Bernoulli **6**(6), 2000, 1081–1107

Exponential-polynomial families and the term structure of interest rates

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Exponential-polynomial families like the Nelson-Siegel or Svenson family are widely used to estimate the current forward rate curve. We investigate whether these methods go well with intertemporal modelling. We characterize the consistent Itô processes which have the property to provide an arbitrage-free interest rate model when representing the parameters of some bounded exponential-polynomial type function. This includes diffusion processes in particular. We show that there is a strong limitation on their choice. Bounded exponential-polynomial families are best not used for modelling the term structure of interest rates.

Keywords: consistent Itô process; diffusion process; exponential-polynomial family; forward rate curve; interest rate model; inverse problem

1. Introduction

The current term structure of interest rates contains all the necessary information for pricing bonds, swaps and forward rate agreements of all maturities. It is used by the central banks as an indicator for their monetary policy.

There are several algorithms for constructing the current forward rate curve from the (finitely many) prices of bonds and swaps observed in the market. Widely used are splines and parametrized families of smooth curves $\{F(\cdot, z)\}_{z\in\mathcal{Z}}$, where $\mathcal{Z} \subset \mathbb{R}^N$, $N \geqslant 1$, denotes some finite-dimensional parameter set. By an optimal choice of the parameter z in $\mathcal{Z}$, an optimal fit of the forward curve $x \mapsto F(x, z)$ to the observerd data is attained. Here $x \geqslant 0$ denotes *time to maturity*. In that sense z represents the current state of the economy taking values in the state space $\mathcal{Z}$.

Examples are the Nelson and Siegel (1987) family with curve shape

$$F_{\mathrm{NS}}(x, z) = z_1 + (z_2 + z_3 x)\,\mathrm{e}^{-z_4 x}$$

and the Svenson (1994) family, an extension of Nelson–Siegel,

$$F_{\mathrm{S}}(x, z) = z_1 + (z_2 + z_3 x)\,\mathrm{e}^{-z_5 x} + z_4 x\,\mathrm{e}^{-z_6 x}.$$

Table 1 gives an overview of the fitting procedures used by some selected central banks. It is taken from the documentation of the Bank for International Settlements (BIS 1999).

Despite the flexibility and low number of parameters of F_{NS} and F_{S}, their choice is somewhat arbitrary. We shall discuss them from an intertemporal point of view: plenty of

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Table 1. Forward rate curve fitting procedures

Central bank	Curve fitting procedure
Belgium	Nelson–Siegel, Svensson
Canada	Svensson
Finland	Nelson–Siegel
France	Nelson–Siegel, Svensson
Germany	Svensson
Italy	Nelson–Siegel
Japan	Smoothing splines
Norway	Svensson
Spain	Nelson–Siegel (before 1995), Svensson
Sweden	Svensson
UK	Svensson
USA	Smoothing splines

cross-sectional data – daily estimations of z – are available. Therefore it would be natural to want to find the stochastic evolution of the parameter z over time. But then there exist economic constraints based on no-arbitrage considerations.

Following Björk and Christensen (1999), instead of F_{NS} and F_{S} we consider general exponential-polynomial families containing curves of the form

$$F(x, z) = \sum_{i=1}^{K} \left(\sum_{\mu=0}^{n_i} z_{i,\mu} x^{\mu} \right) \mathrm{e}^{-z_{i,n_i+1} x},$$

that is, linear combinations of exponential functions $\exp(-z_{i,n_i+1}x)$ over some polynomials of degree $n_i \in \mathbb{N}_0$. Obviously F_{NS} and F_{S} are of this type. We then replace z by an Itô process $Z = (Z_t)_{t \geqslant 0}$ taking values in $\mathscr{Z}$. The following questions arise:

- Does $F(\cdot, Z)$ provide an arbitrage-free interest rate model?
- And what are the conditions on Z for it?

Working in the Heath, Jarrow and Morton (1992) – henceforth HJM – framework with deterministic volatility structure, Björk and Christensen (1999) showed that the exponential-polynomial families are in a certain sense too large to carry an interest rate model. This result has been generalized for the Nelson–Siegel family in Filipović (1999b), including stochastic volatility structure. Expanding the methods used there, we give in this paper the general result for bounded exponential-polynomial families.

The paper is organized as follows. In Section 2 we introduce the class of Itô processes consistent with a given parametrized family of forward rate curves. Consistent Itô processes provide an arbitrage-free interest rate model when driving the parametrized family. They are characterized in terms of their drift and diffusion coefficients by the HJM drift condition.

By solving an inverse problem we obtain the main result for consistent Itô processes, stated in Section 3. It is shown that they are remarkably limited. The proof is divided into several steps, given in Sections 4, 5 and 6.

In Section 7 we extend the notion of consistency to e-consistency when $\mathbb{P}$ is not a martingale measure.

The main result is much clearer when restricting to diffusion processes, as shown in Section 8. It turns out that e-consistent diffusion processes driving bounded exponential-polynomial families like Nelson–Siegel or Svensson are very limited: most of the factors are either constant or deterministic. It is shown in Section 9 that there is no non-trivial diffusion process which is e-consistent with the Nelson–Siegel family. Furthermore, we identify the diffusion process which is e-consistent with the Svensson family. It contains just one non-deterministic component. The corresponding short rate model is shown to be the generalized Vasicek model.

We conclude that bounded exponential-polynomial families, in particular F_{NS} and F_{S}, are best not used for modelling the term structure of interest rates.

2. Consistent Itô processes

For the stochastic background and notation, we refer to Revuz and Yor (1994) and Jacod and Shiryaev (1987). Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{0\leqslant t<\infty}, \mathbb{P})$ be a filtered complete probability space, satisfying the usual conditions, and let $W = (W_t^1, \ldots, W_t^d)_{0\leqslant t<\infty}$ denote a standard d-dimensional $(\mathscr{F}_t)$-Brownian motion, $d \geqslant 1$.

Let $Z = (Z^1, \ldots, Z^N)$ denote an $\mathbb{R}^N$-valued Itô process, $N \geqslant 1$, of the form

$$Z_t^i = Z_0^i + \int_0^t b_s^i \,\mathrm{d}s + \sum_{j=1}^d \int_0^t \sigma_s^{i,j} \,\mathrm{d}W_s^j, \qquad i = 1, \ldots, N, \quad 0 \leqslant t < \infty,$$

where Z_0 is $\mathscr{F}_0$-measurable, and b and σ are progressively measurable processes with values in $\mathbb{R}^N$ and $\mathbb{R}^{N\times d}$ respectively, such that

$$\int_0^t (|b_s| + |\sigma_s|^2)\mathrm{d}s < \infty, \ \mathbb{P}\text{-almost surely}, \qquad \text{for all finite } t.$$

Let $F(x, z)$ be a function in $C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^N)$; that is to say, F and the partial derivatives $\partial F/\partial x$, $\partial F/\partial z_i$, $\partial^2 F/\partial z_i \partial z_j$, which exist for $1 \leqslant i, j \leqslant N$, are continuous functions on $\mathbb{R}_+ \times \mathbb{R}^N$. Interpreting Z_t as the state of the economy at time t, we let $x \mapsto F(x, Z_t)$ stand for the corresponding term structure of interest rates, meaning that $F(x, Z_t)$ denotes the *instantaneous forward rate* at time t for date $t + x$.

Notice that

$$G(x, z) := \exp\left(-\int_0^x F(\eta, z)\mathrm{d}\eta\right)$$

is in $C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^N)$ too. Therefore the price processes for *zero coupon T-bonds*

$$P(t, T) := G(T - t, Z_t), \qquad 0 \leqslant t \leqslant T < \infty, \tag{1}$$

and the process of the *savings account*

$$B(t) := \exp\left(-\int_0^t \frac{\partial}{\partial x} G(0, Z_s)\mathrm{d}s\right), \qquad 0 \leqslant t < \infty,$$

form continuous semimartingales.

Let $\mathscr{Z}$ denote an arbitrary subset of $\mathbb{R}^N$. The function F generates in a canonical way a parametrized set of forward curves $\{F(\cdot, z)\}_{z\in\mathscr{Z}}$. We shall refer to $\mathscr{Z}$ as the state space of the economy.

Defintion 2.1. *Z is called* consistent *with* $\{F(\cdot, z)\}_{z\in\mathscr{Z}}$, *if the support of Z is contained in* $\mathscr{Z}$ *and*

$$\left(\frac{P(t, T)}{B(t)}\right)_{0\leqslant t\leqslant T} \tag{2}$$

is a $\mathbb{P}$*-martingale, for all* $T < \infty$.

Set $a := \sigma\sigma^*$, where σ^* denotes the transpose of σ, that is, $a_t^{i,j} = \sum_{k=1}^d \sigma_t^{i,k}\sigma_t^{j,k}$, for $1 \leqslant i, j \leqslant N$ and $0 \leqslant t < \infty$. Then a is a progressively measurable process with values in the symmetric non-negative definite $N \times N$ matrices.

Using Itô's formula, the dynamics of (2) can be decomposed into finite variation and local martingale parts. Since consistency is required, the former has to vanish. This is the well-known HJM drift condition and is stated explicitly in the following proposition.

Proposition 2.2. *If Z is consistent with* $\{F(\cdot, z)\}_{z\in\mathscr{Z}}$ *then*

$$\begin{aligned}\frac{\partial}{\partial x}F(x, Z) = &\sum_{i=1}^N b^i \frac{\partial}{\partial z_i}F(x, Z)\\ &+ \sum_{i,j=1}^N a^{i,j}\left(\frac{1}{2}\frac{\partial^2}{\partial z_i \partial z_j}F(x, Z) - \frac{\partial}{\partial z_i}F(x, Z)\int_0^x \frac{\partial}{\partial z_j}F(\eta, Z)\mathrm{d}\eta\right),\end{aligned} \tag{3}$$

for all $x \geqslant 0$, $dt \otimes d\mathbb{P}$*-a.s.*

Proof. Analogous to the proof of Filipović (1999b, Proposition 3.2). □

3. Exponential-polynomial families

In this section we introduce a particular class of functions F. Our main result characterizes the corresponding consistent Itô processes.

Let K denote a positive integer and let $n = (n_1, \ldots, n_K)$ be a vector with components $n_i \in \mathbb{N}_0$, for $1 \leqslant i \leqslant K$. Write $|n| := n_1 + \ldots + n_K$. For a point

$$z = (z_{1,0}, \ldots, z_{1,n_1+1}, z_{2,0}, \ldots, z_{2,n_2+1}, \ldots, z_{K,0}, \ldots, z_{K,n_K+1}) \in \mathbb{R}^{|n|+2K}, \tag{4}$$

define the polynomials $p_i(z)$ as

$$p_i(z) = p_i(x, z) := \sum_{\mu=0}^{n_i} z_{i,\mu} x^{\mu}, \qquad 1 \leqslant i \leqslant K.$$

The function F is now defined as

$$F(x, z) := \sum_{i=1}^{K} p_i(x, z)\, \mathrm{e}^{-z_{i,n_i+1} x}. \tag{5}$$

Obviously $F \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^{|n|+2K})$. Hence the previous section applies with $N = |n| + 2K$.

From an economic point of view, it seems reasonable to restrict to bounded forward rate curves. Let, therefore, $\mathscr{Z}$ denote the set of all $z \in \mathbb{R}^N$ such that $\sup_{x \in \mathbb{R}_+} |F(x, z)| < \infty$.

