Between Strassen and Chung normalizations for Lévy's area process*

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Let $\{L(t): t \geq 0\}$ be Lévy's area process, let $\gamma: \mathbf{R}_+ \mapsto \mathbf{R}$, and let $\{Z_t: t \geq 3\}$ be the stochastic process defined by $Z_t(s) = L(ts)/(2t \log \log t)$, $0 \leq s \leq 1$. Conditions on γ are given such that the set of all limit points of $\{\gamma(t)Z_t: t \geq 3\}$ as $t \to \infty$ is a.s. equal to the set of all continuous functions defined on $[0, 1]$ which vanish at 0.

1. Introduction

Let $B = \{B(t): t \geq 0\}$ be an m-dimensional Brownian motion, let $\gamma: \mathbf{R}_+ \mapsto \mathbf{R}$, let $\mathscr{C}_0^m$ denote the set of all $\mathbf{R}^m$-valued continuous functions defined on $[0, 1]$ which vanish at 0, endowed with the uniform topology, and let

$$\phi(t) = \begin{cases} 1, & \text{for } 0 < t < 3, \\ \log \log t, & \text{for } t \geq 3. \end{cases}$$

Let us consider the following conditions.

(A) $\gamma(t) \to \infty$ as $t \to \infty$.
(Bδ_0) There exist $M > 0$ and $0 < \delta < \delta_0 \leq 2$ such that $\gamma(t) \leq M\phi^\delta(t)$ for t sufficiently large.

Baldi and Roynette (1992a) proved the following result:

Theorem 1.1. *Assume* (A) *and* (B1)*, and set*

$$X_t(s) = \frac{B(ts)}{\{2t\phi(t)\}^{1/2}}, \qquad 0 \leq s \leq 1, \qquad t > 0.$$

Then the set of all limit points of $\{\gamma(t)X_t: t > 0\}$ is a.s. equal to $\mathscr{C}_0^m$.

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Consider now Lévy's (1949) area process $L = \{L(t): t \geqslant 0\}$ associated with a two-dimensional Brownian motion $B = (B_1, B_2)$:

$$L(t) = \tfrac{1}{2}\left(\int_0^t B_1(u)B_2(\mathrm{d}u) - \int_0^t B_2(u)B_1(\mathrm{d}u)\right), \qquad t \geqslant 0, \qquad (1)$$

as the stochastic analogue of the area contained in a lens-shaped domain. Some applications of this process in analysis and in statistical inference can be found in the work of Gaveau (1977) and Liptser and Shiryaev (1977, p. 212), respectively. For L the classical law of the iterated logarithm (see, for example, Berthuet 1979) as well as its functional form (see, for example, Baldi 1986 and Helmes *et al.* 1986) have been proved. Chung's version of the law of the iterated logarithm for the area process and related processes was recently established by Rémillard (1994). In view of these results and since the area process shares many properties with the linear Brownian motion (see, for example, Helmes 1986), one would expect that an analogue of Theorem 1.1 holds for L. The aim of this article is to confirm this assertion. More precisely, we prove the following main result.

Theorem 1.2. *Assume* (A) *and* (B2)*, and set*

$$Z_t(s) = \frac{L(ts)}{2t\phi(t)}, \qquad 0 \leqslant s \leqslant 1, \qquad t > 0.$$

Then the set of all limit points of $\{\gamma(t)Z_t: t > 0\}$ *is a.s. equal to* $\mathscr{C}_0^1$.

Remark 1.3. (a) The classical law of the iterated logarithm states that $[-1/\pi, 1/\pi]$ is a.s. the set of all limit points of $\{Z_t(1): t > 0\}$ as $t \to \infty$. Therefore, for every function γ such that $\gamma(t) \to \infty$ as $t \to \infty$, the set of all limit points of $\{\gamma(t)Z_t(1): t > 0\}$ is a.s. equal to **R**. Thus Theorem 1.2 is the analogue of this result when the classical law is replaced by that of Strassen.

(b) Theorem 1.2 is optimal since Chung's form of the law of the iterated logarithm in Rémillard (1994) says that a.s.

$$\liminf_{t\to\infty} \frac{\phi(t)}{2t} \sup_{0\leqslant s\leqslant 1} |L(ts)| = \frac{\pi}{8}. \qquad (2)$$

It follows from (2) that, if $\gamma(t) \sim \phi^2(t)$ as $t \to \infty$, then every point within the ball in $\mathscr{C}_0^1$ of radius $\pi/8$ and centre 0 is not a limit point of $\{\gamma(t)Z_t: t > 0\}$.

