Bernoulli **8**(2), 2002, 195–205

Entrance from 0+ for increasing semi-stable Markov processes

JEAN BERTOIN[1] and MARIA-EMILIA CABALLERO[2]

[1]*Laboratoire de Probabilités et Institut Universitaire de France, Université Pierre et Marie Curie, 175 rue de Chevaleret, F-75013 Paris, France. E-mail: jbe@ccr.jussieu.fr*
[2]*Instituto de Matemáticas, Universidad Nacional Autónoma de México, Mexico 04510 DF, Mexico. E-mail: emilia@servidor.unam.mx*

We consider increasing semi-stable Markov processes starting at $x > 0$ and specify their asymptotic behaviour in law as $x \to 0+$. This can be viewed as an extension of a result of Brennan and Durrett on the asymptotic size of a particle undergoing a certain type of random splitting.

Keywords: entrance boundary; fragmentation; semi-stable Markov process; subordinator

1. Introduction and main result

Motivated by certain limit theorems, Lamperti (1962; 1972) introduced the notion of *semi-stable* Markov process (also called *self-similar* Markov process) as follows. Let $X = (X(t),\ t \geqslant 0)$ be a strong Markov process with values in $]0, \infty[$, and denote by $\mathbb{P}_x$ its law starting at $X(0) = x$. For $\alpha > 0$, call X semi-stable with index $1/\alpha$ whenever it satisfies the following scaling property:

$$\text{the distribution of } (kX(k^{-\alpha} t),\ t \geqslant 0) \text{ under } \mathbb{P}_x \text{ is } \mathbb{P}_{kx}, \tag{1}$$

where $k > 0$ is arbitrary. The main result of Lamperti (1972) is a connection between semi-stable Markov processes and Lévy processes using a time substitution; see Carmona *et al.* (1994; 1997), Vuolle-Apiala (1994) and the references therein for some applications. More precisely, suppose X is a semi-stable Markov process with index $1/\alpha$, starting say at $x > 0$, and set

$$\gamma(t) = \int_0^t \frac{\mathrm{d}s}{X(s)^\alpha}, \qquad t \geqslant 0.$$

Provided that $\gamma(\infty) = \infty$ almost surely, we may consider the inverse functional C of γ, and then

$$\xi_t = \log(X \circ C(t)) - \log x, \qquad t \geqslant 0,$$

is a Lévy process starting at 0. Conversely, if $\xi = (\xi_t,\ t \geqslant 0)$ is a Lévy process starting at 0, define, for any fixed $y \in \mathbb{R}$,

1350–7265

$$C(t) = \int_0^t \exp(\alpha(\xi_s + y))\mathrm{d}s, \qquad t \geqslant 0.$$

Suppose that $C(\infty) = \infty$ a.s., and denote by γ the inverse functional of C. Then

$$X = (\exp\{\xi_{\gamma(t)} + y\},\ t \geqslant 0)$$

is a semi-stable Markov process with index $1/\alpha$ starting at $x = \mathrm{e}^y$.

Lamperti (1972) raised the question of whether one can make sense of X starting at $0+$, which has a crucial importance when one analyses the possible limit processes that can arise from a sequence of normalized Markov processes; see also Vuolle-Apiala (1994). Essentially, this amounts to the problem of entrance from $0+$, that is, to the asymptotic behaviour of $\mathbb{P}_x$ as $x \to 0+$ (of course, by the Markov property, it suffices to consider the behaviour of the one-dimensional distributions). By scaling property (1), this is also equivalent to studying the asymptotic behaviour of $X(t)$ as $t \to \infty$ when the semi-stable process starts at some fixed point.

