

for all $0 < \varepsilon < 1$.

(ii) If $1 < \alpha < 2$ and $1/\alpha < H \leqslant 1$, then there is a $C > 0$ and a $\theta > 0$, both of which

depend only on α and H, such that

$$-\log P(\sup_{0\leqslant t\leqslant 1}|X(t)|\leqslant \varepsilon)\leqslant C\varepsilon^{-1/H}\left(\log\frac{1}{\varepsilon}\right)^{\theta} \tag{3.11}$$

for all $0<\varepsilon<\mathrm{e}^{-1}$.

Proof. Of course, (i) follows from Ryznar's result (1.4). Our approach to (ii) is similar to that of Monrad and Rootzén (1995). By the continuity in probability we have

$$\sup_{0\leqslant t\leqslant 1}|X(t)|\leqslant \sum_{n=1}^{\infty}\max_{i=1,\ldots,2^n}|X(i2^{-n})-X((i-1)2^{-n})|.$$

Therefore, by Lemma 2.1 and the H-sssi property,

$$\begin{aligned}P\left(\sup_{0\leqslant t\leqslant 1}|X(t)|\leqslant \frac{\pi^2}{6}\varepsilon\right) &= P\left(\sup_{0\leqslant t\leqslant 1}|X(t)|\leqslant \sum_{n=1}^{\infty}n^{-2}\varepsilon\right)\\ &\geqslant P(|X(i2^{-n})-X((i-1)2^{-n})|\leqslant n^{-2}\varepsilon,\ i=1,\ldots,n,\ n=1,2,\ldots)\\ &\geqslant \prod_{n=1}^{\infty}(P(|X(2^{-n})|\leqslant n^{-2}\varepsilon))^{2^n}=\prod_{n=1}^{\infty}(P(|Y|\leqslant n^{-2}2^{nH}\varepsilon))^{2^n},\end{aligned} \tag{3.12}$$

where Y is an $S_\alpha(1, 0, 0)$ random variable. Let c_0 be such that, for every $t>0$,

$$P(|Y|>t)\leqslant c_0t^{-\alpha}, \tag{3.13}$$

and let x_0 be such that for every $x>x_0$,

$$\left(1-\frac{1}{x}\right)^{x}\geqslant \mathrm{e}^{-2}. \tag{3.14}$$

Finally, let $c_1=c_1(H)$ and $c_2=c_2(H)$ be such that with

$$n_0=\left[c_1+c_2\log_2\log_2\frac{1}{\varepsilon}+\frac{1}{H}\log_2\frac{1}{\varepsilon}\right] \tag{3.15}$$

we have, for every $0<\varepsilon<\mathrm{e}^{-1}$ and $n\geqslant n_0$,

$$c_0n^{2\alpha}2^{-\alpha Hn}\varepsilon^{-\alpha}\leqslant x_0^{-1}. \tag{3.16}$$

Then, letting c denote a finite positive constant that depends only on α and H, and that may change from line to line, and using, in sequence, the expressions (3.13), (3.16), (3.14), (3.16) again, and finally (3.15), we obtain, for every $0<\varepsilon<\mathrm{e}^{-1}$,

$$
\begin{aligned}
\prod_{n=n_0}^{\infty} (P(|Y| \leqslant n^{-2}2^{nH}\varepsilon))^{2^n} &\geqslant \prod_{n=n_0}^{\infty} (1 - c_0 n^{2\alpha} 2^{-\alpha H n} \varepsilon^{-\alpha})^{2^n} \\
&\geqslant \prod_{n=n_0}^{\infty} \exp(-2c_0 n^{2\alpha} 2^{-\alpha H n} \varepsilon^{-\alpha} 2^n) \\
&= \exp\left(-2c_0 \varepsilon^{-\alpha} \sum_{n=n_0}^{\infty} n^{2\alpha} 2^{-n(\alpha H - 1)}\right) \geqslant \exp(-c\varepsilon^{-\alpha} n_0^{2\alpha} 2^{-n_0(\alpha H-1)}) \\
&\geqslant \exp(-c2^{n_0}) \geqslant \exp\left(-c\varepsilon^{-1/H}\left(\log\frac{1}{\varepsilon}\right)^{c_2}\right). \qquad (3.17)
\end{aligned}
$$