In Section 2 we give three auxiliary lemmas which we need for the proof of our main result in Section 3.

In what follows we shall use the following notation: $|\cdot|$, Euclidean norm in $\mathbf{R}^m$; $\|\cdot\|$, the uniform norm in $\mathscr{C}_0^m$; $\langle\cdot, \cdot\rangle$, the scalar product;

$$H^m = \left\{f \in \mathscr{C}_0^m: f(s) = \int_0^s \dot{f}(u)\,\mathrm{d}u \text{ with } \int_0^1 |\dot{f}(u)|^2\,\mathrm{d}u < \infty\right\}.$$

Further let $\mu: \mathscr{C}_0^m \mapsto \bar{\mathbf{R}}_+$ be the application defined by

$$\mu(f) = \begin{cases} \frac{1}{2} \int_0^1 |\dot{f}(u)|^2 \, \mathrm{d}u, & \text{for } f \in H^m, \\ \infty, & \text{otherwise,} \end{cases}$$

and let $F: H^2 \mapsto \mathscr{C}_0^1$ be the functional associated with the area process: for every $f = (f_1, f_2) \in H^2$,

$$F(f)(t) = \frac{1}{2}\left(\int_0^t f_1(u)\dot{f}_2(u) \, \mathrm{d}u - \int_0^t f_2(u)\dot{f}_1(u) \, \mathrm{d}u \right), \qquad t \geqslant 0.$$

Let us also consider the application $\lambda: \mathscr{C}_0^1 \mapsto \bar{\mathbf{R}}_+$ such that

$$\lambda(g) = \begin{cases} \inf\{\mu(f): F(f) = g\}, & \text{for } F^{-1}(\{g\}) \neq \varnothing, \\ \infty, & \text{otherwise.} \end{cases}$$

It is known that, if $\lambda(g) < \infty$, then there exists $f \in H^2$ such that $F(f) = g$ and $\mu(f) = \lambda(g)$ (see, for example, Azencott 1980, p. 69).

Finally, observe that, if we consider the skew-symmetric matrix

$$J = \begin{pmatrix} 0 & -0.5 \\ 0.5 & 0 \end{pmatrix},$$

then (1) becomes

$$L(t) = \int_0^t \langle JB(u), B(\mathrm{d}u) \rangle, \qquad t \geqslant 0. \tag{3}$$

2. Auxiliary results

We begin with the following lemma.

Lemma 2.1. *Let $g \in \mathscr{C}_0^1$ such that $\lambda(g) = \mu(f) < \infty$ for some f in H^2. Then, for any $r > 0$, we have*

$$P(\|L - g\| \leqslant r) \geqslant \mathrm{e}^{-\lambda(g)} \; P\left(\|L\| \leqslant \frac{r}{2}, \|B\| \leqslant \frac{r}{2c(f)} \right),$$

where $c(f) = 3\{2\mu(f)\}^{1/2}$.

Proof. Let $f \in H^2$ be such that $F(f) = g$ and $\mu(f) = \lambda(g)$. Set $\tilde{B} = B - f$. By virtue of Girsanov's theorem, $\tilde{B}$ is a two-dimensional Brownian motion under the probability measure $\tilde{Q}$ defined by the Radon–Nikodym derivative

$$\frac{\mathrm{d}P}{\mathrm{d}\tilde{Q}} = \exp\left(-\int_0^1 \langle \dot{f}(s), \tilde{B}(\mathrm{d}s) \rangle - \mu(f) \right).$$

Set

$$\tilde{X}(t) = 2\int_0^t \langle J\tilde{B}(s), \dot{f}(s)\rangle \, \mathrm{d}s - \langle J\tilde{B}(t), f(t)\rangle.$$