Defintion 3.1. *The* exponential-polynomial family *$EP(K, n)$ is defined as the set of forward curves $\{F(\cdot, z)\}_{z \in \mathbb{R}^N}$.*

The bounded exponential-polynomial family *$BEP(K, n) \subset EP(K, n)$ is defined as the set of forward curves $\{F(\cdot, z)\}_{z \in \mathscr{Z}}$.*

Clearly $F_{\mathrm{NS}}(x, z) \in BEP(2, (0, 1))$ and $F_{\mathrm{S}}(x, z) \in BEP(3, (0, 1, 1))$, if in each case the parameter z is chosen such that the curve is bounded. From now on, the Nelson–Siegel and Svensson families are considered as subsets of $BEP(2, (0, 1))$ and $BEP(3, (0, 1, 1))$, respectively.

If two exponents z_{i,n_i+1} and z_{j,n_j+1} coincide, the sum (5) defining F reduces to a linear combination of $K - 1$ exponential functions. Thus for $z \in \mathbb{R}^N$ we introduce the equivalence relation

$$i \sim_z j :\Leftrightarrow z_{i,n_i+1} = z_{j,n_j+1} \tag{6}$$

on the set $\{1, \ldots, K\}$ and denote by $[i] = [i]_z$ the equivalence class of i. We will use the notation

$$n_{[i]} = n_{[i]}(z) := \max\{n_j | j \in [i]_z\},$$

$$\mathscr{T}_{[i],\mu} = \mathscr{T}_{[i],\mu}(z) := \{j \in [i]_z | n_j \geqslant \mu\}, \qquad 0 \leqslant \mu \leqslant n_{[i]}(z),$$

$$z_{[i],\mu} = z_{[i],\mu}(z) := \sum_{j \in \mathscr{T}_{[i],\mu}(z)} z_{j,\mu}, \qquad 0 \leqslant \mu \leqslant n_{[i]}(z),$$

$$p_{[i]}(z) := \sum_{j \in [i]_z} p_j(z). \tag{7}$$

In particular $p_{[i]}(z) = \sum_{\mu=0}^{n_{[i]}} z_{[i],\mu} x^{\mu}$, and (5) now reads

$$F(x, z) = \sum_{[i] \in \{1,\ldots,K\}/\sim} p_{[i]}(z)\, \mathrm{e}^{-z_{i,n_i+1} x}.$$

Observe that for $z \in \mathscr{Z}$ we have

$$z_{i,n_i+1}\begin{cases} =0 & \text{only if } \; p_{[i]}(z)=z_{[i],0}, \\ <0 & \text{only if } \; p_{[i]}(z)=0. \end{cases} \tag{8}$$

We shall write the $\mathbb{R}^N$-valued Itô process Z with the same indices as we use for a point $z\in\mathbb{R}^N$ (see (4)),

$$Z_t^{i,\mu}=Z_0^{i,\mu}+\int_0^t b_s^{i,\mu}\,\mathrm{d}s+\sum_{\lambda=1}^d\int_0^t\sigma_s^{i,\mu;\lambda}\,\mathrm{d}W_s^\lambda, \qquad 0\leqslant\mu\leqslant n_i+1, \quad 1\leqslant i\leqslant K. \tag{9}$$

Its diffusion matrix a consists of the components

$$a^{i,\mu;j,\nu}=\sum_{\lambda=1}^d\sigma^{i,\mu;\lambda}\sigma^{j,\nu;\lambda}, \qquad 0\leqslant\mu\leqslant n_i+1, \quad 0\leqslant\nu\leqslant n_j+1, \quad 1\leqslant i,j\leqslant K.$$

Notice that, for $1\leqslant i\leqslant K$,

$$\{z|p_{[i]}(z)=0\}=\bigcup_{\substack{J\subset\{1,\ldots,K\}\\ J\ni i}}\Bigg(\{z|z_{j,n_j+1}=z_{i,n_i+1}\text{ for all } j\in J\} \\ \cap\bigcap_{\mu=0}^{\max\{n_j|j\in J\}}\Bigg\{z\Bigg|\sum_{\substack{j\in J\\ n_j\geqslant\mu}}z_{j,\mu}=0\Bigg\}\Bigg\backslash\bigcup_{l\in J^c}\{z|z_{l,n_l+1}=z_{i,n_i+1}\}\Bigg) \tag{10}$$

is not closed in general but nevertheless a Borel set in $\mathbb{R}^N$. We introduce the optional random sets of singular points (t,ω)

$$\mathscr{A}_i:=\{p_i(Z)=0 \text{ or } p_{[i]}(Z)=0\}, \qquad 1\leqslant i\leqslant K,$$

$$\mathscr{B}:=\bigcup_{\substack{i,j=1\\ i\neq j}}^K\{Z^{i,n_i+1}=Z^{j,n_j+1}\},$$

$$\mathscr{C}:=\bigcup_{\substack{i,j=1\\ i\neq j}}^K\{2Z^{i,n_i+1}=Z^{j,n_j+1}\},$$

and the optional random sets of regular points (t,ω)

$$\mathscr{D}:=(\mathbb{R}_+\times\Omega)\Bigg\backslash\left(\bigcup_{i=1}^K\mathscr{A}_i\cup\mathscr{B}\cup\mathscr{C}\right),$$

$$\mathscr{D}':=(\mathbb{R}_+\times\Omega)\backslash(\mathscr{B}\cup\mathscr{C}).$$

Let us recall that for stopping times S and T, a stochastic interval like $[S,T]$ is a subset of $\mathbb{R}_+ \times \Omega$. Hence $[S] = [S, S]$ is the restriction of the graph of the mapping $S : \Omega \to [0, \infty]$ to the set $\mathbb{R}_+ \times \Omega$.

For any stopping time τ with $[\tau] \in (\mathbb{R}_+ \times \Omega)\backslash \mathscr{A}_i$, we define

$$\tau'(\omega) := \inf\{t > \tau(\omega) | (t, \omega) \in \mathscr{A}_i\},$$

the debut of the optional set $[\tau, \infty[\cap \mathscr{A}_i$. Observe that in general it is not true that $\tau' > \tau$ on $\{\tau < \infty\}$. This can be seen from the following example. For

$$F(x, z) = z_{1,0}\,\mathrm{e}^{-z_{1,1}x} + z_{2,0}\,\mathrm{e}^{-z_{2,1}x} + z_{3,0}\,\mathrm{e}^{-z_{3,1}x} \in BEP(3, (0, 0, 0)),$$

let $Z_t^{1,0} = Z_t^{3,0} = 1$, $Z_t^{2,0} = -1$, $Z_t^{3,1} = 1 + t$ and $Z_t^{1,1} = Z_t^{2,1} = 1$ for $t \in [0, 1]$. Then $p_1(Z_0) = p_{[1]}(Z_0) = 1$ and $p_{[1]}(Z_t) = 0$ for all $t \in (0, 1]$. Hence $[0] \in (\mathbb{R}_+ \times \Omega)\backslash \mathscr{A}_1$, but $\tau' = 0$. However, by continuity of Z we always have

$$\tau < \tau' \ \mathbb{P}\text{-a.s.} \qquad \text{on } \{\omega | (\tau(\omega), \omega) \in \mathscr{D}'\}. \tag{11}$$

Recall the fact that there is a one-to-one correspondence between the Itô processes Z starting in Z_0 (up to indistinguishability) and the equivalence classes of b and σ with respect to the $dt \otimes d\mathbb{P}$-nullsets in $\mathscr{R}_+ \otimes \mathscr{F}$. Hence we may state the following *inverse problem* to equation (3). Given a family of forward curves, for which choices of coefficients b and σ do we obtain a consistent Itô process Z starting at Z_0?

The main result is the following characterization of all consistent Itô processes, which is remarkably restrictive. The proof of the theorem will be given in Sections 5 and 6.

Theorem 3.2. *Let $K \in \mathbb{N}$, $n = (n_1, \ldots, n_K) \in \mathbb{N}_0^K$ and Z be as above. If Z is consistent with $BEP(K, n)$, then necessarily, for $1 \leqslant i \leqslant K$,*

$$a^{i,n_i+1;i,n_i+1} = 0, \qquad \text{on } \{p_i(Z) \neq 0\},\ dt \otimes d\mathbb{P}\text{-a.s.} \tag{12}$$

$$b^{i,n_i+1} = 0, \qquad \text{on } \{p_i(Z) \neq 0\} \cap \{p_{[i]}(Z) \neq 0\},\ dt \otimes d\mathbb{P}\text{-a.s.} \tag{13}$$

Consequently, Z^{i,n_i+1} is constant on intervals where $p_i(Z) \neq 0$ and $p_{[i]}(Z) \neq 0$. That is, for $\mathbb{P}$-almost every ω,

$$Z_t^{i,n_i+1}(\omega) = Z_u^{i,n_i+1}(\omega), \qquad \text{for } t \in [u, v],$$

if $p_i(Z_t(\omega)) \neq 0$ and $p_{[i]}(Z_t(\omega)) \neq 0$ for $t \in (u, v)$.

For a stopping time τ with $[\tau] \subset \mathscr{D}'$, let $\tau'(\omega) := \inf\{t \geqslant \tau(\omega) | (t, \omega) \notin \mathscr{D}'\}$ denote the debut of the optional random set $(\mathscr{B} \cup \mathscr{C}) \cap [\tau, \infty[$. Then we have that $\tau < \tau'$ on $\{\tau < \infty\}$ and

$$Z_{\tau+t}^{i,\mu} = Z_\tau^{i,\mu}\,\mathrm{e}^{-Z_\tau^{i,n_i+1}t} + Z_\tau^{i,\mu+1}\,t\,\mathrm{e}^{-Z_\tau^{i,n_i+1}t}$$

$$Z_{\tau+t}^{i,n_i} = Z_\tau^{i,n_i}\,\mathrm{e}^{-Z_\tau^{i,n_i+1}t}$$

on $[0, \tau' - \tau[$, for $0 \leqslant \mu \leqslant n_i - 1$ and $1 \leqslant i \leqslant K$, up to evanescence.

If $\mathscr{D}'$ *above is replaced by* $\mathscr{D}$ *and* τ' *is the debut of* $(\cup_{i=1}^{K}\mathscr{A}_i \cup \mathscr{B} \cup \mathscr{C}) \cap [\tau, \infty[$, *then* $\tau' = \infty$ *and in addition*

$$Z_{\tau+t}^{i,n_i+1} = Z_{\tau}^{i,n_i+1}$$

for $1 \leqslant i \leqslant K, \mathbb{P}$*-a.s. on* $\{\tau < \infty\}$.

Remark 3.3. It will be made clear in the proof of the theorem that it is actually sufficient to assume Z to be consistent with $EP(K, n)$ for (12) to hold.

As an immediate consequence we may state the following corollaries. The notation is the same as in the theorem.

Corollary 3.4. *If Z is consistent with BEP(K, n) and if the optional random sets* $\{p_i(Z) = 0\}$ *and* $\{p_{[i]}(Z) = 0\}$ *have* $\mathrm{d}t \otimes \mathrm{d}\mathbb{P}$*-measure zero, then the exponent* Z^{i,n_i+1} *is indistinguishable from* Z_0^{i,n_i+1}, $1 \leqslant i \leqslant K$.