Furthermore, using the fact that Y has a positive continuous density and (3.15), we obtain, for every $0 < \varepsilon < \mathrm{e}^{-1}$ (with the same agreement on a generic constant c),

$$
\begin{aligned}
\prod_{n=1}^{n_0-1} (P(|Y| \leqslant n^{-2}2^{nH}\varepsilon))^{2^n} &\geqslant \prod_{n=1}^{n_0-1} (P(|Y| \leqslant c\varepsilon))^{2^n} \\
&= (P(|Y| \leqslant c\varepsilon))^{2^{n_0}-1} \geqslant (c^{-1}\varepsilon)^{2^{n_0}-1} \\
&\geqslant (c^{-1}\varepsilon)^{c\varepsilon^{1/H}(\log\frac{1}{\varepsilon})^{c_2}} \geqslant \exp\left(-c\varepsilon^{1/H}\left(\log\frac{1}{\varepsilon}\right)^{c_2+1}\right),
\end{aligned}
$$

which, together with (3.17), completes the proof of part (ii) of the theorem. □

One of the natural counterparts of fractional Brownian motion in the SαS case, $0 < \alpha < 2$, is the (one-sided) linear fractional SαS motion

$$
X(t) = \int_{-\infty}^{\infty} ((t-x)_+^{H-1/\alpha} - (-x)_+^{H-1/\alpha}) M(\mathrm{d}x), \qquad t \geqslant 0, \qquad (3.18)
$$

$0 < H < 1$, $H \neq 1/\alpha$. Here M is an SαS random measure with Lebesgue control measure. This process is, of course, just fractional Brownian motion when $\alpha = 2$. However, in the case $0 < \alpha < 2$, the one-sided linear fractional SαS motion is only one of the possible H-sssi processes with H in the above range; see, for example, Samorodnitsky and Taqqu (1994). Our theorems above allow one to show that (1.2) essentially extends to this process.

Example 3.2 *One-sided linear fractional SαS motion*. We claim that for every $1 < \alpha < 2$ and $1 > H > 1/\alpha$, there is a finite positive constant $C = C(\alpha, H)$ and a $\theta > 0$ such, that for all $0 < \varepsilon < \mathrm{e}^{-1}$,

$$
C^{-1}\varepsilon^{-1/H} \leqslant -\log P(\sup_{0 \leqslant t \leqslant 1} |X(t)| \leqslant \varepsilon) \leqslant C\varepsilon^{-1/H}\left(\log\frac{1}{\varepsilon}\right)^{\theta}. \qquad (3.19)
$$

The upper bound in (3.19) follows directly from part (ii) of Theorem 3.2, and the lower bound will follow directly from Theorem 3.1, once we check that (3.2) holds in this case. We have

$$\begin{aligned}
&\int_{\mathbb{R}^n} \exp\left(-\int_{-\infty}^{\infty}\left|\sum_{j=1}^n y_j f(j, x)\right|^\alpha m(\mathrm{d}x)\right)\mathrm{d}\mathbf{y} \\
&\quad= \int_{\mathbb{R}^n} \exp\left(-\int_{-\infty}^{\infty}\left|\sum_{j=1}^n y_j((j-x)_+^{H-1/\alpha} - (-x)_+^{H-1/\alpha})\right|^\alpha \mathrm{d}x\right)\mathrm{d}\mathbf{y} \\
&\quad\leqslant \int_{\mathbb{R}^n} \exp\left(-\int_0^{\infty}\left|\sum_{j=1}^n y_j(j-x)_+^{H-1/\alpha}\right|^\alpha \mathrm{d}x\right)\mathrm{d}\mathbf{y} \\
&\quad= \int_{\mathbb{R}^n} \prod_{i=1}^n \exp\left(-\int_{i-1}^{i}\left|\sum_{j=i}^n y_j(j-x)^{H-1/\alpha}\right|^\alpha \mathrm{d}x\right)\mathrm{d}\mathbf{y}.
\end{aligned}$$

