Bernoulli **7**(1), 2001, 119–144

A mixture of the exclusion process and the voter model

VLADIMIR BELITSKY[1*], PABLO A. FERRARI[1**],
MIKHAIL V. MENSHIKOV[1,2] and SERGUEI YU. POPOV[1,3]

[1]*Instituto de Matemática e Estatística, Universidade de São Paulo, Caixa Postal 66281, 05315-970 São Paulo SP, Brazil. E-mail:* [*]*vbel1@ime.usp.br,* [**]*pablo@ime.usp.br*
[2]*Faculty of Mechanics and Mathematics, Moscow State University, Russia. E-mail: menchik@ime.usp.br*
[3]*Institute for Problems of Information Transmission, Russian Academy of Sciences, Moscow, Russia. E-mail: popov@ime.usp.br*

We consider a one-dimensional nearest-neighbour interacting particle system, which is a mixture of the simple exclusion process and the voter model. The state space is taken to be the countable set of the configurations that have a finite number of particles to the right of the origin and a finite number of empty sites to the left of it. We obtain criteria for the ergodicity and some other properties of this system using the method of Lyapunov functions.

Keywords: exclusion process; Lyapunov function; voter model

1. Introduction

In this paper we consider a process that is a mixture of two nearest-neighbour one-dimensional interacting particle systems: the simple exclusion process and the voter model. Let us first define these two processes.

Definition 1.1. *For $\eta \in \{0, 1\}^{\mathbb{Z}}$, denote*

$$\eta_{x,y}(z) = \begin{cases} \eta(y), & \text{if } z = x, \\ \eta(x), & \text{if } z = y, \\ \eta(z), & \text{if } z \neq x, y, \end{cases}$$

and

$$\eta_x(z) = \begin{cases} 1 - \eta(z), & \text{if } z = x, \\ \eta(z), & \text{if } z \neq x. \end{cases}$$

A Markov process $\eta_t \in \{0, 1\}^{\mathbb{Z}}$, $t \in [0, +\infty)$, is called

(a) a simple exclusion process with parameter $0 \leqslant p \leqslant 1$, if its generator Ω_p^{e} has the form

$$\Omega^{e}_{p} f(\eta) = \sum_{x,y} p(x, y)\eta(x)(1 - \eta(y))[f(\eta_{x,y}) - f(\eta)],$$

where

$$p(x, y) = \begin{cases} p, & \text{if } y = x - 1, \\ 1 - p, & \text{if } y = x + 1, \\ 0, & \text{otherwise}; \end{cases}$$

(b) a voter model, if its generator Ω^{v} *is defined by*

$$\Omega^{v} f(\eta) = \sum_{x} c(x, \eta)[f(\eta_x) - f(\eta)],$$

where

$$c(x, \eta) = \begin{cases} \frac{1}{2}(\eta(x-1) + \eta(x+1)), & \text{if } \eta(x) = 0, \\ \frac{1}{2}(2 - \eta(x-1) - \eta(x+1)), & \text{if } \eta(x) = 1. \end{cases} \tag{1.1}$$

The construction of these processes from their generators may be found in Liggett (1985); see the first chapter and the beginning of the chapters on each proces. Harris graphical construction (see Durrett 1988; 1995) is an alternative approach to defining the processes. This will be briefly reviewed and used in Section 2.

Let us call $\eta \in \{0, 1\}^{\mathbb{Z}}$ a *configuration of particles*, and let us interpret $\eta(x) = 1$ as the presence of a particle at the site $x \in \mathbb{Z}$ in the configuration η and $\eta(x) = 0$ as the absence of such a particle. The dynamics of both processes may be interpreted in terms of particles that jump around $\mathbb{Z}$ (the case of the exclusion process) or appear and disappear at the sites of $\mathbb{Z}$ (the case of the voter model).

In the exclusion process, there may be at most one particle at each site of $\mathbb{Z}$. If there is a particle at site x and no particle at site $x + 1$, then the particle at x jumps with rate $1 - p$ to site $x + 1$; similarly, if there is a particle at x and no particle at $x - 1$, then the particle at x jumps with rate p to $x - 1$. This is a conservative dynamics, in the sense that particles neither are created nor disappear. Liggett (1976) described the set of invariant measures for this process. If $p = \frac{1}{2}$, the invariant measures are convex combinations of the translation-invariant product measures parametrized with the density of particles. If $p > \frac{1}{2}$, the set of invariant measures also contains measures with support in the countable state space $\mathscr{D}$, defined as the set of configurations with a finite number of empty sites to the left of the origin and a finite number of particles to the right of it. These measures are called *blocking measures* because, due to the exclusion rule and the accumulation of particles to the left of the origin, the flux of particles is null. Of course there are also blocking measures for $p < \frac{1}{2}$; they are obtained by the reflection $\mathbb{Z} \to -\mathbb{Z}$ of those mentioned above. When an asymmetric exclusion process ($p \neq \frac{1}{2}$) is considered from a random position determined by a so-called second-class particle, a new set of invariant measures arises. Each of these measures is supported by configurations with different asymptotic densities to the right and left of the origin; for this reason, they are called *shock measures*. The existence of shock measures was proved by Ferrari *et al.* (1991) and Ferrari (1992). See Ferrari (1994) and Liggett (1999) for an account of properties of these measures and the asymptotic behaviour of the second-class particle. Derrida *et al.* (1998) propose a nice alternative description of shock measures for this process.

In the voter model, there may also be at most one particle per site, but now with is non-conservative dynamics: a new particle is born at an empty site x at a rate proportional to the number of nearest neighbours of x occupied by particles; and a particle that is present at a site x disappears at a rate proportional to the number of empty neighbours of x. Since only one site changes its value at any given time, this model is a particular case of the so-called *spin-flip* models. There are only two invariant measures for the one-dimensional voter model defined above: one has support on the configuration 'all zeros' and the other has support on the configuration 'all ones'. The basic tool for proving those results is *duality*, a technique that allows properties of the voter model to be expressed as properties of a dual process, a process obtained when one 'looks backwards in time'. There are two dual processes for the voter model: coalescing random walks and annihilating random walks. See Liggett (1985, Chapter V) and Durrett (1995) for accounts of these and many other properties of the voter model.

If the voter model starts from the Heaviside configuration η^0, defined by $\eta^0(x) = \mathbf{1}_{\{x\leqslant 0\}}$, then at any future time it is a random translation of η^0. Indeed, the position of the rightmost particle $X_t = \max\{x : \eta_t(x) = 1\}$ performs a nearest-neighbour symmetric random walk and $\theta_{X_t}\eta_t = \eta_0$, where θ_x is translation by x. This example motivates the introduction of an equivalence relation: we say that two configurations η and η' are equivalent, and write $\eta \sim \eta'$, if one of them is a translation of the other: there exists a $y \in \mathbb{Z}$ such that $\eta(x) = \eta'(x + y)$ for all $x \in \mathbb{Z}$. Let $\tilde{\mathscr{D}} := \mathscr{D}/\sim$ denote the set of equivalence classes induced by $\sim$. Let $\mathscr{D}_0$ denote the set of the configurations in the equivalence class of η^0. In the voter model, $\eta_0 \in \mathscr{D}_0$ implies $\eta_t \in \mathscr{D}_0$ for all t. Hence, writing $\tilde{\eta}_t$ for the equivalence class of η_t, we have that $\tilde{\eta}_0 = \tilde{\eta}^0$ implies $\tilde{\eta}_t \equiv \tilde{\eta}^0$ (nothing moves). The process $\tilde{\eta}_t \in \tilde{\mathscr{D}}$, just defined, is isomorphic to $\theta_{X_t}\eta_t$, the voter model as seen from its rightmost particle.

Cox and Durrett (1995) studied one-dimensional voter models on $\tilde{\mathscr{D}}$ with rate function $\sum_y q(|x - y|)|\eta(x) - \eta(y)|$ for some probability function $q(x)$. They show that if $\sum_x |x|^3 q(x) < \infty$, then the process as seen from the rightmost particle $\tilde{\eta}_t \in \tilde{\mathscr{D}}$ is positive recurrent and hence admits a unique invariant (shock) measure. Calling Y_t the leftmost hole, this implies that under the invariant measure the size of the *hybrid zone* – the region of coexistence of zeros and ones – $X_t - Y_t$ is bigger than -1 and finite with probability one; and of course its distribution is independent of t. They also prove that the expected value of $X_t - Y_t$ under the invariant measure is infinite and that $X_t/\sqrt{t}$ converges as $t \to \infty$ to a centred normal distribution with finite variance. The approach is based on a fine analysis of the (dual) process coalescing random walks. It is also shown there that there are no 'stable' hybrid zones in dimension $d = 2$: if one starts with ones in the negative x half-plane and zeros in the positive half-plane and paints 1s white and 0s black, then the normal distribution with variance approximately t predicts the shade of grey we see at time t in the horizontal direction.

Ferrari (1996) shows the existence of an invariant shock measure for the biased voter model as seen from the rightmost particle. In this model the rate function is given by $c_2(x, \eta) = (a\eta(x) + b(1 - \eta(x)))c(x, \eta)$, with $c(x, \eta)$ as defined in (1.1). The proof in this case is more direct because it is based on straightforward domination by supermartigales.

The goal of this paper is to study the existence of shock measures in a *mixture* of the exclusion process and the voter model.