Proof. If $\{p_i(Z) = 0\}$ and $\{p_{[i]}(Z) = 0\}$ have $dt \otimes d\mathbb{P}$-measure zero, then $\{p_i(Z) \neq 0\} \cap \{p_{[i]}(Z) \neq 0\} = \mathbb{R}_+ \times \Omega$ up to a $dt \otimes d\mathbb{P}$-nullset. The claim follows using (12) and (13). □

Corollary 3.5. *If Z is consistent with BEP(K,n) and if the following three points are* $\mathbb{P}$*-a.s. satisfied:*

(i) $p_i(Z_0) \neq 0$, *for all* $1 \leqslant i \leqslant K$,
(ii) *there exists no pair of indices* $i \neq j$ *with* $Z_0^{i,n_i+1} = Z_0^{j,n_j+1}$,
(iii) *there exists no pair of indices* $i \neq j$ *with* $2Z_0^{i,n_i+1} = Z_0^{j,n_j+1}$,

then Z and hence the interest rate model $F(x, Z)$ *is quasi-deterministic, that is, all randomness remains* $\mathscr{F}_0$*-measurable. In particular, the exponents* Z^{i,n_i+1} *are indistinguishable from* Z_0^{i,n_i+1}, *for* $1 \leqslant i \leqslant K$.

Proof. If (i), (ii) and (iii) hold $\mathbb{P}$-a.s. then $[0] \subset \mathscr{D}$. The claim follows from the second part of the theorem, setting $\tau = 0$. □

4. Auxiliary results

For the proof of the main result we need three auxiliary lemmas, presented in this section. First, there is a result on the identification of the coefficients of Itô processes.

Lemma 4.1. *Let*

$$X_t = X_0 + \int_0^t \beta_s^X \, ds + \sum_{j=1}^d \int_0^t \gamma_s^{X,j} \, dW_s^j,$$

$$Y_t = Y_0 + \int_0^t \beta_s^Y \, ds + \sum_{j=1}^d \int_0^t \gamma_s^{Y,j} \, dW_s^j$$

be two Itô processes. Then $dt \otimes d\mathbb{P}$*-a.s.*

$$1_{\{X=Y\}} \sum_{j=1}^d (\gamma^{X,j})^2 = 1_{\{X=Y\}} \sum_{j=1}^d \gamma^{X,j}\gamma^{Y,j} = 1_{\{X=Y\}} \sum_{j=1}^d (\gamma^{Y,j})^2$$
$$1_{\{X=Y\}}\beta^X = 1_{\{X=Y\}}\beta^Y.$$

Proof. We write $\langle \cdot, \cdot \rangle$ for the scalar product in $\mathbb{R}^d$. Then

$$|\langle \gamma^X, \gamma^X \rangle - \langle \gamma^X, \gamma^Y \rangle| = |\langle \gamma^X, \gamma^X - \gamma^Y \rangle| \leqslant \sqrt{\langle \gamma^X, \gamma^X \rangle}\sqrt{\langle \gamma^X - \gamma^Y, \gamma^X - \gamma^Y \rangle}.$$

By the occupation times formula (see Revuz and Yor 1994, Corollary (1.6), Chapter VI),

$$\int_0^t 1_{\{X_s = Y_s\}} \langle \gamma_s^X - \gamma_s^Y, \gamma_s^X - \gamma_s^Y \rangle \, ds = 0, \qquad \text{for all } t < \infty, \ \mathbb{P}\text{-a.s.}$$

Hence, by the Hölder inequality,

$$\begin{aligned}
&\int_0^t 1_{\{X_s=Y_s\}} |\langle \gamma_s^X, \gamma_s^X \rangle - \langle \gamma_s^X, \gamma_s^Y \rangle| \, ds \\
&\leqslant \int_0^t 1_{\{X_s=Y_s\}} \sqrt{\langle \gamma_s^X, \gamma_s^X \rangle}\sqrt{\langle \gamma_s^X - \gamma_s^Y, \gamma_s^X - \gamma_s^Y \rangle} \, ds \\
&\leqslant \left(\int_0^t 1_{\{X_s=Y_s\}} \langle \gamma_s^X, \gamma_s^X \rangle \, ds \right)^{1/2} \left(\int_0^t 1_{\{X_s=Y_s\}} \langle \gamma_s^X - \gamma_s^Y, \gamma_s^X - \gamma_s^Y \rangle \, ds \right)^{1/2} \\
&= 0, \qquad \text{for all } t < \infty, \ \mathbb{P}\text{-a.s.}
\end{aligned}$$

Thus, by symmetry,

$$1_{\{X=Y\}} \langle \gamma^X, \gamma^X \rangle = 1_{\{X=Y\}} \langle \gamma^X, \gamma^Y \rangle = 1_{\{X=Y\}} \langle \gamma^Y, \gamma^Y \rangle, \qquad dt \otimes d\mathbb{P}\text{-a.s.}$$

By continuity of the processes X and Y, there are sequences of stopping times (S_n) and (T_n), $S_n \leqslant T_n$, with $[S_m, T_m] \cap [S_n, T_n] = \varnothing$ for all $m \neq n$ and

$$\{X = Y\} = \bigcup_{n \in \mathbb{N}} [S_n, T_n], \qquad \text{up to evanescence.}$$

To see this, let $n \in \mathbb{N}$ and let $S(n, 1) := \inf\{t > 0 \,|\, |X_t - Y_t| = 0\}$. Define $T(n, p) := \inf\{t > S(n, p) \,|\, |X_t - Y_t| > 0\}$ and inductively

$$S(n,\ p+1) := \inf\{t > S(n,\ p) | \, |X_t - Y_t| = 0 \text{ and } \sup_{S(n,p)\leqslant s\leqslant t} |X_s - Y_s| > 2^{-n}\}.$$

Then by continuity we have $\lim_{p\to\infty} S(n,\ p) = \infty$ for all $n \in \mathbb{N}$ and it follows that $\{X = Y\} = \cup_{n,p\in\mathbb{N}}[S(n,\ p),\ T(n,\ p)]$. Now proceed as in Jacod and Shiryaev (1987, Lemma I.1.31) to find the sequences (S_n) and (T_n) with the desired properties.

From above we have $1_{\{X=Y\}}(\gamma^X - \gamma^Y)^2 = 0$, $dt \otimes d\mathbb{P}$-a.s. For any $0 \leqslant t < \infty$, therefore, $\int_{S_n\wedge t}^{T_n\wedge t} (\gamma_s^X - \gamma_s^Y)\mathrm{d}W_s = 0$, $\mathbb{P}$-a.s. Hence

$$0 = (X - Y)_{T_n\wedge t} - (X - Y)_{S_n\wedge t} = \int_{S_n\wedge t}^{T_n\wedge t} (\beta_s^X - \beta_s^Y)\mathrm{d}s,\ \mathbb{P}\text{-a.s.}$$

We conclude that

$$\int_0^t 1_{\{X_s=Y_s\}}(\beta_s^X - \beta_s^Y)\mathrm{d}s = \sum_{n\in\mathbb{N}} \int_{S_n\wedge t}^{T_n\wedge t} (\beta_s^X - \beta_s^Y)\mathrm{d}s = 0, \qquad \text{for } 0 \leqslant t < \infty,\ \mathbb{P}\text{-a.s.}$$

Using the same arguments as in the proof of Filipović (1999b, Proposition 3.2), we derive the desired result. □

Next, we list two results in matrix algebra.

Lemma 4.2. *Let $\gamma = (\gamma_{i,j})$ be an $N \times d$ matrix and define the symmetric non-negative definite $N \times N$ matrix $\alpha := \gamma\gamma^*$, that is, $\alpha_{i,j} = \alpha_{j,i} = \sum_{\lambda=1}^d \gamma_{i,\lambda}\gamma_{j,\lambda}$. Let I and J denote two arbitrary subsets of $\{1, \ldots, N\}$. Define*

$$\alpha_{I,J} = \alpha_{J,I} := \sum_{j\in J}\sum_{i\in I} \alpha_{i,j}.$$

Then $\alpha_{I,I} \geqslant 0$ and $|\alpha_{I,J}| \leqslant \sqrt{\alpha_{I,I}}\sqrt{\alpha_{J,J}}$.

Proof. For $1 \leqslant \lambda \leqslant d$, define $\gamma_{I,\lambda} := \sum_{i\in I}\gamma_{i,\lambda}$. Then by definition

$$\alpha_{I,J} = \sum_{j\in J}\sum_{i\in I}\sum_{\lambda=1}^d \gamma_{i,\lambda}\gamma_{j,\lambda} = \sum_{\lambda=1}^d \left(\sum_{i\in I}\gamma_{i,\lambda}\right)\left(\sum_{j\in J}\gamma_{j,\lambda}\right) = \sum_{\lambda=1}^d \gamma_{I,\lambda}\gamma_{J,\lambda}.$$

Hence

$$\alpha_{I,I} = \sum_{\lambda=1}^d (\gamma_{I,\lambda})^2 \geqslant 0,$$

and by the Cauchy–Schwarz inequality

$$|\alpha_{I,J}| \leqslant \sqrt{\sum_{\lambda=1}^d (\gamma_{I,\lambda})^2}\sqrt{\sum_{\lambda=1}^d (\gamma_{J,\lambda})^2} = \sqrt{\alpha_{I,I}}\sqrt{\alpha_{J,J}}.$$

□

Lemma 4.3. *Let* $\alpha = (\alpha_{i,j})$ *be an* $n \times n$ *matrix,* $n \in \mathbb{N}$, *which is diagonally dominant from the right, that is,*

$$|\alpha_{i,i}| \geqslant \sum_{\substack{j=1 \\ j \neq i}}^{n} |\alpha_{i,j}|$$

$$|\alpha_{i,i}| > \sum_{j=i+1}^{n} |\alpha_{i,j}| \qquad \left(set \sum_{j=n+1}^{n} \cdots := 0 \right),$$

for all $1 \leqslant i \leqslant n$. *Then* α *is regular.*

Proof. The proof, involving Gaussian elimination, is a slight modification of an argument given in Schwarz (1986, Theorem 1.5).