Now, for every fixed $y_2, \ldots, y_n$ we have, since $\alpha > 1$,

$$\begin{aligned}
&\int_{\mathbb{R}} \exp\left(-\int_0^1\left|\sum_{j=1}^n y_j(j-x)^{H-1/\alpha}\right|^\alpha \mathrm{d}x\right)\mathrm{d}y_1 \\
&\quad\leqslant \int_{\mathbb{R}} \exp\left(-\left(\int_0^1\left|\sum_{j=1}^n y_j(j-x)^{H-1/\alpha}\right| \mathrm{d}x\right)^\alpha\right)\mathrm{d}y_1 \\
&\quad\leqslant \int_{\mathbb{R}} \exp\left(-\left|\int_0^1\left(y_1(1-x)^{H-1/\alpha} + \sum_{j=2}^n y_j(j-x)^{H-1/\alpha}\right)\mathrm{d}x\right|^\alpha\right)\mathrm{d}y_1 \\
&\quad= \int_{\mathbb{R}} \exp\left(-\left|y_1(H+1-1/\alpha)^{-1} + k(y_2, \ldots, y_n)\right|^\alpha\right)\mathrm{d}y_1 \\
&\quad= \int_{\mathbb{R}} \exp\left(-\left|y_1(H+1-1/\alpha)^{-1}\right|^\alpha\right)\mathrm{d}y_1 := b.
\end{aligned}$$

Here $k(y_2, \ldots, y_n)$ is a finite number (which depends on $y_2, \ldots, y_n$). Therefore, we conclude that

$$\begin{aligned}
&\int_{\mathbb{R}^n} \prod_{i=1}^n \exp\left(-\int_{i-1}^{i}\left|\sum_{j=i}^n y_j(j-x)^{H-1/\alpha}\right|^\alpha \mathrm{d}x\right)\mathrm{d}y_1\,\mathrm{d}y_2 \ldots \mathrm{d}y_n \\
&\quad\leqslant b\int_{\mathbb{R}^{n-1}} \prod_{i=2}^n \exp\left(-\int_{i-1}^{i}\left|\sum_{j=i}^n y_j(j-x)^{H-1/\alpha}\right|^\alpha \mathrm{d}x\right)\mathrm{d}y_2 \ldots \mathrm{d}y_n,
\end{aligned}$$

and continuing in exactly the same manner we obtain that this expression is bounded from above by b^n, checking (3.2), and so completely proving (3.19).

Remarks

(i) Example 3.2 shows that the bound on the lower tail exponent d provided by Theorem 3.1 cannot be improved in general. We do not know whether the logarithmic term is necessary in (3.11), but we conjecture that is not necessary in (3.19).

(ii) The restriction on the parameters $1 < \alpha < 2$ and $1/\alpha < H < 1$ in Example 3.2 is completely natural, and is not due to the methods we employ. Indeed, linear fractional SαS motions with $0 \leqslant \alpha \leqslant 1$ or that with $1 < \alpha < 2$ and $0 < H < 1/\alpha$ are unbounded with probability 1 on any interval of positive length (a completely different behaviour from that of fractional Brownian motion). See, for example, Samorodnitsky and Taqqu (1994).

The cases of Lévy SαS motion and one-sided linear fractional SαS motion may make it appear that (1.2) must be true for SαS H-sssi processes with $0 < \alpha < 2$. The following example shows, however, that this is not the case.