Definition 1.2. *Let $\beta \in [0, 1]$. A Markov process $\eta_t \in \{0, 1\}^{\mathbb{Z}}$, $t \in [0, +\infty)$, is called a* hybrid process *with mixing parameter β and exclusion parameter p, if its generator is*

$$\Omega^{\mathrm{h}}_{\beta,p} := (1 - \beta)\Omega^{\mathrm{e}}_p + \beta\Omega^{\mathrm{v}}. \tag{1.2}$$

The hybrid process $\tilde{\eta}_t$ (the equivalence class of η_t with initial configuration in $\mathscr{D}$) is a Markov process on $\tilde{\mathscr{D}}$. This process is a particular case of a model of random grammars considered by Malyshev (1998). Models consisting of a mixture of a spin-flip dynamics and a symmetric exclusion dynamics are usually called in the literature 'diffusion–reaction processes'. When $\beta \gg 0$, an appropriate space-time rescaling with β produces hydrodynamic limits giving rise to the reaction–diffusion equation $\partial u/\partial t = \partial^2 u/\partial^2 x + f(u)$, where the function f is related to the spin-flip dynamics and $u = u(x, t) \in [0, 1]$, $x, t \in \mathbb{R}_+$ corresponds to the macroscopic density of particles (De Masi *et al.* 1986). In some cases these equations accept travelling-wave solutions – solutions of the type $u(x, t) = u_0(x - vt)$ for some speed v with $\lim_{x\to\infty} u_0(x) = 0$, $\lim_{x\to-\infty} u_0(x) = 1$. This motivates the question about the existence of a microscopic counterpart of the macroscopic travelling-wave solutions. A particular case of the reaction process is the growth model, a process with rate function $c(x, \eta)(1 - \eta(x))$, where $c(x, \eta)$ is as defined in (1.1): 0 flips to 1 at rate proportional to the number of ones in the neighbourhood, but 1 never flips to 0. Bramson *et al.* (1986) showed the existence of an invariant (shock) measure for the process $\tilde{\eta}_t$, where η_t is any non-trivial mixture of the exclusion process and the growth model. Cammarota and Ferrari (1991) proved the normal asymptotic behaviour of $(X_t - \mathrm{E}X_t)/\sqrt{t}$ for this mixture. Machado (1998) studied this process in a strip and in $\mathbb{Z}^d$.

Let $\tilde{\tau}_{\mathrm{c}}(\tilde{\eta})$ be the first time the process $\tilde{\eta}_t$ starting with the configuration $\tilde{\eta} \in \tilde{\mathscr{D}}$ hits $\tilde{\eta}^0$, the Heaviside configuration defined above. The subscript c refers to continuous time (a discrete-time process is introduced below). Let us recall some classical definitions. We say that the process $\tilde{\eta}_t$ is *transient* if $P(\tilde{\tau}_{\mathrm{c}}(\tilde{\eta}) < \infty) < 1$, and *recurrent* if $P(\tilde{\tau}_{\mathrm{c}}(\tilde{\eta}) < \infty) = 1$. In the latter case we say that the process is *positive recurrent* if $\mathrm{E}(\tilde{\tau}_{\mathrm{c}}(\tilde{\eta})) < \infty$ and *null recurrent* if this expectation is infinity. An irreducible countable Markov chain is *ergodic* if it has a unique invariant measure. Since, except for the pure voter model, $\tilde{\eta}_t$ is irreducible, positive recurrence is equivalent to ergodicity in our context. The following theorem contains our results.

Theorem 1.1. *Let η_t be a process in $\mathscr{D}$ with generator $\Omega^{\mathrm{h}}_{\beta,p}$. Let $\tilde{\eta}_t$ be the corresponding process in the space of equivalence classes $\tilde{\mathscr{D}}$.*

(i) *Exclusion process. Assume $\beta = 0$. Then the process $\tilde{\eta}_t$ is ergodic for $p > \frac{1}{2}$ and transient for $p \leqslant \frac{1}{2}$.*

(ii) Hybrid process. Assume $0 < \beta < 1$. *Then,*

(a) there exists $\beta_c < 1$ *such that for any* $\beta > \beta_c$ *and any* $p \in (0, 1)$ *the process* $\tilde{\eta}_t$ *is ergodic*;

(b) for any $p \geq \frac{1}{2}$ *and any* β, *the process is ergodic.*

(iii) Voter model. Assume $\beta = 1$. *Then the process* $\tilde{\eta}_t$ *is positive recurrent. Moreover, for any initial configuration* $\tilde{\eta} \in \tilde{\mathscr{D}}$ *and any* $\varepsilon > 0$,

$$\mathrm{E}(\tilde{\tau}_c(\tilde{\eta}))^{3/2-\varepsilon} < \infty; \qquad \mathrm{E}(\tilde{\tau}_c(\tilde{\eta}))^{3/2+\varepsilon} = \infty. \tag{1.3}$$

The fact that the exclusion process $\tilde{\eta}_t$ in $\tilde{\mathscr{D}}$ is ergodic for $p > \frac{1}{2}$ follows immediately from well-known results of Liggett (1976; 1985) who described the invariant measures for η_t in the irreducible classes of $\mathscr{D}$. Since the system is conservative, ergodicity of η_t on any irreducible class of $\mathscr{D}$ is equivalent to ergodicity of $\tilde{\eta}_t$ on $\tilde{\mathscr{D}}$. Our alternative approach does not use the knowledge of the invariant measure. When $p \leq \frac{1}{2}$, the results of Liggett imply only that the process η_t is not positive recurrent; our result says that it is transient. For $p < \frac{1}{2}$, the transience holds immediately from laws of large numbers for the leftmost hole and the rightmost particle. For $p = \frac{1}{2}$, the transience is a more delicate matter.

The bounds in (1.3) show the velocity of the convergence of the voter model to the invariant measure, which is the singleton supported by $\mathscr{D}_0$. It may be the case that these bounds can be obtained from the duality of the voter model to the coalescing random walks; however, we have not investigated this approach.

Our main results are the conditions for ergodicity for the hybrid model described in Theorem 1.1(ii). This says that if either the voter-model proportion in the hybrid process is large enough or the exclusion process has no drift to the right, then the hybrid process is ergodic. Part (i) of the theorem says that exclusion is transient for $p \leq \frac{1}{2}$, while part (iii) says that the voter model is always positive recurrent. Part (ii)(a) says that the voter model 'wins' if the voter-model proportion is sufficiently large, uniformly on the exclusion asymmetry; and part (ii)(b) says that for the symmetric exclusion, any voter-model proportion guarantees ergodicity.

We are not totally satisfied with this result because sufficient conditions for transience are missing. We would like to be able to show that if the asymmetry of the exclusion process has a tendency to 'escape' from $\mathscr{D}$ then the addition of a small proportion of the voter model will not be able to prevent it from escaping. But for now, it is still very unclear to us whether the process could be transient in this case. We now state a conjecture for the non-ergodicity of the hybrid process. A heuristic argument supporting the conjecture is presented in Section 7.

Conjecture 1.1. *For any* $p < \frac{1}{2}$ *there exists a* $\beta_0(p) > 0$ *such that, for any* $\beta < \beta_0(p)$, *the hybrid process* $\tilde{\eta}_t$ *with parameters* β *and* p *is not ergodic.*

The parameter space $\{(p, \beta) : p, \beta \in [0, 1]\}$ is partitioned into three regions: ergodicity, transience and null-recurrence. Presumably the region of transience satisfies the property that if the hybrid process with parameters (p_0, β_0) is transient, then the one with parameters

(p_1, β_1) will also be transient for $p_1 \leqslant p_0$ and $\beta_1 \leqslant \beta_0$. But we do not have any monotonicity argument to hand to argue this. We know that the transience region is non-empty because it contains the segment $[0, \frac{1}{2}] \times \{0\}$, but we do not know how to prove that it contains points in the interior of the parameter space.

How stable are our results under changes in the dynamics? Can we extend Theorem 1.1 to non-nearest-neighbour processes? When $\beta = 1$, so that only the voter model is present, the answer is given by Cox and Durrett (1995), as described above. When $\beta = 0$, giving an exclusion process, it is known that the process is not ergodic on $\mathscr{D}$ if $p(x, y)$ is symmetric (all invariant measures are translation-invariant in this case), but it is an open problem of Liggett (1985, Section VIII.7, Problem 6) in the case when $p(x, y)$ is asymmetric. The conjecture is that if $p(x, y) = q(y - x)$, for some q, then the system would be ergodic under the condition $\sum_x xq(x) < 0$. In Section 9 we explain where our approach fails to work when extended to the non-nearest-neighbour case.

Motivating examples from real life, a description of shock measures in other one-dimensional models and nice conjectures about the existence of shock measures in other systems can be found in Cox and Durrett (1995, Section 1).

Theorem 1.1 is proven for the discrete-time version of $\tilde{\eta}_t$ and then standard arguments are used to prove the continuous counterpart. The discrete process is a Markov chain in $\tilde{\mathscr{D}}$. The basic tool is a set of theorems from Fayolle *et al.* (1995), which give conditions for ergodicity, recurrence and transience of denumerable Markov chains using so-called Lyapunov functions. The application of these functions to the processes of interest produces sub- or supermartingales, which can be used straightforwardly to show the desired properties. The problem is that these functions are frequently hard to find. One of the contributions of this paper is the exposition of Lyapunov functions that work for the exclusion process, the voter model and mixtures thereof.

The paper is organized in the following manner. In Section 2 we introduce the discrete version of the process $\tilde{\eta}_t$. In Section 3 we state the results we need from Fayolle *et al.* (1995). In Section 4 we introduce the Lyapunov functions of the process that will be relevant in the proofs. In Sections 5, 6 and 7 we state and prove the results for the discrete-time versions of the exclusion process, the voter model and the hybrid process, respectively. In Section 8 we show how to pass from discrete to continuous time and prove Theorem 1.1.

2. Discrete- and continuous-time processes

In this section we introduce discrete-time versions of the exclusion process, the voter model and mixtures thereof, defined in the previous section, and establish their relations with the continuous-time processes.

Let η be a configuration from $\{0, 1\}^{\mathbb{Z}}$. We say that a discrepancy of type 01 occurs in η at the site x if $\eta(x-1) = 0$, $\eta(x) = 1$; similarly, a discrepancy of type 10 occurs in η at x if $\eta(x) = 1$, $\eta(x+1) = 0$. The countable set $\mathscr{D}$ defined above is the set of those configurations of $\{0, 1\}^{\mathbb{Z}}$ in which there are only a finite number of discrepancies, and the number of discrepancies of type 10 minus the number of discrepancies of type 01 is equal to 1. Then it is easy to see that $\mathscr{D} = \{\eta \in \{0, 1\}^{\mathbb{Z}}$: there exist i_0, j_0 such that $\eta(i) = 1$ for $i \leqslant i_0$ and $\eta(j) = 0$ for $j \geqslant j_0\}$ and that $\mathscr{D}$ is countable.

The discrete-time exclusion process with parameter p (which we will call EP(p)) is a Markov process with state space $\mathscr{D}$ and the following dynamics: for every $n \geqslant 0$, if η is the state at time n then η', the state at time $n+1$, is obtained by the following procedure:

(i) Choose one of the discrepancies of η with uniform distribution; say, that at the site x.
(ii) If the discrepancy is 01 (10), then exchange 0 and 1 with probability $0 < p < 1$ $(0 < q := 1 - p < 1)$, and charge nothing with probability q $(1 - q)$.