By assumption $|\alpha_{1,1}| > \sum_{j=2}^{n} |\alpha_{1,j}| \geqslant 0$, in particular $\alpha_{1,1} \neq 0$. If $n = 1$ we are done. If $n > 1$, the elimination step

$$\alpha_{i,j}^{(1)} := \alpha_{i,j} - \frac{\alpha_{i,1}}{\alpha_{1,1}} \alpha_{1,j}, \qquad 2 \leqslant i,\, j \leqslant n,$$

leads to the $(n-1) \times (n-1)$ matrix $\alpha^{(1)} = (\alpha_{i,j}^{(1)})_{2 \leqslant i,j \leqslant n}$. We show that $\alpha^{(1)}$ is diagonally dominant from the right. If $\alpha_{i,1} = 0$, there is nothing to prove for the ith row. Let $\alpha_{i,1} \neq 0$, for some $2 \leqslant i \leqslant n$. We have

$$|\alpha_{i,j}^{(1)}| \geqslant |\alpha_{i,j}| - \left| \frac{\alpha_{i,1}}{\alpha_{1,1}} \right| |\alpha_{1,j}|, \qquad 2 \leqslant j \leqslant n.$$

Therefore

$$\begin{aligned}
\sum_{j=i+1}^{n} |\alpha_{i,j}^{(1)}| \leqslant \sum_{\substack{j=2 \\ j \neq i}}^{n} |\alpha_{i,j}^{(1)}| = \sum_{\substack{j=2 \\ j \neq i}}^{n} \left| \alpha_{i,j} - \frac{\alpha_{i,1}}{\alpha_{1,1}} \alpha_{1,j} \right| &\leqslant \sum_{\substack{j=2 \\ j \neq i}}^{n} |\alpha_{i,j}| + \left| \frac{\alpha_{i,1}}{\alpha_{1,1}} \right| \sum_{\substack{j=2 \\ j \neq i}}^{n} |\alpha_{1,j}| \\
&= \sum_{\substack{j=1 \\ j \neq i}}^{n} |\alpha_{i,j}| - |\alpha_{i,1}| + \left| \frac{\alpha_{i,1}}{\alpha_{1,1}} \right| \left(\sum_{j=2}^{n} |\alpha_{1,j}| - |\alpha_{1,i}| \right) \\
&< |\alpha_{i,i}| - |\alpha_{i,1}| + \left| \frac{\alpha_{i,1}}{\alpha_{1,1}} \right| (|\alpha_{1,1}| - |\alpha_{1,i}|) \\
&= |\alpha_{i,i}| - \left| \frac{\alpha_{i,1}}{\alpha_{1,1}} \right| |\alpha_{1,i}| \leqslant |\alpha_{i,i}^{(1)}|.
\end{aligned}$$

Proceed inductively to $\alpha^{(2)}, \ldots, \alpha^{(n-1)}$. □

5. The case *BEP*(1, *n*)

We will treat the case $K=1$ separately, since it represents a key step in the proof of the general $BEP(K,\, n)$ case. For simplicity we shall skip the index $i=1$ and write $n=n_1\in\mathbb{N}_0$, $p=p_1$, $b^j=b^{1,j}$, $a^{i,j}=a^{1,i;1,j}$, etc. In particular, we use the notation of Section 2 with $N=n+2$.

Lemma 5.1. *Let* $n\in\mathbb{N}_0$ *and* Z *be as above. If* Z *is consistent with* $BEP(1,\, n)$*, then necessarily*

$$Z_t^i = Z_0^i\,\mathrm{e}^{-Z_0^{n+1}t} + Z_0^{i+1}\,t\,\mathrm{e}^{-Z_0^{n+1}t},$$

$$Z_t^n = Z_0^n\,\mathrm{e}^{-Z_0^{n+1}t},$$

$$Z_t^{n+1} = Z_0^{n+1} + \left(\int_0^t b_s^{n+1}\,\mathrm{d}s + \sum_{j=1}^d \int_0^t \sigma_s^{n+1,j}\,\mathrm{d}W_s^j\right)1_{\Omega_0},$$

for $0\leqslant i\leqslant n-1$ *and* $0\leqslant t<\infty$, $\mathbb{P}$-*a.s., where* $\Omega_0:=\{p(Z_0)=0\}$.

Consequently, if Z is consistent with $BEP(1,\, n)$, then $\{p(\mathrm{Z})=0\}=\mathbb{R}_+\times\Omega_0$. Hence $\{Z^{n+1}\neq Z_0^{n+1}\}\subset\{p(Z)=0\}$. Therefore we may state:

Corollary 5.2. *If* Z *is consistent with* $BEP(1,\, n)$*, then* Z *is as in Lemma* 5.1 *and*

$$F(x,\, Z)=p(x,\, Z)\,\mathrm{e}^{-Z_0^{n+1}x}.$$

Hence the corresponding interest rate model is quasi-deterministic, that is, all randomness remains $\mathscr{F}_0$*-measurable.*

Proof of Lemma 5.1. Let $n\in\mathbb{N}_0$ and let Z be an Itô process, consistent with $BEP(1,\, n)$. Fix a point $(t,\,\omega)$ in $\mathbb{R}_+\times\Omega$. For simplicity we write z_i for $Z_t^i(\omega)$, $a_{i,j}$ for $a_t^{i,j}(\omega)$ and b_i for $b_t^i(\omega)$. The proof relies on expanding equation (3) at the point $z=(z_0,\,\ldots,\,z_{n+1})$. The terms involved are

$$\frac{\partial}{\partial x}F(x,\, z)=\left(\frac{\partial}{\partial x}p(x,\, z)-z_{n+1}p(x,\, z)\right)\mathrm{e}^{-z_{n+1}x}, \tag{14}$$

$$\frac{\partial}{\partial z_i}F(x,\, z)=\begin{cases} x^i\,\mathrm{e}^{-z_{n+1}x}, & 0\leqslant i\leqslant n,\\ -xp(x,\, z)\,\mathrm{e}^{-z_{n+1}x}, & i=n+1,\end{cases} \tag{15}$$

$$\frac{\partial^2 F(x,\, z)}{\partial z_i\partial z_j}=\frac{\partial^2 F(x,\, z)}{\partial z_j\partial z_i}=\begin{cases} 0, & 0\leqslant i,\, j\leqslant n,\\ -x^{i+1}\,\mathrm{e}^{-z_{n+1}x}, & 0\leqslant i\leqslant n,\, j=n+1,\\ x^2\,p(x,\, z)\,\mathrm{e}^{-z_{n+1}x}, & i=j=n+1.\end{cases} \tag{16}$$

Finally, the following relation for $m\in\mathbb{N}_0$ is useful:

$$\int_0^x \eta^m \, \mathrm{e}^{-z_{n+1}\eta} \, \mathrm{d}\eta = \begin{cases} -r_m(x) \, \mathrm{e}^{-z_{n+1}x} + \dfrac{m!}{z_{n+1}^{m+1}}, & z_{n+1} \neq 0, \\ \dfrac{x^{m+1}}{m+1}, & z_{n+1} = 0, \end{cases} \tag{17}$$

where

$$r_m(x) = \sum_{k=0}^{m} \frac{m!}{(m-k)!} \frac{x^{m-k}}{z_{n+1}^{k+1}}$$

is a polynomial in x of order m.

Let us suppose first that $z_{n+1} \neq 0$. Thus, subtracting $(\partial/\partial x)F(x, Z)$ from both sides of (3), we obtain a null equation of the form

$$q_1(x) \, \mathrm{e}^{-z_{n+1}x} + q_2(x) \, \mathrm{e}^{-2z_{n+1}x} = 0, \tag{18}$$

which has to hold simultaneously for all $x \geqslant 0$. The polynomials q_1 and q_2 depend on the z_i, b_i and $a_{i,j}$. Equality (18) implies $q_1 = q_2 = 0$. This again yields that all coefficients of the q_i have to be zero.

To proceed we have to distinguish the two cases $p(z) \neq 0$ and $p(z) = 0$. Let us first suppose the former is true. Then there exists an index $i \in \{0, \ldots, n\}$ such that $z_i \neq 0$. Set $m := \max\{i \leqslant n | z_i \neq 0\}$. With regard to (15)–(17), it follows that $\deg q_2 = 2m + 2$. In particular,

$$q_2(x) = a_{n+1,n+1} \frac{z_m^2}{z_{n+1}} x^{2m+2} + \ldots,$$

where $\ldots$ denotes terms of lower order in x. Hence $a_{n+1,n+1} = 0$. But the matrix a has to be non-negative definite, so necessarily

$$a_{n+1,j} = a_{j,n+1} = 0, \qquad \text{for all } 1 \leqslant j \leqslant n+1.$$

In view of Lemma 4.1 (setting $Y = 0$), since we are characterizing a and b up to $dt \otimes d\mathbb{P}$-nullsets, we may assume $a_{i,j} = a_{j,i} = 0$, for $0 \leqslant j \leqslant n+1$, for all $i \geqslant m+1$. Thus the degree of q_2 reduces to $2m$. Explicitly,

$$q_2(x) = \frac{a_{m,m}}{z_{n+1}} x^{2m} + \ldots.$$

Hence $a_{m,m} = 0$ and so $a_{m,j} = a_{j,m} = 0$, for $0 \leqslant j \leqslant n+1$. Proceeding inductively for $i = m-1, m-2, \ldots, 0$, we finally obtain that the diffusion matrix a is equal to zero and hence $q_2 = 0$ is fulfilled.

Now we determine the drift b. By Lemma 4.1, we may assume $b_i = 0$ for $m+1 \leqslant i \leqslant n$. With regard to (14) and (15), q_1 reduces therefore to

$$q_1(x) = -b_{n+1} z_m x^{m+1} + \ldots.$$

It follows that $b_{n+1} = 0$ and we are left with

$$q_1(x) = (b_m + z_{n+1}z_m)x^m + \sum_{i=0}^{m-1}(b_i - z_{i+1} + z_{n+1}z_i)x^i$$

$$= (b_n + z_{n+1}z_n)x^n + \sum_{i=0}^{n-1}(b_i - z_{i+1} + z_{n+1}z_i)x^i.$$

We now turn to the singular cases. If $p(z) = 0$, that is $z_0 = \ldots = z_n = 0$, we may assume $a_{i,j} = a_{j,i} = b_i = 0$, $0 \leqslant j \leqslant n+1$, for all $i \leqslant n$. But this means that $q_1 = q_2 = 0$, independently of the choice of b_{n+1} and $a_{n+1,n+1}$.

For the case where $z_{n+1} = 0$, we need the boundedness assumption $z \in \mathscr{Z}$. By (8) it follows that $z_1 = \ldots = z_n = 0$. So by Lemma 4.1 again $a_{i,j} = a_{j,i} = b_i = 0$, $0 \leqslant j \leqslant n+1$, for all $i \geqslant 1$. Thus in this case equation (3) reduces to

$$0 = b_0 - a_{0,0}x,$$

and therefore $b_0 = a_{0,0} = 0$.

Summarizing all cases, we conclude that necessarily

$$b_i = -z_{n+1}z_i + z_{i+1}, \qquad 0 \leqslant i \leqslant n-1,$$

$$b_n = -z_{n+1}z_n,$$

$$a_{i,j} = 0, \qquad \text{for } (i, j) \neq (n+1, n+1);$$

while b_{n+1} and $a_{n+1,n+1}$ are arbitrary real and non-negative real numbers, respectively, whenever $p(z) = 0$. Otherwise $b_{n+1} = a_{n+1,n+1} = 0$.

The rest of the proof is analogous to the proof of Filipović (1999b, Proposition 4.1). □

6. The general case *BEP*(*K*,*n*)

Using again the notation of Section 3, we give the proof of Theorem 3.2 for the case $K \geqslant 2$. The exposition is somewhat complicated, which is due to the multidimensionality of the problem. The idea, however, is simple. For a fixed point $(t, \omega) \in \mathbb{R}_+ \times \Omega$ we expand equation (3), which turns out to be a linear combination of linearly independent exponential functions, over the ring of polynomials, equalling zero. Consequently, many of the coefficients have to vanish, which leads to our assertion.

The difficulty is that some exponents may coincide. This causes a considerable number of singular cases which require a separate discussion.