Then $\|\tilde{X}\| \leqslant c(f)\|\tilde{B}\|$. Moreover,

$$\tilde{X}(t) = \int_0^t \langle J\tilde{B}(s), \dot{f}(s)\rangle \, \mathrm{d}s + \int_0^t \langle Jf(s), \tilde{B}(\mathrm{d}s)\rangle, \qquad \text{for all } t \geqslant 0, \qquad \tilde{Q} \text{ a.s.,}$$

and

$$L(t) - g(t) = \tilde{L}(t) + \tilde{X}(t), \qquad \text{for all } t \geqslant 0, \qquad \tilde{Q} \text{ a.s.,}$$

where

$$\tilde{L}(t) = \int_0^t \langle J\tilde{B}(s), \tilde{B}(\mathrm{d}s)\rangle.$$

Hence

$$\|L - g\| = \|\tilde{L} + \tilde{X}\| \leqslant \|\tilde{L}\| + c(f)\|\tilde{B}\|.$$

Consequently

$$P(\|L - g\| \leqslant r) \geqslant P\left(\|\tilde{L}\| \leqslant \frac{r}{2}, \|\tilde{B}\| \leqslant \frac{r}{2c(f)}\right)$$

$$= \mathrm{e}^{-\lambda(g)}\int 1_{\{\|\tilde{L}\|\leqslant r/2, \|\tilde{B}\|\leqslant r/2c(f)\}} \mathrm{e}^{-\int_0^1 \langle \dot{f}(s), \tilde{B}(\mathrm{d}s)\rangle} \, \mathrm{d}\tilde{Q}$$

$$= \mathrm{e}^{-\lambda(g)}\int 1_{\{\|L\|\leqslant r/2, \|B\|\leqslant r/2c(f)\}} \mathrm{e}^{-\int_0^1 \langle \dot{f}(s), B(\mathrm{d}s)\rangle} \, \mathrm{d}P,$$

since $\tilde{B}$ is a two-dimensional Brownian motion with respect to $\tilde{Q}$.

The last chain of inequalities also holds for $-f$. Since $c(-f) = c(f)$ and $\cosh x \geqslant 1$ for every real x, it follows that

$$P(\|L - g\| \leqslant r) \geqslant \tfrac{1}{2}\mathrm{e}^{-\lambda(g)}\int 1_{\{\|L\|\leqslant r/2, \|B\|\leqslant r/2c(f)\}} \mathrm{e}^{-\int_0^1 \langle \dot{f}(s), B(\mathrm{d}s)\rangle} \, \mathrm{d}P$$

$$+ \tfrac{1}{2}\mathrm{e}^{-\lambda(g)}\int 1_{\{\|L\|\leqslant r/2, \|B\|\leqslant r/2c(-f)\}} \mathrm{e}^{-\int_0^1 \langle -\dot{f}(s), B(\mathrm{d}s)\rangle} \, \mathrm{d}P$$

$$= \mathrm{e}^{-\lambda(g)}\int 1_{\{\|L\|\leqslant r/2, \|B\|\leqslant r/2c(f)\}} \cosh\left(\int_0^1 \langle \dot{f}(s), B(\mathrm{d}s)\rangle\right) \mathrm{d}P$$

$$\geqslant \mathrm{e}^{-\lambda(g)}P\left(\|L\| \leqslant \frac{r}{2}, \|B\| \leqslant \frac{r}{2c(f)}\right). \qquad \square$$

The next result is inspired by Lemma 2.3 of Rémillard (1994).

Lemma 2.2. *Suppose that $0 < l < \infty$ and $0 < h \leqslant \infty$. Then there exist two finite positive constants δ_1 and δ_2 such that*

$$P\{\sup_{0 \leqslant s \leqslant t} |L(s)| < l, \ \sup_{0 \leqslant s \leqslant t} |B(s)| < h\} \geqslant \delta_1 \, \mathrm{e}^{-\delta_2 t}.$$

Proof. For any infinitely differentiable function u on $\mathbf{R}^{2+1}$, let

$$\mathscr{L}u(x, y) = \tfrac{1}{2}\Delta_x u(x, y) + \tfrac{1}{8}|x|^2 \, \partial_y^2 u(x, y) + \langle Jx, \nabla_x \partial_y u(x, y) \rangle.$$

$\mathscr{L}$ is the generator of the diffusion process $\{B(t), L(t): t \geqslant 0\}$ (see, for example, Rémillard 1994); it is known as the Kohn–Laplace differential operator (see, for example, Gaveau 1977, p. 98).