The exclusion process is a countable Markov chain on $\mathscr{D}$.

Let us now define the discrete-time voter model (which we will call VM) and the discrete-time hybrid process (HP(β, p), where β is the mixing parameter and p is the exclusion parameter). For VM, step (i) is the same, and (ii) is substituted by the following:

(ii′) The chosen discrepancy is substituted by either 11 or 00, each with probability $\frac{1}{2}$.

To construct HP(β, p), we first execute (i), and then with probability $1 - \beta$ execute (ii) (i.e. make an exclusion-process step), and with probability β execute (ii′) (i.e. make a voter-model step). We use the notation $(\xi_n : n \in \mathbb{N})$ for HP(β, p); ξ_n denotes the configuration of the system at time n.

In (2.2) below we shall present the relation between the discrete-time hybrid process $(\xi_n : n \in \mathbb{N})$ and the continuous-time hybrid process $(\eta_t : t \geqslant 0)$ with mixing parameter β and exclusion parameter p. To this end, we shall need the Harris graphical construction for $(\eta_t : t \geqslant 0)$, which we now briefly recall. It is a ‘superposition’ of the graphical construction for the voter model (see Durrett 1995) with that for the exclusion process (see Ferrari 1992) with respective weights β and $1 - \beta$.

Let $\{(\mathscr{N}_t^{x,x+1}, t \geqslant 0)\}_{x\in\mathbb{Z}}$, $\{(\mathscr{N}_t^{x,x-1}, t \geqslant 0)\}_{x\in\mathbb{Z}}$, $\{(\mathscr{M}_t^{x,x+1}, t \geqslant 0)\}_{x\in\mathbb{Z}}$, $\{(\mathscr{M}_t^{x,x-1}, t \geqslant 0)\}_{x\in\mathbb{Z}}$ be four independent families of Poisson point processes with respective rates $(1-\beta)p$, $(1-\beta)q$, $\beta/2$ and $\beta/2$. Given the initial configuration η_0, the dynamics of the process η_t, $t \geqslant 0$, is determined by those Poisson processes in the following manner. If there is a Poisson event at time t in $\mathscr{N}^{x,x+1}$ ($\mathscr{N}^{x,x-1}$), which means $\mathscr{N}_t^{x,x+1} - \mathscr{N}_{t^-}^{x,x+1} = 1$, and if x has a particle while $x+1$ is empty ($x-1$ is empty) in η_{t^-}, then the particle jumps from x to $x+1$ (to $x-1$) at time t. If there is a Poisson event at time t in $\mathscr{M}^{x,x+1}$ ($\mathscr{M}^{x,x-1}$), then the site $x+1$ (the site $x-1$) acquires the same state at time t as the state of x in η_{t^-}.

Let $\tau_0 = 0$ and, for $n \geqslant 1$, set

$$\tau_n = \inf\left\{ t > \tau_{n-1} : \sum_{x,y:|x-y|=1} |\eta_{t^-}(x) - \eta_{t^-}(y)|(\mathscr{N}^{x,y}(\tau_{n-1}, t] + \mathscr{M}^{x,y}(\tau_{n-1}, t]) > 0 \right\}, \tag{2.1}$$

where $\mathscr{N}(s, t]$ denotes the number of Poisson events in the time interval $(s, t]$ for the process $\mathscr{N}$. We call τ_n the instants of *attempted jumps* of the process η_t. It follows from our definitions that if $\eta_0 = \xi_0$, then

$$(\xi_n : n \geqslant 0) = (\eta_{\tau_n} : n \geqslant 0) \qquad \text{in distribution.} \tag{2.2}$$

3. Criteria for recurrence and transience of Markov chains

In this section we state the criteria for ergodicity, recurrence and transience of countable Markov chains to be used later. The next four theorems are Theorems 2.2.3, 2.2.1, 2.2.2, 2.2.7, respectively, of Fayolle *et al.* (1995).

Theorem 3.1. *Let* ξ_t, $t = 0, 1, 2, \ldots,$ *be an irreducible Markov chain with countable state space X. Suppose that there exist a positive function* $f(x)$ *and a finite set* $A \subset X$ *such that*

$$\mathrm{E}(f(\xi_{t+1}) - f(\xi_t)|\xi_t = x) \leqslant -\varepsilon, \tag{3.1}$$

for some $\varepsilon > 0$ *and all* $x \in X \backslash A$, *and that*

$$\mathrm{E}(f(\xi_{t+1})|\xi_t = x) < \infty, \tag{3.2}$$

for $x \in A$. *Then the Markov chain is ergodic.*

Theorem 3.2. *Let* ξ_t, $t = 0, 1, 2, \ldots,$ *be an irreducible Markov chain with countable state space X. Suppose that there exist a positive function* $f(x)$, $f(x) \to \infty$ *as* $x \to \infty$, *and a finite set* $A \subset X$ *such that*

$$\mathrm{E}(f(\xi_{t+1}) - f(\xi_t)|\xi_t = x) \leqslant 0, \tag{3.3}$$

for all $x \in X \backslash A$. *Then the Markov chain is recurrent.*

Theorem 3.3. *Let* ξ_t, $t = 0, 1, 2, \ldots,$ *be an irreducible Markov chain with countable state space X. Suppose that there exist a positive function* $f(x)$ *and a set* $A \subset X$ *such that (3.3) holds for all* $x \in X \backslash A$ *and*

$$f(x_0) < \inf_{x \in A} f(x),$$

for some $x_0 \notin A$. *Then the Markov chain is transient.*

Theorem 3.4. *Let* ξ_t, $t = 0, 1, 2, \ldots,$ *be an irreducible Markov chain with countable state space X. Suppose that there exist a positive function* $f(x)$ *and a constant C such that if* $f(x) > C$, *then*

$$\mathrm{E}(f(\xi_{t+1}) - f(\xi_t)|\xi_t = x) \geqslant \varepsilon, \tag{3.4}$$

for some $\varepsilon > 0$, *and suppose that for some* $K > 0$

$$|f(\xi_{t+1}) - f(\xi_t)| \leqslant K \text{ a.s.} \tag{3.5}$$

Then the Markov chain is transient.

In addition to ergodicity, we shall study the existence of moments of the hitting time of the set $\mathscr{D}_0$. To do this, we shall need the following result of Aspandiiarov *et al.* (1996, Theorem 1).

Theorem 3.5. *Let A be some positive real number. Suppose that we are given an $\{\mathscr{F}_n\}$-adapted stochastic process X_n, $n \geqslant 0$, taking values in an unbounded subset of $\mathbb{R}_+$. Denote by τ_A the moment when the process X_n enters the set $(0, A)$. Assume that there exist $\lambda > 0$, $p_0 \geqslant 1$ such that, for any n, $X_n^{2p_0}$ is integrable and*

$$\mathrm{E}(X_{n+1}^{2p_0} - X_n^{2p_0} | \mathscr{F}_n) \leqslant \lambda X_n^{2p_0 - 2} \tag{3.6}$$

on $\{\tau_A > n\}$. Then there exists a positive constant $C = C(\lambda, p_0)$ such that, for all $x \geqslant 0$, whenever $X_0 = x$ with probability 1

$$\mathrm{E}\tau_A^{p_0} \leqslant Cx^{2p_0}. \tag{3.7}$$

4. Functions of the process

For the sake of brevity we will henceforth write '0-block' for the expression 'block of zeros' and '1-block' for 'block of ones'.

An equivalence class $S \in \tilde{\mathscr{D}}$ can be identified by a finite set of positive numbers in the following form:

$$S = \ldots 111 \overbrace{0000}^{n_1} \overbrace{11111}^{m_1} \overbrace{0000}^{n_2} \overbrace{11111}^{m_2} \ldots \overbrace{00000}^{n_N} \overbrace{1111}^{m_N} 000 \ldots, \tag{4.1}$$

where $n_i = n_i(S)$ is the size of ith 0-block, $m_i = m_i(S)$ is the size of ith 1-block, and $N = N(S)$ is the number of 1-blocks not including the leftmost infinite 1-block. In what follows, the word 'configuration' will usually mean 'equivalence class'. So, for $S \in \tilde{\mathscr{D}}$ we can simply write $S = (n_1, m_1, \ldots, n_N, m_N)$.

Denote $r_0 = 0$, $r_i = \sum_{j=1}^{i}(m_j + n_j)$, $l_i = \sum_{j=1}^{i-1}(m_j + n_j) + n_i + 1$, $i = 1, \ldots, N$. Let η be the configuration from the equivalence class S such that $\eta(x) = 1$ for $x \leqslant 0$ and $\eta(1) = 0$. Define the configurations $\eta_k^{\rightarrow}$, $\eta_k^{\leftarrow}$, η_k^{+r}, η_k^{+l}, η_k^{-r}, η_k^{-l} in the following way:

- $\eta_k^{\rightarrow}(x) = \eta(x)$ for $x \neq r_k$, $r_k + 1$, $\eta_k^{\rightarrow}(r_k) = 0$, $\eta_k^{\rightarrow}(r_k + 1) = 1$, $k = 0, \ldots, N$;
- $\eta_k^{\leftarrow}(x) = \eta(x)$ for $x \neq l_k$, $l_k - 1$, $\eta_k^{\leftarrow}(l_k) = 0$, $\eta_k^{\leftarrow}(l_k - 1) = 1$, $k = 1, \ldots, N$;
- $\eta_k^{+r}(x) = \eta(x)$ for $x \neq r_k + 1$, $\eta_k^{+r}(r_k + 1) = 1$, $k = 0, \ldots, N$;
- $\eta_k^{+l}(x) = \eta(x)$ for $x \neq l_k - 1$, $\eta_k^{+l}(l_k - 1) = 1$, $k = 1, \ldots, N$;
- $\eta_k^{-r}(x) = \eta(x)$ for $x \neq r_k$, $\eta_k^{-r}(r_k) = 0$, $k = 0, \ldots, N$;
- $\eta_k^{-l}(x) = \eta(x)$ for $x \neq l_k$, $\eta_k^{-l}(l_k) = 0$, $k = 1, \ldots, N$.

and $S_k^{\rightarrow}$, $S_k^{\leftarrow}$, S_k^{+r}, S_k^{+l}, S_k^{-r}, S_k^{-l} are the corresponding classes of equivalence. Informally speaking,

- $S_k^{\rightarrow}$ is the configuration obtained from S by moving the rightmost 1 of the kth 1-block by one unit to the right, $k = 0, \ldots, N$;
- $S_k^{\leftarrow}$ is the configuration obtained from S by moving the leftmost 1 of the kth 1-block by one unit to the left, $k = 1, \ldots, N$;
- S_k^{+r} is the configuration obtained from S by adding an extra 1 to the right of the kth 1-block, $k = 0, \ldots, N$;

- S_k^{+l} is the configuration obtained from S by adding an extra 1 to the left of the kth 1-block, $k = 1, \dots, N$;
- S_k^{-r} is the configuration obtained from S by removing the rightmost 1 from the kth 1-block, $k = 0, \dots, N$;
- S_k^{-l} is the configuration obtained from S by removing the leftmost 1 from the kth 1-block, $k = 1, \dots, N$.