Let $K \geqslant 2$, $n = (n_1, \ldots, n_K) \in \mathbb{N}_0^K$, and let Z be consistent with $BEP(K, n)$. As in the proof of Lemma 5.1, we fix a point (t, ω) in $\mathbb{R}_+ \times \Omega$ and use the shorthand notation $z_{i,\mu}$ for $Z_t^{i,\mu}(\omega)$, $a_{i,\mu;j,\nu}$ for $a_t^{i,\mu;j,\nu}(\omega)$ and $b_{i,\mu}$ for $b_t^{i,\mu}(\omega)$, etc. Since we are characterizing a and b up to a $dt \otimes d\mathbb{P}$-nullset, we assume that (t, ω) is chosen outside an exceptional $dt \otimes d\mathbb{P}$-nullset. In particular, the lemmas from Section 4 shall apply each time this choice is made.

The strategy is the same as for the case $K = 1$. Thus we expand equation (3) at the point

$z=(z_{1,0},\dots,z_{K,n_K+1})$ to obtain a linear combination of (ideally) linearly independent exponential functions over the ring of polynomials

$$\sum_{i=1}^{K} q_i(x)\,\mathrm{e}^{-z_{i,n_i+1}x}+\sum_{1\leqslant i\leqslant j\leqslant K} q_{i,j}(x)\,\mathrm{e}^{-(z_{i,n_i+1}+z_{j,n_j+1})x}=0. \tag{19}$$

Consequently, all polynomials q_i and $q_{i,j}$ have to be zero. The main difference between this case and the case $K=1$ is that representation (19) may not be unique due to the possibly multiple occurrence of the following singular cases:

(i) $z_{i,n_i+1}=z_{j,n_j+1}$, for $i\neq j$,
(ii) $2z_{i,n_i+1}=z_{j,n_j+1}+z_{k,n_k+1}$,
(iii) $2z_{i,n_i+1}=z_{j,n_j+1}$,
(iv) $z_{i,n_i+1}=z_{j,n_j+1}+z_{k,n_k+1}$,

for some indices $1\leqslant i, j, k\leqslant K$. However, the lemmas in Section 4 and the boundedness assumption $z\in\mathscr{Z}$ are good enough to settle these four cases.

Let us suppose first that $p_i(z)\neq 0$, for all $i\in\{1,\dots,K\}$. To settle case (i), let $\sim$ denote the equivalence relation defined in (6). After reparametrization if necessary, we may assume that

$$\{1,\dots,K\}/_\sim=\{[1],\dots,[\tilde{K}]\}$$

and $z_{1,n_1+1}<\dots<z_{\tilde{K},n_{\tilde{K}}+1}$ for some integer $\tilde{K}\leqslant K$. Write $I:=\{1,\dots,\tilde{K}\}$. In view of Lemma 4.1 we may assume

$$a_{j,n_j+1;j,n_j+1}=a_{i,n_i+1;i,n_i+1}\quad\text{and}\quad b_{j,n_j+1}=b_{i,n_i+1}\qquad\text{for all } j\in[i],\ i\in I. \tag{20}$$

The proof of (12) and (13) is divided into four lemmas.

Lemma 6.1. $a_{i,n_i+1;i,n_i+1}=0$, *for all* $i\in I$.

Proof. Expression (19) takes the form

$$\sum_{i\in I}\tilde{q}_i(x)\,\mathrm{e}^{-z_{i,n_i+1}x}+\sum_{\substack{i,j\in I\\ i\leqslant j}}\tilde{q}_{i,j}(x)\,\mathrm{e}^{-(z_{i,n_i+1}+z_{j,n_j+1})x}=0, \tag{21}$$

for some polynomials $\tilde{q}_i$ and $\tilde{q}_{i,j}$. Taking into account cases (ii)–(iv), this representation may still not be unique. However, if for an index $i\in I$ there exist no $j, k\in I$ such that $2z_{i,n_i+1}=z_{j,n_j+1}+z_{k,n_k+1}$ or $2z_{i,n_i+1}=z_{j,n_j+1}$ (in particular, $z_{i,n_i+1}\neq 0$) then we have

$$\tilde{q}_{i,i}(x)=a_{i,n_i+1;i,n_i+1}\frac{\sum_{j\in\mathscr{I}_{[i],\mu_m}} z_{j,\mu_m}^2}{z_{i,n_i+1}}x^{2\mu_m+2}+\dots,$$

where $\mu_m:=\max\{\nu\,|\,\nu\leqslant n_j \text{ and } z_{j,\nu}\neq 0 \text{ for some } j\in[i]\}\in\mathbb{N}_0$. Hence $a_{i,n_i+1;i,n_i+1}=0$ and the lemma is proved for the regular case.

For the singular cases observe first that $z_{i,n_i+1}=0$ implies $a_{i,n_i+1;i,n_i+1}=0$, which follows from Lemma 4.1. Now we split I into two disjoint subsets I_1 and I_2, where

$$I_1 := \{i \in I | z_{i,n_i+1} \neq 0, \text{ and there exist } j,\ k \in I \text{ such that } 2z_{i,n_i+1} = z_{j,n_j+1} + z_{k,n_k+1} \text{ or } 2z_{i,n_i+1} = z_{j,n_j+1}\},$$

$$I_2 := I \backslash I_1.$$

Observe that $z_{\tilde{K},n_{\tilde{K}}+1} > 0$ implies $\tilde{K} \in I_2$ and $z_{1,n_1+1} < 0$ implies $1 \in I_2$. Since at least one of these events has to happen, the set I_2 is not empty. We have shown above that $a_{i,n_i+1;i,n_i+1} = 0$, for $i \in I_2$. If I_1 is not empty, we will show that for each $i \in I_1$, the parameter z_{i,n_i+1} can be written as a linear combination of z_{j,n_j+1}s with $j \in I_2$. From this it follows by Lemmas 4.1 and 4.2 that $a_{i,n_i+1;i,n_i+1} = 0$ for all $i \in I_1$ and the lemma is completely proved. We proceed as follows. Write $I_1 = \{i_1, \ldots, i_r\}$ with $z_{i_1,n_{i_1}+1} < \ldots < z_{i_r,n_{i_r}+1}$. For each $i_k \in I_1$ there exists one linear equation of the form

$$(*, \ldots, *, 2, *, \ldots, *) \begin{pmatrix} z_{i_1,n_{i_1}+1} \\ \vdots \\ z_{i_k,n_{i_k}+1} \\ \vdots \\ z_{i_r,n_{i_r}+1} \end{pmatrix} = \alpha_k,$$

where $*$ stands for 0 or -1, but with at most one -1 on each side of 2. The α_k on the right-hand side is 0 or z_{i,n_i+1} or $z_{i,n_i+1} + z_{j,n_j+1}$ for some indices $i, j, \in I_2$. Hence we obtain the system of linear equations

$$\begin{pmatrix} 2 & * & \ldots & * \\ * & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ * & \ldots & * & 2 \end{pmatrix} \begin{pmatrix} z_{i_1,n_{i_1}+1} \\ \vdots \\ \vdots \\ z_{i_r,n_{i_r}+1} \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \vdots \\ \alpha_r \end{pmatrix}.$$

By Lemma 4.3, the matrix on the left-hand side is invertible, from which our assertion follows. □

Lemma 6.2. $a_{j,n_j+1;k,\nu} = a_{k,\nu;j,n_j+1} = 0$, *for* $0 \leqslant \nu \leqslant n_k$, *for all* $1 \leqslant j,\ k \leqslant K$.

Proof. In view of (20), the lemma follows immediately from Lemmas 6.1 and 4.2. □

Analogous to the notation introduced in (7), we set

$$b_{[i],\mu} := \sum_{j \in \mathcal{T}_{[i],\mu}} b_{j,\mu},$$

$$\sigma_{[i],\mu;\lambda} := \sum_{j \in \mathcal{T}_{[i],\mu}} \sigma_{j,\mu;\lambda},$$

$$a_{[i],\mu;k,\nu} := \sum_{j \in \mathcal{T}_{[i],\mu}} a_{j,\mu;k,\nu},$$

for $0 \leqslant \mu \leqslant n_{[i]}$, $0 \leqslant \nu \leqslant n_k$, $1 \leqslant k \leqslant K$, $1 \leqslant \lambda \leqslant d$, $i \in I$, and

$$a_{[i],\mu;[k],\nu} := \sum_{l \in \mathcal{T}_{[k],\nu}} \sum_{j \in \mathcal{T}_{[i],\mu}} a_{j,\mu;l,\nu},$$

for $0 \leqslant \mu \leqslant n_{[i]}$, $0 \leqslant \nu \leqslant n_{[k]}$, $i, k \in I$.

Lemma 6.3. *If $z_{[i],\mu} = 0$, for $i \in I$ and $\mu \in \{0, \ldots, n_{[i]}\}$, then*

$$b_{[i],\mu} = a_{[i],\mu;[i],\mu} = a_{[i],\mu;k,\nu} = a_{k,\nu;[i],\mu} = 0,$$

for all $0 \leqslant \nu \leqslant n_k$, $1 \leqslant k \leqslant K$.

Proof. Notice that $a_{[i],\mu;[i],\mu} = \sum_{\lambda=1}^{d} \sigma_{[i],\mu;\lambda}^2$. Hence Lemma 6.3 follows by Lemmas 4.1 and 4.2. □

Lemma 6.4. *$b_{i,n_i+1} = 0$, for all $i \in I$ such that $p_{[i]}(z) \neq 0$.*

Proof. Suppose first that $z_{i,n_i+1} \neq 0$, for all $i \in I$. Let $i \in I$ such that $p_{[i]}(z) \neq 0$, and let us assume there exist no $j, k \in I$ with $z_{i,n_i+1} = z_{j,n_j+1} + z_{k,n_k+1}$. What does the polynomial $\tilde{q}_i$ in (21) look like? With regard to (20), Lemmas 6.2 and 4.1, and equalities (14)–(17), the contributing terms are

$$\frac{\partial}{\partial x} p_j(x, z)\, \mathrm{e}^{-z_{j,n_j+1}x} = \left(\left(\sum_{\mu=1}^{\mu_m \wedge n_j} z_{j,\mu} x^{\mu-1} \right) - z_{i,n_i+1} \left(\sum_{\mu=0}^{\mu_m \wedge n_j} z_{j,\mu} x^{\mu} \right) \right) \mathrm{e}^{-z_{i,n_i+1}x}, \tag{22}$$

$$\sum_{\mu=0}^{n_j+1} b_{j,\mu} \frac{\partial}{\partial z_{j,\mu}} F(x, z) = \left(\left(\sum_{\mu=0}^{\mu_m \wedge n_j} b_{j,\mu} x^{\mu} \right) - b_{i,n_i+1} \left(\sum_{\mu=0}^{\mu_m \wedge n_j} z_{j,\mu} x^{\mu+1} \right) \right) \mathrm{e}^{-z_{i,n_i+1}x} \tag{23}$$

and

$$- \left(\sum_{\mu=0}^{n_j} a_{j,\mu;k,\nu} \frac{\partial}{\partial z_{j,\mu}} F(x, z) \int_0^x \frac{\partial}{\partial z_{k,\nu}} F(\eta, z) \mathrm{d}\eta \right)$$

$$= - \left(\sum_{\mu=0}^{\mu_m \wedge n_j} a_{j,\mu;k,\nu} \frac{n_k!}{z_{k,n_k+1}^{n_k+1}} x^\mu \right) \mathrm{e}^{-z_{i,n_i+1}x} - \begin{pmatrix} \text{polynomial} \\ \text{in } x \end{pmatrix} \mathrm{e}^{-(z_{i,n_i+1}+z_{k,n_k+1})x}, \qquad (24)$$

for $0 \leqslant \nu \leqslant n_k$, for all $1 \leqslant k \leqslant K$ and $j \in [i]$. We have used the integer

$$\mu_m := \max\{\lambda | \lambda \leqslant n_l \text{ and } z_{l,\lambda} \neq 0 \text{ for some } l \in [i]\}.$$

Define $\tilde{\mu}_m := \max\{\lambda | \lambda \leqslant n_{[i]} \text{ and } z_{[i],\lambda} \neq 0\} \in \mathbb{N}_0$. Obviously $\tilde{\mu}_m \leqslant \mu_m$. By Lemma 6.3 we have $a_{[i],\mu;k,\nu} = 0$, for all $\tilde{\mu}_m < \mu \leqslant n_{[i]}$. Thus summing the above expressions over $j \in [i]$, we obtain

$$\tilde{q}_i(x) = -b_{i,n_i+1} z_{[i],\tilde{\mu}_m} x^{\tilde{\mu}_m+1} + \dots . \qquad (25)$$

Consequently, $b_{i,n_i+1} = 0$ in the regular case.