We focus here on the case where X has non-decreasing sample paths, that is, when ξ is a subordinator – see Chapter III in Bertoin (1996) for background. This is quite a special case; however, it has some interesting applications that will be discussed at the end of this section. To state our main result, we need to introduce some notation. We assume henceforth that ξ is a subordinator and write Φ for its Laplace exponent, that is,

$$\mathbb{E}(\exp(-q\xi_t)) = \exp(-t\Phi(q)), \qquad t, q \geqslant 0.$$

The celebrated Lévy–Khinchine formula enables us to express the Laplace exponent in the form

$$\Phi(q) = dq + \int_{]0,\infty[} (1 - \mathrm{e}^{-qx})\Pi(\mathrm{d}x),$$

where $d \geqslant 0$ is the drift coefficient and Π the Lévy measure. We shall implicitly exclude the case where ξ is arithmetic, that is, where the drift is $d = 0$ and the Lévy measure Π is supported by $r\mathbb{N}$ for some $r > 0$ (this case can be treated by a simple variation of our argument). Finally, we write m for the mean of the subordinator,

$$m = \mathbb{E}(\xi_1) = d + \int_{]0,\infty[} x\Pi(\mathrm{d}x) = \Phi'(0+) \in]0, \infty].$$

We are now able to state our result.

Theorem 1. (i) *If $m < \infty$, then for every $t > 0$, $\mathbb{P}_x(X(t) \in \cdot)$ converges as $x \to 0+$ to the entrance law at $0+$, $\mathbb{P}_{0+}(X(t) \in \cdot)$. More precisely, the latter is determined by the following identities:*

$$\mathbb{E}_{0+}(X(t)^{-\alpha}) = \frac{1}{\alpha t m},$$

$$\mathbb{E}_{0+}(X(t)^{-\alpha k}) = \frac{(k-1)!}{\alpha t^k m\Phi(\alpha)\ \ldots\ \Phi(\alpha(k-1))}, \qquad \text{for } k = 2, 3, \ldots .$$

(ii) *If* $m = \infty$*, then for every* $t > 0$*,* $\mathbb{P}_x(X(t) \in \cdot)$ *converges as* $x \to 0+$ *to the Dirac point mass at* ∞*.*

The formulae for the moments in Theorem 1 are reminiscent of those obtained by Brennan and Durrett (1987, p. 114) – see their equation (3). Let us now explain the connection. Motivated by a model for polymer degradation, Brennan and Durrett considered the evolution of a particle system in which each particle of size ℓ waits an exponential length of time with parameter ℓ^α and then undergoes binary splitting into a left particle of size ℓV and a right particle of size $\ell(1 - V)$. Here, V is a random variable with values in $]0, 1[$ which has a fixed distribution and is independent of the past behaviour of the system. Suppose now that we start at the initial time with a single particle of unit size; we focus on the leftmost particle and write $\hat{\ell}_t$ for its length at time t.

From the preceding verbal description, one can easily see that the process $X = (1/\hat{\ell}_t, t \geqslant 0)$ is an increasing semi-stable Markov process with index $1/\alpha$, starting at 1. More precisely, the subordinator ξ resulting from Lamperti's transformation is a compound Poisson process with Lévy measure (i.e. jump intensity) the distribution of $-\log V$, that is, $\Pi(\cdot) = \mathbb{P}(-\log V \in \cdot)$. In other words, its Laplace exponent is given by

$$\Phi(q) = \mathbb{E}(1 - \exp\{q \log V\}) = 1 - \mathbb{E}(V^q), \qquad q > 0. \tag{2}$$

It follows from scaling property (1) that for every $t > 0$, the distribution of $t\hat{\ell}_t^\alpha$ is the same as that of $X(1)^{-\alpha}$ under $\mathbb{P}_x$ for $x = t^{-1/\alpha}$. Provided that the variable $-\log V$ is not arithmetic and has finite mean $\mathbb{E}(-\log V) := m$, we then deduce from Theorem 1 that

$$t\hat{\ell}_t^\alpha \text{ converges in distribution as } t \to \infty \text{ towards a variable } Y, \tag{3}$$

where the distribution of Y is specified by its integer moments,

$$\mathbb{E}(Y) = \mathbb{E}_{0+}(X(1)^{-\alpha}) = \frac{1}{\alpha m},$$

$$\mathbb{E}(Y^k) = \mathbb{E}_{0+}(X(1)^{-k\alpha}) = \frac{(k-1)!}{\alpha m \Phi(\alpha) \ \ldots \ \Phi(\alpha(k-1))}, \qquad \text{for } k = 2, 3, \ldots,$$

in which Φ is given in terms of the splitting variable V by (2). This is precisely the result observed by Brennan and Durrett. In turn, a continuous-time version of the argument developed by Brennan and Durrett provides key steps to establishing Theorem 1.