Now, for any $n \geqslant 0$, let R_n be the simple Laguerre polynomial, i.e.,

$$R_n(t) = \sum_{j=0}^{n} \binom{n}{j} \frac{(-t)^j}{j!}.$$

Then R_n satisfies the differential equation

$$tR_n''(t) + (1 - t)R_n'(t) + nR_n(t) = 0, \qquad \text{for all } t \in \mathbf{R},$$

and let z_n be the smallest positive zero of R_n. Then $z_n \to 0$ as $n \to \infty$ because

$$\lim_{n \to \infty} R_n\left(\frac{t}{n}\right) = \sum_{j=0}^{\infty} \frac{(-t)^j}{(j!)^2} = J_0(2t^{1/2}),$$

where J_0 is the Bessel function of the first kind of order 0. In fact, if a is the smallest positive zero of J_0 and if $0 < a_1 < a < a_2$ are chosen such that $J_0(a_1) > 0$ and $J_0(a_2) < 0$, it follows that $R_n(a_1^2/4n) > 0$ and $R_n(a_2^2/4n) < 0$ for n large enough. Hence there is at least one zero in the interval $(a_1^2/4n, a_2^2/4n)$ proving that $z_n < a_2^2/4n$. Moreover, $R_n(t) > 0$ for all $t \in [0, z_n)$. Set $a_n = \sup_{0 \leqslant t \leqslant z_n} R_n(t)$.

Further, if $\theta = \pi/4l$, the function

$$u_n(x, y) = \mathrm{e}^{-\theta|x|^2/2} R_n(\theta|x|^2) \cos(\theta y)$$

satisfies the equation

$$\mathscr{L}u_n = -\theta(2n + 1)u_n$$

for any $(x, y) \in \mathbf{R}^{2+1}$. Since R_n is a polynomial of degree n, there exists a constant D_n such that $|R_n(t)| \leqslant D_n|t|^n$, such that $|u_n(x, y)| \leqslant D_n \theta^n |x|^{2n} \mathrm{e}^{-\theta|x|^2/2}$. Therefore $M_n(t) = \mathrm{e}^{t\theta(2n+1)} u_n(B_t, L_t)$ is integrable and is a martingale. Moreover, $u_n(x, y) = 0$ if $|y| = l$ or if $|x| = h_n = (z_n/\theta)^{1/2}$.

Let now $0 < h \leqslant \infty$ and $l > 0$ be given and let $C = C(l, h)$ be the cylinder:

$$C(l, h) = \{(x, y) \in \mathbf{R}^{2+1}: |x| \leqslant h, |y| \leqslant l\}.$$

Next, let $T_{l,h} = \inf\{t > 0: (B_t, L_t) \notin C(l, h)\}$. $T_{l,h}$ is a stopping time. Our aim is to show that there exist two finite positive constants δ_1 and δ_2 such that

$$P(T_{l,h} > t) \geqslant \delta_1 \, \mathrm{e}^{-\delta_2 t}.$$

Indeed, choose n sufficiently large that $h_n = (z_n/\theta)^{1/2} \leqslant h$. If $h = \infty$, take $n = 0$ and in this case $z_0 = +\infty$. For simplicity, set $T_n = T_{l,h_n}$. Then $M_n(\min\{t, T_n\})$ is a martingale and

$$1 = M_n(0) = E(M_n(\min\{t, T_n\})) = E(\mathrm{e}^{t\theta(2n+1)}u_n(B_t, L_t)1_{\{T_n > t\}}),$$

since $M_n(T_n) = 0$ if $T_n \leqslant t$. Therefore

$$1 \leqslant \mathrm{e}^{t\theta(2n+1)} \sup_{(x,y)\in C(l,h_n)} u_n(x, y)P(T_n > t) \leqslant a_n \, \mathrm{e}^{t\theta(2n+1)} P(T_n > t),$$

yielding

$$P(T_n > t) \geqslant a_n^{-1} \, \mathrm{e}^{-t\theta(2n+1)}$$

and the proof is completed.   □

Next, for every integer $k \geqslant 2$, define $t_k = k^k$ and let L^k be the area process associated with the two-dimensional Brownian motion $B^k = \{B^k(s) = B(s + t_{k-1}) - B(t_{k-1}): s \geqslant 0\}$. Lemmas 2.1 and 2.2 are used to prove the following.