Clearly, EP can transform S to $S_k^{\rightarrow}$ or $S_k^{\leftarrow}$, while by using VM we can obtain $S_k^{\pm r}$ or $S_k^{\pm l}$.

Denote also $R_i = \sum_{j=1}^{i} n_j$, $T_i = \sum_{j=i}^{N} m_j$, and let

$$|S| = \sum_{j=1}^{N} (m_j + n_j) = R_N + T_1$$

stand for the length of the 'non-trivial' part of configuration S. We take $R_0 = T_{N+1} = 0$.

We define two functions $f_1, f_2 : \tilde{\mathscr{D}} \mapsto \mathbb{R}$, which will play the crucial role in our arguments:

$$\begin{aligned} f_1(S) &= \frac{1}{2}\left(\sum_{k:S(k)=1} \left(\sum_{m<k} \mathbf{1}_{\{S(m)=0\}} \right) + \sum_{k:S(k)=0} \left(\sum_{m>k} \mathbf{1}_{\{S(m)=1\}} \right) \right) \\ &= \frac{1}{2}\left(\sum_{i=1}^{N} m_i R_i + \sum_{i=1}^{N} n_i T_i \right) \\ &= \sum_{i=1}^{N} m_i R_i = \sum_{i=1}^{N} n_i T_i \end{aligned}$$

and

$$\begin{aligned} f_2(S) &= \frac{1}{2}\left(\sum_{k:S(k)=1} \left(\sum_{m<k} \mathbf{1}_{\{S(m)=0\}} \right)^2 + \sum_{k:S(k)=0} \left(\sum_{m>k} \mathbf{1}_{\{S(m)=1\}} \right)^2 \right) \\ &= \frac{1}{2}\left(\sum_{i=1}^{N} m_i R_i^2 + \sum_{i=1}^{N} n_i T_i^2 \right), \end{aligned}$$

for all $S \in \tilde{\mathscr{D}}$.

At this point some remarks about f_1, f_2 are in order. The value $f_1(S)$ is equal exactly to the number of nearest-neighbour transpositions needed to pass from S to $\mathscr{D}_0$, that is, $f_1(S)$ is in some sense the 'distance' from S to the trivial configuration. Unfortunately, as we will see later, the function f_1 does not 'work' well for some configurations S (namely, for S such that $N(S)$ is small with respect to $|S|$). The function f_2 is the result of our attempts to modify f_1 in order to eliminate this disadvantage; we cannot attach any intuitive meaning to $f_2(S)$.

Let us obtain some relations between $|S|$, $f_1(S)$ and $f_2(S)$.

Lemma 4.1. *For any $S \in \mathscr{D}$, the following hold:*

(i) $|S|/2 \leqslant f_1(S) \leqslant |S|^2/4$;
(ii) $|S|^2/4 \leqslant f_2(S) \leqslant |S|^3/8$;
(iii) $f_1(S) \leqslant (f_2(S))^{3/4}$.

Proof. Parts (i) and (ii) are simple to prove. We have

$$f_1(S) = \frac{1}{2}\left(\sum_{i=1}^{N} m_i R_i + \sum_{i=1}^{N} n_i T_i\right) \geqslant \frac{1}{2}(R_N + T_1) = \frac{|S|}{2},$$

$$f_1(S) = \sum_{i=1}^{N} m_i R_i \leqslant R_N \sum_{i=1}^{N} m_i = R_N T_1 \leqslant \frac{(R_N + T_1)^2}{4} = \frac{|S|^2}{4};$$

and analogously,

$$f_2(S) \geqslant \frac{1}{2}(R_N^2 + T_1^2) \geqslant \frac{1}{4}(R_N + T_1)^2 = \frac{|S|^2}{4},$$

$$f_2(S) \leqslant \frac{1}{2}\left(R_N^2 \sum_{i=1}^{N} m_i + T_1^2 \sum_{i=1}^{N} n_i\right) = \frac{1}{2} R_N T_1 (R_N + T_1) \leqslant \frac{|S|^3}{8}.$$

Let us prove (iii). We shall make use of the following simple consequence of the Jensen inequality: if we have n positive numbers $\gamma_1, \ldots, \gamma_n$ such that $\sum_{i=1}^{n} \gamma_i = 1$, then, for any $x_1, \ldots, x_n$,

$$\gamma_1 x_1 + \ldots + \gamma_n x_n \leqslant (\gamma_1 x_1^2 + \ldots + \gamma_n x_n^2)^{1/2}. \tag{4.2}$$

Denote $\alpha_i = m_i/|S|$, $\beta_i = n_i/|S|$, so $\sum_{i=1}^{N}(\alpha_i + \beta_i) = 1$. Using (4.2) and (ii), we obtain

$$\begin{aligned} f_1(S) &= \frac{1}{2}\sum_{i=1}^{N}(m_i R_i + n_i T_i) = \frac{|S|}{2}\sum_{i=1}^{N}(\alpha_i R_i + \beta_i T_i) \\ &\leqslant \frac{|S|}{2}\left(\sum_{i=1}^{N}(\alpha_i R_i^2 + \beta_i T_i^2)\right)^{1/2} = \frac{\sqrt{|S|}}{\sqrt{2}}(f_2(S))^{1/2} \\ &\leqslant \frac{\sqrt{2}(f_2(S))^{1/4}}{\sqrt{2}}(f_2(S))^{1/2} = (f_2(S))^{3/4}, \end{aligned}$$

thus completing the proof of Lemma 4.1. □

As usual, P and E stand for probability and expectation. If there is any possibility of ambiguity, we use $\mathrm{E}_p^{\mathrm{e}}$ (P_p^{e}) to denote expectation (probability) with respect to EP(p), E^{v} (P^{v}) for expectation (probability) with respect to VM, and $\mathrm{E}_{\beta,p}^{\mathrm{h}}$ ($P_{\beta,p}^{\mathrm{h}}$) for expectation (probability) with respect to HP(β, p).

5. Exclusion process

In this section we shall study the EP using the method of Lyapunov functions.

Theorem 5.1. *If $p > q$, then the exclusion process is ergodic.*

Proof. As we noticed before, the EP can transform a configuration S only to $S_k^{\rightarrow}$ or to $S_k^{\leftarrow}$ symbols (see Section 4). Then, it is easy to establish that

$$\begin{aligned} f_2(S_k^{\rightarrow}) - f_2(S) &= \frac{1}{2}((R_k+1)^2 - R_k^2 + (T_{k+1}+1)^2 - T_{k+1}^2) \\ &= 1 + R_k + T_{k+1} \end{aligned} \tag{5.1}$$

and

$$\begin{aligned} f_2(S_k^{\leftarrow}) - f_2(S) &= \frac{1}{2}((R_k-1)^2 - R_k^2 + (T_k-1)^2 - T_k^2) \\ &= 1 - R_k - T_k. \end{aligned} \tag{5.2}$$

Combining (5.1) and (5.2), we have that

$$\mathrm{E}(f_2(\xi_{t+1}) - f_2(\xi_t) | \xi_t = S) = \frac{N+q}{2N+1} - \frac{p-q}{2N+1} \sum_{i=1}^{N} (R_i + T_i). \tag{5.3}$$

Since $R_N + T_1 = |S|$, $R_i \geqslant i$ and $T_i \geqslant N - i + 1$, it is straightforward to obtain that

$$\sum_{i=1}^{N} (R_i + T_i) \geqslant \max\{|S|, N(N+1)\}.$$

Using this fact, we obtain from (5.3) that, for any $\varepsilon > 0$,

$$\mathrm{E}(f_2(\xi_{t+1}) - f_2(\xi_t) | \xi_t = S) < -\varepsilon \tag{5.4}$$

for all but finitely many S. So, by Theorem 3.1, EP(p) is ergodic when $p > \frac{1}{2}$. □

Theorem 5.2. *When $p \leqslant q$ the exclusion process is transient.*

Proof. First we consider the case $p < q$. With $S_k^{\rightarrow}$ and $S_k^{\leftarrow}$ as defined above, we have that

$$f_1(S_k^{\leftarrow}) - f_1(S) = -1 \tag{5.5}$$

and

$$f_1(S_k^{\rightarrow}) - f_1(S) = 1, \tag{5.6}$$

so that, for some $\varepsilon = \varepsilon(p, q) > 0$,

$$\mathrm{E}_p^{\mathrm{e}}(f_1(\xi_{t+1}) - f_1(\xi_t) | \xi_t = S) = \frac{N(q-p)}{2N+1} + \frac{q}{2N+1} \geqslant \varepsilon \tag{5.7}$$

and, clearly, $|f_1(\xi_{t+1}) - f_1(\xi_t)| \leqslant 1$ almost surely. Then by Theorem 3.4, the process ξ_t is transient.

Let us turn now to the case $p = q = \frac{1}{2}$. Using the function $f_1(S)$ defined above and (5.5), (5.6), we have that

$$\mathrm{E}^{\mathrm{e}}_{1/2}(f_1(\xi_{t+1}) - f_1(\xi_t)|\xi_t = S) = \frac{1}{2(2N+1)}, \tag{5.8}$$

so Theorem 3.3 does not apply. Therefore, we need a different approach.

We fix an arbitrary $\alpha > 0$ and define the function $\psi : \mathscr{D} \backslash \mathscr{D}_0 \mapsto \mathbb{R}$ by

$$\psi(S) := (f_1(S))^{-\alpha}.$$

Note that the definition is correct because $f_1(S) > 0$ for $S \notin \mathscr{D}_0$. (Actually, for the purposes of this proof it is sufficient to take $\alpha = 1$, but, since we will need analogous calculations later in this paper, at this point we prefer to do the calculations for arbitrary $\alpha > 0$.) To study the properties of the process $\psi(\xi_t)$, we need the following lemma.