We stress that Theorem 1 is also relevant to the investigation of a more general type of self-similar fragmentation considered in Bertoin (2001) – roughly speaking, the situation treated there includes cases in which particles undergo splitting infinitely many times on any non-trivial time interval. Indeed, the process giving the mass of a tagged fragment as time passes is then a decreasing semi-stable process whose parameters are specified in terms of the characteristics of the self-similar fragmentation (for details, see Bertoin 2001), and thus its asymptotic behaviour as time goes to infinity can be deduced from Theorem 1.

The proof of Theorem 1 will be presented in the next section. Some partial results on the entrance law for semi-stable Markov processes having only negative jumps are presented in Section 3.

2. Proof of Theorem 1

To start with, we introduce the so-called age and residual lifetime processes

$$A_t = t - \xi_{L(t)-}, \qquad R_t = \xi_{L(t)} - t,$$

where L is the local time, that is,

$$L(t) = \inf\{s \geqslant 0 : \xi_s \geqslant t\}, \qquad t \geqslant 0.$$

We recall a well-known consequence of the renewal theorem (see, for instance, Bertoin *et al*. 1999).

Lemma 2. (i) *If $m < \infty$, then (A_t, R_t) converges in distribution towards $(UZ, (1-U)Z)$ as $t \to \infty$, where the random variables U and Z are independent, U is uniformly distributed on* $[0, 1]$ *and the law of Z is given by*

$$\mathbb{P}(Z \in \mathrm{d}z) = m^{-1}(d\delta_0(\mathrm{d}z) + z\Pi(\mathrm{d}z)), \qquad z \geqslant 0,$$

with δ_0 standing for the Dirac point mass at 0.

(ii) *If $m = \infty$, then (A_t, R_t) converges in probability towards* (∞, ∞).

We will deduce Theorem 1 from the following limit theorem, which in turn is essentially a consequence of Lemma 2. Recall that we denote by X the (right-continuous) semi-stable Markov process associated with ξ by Lamperti's time substitution, and that $\mathbb{P}_x$ stands for the law of X started from x.

Proposition 3. *Denote the first passage time of X above a level $b > 0$ by*

$$T_b = \inf\{t \geqslant 0 : X(t) > b\}.$$

(i) *If $m < \infty$, then the distribution of $(T_b, X(T_b))$ under $\mathbb{P}_x$ converges as $x \to 0+$ to the law of the pair*

$$\left(b^\alpha \exp(-\alpha UZ)\int_0^\infty \exp(-\alpha\xi_s)\mathrm{d}s,\ b\exp((1-U)Z)\right),$$

where ξ, U and Z are independent, and U and Z are as in Lemma 2.

(ii) *If $m = \infty$, then the distribution of $(T_b, X(T_b))$ under $\mathbb{P}_x$ converges as $x \to 0+$ to the Dirac point mass at* $(0, \infty)$.

The proof of Proposition 3 relies on four simple results on subordinators.

Lemma 4. *For every $t > 0$, let $\tilde{\xi}^{(t)}$ be the time-reversed process given by*

$$\tilde{\xi}^{(t)}_s = \begin{cases} \xi_{L(t)-} - \xi_{(L(t)-s)-} & \text{if } s < L(t), \\ \xi_s & \text{if } s \geqslant L(t). \end{cases}$$

Then ξ and $\tilde{\xi}^{(t)}$ have the same distribution, and moreover the local time $\tilde{L}^{(t)}$, the age $\tilde{A}^{(t)}$ and the residual lifetime $\tilde{R}^{(t)}$ of $\tilde{\xi}^{(t)}$ satisfy

$$\tilde{L}^{(t)}(t) = L(t), \qquad \tilde{A}_t^{(t)} = A_t, \qquad \tilde{R}_t^{(t)} = R_t.$$