Lemma 2.3. *For every $r > 0$, every integer $k \geqslant 2$, and every $g \in \mathscr{C}_0^1$ such that $\lambda(g) < \infty$, let C_k^r denote the event*

$$\left\{ \sup_{0\leqslant s\leqslant(t_k-t_{k-1})/t_k} \left| \gamma(t_k)\frac{L^k(t_k s)}{2t_k\phi(t_k)} - g(s) \right| \leqslant r \right\}.$$

Then

$$\liminf_{k\to\infty} \frac{1}{\phi(t_k)} \log P(C_k^r) = 0.$$

In particular $\sum_k P(C_k^r) = \infty$.

Proof. We have

$$P(C_k^r) \geqslant P\left(\left\| \gamma(t_k)\frac{L^k(t_k\cdot)}{2t_k\phi(t_k)} - g \right\| \leqslant r \right).$$

Since $L^k(t_k\cdot)/t_k$ has the same law as L (see, for example, Proposition 2.1 of Helmes 1986), we deduce that

$$P(C_k^r) \geqslant P\left(\left\| \gamma(t_k)\frac{L}{2\phi(t_k)} - g \right\| \leqslant r \right) = P\left(\left\| L - \frac{2\phi(t_k)}{\gamma(t_k)}g \right\| \leqslant \frac{2\phi(t_k)}{\gamma(t_k)}r \right).$$

For simplicity, set $\theta_k = 2\phi(t_k)/\gamma(t_k)$. Let $f \in H^2$ be such that $F(f) = g$ and $\lambda(g) = \mu(f)$. Then $F(\theta_k^{1/2}f) = \theta_k g$, $\lambda(\theta_k g) = \theta_k \lambda(g)$ and $c(\theta_k^{1/2}f) = \theta_k^{1/2}c(f)$. Since the law of

$\{(B(st),\, L(st))\colon s \geqslant 0\}$ is the same as that of $\{(t^{1/2}B(s),\, tL(t))\colon s \geqslant 0\}$ for any $t > 0$ and using also Lemma 2.1, and Lemma 2.2 with $t = 1/\theta_k$, $l = r/2$ and $h = r/2c(f)$, we obtain

$$P(C_k^r) \geqslant e^{-\theta_k \lambda(g)} P\left(\|L\| \leqslant \frac{\theta_k r}{2},\, \|B\| \leqslant \frac{\theta_k^{1/2} r}{2c(f)} \right)$$

$$= e^{-\theta_k \lambda(g)} P\left\{ \sup_{0 \leqslant s \leqslant 1/\theta_k} |L(s)| \leqslant \frac{r}{2},\, \sup_{0 \leqslant s \leqslant 1/\theta_k} |B(s)| \leqslant \frac{r}{2c(f)} \right\}$$

$$\geqslant e^{-\theta_k \lambda(g)} \delta_1\, e^{-\delta_2/\theta_k}.$$

Therefore, in view of Conditions (A) and (B2), we obtain

$$\liminf_{k \to \infty} \frac{1}{\phi(t_k)} \log P(C_k^r) \geqslant \liminf_{k \to \infty} \left(\frac{-2\lambda(g)}{\gamma(t_k)} + \frac{\log \delta_1}{\phi(t_k)} - \frac{\delta_2 \gamma(t_k)}{2\phi(t_k)^2} \right) = 0.$$

Thus

$$\liminf_{k \to \infty} \frac{1}{\phi(t_k)} \log P(C_k^r) = 0.$$

Hence, if k is large enough,

$$P(C_k^r) \geqslant e^{-0.5\phi(t_k)} = \frac{1}{(k \log k)^{1/2}},$$

proving that $\sum_k P(C_k^r) = \infty$. $\qquad\square$

3. Proof of the main result

We now prove Theorem 1.2. At a certain moment we shall use Lemma 2.3.