Lemma 5.1. *For any* $C > 0$ *the set*

$$A_C = \{S : f_1(S) < CN(S)\} \tag{5.9}$$

is finite.

Proof. Clearly, $R_i \geqslant i$ and $m_i \geqslant 1$, so $f_1(S) \geqslant N(S)(N(S)+1)/2$. Thus, for a configuration S to belong to A_C, it is necessary that the number of 1-blocks be less than $2C - 1$, so A_C is a subset of

$$\{S : f_1(S) < C(2C-1)\},$$

which is obviously finite. □

It follows from (5.5) and (5.6) that

$$\mathrm{E}^{\mathrm{e}}_{1/2}((f_1(\xi_{t+1}) - f_1(\xi_t))^2|\xi_t = S) = \frac{1}{2}. \tag{5.10}$$

By elementary calculations, we obtain that for any $\alpha > 0$ there exist two positive numbers $C_1 = C_1(\alpha)$, $C_2 = C_2(\alpha)$ such that

$$(x+1)^{-\alpha} - 1 \leqslant -\alpha x + C_1 x^2, \tag{5.11}$$

when $|x| < C_2$.

Using (5.8), (5.10), (5.11) and Lemma 5.1, we obtain

$$
\begin{aligned}
&\mathrm{E}^{\mathrm{e}}_{1/2}(\psi(\xi_{t+1})-\psi(\xi_t)|\xi_t=S)\\
&=f_1^{-\alpha}(S)\mathrm{E}^{\mathrm{e}}_{1/2}\left(\left(\frac{f_1(\xi_{t+1})}{f_1(\xi_t)}\right)^{-\alpha}-1|\xi_t=S\right)\\
&=f_1^{-\alpha}(S)\mathrm{E}^{\mathrm{e}}_{1/2}\left(\left(\frac{f_1(\xi_{t+1})-f_1(\xi_t)}{f_1(\xi_t)}+1\right)^{-\alpha}-1|\xi_t=S\right)\\
&\leqslant f_1^{-\alpha}(S)\left(-\frac{\alpha}{f_1(S)}\cdot\frac{1}{2(2N+1)}+\frac{C_1}{2f_1^2(S)}\right)\\
&=f_1^{-\alpha-2}(S)\left(-\frac{\alpha f_1(S)}{2(2N+1)}+\frac{C_1}{2}\right)<0
\end{aligned}
\tag{5.12}
$$

on $\{S: f_1(S)>\max\{1/C_2,\ C_1(2N(S)+1)/\alpha\}\}$, and hence for all but finitely many S. Applying Theorem 3.4, we conclude the proof of Theorem 5.2. □

6. Voter model

The subject of this section is the discrete-time voter model. For a process starting from a configuration S, denote by $\tau(S)$ the moment of hitting the set $\mathscr{D}_0$. The main result of this section is the following.

Theorem 6.1. *The discrete-time voter model is positive recurrent. Moreover, for any initial configuration S_0 and any $\varepsilon>0$,*

$$\mathrm{E}(\tau(S_0))^{3/2-\varepsilon}<\infty \tag{6.1}$$

and

$$\mathrm{E}(\tau(S_0))^{3/2+\varepsilon}=\infty. \tag{6.2}$$

Proof. Since positive recurrence means just the existence of $\mathrm{E}\tau(S_0)$, we shall turn directly to the proof of (6.1). The idea is to apply Theorem 3.5 to the process $f_2^{\alpha}(\xi_t)$ for some $\alpha<1$.

First, we need the following important fact.

Lemma 6.1. *We have*

$$\mathrm{E}^{\mathrm{v}}(f_2(\xi_{t+1})-f_2(\xi_t)|\xi_t=S)=0 \tag{6.3}$$

for any $S\in\mathscr{D}$.

Proof. If $S\in\mathscr{D}_0$, then (6.3) is trivial. For $S\notin\mathscr{D}_0$ a direct computation gives

$$f_2(S_k^{+r}) - f_2(S) = \frac{1}{2}(R_k + T_{k+1} + R_k^2 - T_{k+1}^2) - \sum_{i=k+1}^{N} m_i R_i + \sum_{i=1}^{k} n_i T_i, \tag{6.4}$$

$$f_2(S_k^{-r}) - f_2(S) = \frac{1}{2}(R_k + T_{k+1} - R_k^2 + T_{k+1}^2) + \sum_{i=k+1}^{N} m_i R_i - \sum_{i=1}^{k} n_i T_i, \tag{6.5}$$

for $k = 0, \ldots, N$, and

$$f_2(S_k^{+l}) - f_2(S) = \frac{1}{2}(-R_k - T_k + R_k^2 - T_k^2) - \sum_{i=k}^{N} m_i R_i + \sum_{i=1}^{k} n_i T_i, \tag{6.6}$$

$$f_2(S_k^{-l}) - f_2(S) = \frac{1}{2}(-R_k - T_k - R_k^2 + T_k^2) + \sum_{i=k}^{N} m_i R_i - \sum_{i=1}^{k} n_i T_i, \tag{6.7}$$

for $k = 1, \ldots, N$.

The right-hand sides of (6.4)–(6.7) sum to 0, thus concluding the proof of Lemma 6.1. □

Then, from (6.5), we note that

$$|f_2(S_0^{-r}) - f_2(S)| \geqslant \frac{T_1^2}{2}; \tag{6.8}$$

and from (6.4),

$$|f_2(S_N^{+r}) - f_2(S)| \geqslant \frac{R_N^2}{2}. \tag{6.9}$$

These two inequalities give us that there exists a constant $C > 0$ such that

$$\mathrm{E}^{\mathrm{v}}((f_2(\xi_{t+1}) - f_2(\xi_t))^2 | \xi_t = S) \geqslant \frac{C|S|^4}{N} \tag{6.10}$$

for all S. Now, a very important observation is that the VM does not increase the number of blocks $N_t = N(\xi_t)$. So we have, for all S,

$$\mathrm{E}^{\mathrm{v}}((f_2(\xi_{t+1}) - f_2(\xi_t))^2 | \xi_t = S) \geqslant C_0 |S|^4 \tag{6.11}$$

with $C_0 = C_0(S_0) = C/N(S_0)$.

Elementary calculus gives us that, for $0 < \alpha < 1$ and for $|x| \leqslant 1$, there exists a positive constant C_1 such that

$$(x+1)^\alpha - 1 \leqslant \alpha x - C_1 x^2. \tag{6.12}$$

Employing considerations analogous to (5.12) and applying (6.12), Lemma 6.1 and (6.11), we obtain

$$\mathrm{E}^{\mathrm{v}}((f_2(\xi_{t+1}))^\alpha - (f_2(\xi_t))^\alpha | \xi_t = S) \leqslant -C_0 C_1 (f_2(S))^{\alpha-2} |S|^4. \tag{6.13}$$

Applying Lemma 4.1(ii) to the last inequality gives

$$\mathrm{E}^{\mathrm{v}}((f_2(\xi_{t+1}))^\alpha - (f_2(\xi_t))^\alpha | \xi_t = S) \leqslant -16C_0C_1(f_2(S))^{\alpha-2/3}.$$

We apply Theorem 3.5 to the process $X_t = (f_2(\xi_t))^{1/3}$, taking α to be close to 1 to finish the proof of (6.1).

Let us turn now to the proof of (6.2). We let the process start from configuration S_0 such that $N(S_0) = 1$. Since this configuration is reachable from any other configuration, it is sufficient to prove (6.2) for this S_0. As already mentioned, the voter model does not increase the number of blocks N, so the process can be represented as $\xi_t = (n_t, m_t)$, which clearly is a random walk in $\mathbb{Z}_+^2$, and we are interested in the moment of hitting the boundary. Note that the transition probabilities of this random walk can be described like this: from the state (n, m) a transition can occur to the states $(n+1, m)$, $(n-1, m)$, $(n, m+1)$, $(n, m-1)$, $(n+1, m-1)$ and $(n-1, m+1)$, each with probabilities $\frac{1}{6}$.

Denote by $\tau_{n,m}$ the moment of hitting $\mathscr{D}_0$ (i.e. the boundary) provided that the starting point was (n, m). To proceed, we need the following lemma.

Lemma 6.2. *There exist two positive constants δ, C, such that, for any n, m,*

$$P\{\tau_{n,m} > \delta n^2\} \geqslant \frac{Cm}{m+n} \tag{6.14}$$

and

$$P\{\tau_{n,m} > \delta m^2\} \geqslant \frac{Cn}{m+n}. \tag{6.15}$$

Remark 6.1. It can be shown that Lemma 6.2 holds for any homogeneous random walk in $\mathbb{Z}_+^2$ with bounded jumps and zero drift in the interior of $\mathbb{Z}_+^2$.

Proof. Without loss of generality we can suppose that $n \leqslant m$. Then, to prove (6.14), we will prove a stronger fact:

$$P\{\tau_{n,m} > \delta n^2\} \geqslant C_0 \tag{6.16}$$

for some C_0. In fact, it is a classical result that a homogeneous random walk in $\mathbb{Z}_+^2$ with bounded jumps and zero drift in the interior with some uniformly positive probability cannot deviate by the distance n from its initial position during the time n^2. To show how this can be formally proved, we denote by $\rho((n_1, m_1), (n_2, m_2))$ the Euclidean distance between the points (n_1, m_1) and (n_2, m_2). Let the process start from (n, m), and denote $Y_t = \rho(\xi_t, (n, m))$. Then, it is straightforward to show that the process Y_t satisfies the hypothesis of Lemma 2 from Aspandiiarov *et al.* (1996), so we apply the lemma and (6.14) is proved.

To prove (6.15), we need some additional notation. Denote

$$W^{\mathrm{i}}(m) = \{(n', m') : \rho((n', m'), (m, m)) \leqslant m/2 + \sqrt{2}\},$$

$$V^{\mathrm{i}}(m) = \{(n', m') : m/2 < \rho((n', m'), (m, m)) \leqslant m/2 + \sqrt{2}\},$$

$$W^{\mathrm{e}}(m) = \{(n', m') : m/2 + \sqrt{2} < \rho((n', m'), (m, m)) \leqslant m\},$$

$$V^{\mathrm{e}}(m) = \{(n', m') : m - \sqrt{2} < \rho((n', m'), (m, m)) \leqslant m\}.$$

Clearly, the set $V^{\mathrm{i}}(m)$ is the boundary of $W^{\mathrm{i}}(m)$, and the set $V^{\mathrm{e}}(m)$ is the external boundary of $W^{\mathrm{e}}(m)$.