Proof. The law of $L(t)$ has at most countably many atoms, none of which can be 0. Thus the set of real numbers which can be expressed in the form $(x/k)^{-1/n}$, where x is an atom and k and n positive integers, is at most countable. Hence there exists $a > 1$ which cannot be expressed in the preceding form so that $\mathbb{P}(L(t) = ka^{-n}) = 0$ for every $k, n \in \mathbb{N}$. Pick such an $a > 1$, and set $k(n) = [a^n L(t)]$, where $[\cdot]$ refers to the integer part. We thus have a.s.

$$a^{-n}k(n) < L(t) \quad \text{and} \quad \lim_{n\to\infty} a^{-n}k(n) = L(t). \tag{4}$$

Next, let

$$\tilde{\xi}_s^{(k,n)} = \begin{cases} \xi_{ka^{-n}} - \xi_{(ka^{-n}-s)-} & \text{if } s < ka^{-n}, \\ \xi_s & \text{if } s \geqslant ka^{-n}. \end{cases}$$

By the duality lemma (see, for example, Bertoin 1996, p. 45), the process $\tilde{\xi}^{(k,n)}$ has the same law as ξ for each fixed k, n. On the other hand, we have the equivalence

$$k = k(n) \Leftrightarrow \xi_{ka^{-n}} < t \quad \text{and} \quad \xi_{(k+1)a^{-n}} \geqslant t \Leftrightarrow \tilde{\xi}^{(k,n)}_{ka^{-n}} < t \text{ and } \tilde{\xi}^{(k,n)}_{(k+1)a^{-n}} \geqslant t,$$

so that, for every measurable functional $F \geqslant 0$,

$$\begin{aligned} \mathbb{E}(F(\tilde{\xi}^{(k(n),n)})) &= \sum_{k=0}^{\infty} \mathbb{E}(F(\tilde{\xi}^{(k,n)}),\, \tilde{\xi}^{(k,n)}_{ka^{-n}} < t \leqslant \tilde{\xi}^{(k,n)}_{(k+1)a^{-n}}) \\ &= \sum_{k=0}^{\infty} \mathbb{E}(F(\xi),\, \xi_{ka^{-n}} < t \leqslant \xi_{(k+1)a^{-n}}) \\ &= \mathbb{E}(F(\xi)). \end{aligned}$$

We have thus proved that for every n, $\tilde{\xi}^{(k(n),n)}$ has the same law as ξ. When $n \to \infty$, we see from (4) that $\tilde{\xi}_s^{(k(n),n)}$ converges a.s. to $\tilde{\xi}_s^{(t)}$ for every $s, t \geqslant 0$. We thus obtain that ξ and $\tilde{\xi}^{(t)}$ have the same finite-dimensional distributions, and hence the same law. Finally, the identities for the local time, age and residual lifetime are clear from the construction. □

Corollary 5. *Fix $0 < x < b$ and set $t = \log(b/x)$. The distribution of $(T_b, X(T_b))$ under $\mathbb{P}_x$ is the same as that of*

$$\left(b^{\alpha} \exp(-\alpha A_t) \int_0^{L(t)} \exp(-\alpha \xi_s) \mathrm{d}s,\; b \exp(R_t) \right).$$

Proof. Recall the identities $X \circ C(s) = \exp(\xi_s + \log x)$ and $C(s) = \int_0^s \exp(\alpha(\xi_u + \log x))\mathrm{d}u$. It follows that

$$T_b = \int_0^{L(t)} \exp(\alpha(\xi_s + \log x))\mathrm{d}s, \qquad X(T_b) = b\exp(R_t).$$

Next, recall Lemma 4. We have on the one hand

$$X(T_b) = b\exp(\tilde{R}_t^{(t)}),$$

and on the other hand, by straightforward calculations,

$$\begin{aligned} T_b &= \int_0^{L(t)} \exp(\alpha(\xi_s + \log x))\mathrm{d}s \\ &= \exp(\alpha\log b - \alpha(t - \xi_{L(t)-}))\int_0^{L(t)} \exp(-\alpha(\xi_{L(t)-} - \xi_{(L(t)-s)-}))\mathrm{d}s \\ &= b^\alpha \exp(-\alpha\tilde{A}_t^{(t)})\int_0^{L(t)_{(t)}} \exp(-\alpha\tilde{\xi}_s^{(t)})\mathrm{d}s. \end{aligned}$$