Example 3.3 *Sub-fractional Brownian motion*. We consider the SαS H-sssi process $\{X(t),\ t \geqslant 0\}$ defined by

$$X(t) = \int_\Omega f(t, x) M(\mathrm{d}x), \qquad t \geqslant 0, \tag{3.20}$$

where $(\Omega, \mathscr{F}, P)$ is a probability space, M has control measure P, and $\{f((t, \cdot),\ t \geqslant 0\}$ is, under P, a fractional Brownian motion. Alternatively, one can represent our process in the form

$$X(t) = c(\alpha, H) A^{1/2} Y(t), \qquad t \geqslant 0, \tag{3.21}$$

where A is an $S_{\alpha/2}(1, 1, 0)$ random variable, independent of the fractional Brownian motion $\{Y(t),\ t \geqslant 0\}$, and $c(\alpha, H)$ is a finite positive constant. The process $\{X(t),\ t \geqslant 0\}$ defined by either (3.20) or (3.21) is called sub-fractional Brownian motion. It is clearly an SαS H-sssi process, with the same exponent of self-similarity H as the underlying fractional Brownian motion $\{Y(t),\ t \geqslant 0\}$. We will see that this process satisfies

$$C^{-1} \varepsilon^{-\frac{2\alpha}{2-\alpha+2\alpha H}} \leqslant -\log P(\sup_{0 \leqslant t \leqslant 1} |X(t)| \leqslant \varepsilon) \leqslant C \varepsilon^{-\frac{2\alpha}{2-\alpha+2\alpha H}}, \qquad 0 < \varepsilon < 1, \tag{3.22}$$

for some $C > 0$.

Indeed, (3.22) follows immediately from (1.2) and the fact that

$$-\log P(A < \varepsilon) \sim c\varepsilon^{-\frac{\alpha}{2-\alpha}},\ \varepsilon \to 0$$

(recall that c stands for a finite positive constant that may change from line to line), by noticing that, on one hand, for every $0 < \varepsilon < 1$ and $0 < \theta < 1$,

$$\begin{aligned} P(\sup_{0 \leqslant t \leqslant 1} |X(t)| \leqslant \varepsilon) &= P(A^{1/2} \sup_{0 \leqslant t \leqslant 1} |Y(t)| \leqslant \varepsilon) \\ &\geqslant P(A^{1/2} \leqslant \varepsilon^\theta) P(\sup_{0 \leqslant t \leqslant 1} |Y(t)| \leqslant \varepsilon^{1-\theta}) \\ &\geqslant \exp(-c(\varepsilon^{2\theta})^{-\frac{\alpha}{2-\alpha}}) \exp(-c(\varepsilon^{1-\theta})^{-\alpha}) \end{aligned}$$

and then choosing $\theta = (2-\alpha)/(4-\alpha)$, and on the other hand

$$P(\sup_{0\leqslant t\leqslant 1} |X(t)| \leqslant \varepsilon) \leqslant P(A^{1/2} \leqslant \varepsilon^{\theta}) + P(\sup_{0\leqslant t\leqslant 1} |Y(t)| \leqslant \varepsilon^{1-\theta})$$

$$\leqslant \exp(-c(\varepsilon^{2\theta})^{-\frac{\alpha}{2-\alpha}}) + \exp(-c(\varepsilon^{1-\theta})^{-\alpha})$$

and choosing the same θ as above.

Note that the lower tail exponent d in (3.22) is

$$\frac{2\alpha}{2-\alpha+2\alpha H} < \frac{1}{H}$$

for $0<\alpha<2$. Therefore, the bound of Theorem 3.2(ii) (in the case $1/\alpha < H < 1$) is not sharp for sub-fractional Brownian motion. The dependence structure of SαS H-sssi processes with $0<\alpha<2$ varies significantly from one class of such processes to another, causing the effect demonstrated in this paper: even though there are bounds on the lower tail exponent d that depend only on H and α, d may actually depend on a particular chosen class of the processes.

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