We consider the two possible cases: (a) $(n, m) \in W^{\mathrm{i}}(m)$; and (b) $(n, m) \in W^{\mathrm{e}}(m)$.

Case (a). First, we denote $Y_t = \rho(\xi_t, (m, m))$. Then, we apply Lemma 2 from Aspandiiarov *et al.* (1996) to obtain that $P\{\tau_{n,m} > \delta n^2\} \geqslant C_1$ for some C_1, and thus (6.15).

Case (b). We keep the notation Y_t from the previous paragraph. Denote by $p_{n,m}$ the probability of hitting the set $V^{\mathrm{i}}(m)$ before the set $V^{\mathrm{e}}(m)$, provided that the starting point is (n, m). Our goal is to estimate this probability from below.

For $C > 0$, consider the process Z_t^C, $t = 0, 1, 2, \ldots,$ defined in the following way:

$$Z_t^C = \exp\left\{C\left(1 - \frac{Y_t}{m}\right)\right\} = \exp\left\{C\left(1 - \frac{\rho(\xi_t, (m, m))}{m}\right)\right\},$$

$Z_0^C = \exp\{Cn/m\}$. One can prove the following technical fact: there exists a constant C (not depending on m) such that

$$\mathrm{E}(Z_{t+1}^C - Z_t^C | \xi_t = (n', m')) \geqslant 0, \tag{6.17}$$

for any point $(n', m') \in W^{\mathrm{e}}(m)$ and if m is large enough. Indeed, using the fact that there exist positive constants C_i, $i = 1, 2$, such that

$$\mathrm{e}^{-x} - 1 \geqslant -x + C_1 x^2$$

on $|x| < C_2$, we write

$$\begin{aligned}
&\mathrm{E}(Z_{t+1}^C - Z_t^C | \xi_t = (n', m')) \\
&\quad = \exp\left\{C\left(1 - \frac{|n' - m|}{m}\right)\right\}\mathrm{E}\left(\exp\left\{-\frac{C}{m}(Y_{t+1} - Y_t)\right\} - 1 | \xi_t = (n', m')\right) \\
&\quad \geqslant \frac{C}{m}\exp\left\{C\left(1 - \frac{|n' - m|}{m}\right)\right\}\mathrm{E}\left(-(Y_{t+1} - Y_t) + \frac{C_1 C}{m}(Y_{t+1} - Y_t)^2 | \xi_t = (n', m')\right).
\end{aligned}$$

Then, using properties of the process Y_t, one can complete the proof of (6.17).

Now, to estimate $p_{n,m}$, we make the sets $V^{\mathrm{i}}(m)$ and $V^{\mathrm{e}}(m)$ absorbing. Using this property, the process Z_t^C converges as $t \to \infty$ to Z_∞^C, so

$$\begin{aligned} \mathrm{E} Z_\infty^C &\geqslant p_{n,m} \exp\left\{\frac{C}{2}\right\} + (1 - p_{n,m}) \\ &\geqslant \mathrm{E} Z_0^C = \exp\left\{\frac{Cn}{m}\right\}, \end{aligned}$$

and thus

$$\begin{aligned} p_{n,m} &\geqslant \frac{\exp\{Cn/m\} - 1}{\exp\{C/2\} - 1} \\ &\geqslant \frac{C}{\exp\{C/2\} - 1} \cdot \frac{n}{m} \geqslant \frac{2C}{\exp\{C/2\} - 1} \cdot \frac{n}{m+n}. \end{aligned} \tag{6.18}$$

So, starting from the point (n, m), with probability given by (6.18) the random walk hits the set $V^{\mathrm{i}}(m)$. Then, from the case (a) it follows that with uniformly positive probability it will take at least δm steps to reach the external boundary $V^{\mathrm{e}}(m)$, so we complete the proof of (6.15), and thus of Lemma 6.2. □

Now, supposing that (6.2) does not hold, we have (denoting $\tau := \tau(S_0)$ and $a \wedge b := \min\{a, b\}$)

$$\begin{aligned} \mathrm{E}\tau^{3/2+\varepsilon} &\geqslant \mathrm{E}(\tau^{3/2+\varepsilon} \mathbf{1}_{\{\tau \geqslant t\}}) = \mathrm{E}((t + \tau\xi_t)^{3/2+\varepsilon} \mathbf{1}_{\{\xi_s \notin \mathscr{D}_0 \text{ for all } s \leqslant t\}}) \\ &\geqslant \frac{1}{2}\mathrm{E}\left((t + \delta n_t^2)^{3/2+\varepsilon} \frac{Cm_t}{m_t + n_t} \mathbf{1}_{\{\xi_s \notin \mathscr{D}_0 \text{ for all } s \leqslant t\}}\right) \\ &\quad + \frac{1}{2}\mathrm{E}\left((t + \delta m_t^2)^{3/2+\varepsilon} \frac{Cn_t}{m_t + n_t} \mathbf{1}_{\{\xi_s \notin \mathscr{D}_0 \text{ for all } s \leqslant t\}}\right) \\ &\geqslant \delta' C' \mathrm{E}((n_t^{2+\varepsilon'} m_t + m_t^{2+\varepsilon'} n_t) \mathbf{1}_{\{\xi_s \notin \mathscr{D}_0 \text{ for all } s \leqslant t\}}) \\ &= \delta' C' \mathrm{E}((n_{t\wedge\tau}^{2+\varepsilon'} m_{t\wedge\tau} + m_{t\wedge\tau}^{2+\varepsilon'} n_{t\wedge\tau})) \\ &= C'' \mathrm{E}(f_2(\xi_{t\wedge\tau}))^{1+\varepsilon''}. \end{aligned} \tag{6.19}$$

for some constants δ', ε', C', ε'' and C''.

From (6.19) we obtain that the family $\{f_2(\xi_{t\wedge\tau})\}$ is uniformly integrable as $t \to \infty$, so $\mathrm{E}f_2(\xi_t) \to \mathrm{E}f_2(\xi_\tau) = 0$. But this obviously contradicts Lemma 6.1. □

7. Hybrid process

As already proved, EP(p) is transient when $p \leqslant \frac{1}{2}$, and VM is ergodic. Now, what happens if we combine them? The following theorems give an (incomplete) answer to this question.

Theorem 7.1. *There exists $\beta_0 < 1$ such that, for any p, the process* HP(β, p) *is ergodic for all* $\beta > \beta_0$.

Theorem 7.2. *For any $\beta > 0$ and $p \geqslant \frac{1}{2}$ the process* HP(β, $\frac{1}{2}$) *is ergodic.*

We also formulate the following plausible conjecture. Its not completely rigorous proof will be presented in Section 7.3.

Conjecture 7.1. *For any $p < \frac{1}{2}$ there exists $\beta_0 = \beta_0(p) > 0$ such that the process* HP(β, p) *is not ergodic for $\beta < \beta_0$.*

7.1. Proof of Theorem 7.1

We use the notation introduced in Section 4. Direct computations yield

$$f_1(S_k^{+l}) - f_1(S) = R_k - T_k - 1,$$

$$f_1(S_k^{-l}) - f_1(S) = -R_k + T_k - 1,$$

$$f_1(S_k^{+r}) - f_1(S) = R_k - T_{k+1},$$

$$f_1(S_k^{+r}) - f_1(S) = -R_k + T_{k+1},$$

so

$$\mathrm{E}^{\mathrm{v}}(f_1(\xi_{t+1}) - f_1(\xi_t) | \xi_t = S) = -\frac{N}{2N+1}. \tag{7.1}$$

Combining this with (5.7), we obtain that there exists a positive number $C = C(\beta)$ such that

$$\begin{aligned} \mathrm{E}^{\mathrm{h}}_{\beta,p}(f_1(\xi_{t+1}) - f_1(\xi_t) | \xi_t = S) &= -\frac{1}{2N+1}(\beta N - (1-\beta)((q-p)N + q)) \\ &< -C(\beta) \end{aligned} \tag{7.2}$$

for $\beta > \frac{2}{3}$. Applying Theorem 3.1, we finish the proof.

7.2. Proof of Theorem 7.2

To prove the desired result, we apply Theorem 3.1 to the function $\varphi(S) := (f_2(S))^\alpha$ for some $\alpha < 1$.

First, we prove the theorem for the case $p = \frac{1}{2}$. Inserting $p = q = \frac{1}{2}$ into (5.3), we obtain for the EP($\frac{1}{2}$) step

$$\mathrm{E}^{\mathrm{e}}_{1/2}(f_2(\xi_{t+1}) - f_2(\xi_t) | \xi_t = S) = \frac{1}{2}. \tag{7.3}$$

It is elementary that, for $\alpha \in (0, 1)$,

$$(x+1)^\alpha - 1 \leqslant \alpha x, \tag{7.4}$$

for all $x \geqslant -1$. Using (7.3), (7.4) and (6.12), we obtain

$$\mathrm{E}^{\mathrm{e}}_{1/2}((f_2(\xi_{t+1}))^\alpha - (f_2(\xi_t))^\alpha | \xi_t = S) \leqslant \frac{\alpha(f_2(S))^{\alpha-1}}{2}. \tag{7.5}$$

Now let us make the necessary estimate for the VM step. Here we will need a bound which is more accurate than (6.10).