Since H^1 is dense in $\mathscr{C}_0^1$, it suffices to prove that the set of all limit points of $\{\gamma(t)Z_t\colon t > 0\}$ contains a.s. H^1. To this purpose we only have to show that, for every $r > 0$ and every $g \in H^1$,

$$P(\liminf_{t \to \infty} \|\gamma(t)Z_t - g\| \leqslant r) = 1. \tag{4}$$

First we prove that

$$P(\liminf_{k \to \infty} \|\gamma(t_k)Z_{t_k} - g\| \leqslant r) = 1. \tag{5}$$

Let us note that

$$\|\gamma(t_k)Z_{t_k} - g\| \leqslant \sup_{0 \leqslant s \leqslant t_{k-1}/t_k} |\gamma(t_k)Z_{t_k}(s) - g(s)| + \sup_{t_{k-1}/t_k \leqslant s \leqslant 1} |\gamma(t_k)Z_{t_k}(s) - g(s)|$$

$$= I_1^k + I_2^k.$$

We have

$$I_1^k = \sup_{0 \leqslant s \leqslant 1} \left| \gamma(t_k) \frac{L(t_{k-1}s)}{2t_k\phi(t_k)} - g\left(\frac{t_k}{t_{k-1}}s\right) \right| \leqslant \gamma(t_k) \left\| \frac{L(t_{k-1}\cdot)}{2t_k\phi(t_k)} \right\| + \left\| g\left(\frac{t_k}{t_{k-1}}\cdot\right) \right\|.$$

Therefore

$$I_1^k \leqslant \gamma(t_k) \frac{t_{k-1}}{t_k} \frac{\phi(t_{k-1})}{\phi(t_k)} \left\| \frac{L(t_{k-1}\cdot)}{2t_{k-1}\phi(t_{k-1})} \right\| + \left(\frac{t_{k-1}}{t_k}\right)^{1/2} \left(\int_0^1 |\dot{g}(u)|^2 \, du\right)^{1/2}.$$

In view of Condition (A) and since $t_{k-1}/t_k \leqslant 1/k$, there exists $0 < C_1 < \infty$ such that, for k large enough,

$$I_1^k \leqslant \frac{C_1(\log k)^\delta}{k} \left\| \frac{L(t_{k-1}\cdot)}{2t_{k-1}\phi(t_{k-1})} \right\| + \frac{1}{k^{1/2}} \left(\int_0^1 |\dot{g}(u)| \, du\right)^{1/2}. \tag{6}$$

By virtue of Strassen's law of the iterated logarithm for the area process, the sequence

$$\left\{ \left\| \frac{L(t_{k-1}\cdot)}{2t_{k-1}\phi(t_{k-1})} \right\| : k \geqslant 2 \right\}$$

is bounded. Combining this fact with (6), we obtain that

$$\lim_{k\to\infty} I_1^k = 0. \tag{7}$$

By virtue of (7), the proof of (5) reduces to that of $P((\liminf_{k\to\infty} I_2^k \leqslant r) = 1$.
   We have

$$I_2^k = \sup_{t_{k-1}/t_k \leqslant s \leqslant 1} \left| \gamma(t_k) \frac{L(t_k s)}{2t_k\phi(t_k)} - g(s) \right| = \sup_{0 \leqslant s \leqslant t_k - t_{k-1}} \left| \gamma(t_k) \frac{L(s + t_{k-1})}{2t_k\phi(t_k)} - g\left(\frac{s + t_{k-1}}{t_k}\right) \right|$$

$$\leqslant \sup_{0 \leqslant s \leqslant t_k - t_{k-1}} \left| \gamma(t_k) \frac{L(s + t_{k-1}) - L(t_{k-1})}{2t_k\phi(t_k)} - g\left(\frac{s + t_{k-1}}{t_k}\right) \right| + \gamma(t_k) \left| \frac{L(t_{k-1})}{2t_k\phi(t_k)} \right|$$

$$= J_1^k + J_2^k.$$

By similar arguments as the proof of (7), one may prove that $\lim_{k\to\infty} J_2^k = 0$. Therefore, it remains to prove that $P(\liminf_{k\to\infty} J_1^k \leqslant r) = 1$.
   Now, by virtue of Proposition 2.1 of Helmes (1986), we have

$$L(s + t_{k-1}) - L(t_{k-1}) = L^k(s) + \langle JB(t_{k-1}), B^k(s) \rangle, \qquad s \geqslant 0.$$