For the singular cases the boundedness assumption $z \in \mathscr{Z}$ is essential. We split I into two disjoint subsets J_1 and J_2, where

$$J_1 := \{i \in I | \text{ there exist } j, k \in I, \text{ such that } z_{i,n_i+1} = z_{j,n_j+1} + z_{k,n_k+1} \text{ and } z_{j,n_j+1} > 0 \text{ and } z_{k,n_k+1} > 0\},$$

$$J_2 := I \backslash J_1.$$

Notice that in any case $1 \in J_2$. We have shown above that for each $i \in J_2$ such that z_{i,n_i+1} is not the sum of two other z_{j,n_j+1}s it follows that $b_{i,n_i+1} = 0$. We will now show that $b_{i,n_i+1} = 0$ for all $i \in J_2$. Let $i \in J_2$ and assume there exist $j, k \in I$ with $z_{i,n_i+1} = z_{j,n_j+1} + z_{k,n_k+1}$. Then necessarily one of the summands is strictly less than zero. Without loss of generality $z_{j,n_j+1} < 0$. Since $z \in \mathscr{Z}$, we have $p_{[j]}(z) = 0$ (see (8)). Thus $a_{[j],\mu;[j],\mu} = 0$ by Lemma 6.3 and therefore $a_{[j],\mu;k,\nu} = 0$, for all $0 \leqslant \mu \leqslant n_{[j]}$, $0 \leqslant \nu \leqslant n_k$, $1 \leqslant k \leqslant K$. The terms contributing to the polynomial in front of $\mathrm{e}^{-z_{i,n_i+1}x}$, that is $\tilde{q}_i + \tilde{q}_{j,k} + \dots$, are those in (22)–(24) and also

$$- a_{l,\mu;m,\nu} \frac{\partial}{\partial z_{l,\mu}} F(x, z) \int_0^x \frac{\partial}{\partial z_{m,\nu}} F(\eta, z)\, \eta = -a_{l,\,\mu;m,\nu} x^\mu \,\mathrm{e}^{-z_{j,n_j+1}x} \int_0^x \eta^\nu \,\mathrm{e}^{-z_{k,n_k+1}\eta}\, \mathrm{d}\eta, \qquad (26)$$

for $0 \leqslant \mu \leqslant n_l$, $0 \leqslant \nu \leqslant n_m$, $l \in [j]$, $m \in [k]$. However, summing – for fixed μ, m and ν – the right-hand side of (26) over $l \in \mathscr{T}_{[j],\mu}$ gives zero. Hence the terms in (26) do not actually contribute to the polynomial. The same conclusion can be drawn for all $j, k \in I$ with the property that $z_{i,n_i+1} = z_{j,n_j+1} + z_{k,n_k+1}$. It finally follows, as in the regular case, that $b_{i,n_i+1} = 0$ for all $i \in J_2$.

If J_1 is not empty, we show that for each $i \in J_1$, the parameter z_{i,n_i+1} can be written as a linear combination of z_{j,n_j+1}s with $j \in J_2$. From this it follows by Lemma 4.1 that $b_{i,n_i+1} = 0$ for all $i \in J_1$. We proceed as follows. Write $J_1 = \{i_1, \dots, i_{r'}\}$ with $z_{i_1,n_{i_1}+1} < \dots < z_{i_{r'},n_{i_{r'}}+1}$. For each $i_k \in J_1$ there exists one linear equation of the form

$$(*, \ldots, *, 1, 0 \ldots, 0)\begin{pmatrix} z_{i_1,n_{i_1}+1} \\ \vdots \\ z_{i_k,n_{i_k}+1} \\ \vdots \\ z_{i_{r'},n_{i_{r'}}+1} \end{pmatrix} = \alpha'_k,$$

where the $*$ stand for 0 or -1, but at most two of them are -1. The α'_k on the right-hand side is 0 or z_{i,n_i+1} or $z_{i,n_i+1} + z_{j,n_j+1}$ for some indices $i, j \in J_2$ with $z_{i,n_i+1} > 0$ and $z_{j,n_j+1} > 0$. Obviously α'_1 is of the latter form. Hence we obtain the system of linear equations

$$\begin{pmatrix} 1 & 0 & \ldots & 0 \\ * & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ * & \ldots & * & 1 \end{pmatrix}\begin{pmatrix} z_{i_1,n_{i_1}+1} \\ \vdots \\ \vdots \\ z_{i_{r'},n_{i_{r'}}+1} \end{pmatrix} = \begin{pmatrix} z_{i,n_i+1} + z_{j,n_j+1} \\ \alpha'_2 \\ \vdots \\ \alpha'_{r'} \end{pmatrix},$$

for some $i, j \in J_2$. On the left-hand side is a lower-triangular matrix, which is therefore invertible. Hence the lemma is proved in the case where $z_{i,n_i+1} \neq 0$ for all $i \in I$.

Assume now that there exists $i \in I$ with $z_{i,n_i+1} = 0$. Then $i \in J_2$. We have to make sure that also in this case $b_{j,n_j+1} = 0$, for all $j \in J_2$. Clearly b_{i,n_i+1} is zero by Lemma 4.1. The problem is that $z_{j,n_j+1} = z_{i,n_i+1} + z_{j,n_j+1}$ for all $j \in J_2$. But following the lines above, it is enough to show $a_{[i],\mu;[i],\mu} = 0$, for all $0 \leqslant \mu \leqslant n_{[i]}$. From the boundedness assumption $z \in \mathscr{Z}$ we know that $p_{[i]}(z) = z_{[i],0}$ (see (8)). Hence $a_{[i],\mu;[i],\mu} = 0$, for $1 \leqslant \mu \leqslant n_{[i]}$. Suppose there is no pair of indices $j, k \in I\backslash\{i\}$ with $z_{j,n_j+1} + z_{k,n_k+1} = 0$. Summing the contributing terms in (22)–(24) over $j \in [i]$, we obtain the polynomial in front of e^0, i.e.

$$\tilde{q}_i(x) + \tilde{q}_{i,i}(x) = -a_{[i],0;[i],0}x + \ldots, \tag{27}$$

hence $a_{[i],0;[i],0} = 0$. If there exist a pair of indices $j, k \in I\backslash\{i\}$ with $z_{j,n_j+1} + z_{k,n_k+1} = 0$, then one of these summands is strictly less than zero. Arguing as before, the polynomial in front of e^0 remains of the form (27) and again $a_{[i],0;[i],0} = 0$. Thus the lemma is completely proved. □

So far we have established (12) and (13) under the hypothesis that $p_i(z) \neq 0$, for all $i \in \{1, \ldots, K\}$. Suppose now that there is an index $i \in \{1, \ldots, K\}$ with $p_i(z) = 0$. By Lemma 4.1, we may assume $a_{i,\mu;i,\mu} = b_{i,\mu} = 0$, for all $0 \leqslant \mu \leqslant n_i$. But then Lemma 4.2 tells us that none of the terms including the index i appears in (19). In particular, $a_{i,n_i+1;i,n_i+1}$ and b_{i,n_i+1} can be chosen arbitrarily without affecting equation (19). This means that we may skip i and proceed, after a reparametrization if necessary, with the remaining index set $\{1, \ldots, K-1\}$ to establish Lemmas 6.1–6.4 as above.

This all has to hold for $dt \otimes d\mathbb{P}$-a.e. (t, ω). Hence (12) and (13) are fully proved. A closer look to the proof of (12), that is, Lemma 6.1, shows that the boundedness assumption $z \in \mathscr{Z}$ was not explicitly used there – whence Remark 3.3.

Next we prove that the exponents Z^{i,n_i+1} are locally constant on intervals where $p_i(Z)$

and $p_{[i]}(Z)$ do not vanish. Let $v \geqslant 0$ be a rational number and let $T_v := \inf\{t > v | p_i(Z_t) = 0 \text{ or } p_{[i]}(Z) = 0\}$ denote the debut of the optional set $[v, \infty[\cap \mathscr{A}_i$. By (12) and (13) and the continuity of Z we have that Z^{i,n_i+1} is $\mathbb{P}$-a.s. constant on $[v, T_v]$, hence $\mathbb{P}$-a.s. constant on every such interval $[v, T_v]$. Since every open interval where $p_i(Z_t) \neq 0$ or $p_{[i]}(Z_t) \neq 0$ is covered by a countable union of intervals $[v, T_v]$ and by continuity of Z, the assertion follows and the first part of the theorem is proved.

To establish the second part of the theorem, let τ be a stopping time with $[\tau] \in \mathscr{D}'$ and $\mathbb{P}(\tau < \infty) > 0$. Define the stopping time $\tau'(\omega) := \inf\{t \geqslant \tau(\omega) | (t, \omega) \notin \mathscr{D}'\}$. By continuity of Z, we conclude that $\tau < \tau'$ on $\{\tau < \infty\}$. Choose a point (t, ω) in $[\tau, \tau'[$. We use shorthand notation as above.

By definition of $\mathscr{D}'$ we can exclude the singular cases $z_{i,n_i+1} = z_{j,n_j+1}$ or $2z_{i,n_i+1} = z_{j,n_j+1}$, for $i \neq j$. In particular, $\tilde{K} = K$, hence $I = \{1, \ldots, K\}$. First, we show that the diffusion matrix for the coefficients of the polynomials $p_i(z)$ vanishes.