Lemma 4 now completes the proof. □

When $m = \infty$, Proposition 3(ii) follows immediately from Corollary 5 and Lemma 2, and so does Theorem 1(ii). Therefore, we shall now focus on the finite-mean case. The third lemma is a standard consequence of the renewal theorem. For every $s > 0$, write $\mathbb{D}_{[0,s]}$ ($\mathbb{D}$) for Skorohod's space of cadlag functions on $[0, s]$ (on $[0, \infty[$).

Lemma 6. *Let $F : \mathbb{D}_{[0,s]} \to \mathbb{R}$ and $G : [0, \infty[^2 \to \mathbb{R}$ be measurable bounded functions. If $m < \infty$, then*

$$\lim_{t\to\infty} \mathbb{E}(F(\xi_r, t \leqslant s)G(A_t, R_t), s \leqslant L(t)) = \mathbb{E}(F(\xi_r, r \leqslant s))\mathbb{E}(G(UZ, (1-U)Z))$$

where the pair (U, Z) is as in Lemma 2.

Proof. By the simple Markov property at time s, we have

$$\mathbb{E}(F(\xi_r, r \leqslant s)G(A_t, R_t), s \leqslant L(t)) = \int_{[0,t]} \mathbb{P}(\xi_s \in \mathrm{d}x)\mathbb{E}(F(\xi_r, r \leqslant s)|\xi_s = x)\mathbb{E}(G(A_{t-x}, R_{t-x})).$$

According to Lemma 2, we have, for each fixed x, that

$$\lim_{t\to\infty} \mathbb{E}(G(A_{t-x}, R_{t-x})) = \mathbb{E}(G(UZ, (1-U)Z)).$$

We now deduce by dominated convergence that

$$\lim_{t\to\infty} \mathbb{E}(F(\xi_r, r \leqslant s)G(A_t, R_t), s \leqslant L(t)) = \mathbb{E}(F(\xi_r, r \leqslant s))\mathbb{E}(G(UZ, (1-U)Z)),$$

which establishes our claim. □

Finally, we derive from Lemma 6 the following limit result.

Corollary 7. *Suppose $m < \infty$. Then as $t \to \infty$, the triplet*

$$\left(\int_0^{L(t)} \exp(-\alpha\xi_s)\mathrm{d}s,\ A_t,\ R_t\right)$$

converges in distribution towards

$$\left(\int_0^{\infty} \exp(-\alpha\xi_s)\mathrm{d}s,\ UZ,\ (1-U)Z\right),$$

where ξ, U, Z *are independent and* U *and* Z *are as in Lemma* 2.

Proof. On the one hand, for every $c \geqslant 0$ and $x, y > 0$, we have, for every $s > 0$,

$$\rho(t) := \mathbb{P}\left(\int_0^{L(t)} \exp(-\alpha\xi_r)\mathrm{d}r > c,\ A_t > x,\ R_t > y\right)$$

$$\geqslant \mathbb{P}\left(\int_0^{s} \exp(-\alpha\xi_r)\mathrm{d}r > c,\ A_t > x,\ R_t > y,\ s \leqslant L(t)\right).$$

We deduce from Lemma 6 that

$$\liminf_{t\to\infty} \rho(t) \geqslant \mathbb{P}\left(\int_0^{s} \exp(-\alpha\xi_r)\mathrm{d}r > c\right)\mathbb{P}(UZ > x,\ (1-U)Z > y),$$

and since s can be chosen arbitrarily large,

$$\liminf_{t\to\infty} \rho(t) \geqslant \mathbb{P}\left(\int_0^{\infty} \exp(-\alpha\xi_r)\mathrm{d}r > c\right)\mathbb{P}(UZ > x,\ (1-U)Z > y).$$