Lemma 7.1. *There exists $C' > 0$ such that*

$$\mathrm{E}^{\mathrm{v}}((f_2(\xi_{t+1}) - f_2(\xi_t))^2 | \xi_t = S) \geqslant C' |S|^{16/5}. \tag{7.6}$$

Proof. To calculate exactly the left-hand side of (7.6), one has to square (6.4)–(6.5), sum them and divide by $4N+2$. But this calculation appears to be too difficult; so we will only obtain a lower bound. Denote $\Delta_k = f_2(S_k^{+r}) - f_2(S)$, so that

$$\mathrm{E}^{\mathrm{v}}((f_2(\xi_{t+1}) - f_2(\xi_t))^2 | \xi_t = S) \geqslant \frac{1}{4N+2} \sum_{i=1}^{N} \Delta_i^2. \tag{7.7}$$

Simple algebraic calculations using (6.4) give that

$$\Delta_{i+1} - \Delta_i \geqslant N, \tag{7.8}$$

for $i = 0, \ldots, N-1$. From (6.4) one also obtains $\Delta_0 < 0$ and $\Delta_N > 0$, so denote $L = \min\{k : \Delta_{k-1} < 0,\ \Delta_k \geqslant 0\}$. Using (7.8) gives

$$\sum_{i=1}^{N} \Delta_i^2 \geqslant \sum_{i=0}^{L-1} (N(L-i-1))^2 + \sum_{i=L}^{N} (N(i-L))^2$$

$$\geqslant N^2 \sum_{i=1}^{N/2} i^2 \geqslant C_1 N^5,$$

for some C_1, so by (7.7) we obtain that

$$\mathrm{E}^{\mathrm{v}}((f_2(\xi_{t+1}) - f_2(\xi_t))^2 | \xi_t = S) \geqslant C_2 N^4,$$

for some C_2. Combining this with (6.10) gives

$$\mathrm{E}^{\mathrm{v}}((f_2(\xi_{t+1}) - f_2(\xi_t))^2 | \xi_t = S) \geqslant \max\left\{ C_2 N^4, \frac{C|S|^4}{N} \right\} \geqslant C^{4/5} C_2^{1/5} |S|^{16/5},$$

thus proving Lemma 7.1. □

Remark 7.1. The exponent $16/5$ in Lemma 7.1 is the best possible; to see this, one may take a configuration S with $n_1 = m_N = N^{5/4}$ and $n_2 = \ldots = n_N = m_1 = \ldots = m_{N-1} = 1$ and compute the left-hand side of (7.6).

We now resume our proof of Theorem 7.2. Analogously to (5.12), using (6.12) together with Lemmas 6.1 and 7.1, we obtain, for some positive constant C_2,

$$\mathrm{E}^{\mathrm{v}}((f_2(\xi_{t+1}))^{\alpha} - (f_2(\xi_t))^{\alpha}|\xi_t = S) \leqslant -C_2(f_2(S))^{\alpha-2}|S|^{16/5}. \tag{7.9}$$

So, for HP($\beta, \frac{1}{2}$), combining (7.5) with (7.9) and using the fact that $|S|^{16/5} \geqslant 2^{16/5}(f_2(S))^{16/15}$ by virtue of Lemma 4.1(ii), we obtain, for $14/15 < \alpha < 1$

$$\begin{aligned}
&\mathrm{E}^{\mathrm{h}}_{\beta,1/2}(\varphi(\xi_{t+1}) - \varphi(\xi_t)|\xi_t = S)\\
&\quad = \mathrm{E}^{\mathrm{h}}_{\beta,1/2}((f_2(\xi_{t+1}))^{\alpha} - (f_2(\xi_t))^{\alpha}|\xi_t = S)\\
&\quad \leqslant (1-\beta)\frac{\alpha(f_2(S))^{\alpha-1}}{2} - \beta C_2(f_2(S))^{\alpha-2}|S|^{16/5}\\
&\quad \leqslant (1-\beta)\frac{\alpha(f_2(S))^{\alpha-1}}{2} - 2^{16/5}\beta C_2(f_2(S))^{\alpha-14/15}\\
&\quad = -(f_2(S))^{\alpha-14/15}\left[2^{16/5}\beta C_2 - \frac{(1-\beta)\alpha}{2}(f_2(S))^{-1/15}\right] < -1
\end{aligned} \tag{7.10}$$

for all but a finite number of Ss (indeed, the expression in the square brackets is positive for all but finitely many S, and the fact that $\alpha > 14/15$ guarantees that the absolute value of the left-hand side of (7.10) is large enough for all but a finite number of Ss). Applying Theorem 3.1, we conclude the proof of Theorem 7.2 for $p = \frac{1}{2}$.

When $p > \frac{1}{2}$, using (5.4) and Lemma 6.1 gives that, for any $\varepsilon > 0$,

$$\mathrm{E}^{\mathrm{h}}_{\beta,p}(f_2(\xi_{t+1}) - f_2(\xi_t)|\xi_t = S) = (1-\beta)\mathrm{E}^{\mathrm{e}}_{p}(f_2(\xi_{t+1}) - f_2(\xi_t)|\xi_t = S) < -(1-\beta)\varepsilon$$

for all but finite number of S, and we apply Theorem 3.1 again. □

Remark 7.2. Using the technique of Section 6, it is possible to establish that there exists some $p_0 = p_0(\beta) > 1$ such that for the process HP($\beta, \frac{1}{2}$) we have that $\mathrm{E}(\tau(S_0))^p < \infty$ for all $p < p_0$. By using the technique of Menshikov and Popov (1995), one can obtain polynomial bounds on the decay of the stationary measure.

7.3. Nonergodicity

Here we will present an argument in favour of the validity of Conjecture 7.1.

We rewrite (7.2) as

$$\begin{aligned}
\mathrm{E}^{\mathrm{h}}_{\beta,p}(f_1(\xi_{t+1}) - f_1(\xi_t)|\xi_t = S) &= \frac{1}{2N+1}(-\beta N + (1-\beta)((q-p)N + q))\\
&\geqslant \frac{N}{2N+1}((1-\beta)(q-p) - \beta) > 0
\end{aligned} \tag{7.11}$$

when $\beta < (q-p)/2q$. Unfortunately, because VM does not possess property (3.5), we cannot apply Theorem 3.4. Moreover, it is still very unclear whether the process is transient in this case. So instead we shall explain why we believe it is not ergodic. We need the following three lemmas.

Lemma 7.2. *Let ξ_t, $t = 0, 1, 2, \dots,$ be a Markov chain on a countable state space X, let $0 \in X$ be an absorbing state, and define $\tau := \min\{t : \xi_t = 0\}$ to be the hitting time of 0. Suppose that for any starting point x we have $f(x) := \mathrm{E}_x\tau < \infty$. Then*

$$\mathrm{E}f(\xi_t) \to 0, \tag{7.12}$$

as $t \to \infty$.

Proof. Let x_0 be the starting position of the Markov chain. It is straightforward to see that

$$\mathrm{E}(f(\xi_{t+1}) - f(\xi_t)|\xi_t = x) = -\mathbf{1}_{\{x\neq 0\}}, \tag{7.13}$$

so, taking expectations in (7.13), we obtain

$$\mathrm{E}_{x_0}f(\xi_{t+1}) - \mathrm{E}_{x_0}f(\xi_t) = -P_{x_0}\{\tau > t\}. \tag{7.14}$$

Summing with respect to t in (7.14), we obtain

$$\mathrm{E}_{x_0}f(\xi_{t+1}) = f(x_0) - \sum_{i=0}^{t} P_{x_0}\{\tau > i\} \to 0$$

as $t \to \infty$, thus completing the proof. □

Lemma 7.3. *Let ξ_t, $t = 0, 1, 2, \dots,$ be a Markov chain on a countable state space X, and let $0 \in X$ be an absorbing state. Let x_0 be the starting position of the Markov chain, τ be the moment of hitting 0, and suppose that $\mathrm{E}_{x_0}\tau < \infty$ for all x_0. Let $f(x)$ be some non-negative function on X such that, for some constant K,*

$$\mathrm{E}(f(\xi_{t+1}) - f(\xi_t)|\xi_t = x) \leqslant K, \tag{7.15}$$

for all $x \neq 0$. Then there exists a constant M such that $\mathrm{E}f(\xi_t) < M$ for all t.

Proof. The proof is analogous to that of Lemma 7.2: first, we rewrite (7.15) as

$$\mathrm{E}(f(\xi_{t+1}) - f(\xi_t)|\xi_t = x) \leqslant K\mathbf{1}_{\{\tau > t\}}, \tag{7.16}$$

for all x. So

$$\mathrm{E}_{x_0}f(\xi_{t+1}) - \mathrm{E}_{x_0}f(\xi_t) \leqslant KP_{x_0}\{\tau > t\}, \tag{7.17}$$

and, summing with respect to t in (7.17), we get

$$\mathrm{E}_{x_0}f(\xi_{t+1}) \leqslant f(x_0) + K\sum_{i=0}^{t} P_{x_0}\{\tau > i\} \leqslant f(x_0) + K\mathrm{E}_{x_0}\tau.$$

Defining $M := f(x_0) + K\mathrm{E}_{x_0}\tau$, we conclude the proof. □

Analogously to Lemmas 7.2 and 7.3, we can prove the following lemma, which is, in fact, an adaptation of Lemma 2.2 from Menshikov and Popov (1995) to our situation.

Lemma 7.4. *Let ξ_t, $t = 0, 1, 2, \ldots,$ be a Markov chain on a countable state space X, let $0 \in X$ be an absorbing state, x_0 be the starting point, and τ be the moment of hitting 0, and suppose that $\mathrm{E}_{x_0}\tau < \infty$ for all x_0. Let $f(x)$ be some non-negative function on X such that $\mathrm{E}_{x_0} f(\xi_t) \to 0$ as $t \to \infty$, and, for some positive constant K,*

$$\mathrm{E}(f(\xi_{t+1}) - f(\xi_t)|\xi_t = x) \geqslant -K, \tag{7.18}$$

for all x. Then $\mathrm{E}_{x_0}\tau \geqslant f(x_0)/K$.

So let us take a hybrid process $\mathrm{HP}(\beta, p)$ which satisfies (7.11). We suppose that it is ergodic and try to obtain a contradiction. Define $\varphi_{\beta,p}(S)$ to be the mean hitting time of $\mathscr{D}_0$ starting from S, i.e. $\varphi_{\beta,p}(S) = \mathrm{E}^{\mathrm{h}}_{\beta,p}\tau(S)$. We will prove the following lemma.