Lemma 6.5. *$a_{i,\mu;j,\nu} = a_{j,\nu;i,\mu} = 0$, for $0 \leqslant \mu \leqslant n_i$ and $0 \leqslant \nu \leqslant n_j$, for all $i, j \in I$.*

Proof. By Lemma 4.1 it is enough to prove that the diagonal $a_{i,\mu;i,\mu}$ vanishes for $0 \leqslant \mu \leqslant n_i$ and $i \in I$. If there is an index $i \in I$ with $p_i(z) = 0$ then, arguing as above, $a_{i,\mu;i,\mu} = b_{i,\mu} = 0$, for all $0 \leqslant \mu \leqslant n_i$, and we may skip the index i. Hence we assume now that there is a $K' \leqslant K$ such that $p_i(z) \neq 0$ (and thus $z_{i,n_i+1} \geqslant 0$, since $z \in \mathscr{Z}$) for all $1 \leqslant i \leqslant K'$. Let $I' := \{1, \ldots, K'\}$. To handle the singular cases, we split I' into two disjoint subsets I'_1 and I'_2, where

$$I'_1 := \{i \in I' | z_{i,n_i+1} > 0, \text{ and there exist } j, k \in I' \text{ such that } 2z_{i,n_i+1} = z_{j,n_j+1} + z_{k,n_k+1}\},$$

$$I'_2 := I' \setminus I'_1.$$

Hence $z_{i,n_i+1} = 0$ for $i \in I'$ implies $i \in I'_2$. We have already shown in the proof of Lemma 6.4 that in this case $a_{i,\mu;i,\mu} = 0$, for all $0 \leqslant \mu \leqslant n_i$. The same follows for $i \in I'_2$ with $z_{i,n_i+1} > 0$, as was demonstrated for the case $K = 1$.

Now let $i \in I'_1$ and let $l, m \in I'$, such that $l \leqslant m$ and $2z_{i,n_i+1} = z_{l,n_l+1} + z_{m,n_m+1}$. Thus the polynomial in front of $\mathrm{e}^{-2z_{i,n_i+1}x}$ is $q_{i,i} + q_{l,m} + \ldots$, and among the contributing terms are also those in (26). If l or m is in I'_2, those are all zero. Write $I'_1 = \{i_1, \ldots, i_{r''}\}$ with $z_{i_1,n_{i_1}+1} < \ldots < z_{i_{r''},n_{i_{r''}}+1}$. Then necessarily $l \in I'_2$ in the above representation for $z_{i_1,n_{i_1}+1}$. Thus the polynomial in front of $\mathrm{e}^{-2z_{i_1,n_{i_1}+1}x}$ is q_{i_1,i_1}. It follows that $a_{i_1,\mu;i_1,\mu} = 0$, for all $0 \leqslant \mu \leqslant n_{i_1}$, as was demonstrated for the case $K = 1$. Proceeding inductively for $i_2, \ldots, i_{r''}$, we eventually derive that $a_{i,\mu;i,\mu} = 0$, for all $0 \leqslant \mu \leqslant n_i$ and $i \in I'$. This establishes the lemma. □

We are left with the task of determining the drift of the coefficients in $p_i(z)$. By (13), we have $b_{i,n_i+1} = 0$ for all $i \in I'$. Straightforward calculations show that (19) reduces to

$$\sum_{i=1}^{K'} q_i(x)\,\mathrm{e}^{-z_{i,n_i+1}x} = 0,$$

with

$$q_i(x) = (b_{i,n_i} + z_{i,n_i+1}z_{i,n_i})x^{n_i} + \sum_{\mu=0}^{n_i-1}(b_{i,\mu} - z_{i,\mu+1} + z_{i,n_i+1}z_{i,\mu})x^{\mu}.$$

We conclude that for all $1 \leqslant i \leqslant K$ (in particular, if $p_i(z) = 0$)

$$\begin{aligned} b_{i,\mu} &= z_{i,\mu+1} - z_{i,n_i+1}z_{i,\mu}, \qquad 0 \leqslant \mu \leqslant n_i - 1, \\ b_{i,n_i} &= -z_{i,n_i+1}z_{i,n_i}. \end{aligned} \tag{28}$$

By continuity of Z, Lemma 6.5 and (28) hold pathwise on the semi-open interval $[\tau(\omega), \tau'(\omega)[$ for almost every ω. Therefore $Z_{\tau+\cdot}$ is of the claimed form on $[0, \tau' - \tau[$.

Now replace $\mathscr{D}'$ by $\mathscr{D}$ and proceed as above. By (11) we have $\tau < \tau'$ on $\{\tau < \infty\}$, and, since $\mathscr{D} \subset \mathscr{D}'$, all the above results remain valid. In addition, $p_i(z) = p_{[i]}(z) \neq 0$ and thus $a_{i,n_i+1;i,n_i+1} = b_{i,n_i+1} = 0$, for all $1 \leqslant i \leqslant K$, by (12) and (13). Hence $Z^{i,n_i+1}_{\tau+\cdot} = Z^{i,n_i+1}_{\tau}$ on $[0, \tau' - \tau[$, for all $1 \leqslant i \leqslant K$, up to evanescence. But this again implies $\tau' = \infty$ by the continuity of Z.

7. E-consistent Itô processes

An Itô process Z is by definition consistent with a family $\{F(\cdot, z)\}_{z\in\mathscr{Z}}$ if and only if $\mathbb{P}$ is a martingale measure for the discounted bond price processes. We could generalize this definition and call a process Z *e-consistent* with $\{F(\cdot, z)\}_{z\in\mathscr{Z}}$ if there exists an equivalent martingale measure $\mathbb{Q}$. Then obviously consistency implies e-consistency, and e-consistency implies the absence of arbitrage opportunities as is well known.

Where the filtration is generated by the Brownian motion W, that is, $(\mathscr{F}_t) = (\mathscr{F}^W_t)$, we can give the following stronger result:

Proposition 7.1. *Let* $K \in \mathbb{N}$ *and* $n = (n_1, \ldots, n_K) \in \mathbb{N}_0^K$. *If* $(\mathscr{F}_t) = (\mathscr{F}^W_t)$, *then any Itô process* Z *which is e-consistent with* $BEP(K, n)$ *is of the form stated in Theorem* 3.2.

Proof. Let Z be an e-consistent Itô process under $\mathbb{P}$, and let $\mathbb{Q}$ be an equivalent martingale measure. Since $(\mathscr{F}_t) = (\mathscr{F}^W_t)$, we know that all $\mathbb{P}$-martingales have the representation property relative to W. By Girsanov's theorem it follows, therefore, that Z remains an Itô process under $\mathbb{Q}$, which is consistent with $BEP(K, n)$. The drift coefficients $b^{i,\mu}$ change under $\mathbb{Q}$ into $\tilde{b}^{i,\mu}$; while $b^{i,\mu} = \tilde{b}^{i,\mu}$ on $\{a^{i,\mu;i,\mu} = 0\}$, $dt \otimes d\mathbb{P}$-a.s. The diffusion matrix a remains the same. Therefore, and since the measures $dt \otimes d\mathbb{Q}$ and $dt \otimes d\mathbb{P}$ are equivalent on $\mathbb{R}_+ \times \Omega$, the Itô process Z is of the form stated in Theorem 3.2. □

Notice that in this case the quasi-deterministic, that is, $\mathscr{F}_0$-measurable, expression in Corollaries 3.5 and 5.2 is a purely deterministic one.

8. The diffusion case

The main result from Section 3 is much clearer for diffusion processes. In all applications the generic Itô process Z on $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{0\leqslant t<\infty}, \mathbb{P})$ given by (9) is the solution of a stochastic differential equation

$$Z_t^{i,\mu} = Z_0^{i,\mu} + \int_0^t b^{i,\mu}(s,\ Z_s)\mathrm{d}s + \sum_{\lambda=1}^{d} \int_0^t \sigma^{i,\mu;\lambda}(s,\ Z_s)\mathrm{d}W_s^{\lambda}, \tag{29}$$

for $0 \leqslant \mu \leqslant n_i + 1$ and $1 \leqslant i \leqslant K$, where b and σ are some Borel measurable mappings from $\mathbb{R}_+ \times \mathbb{R}^N$ to $\mathbb{R}^N$ and $\mathbb{R}^{N\times d}$, respectively.

The coefficients b and σ could be derived by statistical inference methods from the daily observations of the diffusion Z made by some central bank. These observations are of course made under the objective probability measure. Hence $\mathbb{P}$ is not a martingale measure in applications of this kind.

On the other hand, we want a model for pricing interest rate sensitive securities. Thus the diffusion has to be e-consistent. If we assume that $(\mathscr{F}_t) = (\mathscr{F}_t^W)$, the previous section applies. To stress the fact that $\mathscr{F}_0^W$-measurable functions are deterministic, we denote the initial values of the diffusion in (29) with small letters $z_0^{i,\mu}$.

Since all reasonable theory for stochastic differential equations requires continuity properties of the coefficients, we shall assume in the following that $b(t, z)$ and $\sigma(t, z)$ are continuous in z. The main result for e-consistent diffusion processes is divided into the two following theorems. The first one only requires consistency with $EP(K, n)$.

Theorem 8.1. *Let $K \in \mathbb{N}$, $n = (n_1, \ldots, n_K) \in \mathbb{N}_0^K$, $(\mathscr{F}_t) = (\mathscr{F}_t^W)$, and let the diffusion Z, b and σ be as above. If Z is e-consistent with $BEP(K, n)$ or with $EP(K, n)$, then necessarily the exponents are constant,*

$$Z^{i,n_i+1} \equiv z_0^{i,n_i+1},$$

for all $1 \leqslant i \leqslant K$.

Proof. The significant difference between this proof and that of Theorem 3.2 is that now the diffusion matrix a and the drift b depend continuously on z.

First, observe that the following sets of singular values,

$$\mathscr{M} := \bigcup_{i=1}^{K} \{z \in \mathbb{R}^N \,|\, p_i(z) = 0 \text{ or } p_{[i]}(z) = 0\}$$

and

$$\mathscr{N} := \{z \in \mathbb{R}^N | z_{i,n_i+1} = z_{j,n_j+1} + z_{k,n_k+1} \text{ for some } 1 \leqslant i,\ j,\ k \leqslant K\},$$

are contained in a finite union of hyperplanes of $\mathbb{R}^N$ (see (10)). Hence $(\mathscr{M}\cup\mathscr{N})\subset\mathbb{R}^N$ has Lebesgue measure zero. Thus the topological closure of $\mathscr{G}:=\mathbb{R}^N\backslash(\mathscr{M}\cup\mathscr{N})$ is $\mathbb{R}^N$.