On the other hand, we have, for every $\varepsilon > 0$ and $s > 0$,

$$\rho(t) \leqslant \mathbb{P}\left(\int_0^{s} \exp(-\alpha\xi_r)\mathrm{d}r > c-\varepsilon,\ A_t > x,\ R_t > y,\ s \leqslant L(t)\right)$$

$$+ \mathbb{P}(s > L(t)) + \mathbb{P}\left(\int_s^{\infty} \exp(-\alpha\xi_r)\mathrm{d}r \geqslant \varepsilon\right).$$

Because $\int_0^\infty \exp(-\alpha\xi_r)\mathrm{d}r < \infty$ a.s., the third term in the sum can be made as small as we wish by choosing s sufficiently large. We then apply Lemma 6 again and deduce (since $\varepsilon > 0$ can be taken arbitrarily small) that

$$\limsup_{t\to\infty} \rho(t) \leqslant \mathbb{P}\left(\int_0^{\infty} \exp(-\alpha\xi_r)\mathrm{d}r \geqslant c\right)\mathbb{P}(UZ > x,\ (1-U)Z > y).$$

The proof of Corollary 7 is now complete. □

Proposition 3(i) now follows immediately from Corollaries 5 and 7; so all that is needed is to establish Theorem 1(i). But since, for every $t, b > 0$, one has

$$\mathbb{P}_x(T_b < t) \leqslant \mathbb{P}_x(X(t) > b) \leqslant \mathbb{P}_x(T_b \leqslant t),$$

it follows from Proposition 3 that $\mathbb{P}_x(X(t) \in \cdot)$ converges weakly as $x \to 0+$ towards $\mathbb{P}_{0+}(X(t) \in \cdot)$, where the latter is the law of the variable

$$t^{1/\alpha} \exp(UZ)\left(\int_0^\infty \exp(-\alpha\xi_s)\mathrm{d}s\right)^{-1/\alpha},$$

in the notation of Proposition 3. The distribution of the exponential integral $\int_0^\infty \exp(-\alpha\xi_s)\mathrm{d}s$ has been determined by Carmona *et al*. (1997) – see their Proposition 3.3. Its integer moments characterize its law and are given in terms of the Laplace exponent Φ by

$$\mathbb{E}\left(\left(\int_0^\infty \exp(-\alpha\xi_s)\mathrm{d}s\right)^k\right) = \frac{k!}{\Phi(\alpha)\ \ldots\ \Phi(\alpha k)}, \qquad k = 1, 2, \ldots. \tag{5}$$

As a consequence, we can calculate the moments $\mathbb{E}_{0+}(X(t)^{-k\alpha})$ by applying (5) and the simple identity

$$\mathbb{E}(\exp(-qUZ)) = \frac{\Phi(q)}{mq}, \qquad \text{for } q > 0.$$

The assertion that these moments determine the entrance law $\mathbb{P}_{0+}(X(t) \in \cdot)$ is immediate from Carleman's condition (see, for example, Carmona *et al*. 1997, Proposition 3.3).

Remark. It is interesting to compare Theorem 1 and equation (5) to derive the identity

$$\mathbb{P}_{0+}(X(1)^{-\alpha} \in \mathrm{d}x) = \frac{1}{\alpha m x}\mathbb{P}\left(\int_0^\infty \exp(-\alpha\xi_s)\mathrm{d}s \in \mathrm{d}x\right).$$

This result has been recently extended by Bertoin and Yor (2001) to general (i.e. not necessarily increasing) semi-stable processes.

3. On the case without positive jumps

To conclude this paper, let us briefly discuss the case of a semi-stable process having only negative jumps, that is, when ξ is no longer a subordinator, but rather a Lévy process with no positive jumps. The situation where ξ drifts to $-\infty$ is of no interest in the framework of this paper, because then it is easy to see that the semi-stable process reaches 0 in a finite time a.s., and that this absorption time tends to 0 when the starting point tends to 0. So, we henceforth assume that ξ is a Lévy process with no positive jumps that does not drift to $-\infty$, and refer to Chapter VII in Bertoin (1996) for background.