Lemma 7.5. *For any $p > \frac{1}{2}$, β, there exists a positive constant $C = C(\beta, p)$ such that*

$$\varphi_{\beta,p}(S) \geqslant Cf_1(S). \tag{7.19}$$

Proof. From (5.3) and Lemma 6.1 we obtain that, for $p > \frac{1}{2}$ and any β,

$$\mathrm{E}^{\mathrm{h}}_{\beta,p}(f_2(\xi_{t+1}) - f_2(\xi_t)|\xi_t = S) \leqslant \frac{1}{2}. \tag{7.20}$$

Applying Lemma 7.3 gives that there exists a constant M such that $\mathrm{E}^{\mathrm{h}}_{\beta,p} f_2(\xi_t) < M$ for all t. Using Lemma 4.1(iii), we see that $\mathrm{E}^{\mathrm{h}}_{\beta,p} f_1(\xi_t) \to 0$ as $t \to \infty$. An application of Lemma 7.4 completes the proof. □

Conjecture 7.2. *Lemma 7.5 holds for any p.*

We have been unable to prove the above conjecture. Intuitively, $\varphi_{\beta,p}(S)$ grows when p decreases and the monotonicity argument might be applicable to prove this fact. For the pure exclusion process, this argument follows from the basic coupling (see Liggett 1985, Section VIII.2). When the voter model is added, this coupling does not work.

Now, if we suppose this to be true, the rest of the proof is straightforward. If the process $\mathrm{HP}(\beta, p)$ is ergodic, then the function $\varphi_{\beta,p}(S)$ is well defined, so, by Lemma 7.2, $\mathrm{E}^{\mathrm{h}}_{\beta,p}\varphi_{\beta,p}(\xi_t) \to 0$ as $t \to \infty$. Thus, using (7.19), we obtain that $\mathrm{E}^{\mathrm{h}}_{\beta,p} f_1(\xi_t) \to 0$. But this obviously contradicts (7.11). □

8. Continuous time

In this section we show how Theorem 1.1 follows from the theorems proved in the previous three sections.

Observe first that the transience is a property of the skeleton of a Markov process. In our case the skeleton is the process $\xi_n = \eta_{\tau_n}$, as defined in Section 2. Notice that according to this definition, ξ_{n+1} may be the same as ξ_n; this deviates slightly from the usual notion of the skeleton. In our version of the skeleton the exit time of a configuration is a geometric random variable with parameter greater than $\beta + (1-\beta)\min\{p, q\}$. This implies that the skeleton cannot get stacked. Hence Theorem 5.2, which states the transience for the discrete-time exclusion process with $p \leqslant \frac{1}{2}$, implies the same for the continuous-time process. This shows the transient part of Theorem 1.1(i).

To prove that ergodicity for the discrete-time process implies ergodicity for the continuous-time one, let $\eta \in \mathscr{D}_0$, let S be the equivalence class of η and write

$$\tau_c(\eta) = \sum_{n=1}^{\tau(S)} (\tau_n - \tau_{n-1}) \tag{8.1}$$

where we recall that $\tau_c(\eta)$ and $\tau(S)$ are the hitting times of $\mathscr{D}_0$ for the continuous- and discrete-time processes starting from η and S respectively, and τ_n is the instant of the nth attempted jump of the continuous process η_t, as defined in (2.1). Given the past up to τ_n, $\tau_{n+1} - \tau_n$ is an exponential random variable with rate greater than 1 – the worst case obtained when the configurations belong to $\mathscr{D}_0$. Hence, $\tau_{n+1} - \tau_n$ is stochastically bounded below by an exponential random variable of rate 1 independent of everything. This implies that

$$\mathrm{E}\tau_c(\eta) \leqslant \mathrm{E}\tau(\eta). \tag{8.2}$$

Since the ergodicity is equivalent to the finiteness of the expected return time for any given configuration, ergodicity for the discrete-time process implies the same for the continuous-time one. With this argument Theorem 5.1 implies the ergodic part of Theorem 1.1(i) and Theorems 7.1 and 7.2 imply Theorem 1.1(ii).

The argument above implies a stronger statement for the pure voter model. Let $\beta = 0$ and observe that, for any $\eta \notin \mathscr{D}_0$, there are at least three discrepancies. Hence, for the voter model,

$$\tau_c(\eta) \leqslant \sum_{n=1}^{\tau(S)} (\tau'_n - \tau'_{n-1}), \tag{8.3}$$

where $(\tau'_n - \tau'_{n-1})$ are independently exponentially distributed with parameter 3 and independent of $\tau(S)$. This, together with Lemma 8.1 below, shows that the first part of (1.3) follows from (6.1).

We now show how to obtain the second part of (1.3) from (6.2). Let $\mathscr{D}_1$ be the set of configurations on $\mathscr{D}$ having exactly three discrepancies (that is, 10, 01 and 10). This means that in the representation (4.1) the configurations belonging to $\mathscr{D}_1$ have $N = 1$, that is only one finite 0-block and one finite 1-block. The transition rate for configurations in $\mathscr{D}_1$ in the voter model is exactly 3. If the process is in $\mathscr{D}_1$ then it can only either stay in $\mathscr{D}_1$ or jump to $\mathscr{D}_0$. Hence, for $\eta \in \mathscr{D}_1$,

$$\tau_c(\eta) = \sum_{n=1}^{\tau(S)} (\tau'_n - \tau'_{n-1}). \tag{8.4}$$

Lemma 8.1 below shows that (6.2) implies that the left-hand side of (8.4) is infinite for any $\eta \in \mathscr{D}_1$. As argued before, any configuration in $\mathscr{D}_1$ is reachable from any other, hence the same is valid for any $\eta \in \mathscr{D} \backslash \mathscr{D}_0$. This shows the second part of (1.3).

Lemma 8.1. *Let τ be a positive integer random variable and τ_i be non-negative independent random variables with the exponential distribution and independent of τ. Then, for any $p > 0$,*

$$\mathrm{E}\left(\sum_{n=1}^{\tau} \tau_n\right)^p < \infty \text{ if and only if } \mathrm{E}\tau^p < \infty. \tag{8.5}$$

Proof. By independence,

$$\mathrm{E}\left(\sum_{n=1}^{\tau} \tau_n\right)^p = \sum_n \mathrm{E}\left(\sum_{i=1}^{n} \tau_i\right)^p P(\tau = n). \tag{8.6}$$

But

$$\mathrm{E}\left(\sum_{i=1}^{n} \tau_i\right)^p = \frac{\Gamma(n+p)}{\Gamma(n)} \tag{8.7}$$

which is of the order of n^p. □

9. Final remarks

Let us make several remarks with respect to extensions of our results to the non-nearest-neighbor case. For a non-nearest-neighbor voter model, we failed to find an analogue of Lemma 6.1. To be precise, in this case, Lemma 6.1 is incorrect for f_2 as stated, and we could not find a substitute for f_2 that would provide relevant information both for this voter model and for the hybrid process constructed by mixing this voter model with an exclusion process of any range. When the hybrid process consists of a nearest-neighbor voter model and a finite-range exclusion process, an analogue of Theorem 1.1(ii)(a) may be obtained by an appropriate, straightforward, modification of our arguments. Anything beyond this result was not possible. A reason for this was again our failure to find substitutes for f_1 and f_2 that would work for this case as well as f_1 and f_2 have worked for the nearest-neighbour system.

The results presented above show that, to a certain extent, the success of our methods depends on the correct choice of Lyapunov function.

Acknowledgements

The authors thank the Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP) and the Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) for financial support.

References

Aspandiiarov, S., Iasnogorodski, R. and Menshikov, M.V. (1996) Passage-time moments for nonnegative stochastic processes and an application to reflected random walks in a quadrant. *Ann. Probab.*, **24**, 932–960.

Bramson, M., Calderoni, P., De Masi, A., Ferrari, P.A., Lebowitz, J. and Schonmann, R. (1986) Microscopic selection principle for a diffusion-reaction equation. *J. Statist. Phys.*, **45**, 905–920.

Cammarota, C. and Ferrari, P.A. (1991) Invariance principle for the branching exclusion process. *Stochastic Process. Appl.*, **38**, 1–11.

Cox, J.T. and Durrett, R. (1995) Hybrid zones and voter model interfaces. *Bernoulli*, **1**, 343–370.

De Masi, A., Ferrari, P.A. and Lebowitz, J. (1986) Reaction-diffusion equations for interacting particle systems. *J. Statist. Phys.*, **44**, 589–644.

Derrida, B., Goldstein, S., Lebowitz, J.L. and Speer, E. (1998) Shift equivalence of measures and the intrinsic structure of shocks in the asymmetric simple exclusion process. *J. Statist. Phys.*, **93**, 547–571.

Durrett, R. (1988) *Lecture Notes on Particle Systems and Percolation.* Belmont, CA: Wadsworth.

Durrett, R. (1995) *Ten lectures on particle systems.* In P. Bernard (ed.), *Lectures on Probability Theory: École d'Éte de Probabilités de Saint Flour XXIII – 1993*, Lecture Notes in Math. 1608. New York: Springer–Verlag.

Fayolle, G., Malyshev, V.A. and Menshikov, M.V. (1995) *Topics in the Constructive Theory of Countable Markov Chains.* Cambridge: Cambridge University Press.

Ferrari, P.A. (1992) Shock fluctuations in asymmetric simple exclusion. *Probab. Theory Related Fields*, **91**, 81–101.

Ferrari, P.A. (1994) Shocks in one-dimensional processes with drift. In G. Grimmett (ed.), *Probability and Phase Transition*, NATO Adv. Sci. Inst. Ser. C, Math. Phys. Sci. **420**, pp. 35–48. Dordrecht: Kluwer Academic.

Ferrari, P.A. (1996) Growth processes on a strip. In R. Bamon, J.-M. Gambaudo, S. Martinez (eds), *Disordered Systems*, Travaux en Cours, 53, pp. 87–111. Paris: Hermann.

Ferrari, P.A., Kipnis, C. and Saada, S. (1991) Microscopic structure of travelling waves in the asymmetric simple exclusion process. *Ann. Probab.*, **19**, 226–244.

Liggett, T.M. (1976) Coupling the simple exclusion process. *Ann. Probab.*, **4**, 339–356.

Liggett, T.M. (1985) *Interacting Particle Systems.* Berlin: Springer-Verlag.

Liggett, T.M. (1999) *Stochastic Interacting Systems. Contact, Voter and Exclusion Processes.* Berlin: Springer-Verlag.

Machado, F.P. (1998) Asymptotic shape for the branching exclusion process. *Markov Process. Related Fields*, **4**, 535–547.

Malyshev, V.A. (1998) Random grammars. *Uspekhi Mat. Nauk* **53**(2), 107–134 (in Russian). Translated in *Russian Math. Surveys*, **53**(2), 345–370.

Menshikov, M.V. and Popov, S.Yu. (1995) Exact power estimates for countable Markov chains. *Markov Process. Related Fields*, **1**, 57–78.

Received April 1999 and revised February 2000