Now let Z be the diffusion in (29), which is e-consistent with either $BEP(K, n)$ or $EP(K, n)$. A closer look at the proof of Lemma 6.4 shows that the boundedness assumption $z\in\mathscr{Z}$ was not used for the regular case $z\in\mathscr{G}$ (see (25)). Combining this with (12), (13) and Remark 3.3, we conclude that, for any $1\leqslant i\leqslant K$ and $1\leqslant\lambda\leqslant d$,

$$b^{i,n_i+1}(t, Z_t(\omega))=\sigma^{i,n_i+1;\lambda}(t, Z_t(\omega))=0, \qquad \text{for } (t, \omega)\in\{Z\in\mathscr{G}\}\backslash N,$$

where N is an $\mathscr{B}_+\otimes\mathscr{F}$-measurable $dt\otimes d\mathbb{P}$-nullset. By the very definition of the product measure,

$$0=\int_N 1\,\mathrm{d}t\otimes d\mathbb{P}=\int_{\mathbb{R}_+}\mathbb{P}[N_t]\,\mathrm{d}t,$$

where $N_t:=\{\omega|(t, \omega)\in N\}\in\mathscr{F}$. Consequently $\mathbb{P}[N_t]=0$ for almost every $t\in\mathbb{R}_+$. Hence, by continuity of $b(t, \cdot)$ and $\sigma(t, \cdot)$,

$$b^{i,n_i+1}(t, \cdot)=\sigma^{i,n_i+1;\lambda}(t, \cdot)=0 \tag{30}$$

on $\operatorname{supp}(Z_t)\cap\mathscr{G}$, for almost every $t\in\mathbb{R}_+$. Here $\operatorname{supp}(Z_t)$ denotes the support of the (regular) distribution of Z_t, which is by definition the smallest closed set $A\subset\mathbb{R}^N$ with $\mathbb{P}[Z_t\in A]=1$. Thus, again by continuity of $b(t, \cdot)$ and $\sigma(t, \cdot)$, equality (29) holds for almost every $t\in\mathbb{R}_+$ on the closure of $\operatorname{supp}(Z_t)\cap\mathscr{G}$, which is $\operatorname{supp}(Z_t)$. Hence we may replace the functions $b^{i,n_i+1}(t, \cdot)$ and $\sigma^{i,n_i+1;\lambda}(t, \cdot)$ by zero for almost every t without changing the diffusion Z, whence the assertion follows. □

The sum of two real-valued diffusion processes with coefficients continuous in some argument is again a real-valued diffusion with coefficients continuous in that argument. Consequently, we may assume that the exponents z_0^{i,n_i+1} of the above e-consistent diffusion are mutually distinct. Since otherwise we add the corresponding polynomials to obtain in a canonical way an $\mathbb{R}^{\tilde{N}}$-valued diffusion $\tilde{Z}$ which is e-consistent with $BEP(\tilde{K}, \tilde{n})$ or $EP(\tilde{K}, \tilde{n})$, for some $\tilde{K}<K$, $\tilde{N}<N$ and some $\tilde{n}\in\mathbb{N}_0^{\tilde{K}}$. Clearly $\tilde{Z}$ provides the same interest rate model as Z and its coefficients are continuous in z.

For the second theorem we have to require e-consistency with $BEP(K, n)$. After a reparametrization if necessary, we may thus assume that

$$0\leqslant z_0^{1,n_1+1}<\ldots<z_0^{K,n_K+1}$$

(see (8)). The continuation of Theorem 8.1 is now as follows:

Theorem 8.2. *If Z is e-consistent with $BEP(K, n)$, then it is non-trivial only if there exists a pair of indices $1\leqslant i<j\leqslant K$, such that*

$$2z_0^{i,n_i+1}=z_0^{j,n_j+1}.$$

Proof. If there is no pair of indices $1\leqslant i<j\leqslant K$ such that $2z_0^{i,n_i+1}=z_0^{j,n_j+1}$, then $\mathscr{D}'=\mathbb{R}_+\times\Omega$. But then Z is deterministic by the second part of Theorem 3.2. □

The message of Theorem 8.1 is the following: there is no possibility of modelling the term structure of interest rates by exponential-polynomial families with varying exponents driven by diffusion processes. From this point of view there is no use for daily estimations of the exponents of exponential-polynomial type functions such as F_{NS} or F_S. Once the exponents are chosen, they have to be kept constant. Furthermore, there is a strong restriction on this choice by Theorem 8.2. It will be shown in the next section what this means for F_{NS} and F_S in particular.

Remark 8.3. The boundedness assumption in Theorem 8.2 – that is, e-consistency with $BEP(K, n)$ – is essential for the strong (negative) result to be valid. It can easily be checked that $F(x, z) = z_0 + z_1 x \in EP(1, 1)$ allows for a non-trivial consistent diffusion process (see Filipović 1999a).

Remark 8.4. The choice of an infinite time horizon for traded bonds is not a restriction (see (1)). Indeed, we can limit our considerations to bonds $P(t, T)$ which mature within a given finite time interval $[0, T^*]$. Consequently, the HJM drift condition (3) can only be deduced for $x \in [0, T^* - t]$, for $dt \otimes d\mathbb{P}$-a.e. $(t, \omega) \in [0, T^*] \times \Omega$. But the functions appearing in (3) are analytic in x. Hence, whenever $t < T^*$, relation (3) extends to all $x \geqslant 0$. All conclusions on e-consistent Itô processes $(Z_t)_{0 \leqslant t \leqslant T^*}$ can now be drawn as before.

9. Applications

In this section we apply the results on e-consistent diffusion processes to the Nelson–Siegel and Svensson families, whose curve shapes were given in Section 1.

9.1. The Nelson–Siegel family

In view of Theorem 8.1 we have $z_4 > 0$. Hence it is immediate from Theorem 8.2 that there is no non-trivial e-consistent diffusion. This result has already been obtained in Filipović (1999b) for e-consistent Itô processes.

9.2. The Svensson family

By Theorems 8.1 and 8.2 there remain the two choices

(i) $2z_6 = z_5 > 0$,
(ii) $2z_5 = z_6 > 0$.

We shall identify the e-consistent diffusion process $Z = (Z^1, \ldots, Z^6)$ in both cases. Let $\mathbb{Q}$ be an equivalent martingale measure. Under $\mathbb{Q}$ the diffusion Z transforms into a consistent one. Now we proceed as in the proof of Theorem 3.2. The expansion (19) is given by

$$Q_1(x) + Q_2(x)\,e^{-z_5 x} + Q_3(x)\,e^{-z_6 z} + Q_4(x)\,e^{-2z_5 x} + Q_5(x)\,e^{-(z_5+z_6)x} + Q_6(x)\,e^{-2z_6 x} = 0,$$

for polynomials $Q_1 \ldots, Q_6$. Explicitly,

$$Q_1(x) = -a_{1,1}x + \ldots$$

$$Q_2(x) = -a_{1,3}x^2 + \ldots$$

$$Q_3(x) = -a_{1,4}x^2 + \ldots$$

$$Q_4(x) = \frac{a_{3,3}}{z_5}x^2 + \ldots$$

$$Q_6(x) = \frac{a_{4,4}}{z_6}x^2 + \ldots$$

where $\ldots$ denotes terms of lower order in x. Hence $a_{1,1} = 0$ in any case. By the usual arguments (the matrix a is non-negative definite) the degree of Q_2 and Q_3 reduces to at most 1. Thus in both cases (i) and (ii) it follows that $a_{3,3} = a_{4,4} = 0$. We are left with

$$\begin{aligned} Q_1(x) &= b_1, \\ Q_2(x) &= (b_3 + z_3 z_5)x + b_2 - z_3 - \frac{a_{2,2}}{z_5} + z_2 z_5, \\ Q_3(x) &= (b_4 + z_4 z_6)x - z_4, \\ Q_4(x) &= \frac{a_{2,2}}{z_5} \end{aligned} \tag{31}$$

while $Q_5 = Q_6 = 0$. Since in case (i) Q_4 must be 0, we have $a_{2,2} = 0$ and Z is deterministic. We conclude that there is no non-trivial e-consistent diffusion in case (i).

In case (ii) the condition $Q_3 + Q_4 = 0$ leads to

$$a_{2,2} = z_4 z_5. \tag{32}$$

Hence there is a possibility of a non-deterministic consistent diffusion Z. We derive, from (31) and (32), that

$$b_1 = 0,$$

$$b_2 = z_3 + z_4 - z_5 z_2,$$

$$b_3 = -z_5 z_3,$$

$$b_4 = -2z_5 z_4.$$

Therefore the dynamics of $Z^1, Z^3, \ldots, Z^6$ are deterministic. In particular,

$$\begin{aligned} Z_t^1 &\equiv z_0^1, \\ Z_t^3 &= z_0^3 \, \mathrm{e}^{-z_0^5 t}, \\ Z_t^4 &= z_0^4 \, \mathrm{e}^{-2z_0^5 t}, \end{aligned} \tag{33}$$

while $Z_t^5 \equiv z_0^5$ and $Z_t^6 \equiv 2z_0^5$. Denoting by $\tilde{W}$ the Girsanov transform of W, we have under the equivalent martingale measure $\mathbb{Q}$

$$Z_t^2 = z_0^2 + \int_0^t \left(\Phi(s) - z_0^5 Z_s^2\right) \mathrm{d}s + \sum_{\lambda=1}^{d} \int_0^t \sigma^{2,\lambda}(s) \mathrm{d}\tilde{W}_s^\lambda, \tag{34}$$

where $\Phi(t)$ and $\sigma^{2,\lambda}(t)$ are deterministic functions in t, namely

$$\Phi(t) := z_0^3\, \mathrm{e}^{-z_0^5 t} + z_0^4\, \mathrm{e}^{-2z_0^5 t}$$

and

$$\sum_{\lambda=1}^{d} \left(\sigma^{2,\lambda}(t)\right)^2 = z_0^4 z_0^5\, \mathrm{e}^{-2z_0^5 t}.$$

By Lévy's characterization theorem (see Revuz and Yor 1994, Theorem (3.6), Chapter IV), the real-valued process

$$W_t^* := \sum_{\lambda=1}^{d} \int_0^t \frac{\sigma^{2,\lambda}(s)}{\sqrt{z_0^4 z_0^5\, \mathrm{e}^{-z_0^5 s}}}\, \mathrm{d}\tilde{W}_s^\lambda, \qquad 0 \leqslant t < \infty,$$

is an $(\mathscr{F}_t)$-Brownian motion under $\mathbb{Q}$. Hence the corresponding *short rates* $r_t = F_S(0, Z_t) = z_0^1 + Z_t^2$ satisfy

$$\mathrm{d}r_t = (\phi(t) - z_0^5 r_t)\mathrm{d}t + \tilde{\sigma}(t)\mathrm{d}W_t^*,$$

where $\phi(t) := \Phi(t) + z_0^1 z_0^5$ and $\tilde{\sigma}(t) := \sqrt{z_0^4 z_0^5}\, \mathrm{e}^{-z_0^5 t}$. This is just the generalized Vasicek model. It can be easily given a closed-form solution for r_t (see Musiela and Rutkowski 1987, p. 293).

Summarizing case (ii), we have found a non-trivial e-consistent diffusion process, which is identified by (33) and (34). Actually Φ has to be replaced by a predictable process $\tilde{\Phi}$ due to the change of measure. Nevertheless, this is just a one-factor model. The corresponding interest rate model is the generalized Vasicek short rate model. This is very unsatisfactory since Svensson-type functions $F_S(x, z)$ have six factors $z_1, \ldots, z_6$ which are observed. And it is evident that just one of them, z_2, can be chosen to be non-deterministic.

10. Conclusions

Bounded exponential-polynomial families such as the Nelson–Siegel and Svensson families may be well suited for daily estimations of the forward rate curve. They are best not used for intertemporal interest rate modelling by diffusion processes. This is because the exponents have to be kept constant and, moreover, this choice is very restrictive whenever you want to exclude arbitrage possibilities. It has been shown for the Nelson–Siegel family in particular that there exists no non-trivial diffusion process providing an arbitrage-free model. However, there is a choice for the Svensson family, albeit a very limited one, since all parameters but one have to be kept either constant or deterministic.

Acknowledgement

I am greatly indebted to T. Björk for pointing out this extension of my earlier work. I thank B.J. Christensen for his interest in these results and for discussion. I am also grateful to H. Pages from the BIS for his comments and for making available the technical documentation. This paper is part of my doctoral thesis, written under the supervision of F. Delbaen. Support from Credit Suisse is gratefully acknowledged.

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Received November 1998 and revised February 2000