It follows that

$$J_1^k \leqslant \sup_{0 \leqslant s \leqslant t_k - t_{k-1}} \left| \gamma(t_k) \frac{L^k(s)}{2 t_k \phi(t_k)} - g\left(\frac{s}{t_k}\right) \right| + \sup_{0 \leqslant s \leqslant t_k - t_{k-1}} \left| \gamma(t_k) \frac{\langle JB(t_{k-1}), B^k(s) \rangle}{2 t_k \phi(t_k)} \right|$$

$$+ \sup_{0 \leqslant s \leqslant t_k - t_{k-1}} \left| g\left(\frac{s}{t_k}\right) - g\left(\frac{s + t_{k-1}}{t_k}\right) \right|$$

$$= K_1^k + K_2^k + K_3^k.$$

By virtue of the Cauchy–Schwarz inequality, we have

$$K_3^k \leqslant \left(\frac{t_{k-1}}{t_k}\right)^{1/2} \left(\int_0^1 |\dot{g}(u)|^2 \, \mathrm{d}u\right)^{1/2}.$$

So, $\lim_{k \to \infty} K_3^k = 0$.

Now, let $0 < r_1 < r$ and consider the events

$$A_k = \left\{ \sup_{0 \leqslant s \leqslant t_k - t_{k-1}} \gamma(t_k) \left| \frac{\langle JB(t_{k-1}), B^k(s) \rangle}{2 t_k \phi(t_k)} \right| \leqslant r - r_1 \right\},$$

$$B_k = \left\{ \sup_{0 \leqslant s \leqslant t_k - t_{k-1}} \{\phi(t_k)\}^2 \left| \frac{\langle JB(t_{k-1}), B^k(s) \rangle}{2 t_k \phi(t_k)} \right| \leqslant \frac{r - r_1}{M} \right\},$$

$$C_k = \left\{ \sup_{0 \leqslant s \leqslant t_k - t_{k-1}} \left| \gamma(t_k) \frac{L^k(s)}{2 t_k \phi(t_k)} - g\left(\frac{s}{t_k}\right) \right| \leqslant r_1 \right\},$$

$$D_k = \{K_1^k + K_2^k \leqslant r\}.$$

In view of Condition (B2), for k large enough, $B_k \subset A_k$. It follows that $P(A_k^c \text{ i.o.}) \leqslant P(B_k^c \text{ i.o.})$. By virtue of Lemma 3.2 of Rémillard (1994), we have $P(B_k^c \text{ i.o.}) = 0$. Therefore $P(A_k^c \text{ i.o.}) = 0$. Now, $A_k \cap C_k$ being a subset of D_k, we have $P(A_k \cap C_k \text{ i.o.}) \leqslant P(D_k \text{ i.o.})$. Moreover $P(A_k^c \text{ i.o.}) = 0$. It follows that

$$P(D_k \text{ i.o.}) \geqslant P(C_k \text{ i.o.}). \tag{8}$$

Next, let $\mathscr{F}_t = \sigma(B_s: 0 \leqslant s \leqslant t)$. It is easy to see that the event C_k belongs to $\mathscr{F}_{t_k}$ and is independent of $\mathscr{F}_{t_{k-1}}$. By virtue of Lemma 2.3 and Lévy's extension of the Borel–Cantelli lemma (Lévy 1954, Corollary 68, p. 249), we deduce that $P(C_k \text{ i.o.}) = 1$. Therefore, by using (8), we obtain $P(D_k \text{ i.o.}) = 1$ which implies that $P(\liminf_{k \to \infty} J_1^k \leqslant r) = 1$. Thus we proved (5). Since

$$P(\liminf_{k \to \infty} \|\gamma(t_k) Z_{t_k} - g\| \leqslant r) \leqslant P(\liminf_{t \to \infty} \|\gamma(t) Z_t - g\| \leqslant r),$$

we are led to (4). By letting $r \to 0$ in (4), we obtain

$$P(\liminf_{t \to \infty} \|\gamma(t)Z_t - g\| = 0) = 1.$$

So, we proved that a.s. every point of H^1 is a limit point of $\{\gamma(t)Z_t: t > 0\}$ which completes the proof of Theorem 1.2.

Remark 3.1. (a) Theorem 1.2 generalizes easily to processes $L = \{L(t): t \geqslant 0\}$ of the form (3), where $J \neq 0$ is an arbitrary skew-symmetric $m \times m$ matrix and B an m-dimensional Brownian motion.

(b) The analogue of Theorem 1.1 when $\mathscr{C}_0^1$ is endowed with a Hölder norm has been proved by Baldi and Roynette (1992b). It might be interesting to investigate such a setting for the assertion of Theorem 1.2.

Acknowledgements

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