A natural way to tackle entrance from 0+ is to search for an extension of Proposition 3. Indeed, if one can establish the convergence as $x \to 0+$ of the distribution under $\mathbb{P}_x$ of the pair $(T_b, X(T_b))$ of the first passage time above a level $b > 0$ and the position at this passage time, then an application of the strong Markov property at time T_b yields the existence of a unique entrance law at 0+, that is, the one-dimensional distribution of the self-similar Markov process that starts at 0, and continuously enters $]0, \infty[$. The absence of positive jumps ensures that $X(T_b) = b$ provided that $X(0) = x < b$, so we need only study T_b. More precisely, we obtain

$$\mathbb{P}_{0+}(X(t) \geqslant b) = \mathbb{P}_{0+}(T_b \leqslant t, X(t) \geqslant b)$$
$$= \int_{[0,t]} \mathbb{P}_b(X(t-u) \geqslant b)\mathbb{P}_{0+}(T_b \in \mathrm{d}u).$$

By Lamperti's construction, the law under $\mathbb{P}_x$ of T_b is the same as that of

$$\int_0^{\tau(\log(b/x))} \exp(\alpha(\xi_s + \log x))\mathrm{d}s,$$

where $\tau(t) = \inf\{s : \xi_s > t\}$. Properties of Lévy processes conditioned to stay positive (for details, see Bertoin 1996, Section VII.3) can be used to determine the asymptotic behaviour of this quantity as $x \to 0+$. Specifically, let $\xi^{\uparrow} = (\xi^{\uparrow}_s, s \geqslant 0)$ be distributed as ξ conditioned to stay positive, and introduce its last passage-time process $\lambda(\cdot) = \sup\{s \geqslant 0 : \xi^{\uparrow}_s \leqslant \cdot\}$. According to Theorem VII.18 in Bertoin (1996), the processes $(\xi_s, 0 \leqslant s < \tau(t))$ and $(t - \xi^{\uparrow}_{(\lambda(t)-s)-}, 0 \leqslant s < \lambda(t))$ have the same distribution, so the law under $\mathbb{P}_x$ of T_b is also the same as

$$\int_0^{\lambda(\log(b/x))} \exp(\alpha(\log b - \xi^{\uparrow}_s))\mathrm{d}s.$$

As $x \to 0+$, the latter converges to $b^{\alpha}I$, with

$$I := \int_0^{\infty} \exp(-\alpha\xi^{\uparrow}_s)\mathrm{d}s.$$

It is interesting to point out that the variable I is self-decomposable, that is, for every $0 < c < 1$, there exists a variable J_c which is independent of I and such that $J_c + cI$ has the same law as I (see, for example, Sato 1999). Specifically, according to Corollary VII.19 in Bertoin (1996), the process shifted by the last passage time $\lambda(-(1/\alpha)\log c)$,

$$\xi'_s := \xi^{\uparrow}_{s+\lambda(-(1/\alpha)\log c)} - \log c^{-1/\alpha}, \qquad s \geqslant 0,$$

is distributed as $\xi^{\uparrow}$ and is independent of $(\xi^{\uparrow}_s, 0 \leqslant s \leqslant \lambda(-(1/\alpha)\log c))$. As we can re-express I in the form

$$I = \int_0^{\lambda(-(1/\alpha)\log c)} \exp(-\alpha\xi^{\uparrow}_s)\mathrm{d}s + c\int_0^{\infty} \exp(-\alpha\xi'_s)\mathrm{d}s,$$

the self-decomposability follows.

One can also calculate the moments of I using the expression for the potential density of $\xi^{\uparrow}$ (see Bertoin 1996, p. 206) and a standard formula for the moments of additive functionals of Markov processes (see, for example, formula (3.4) in Fitzsimmons and Getoor 1992). For instance, elementary computations show that the first three moments are given by

$$\mathbb{E}(I) = \frac{1}{\Psi(\alpha)},$$

$$\mathbb{E}(I^2) = 2\left(\frac{1}{\Psi(\alpha)^2} - \frac{1}{\Psi(\alpha)\Psi(2\alpha)}\right),$$

$$\mathbb{E}(I^3) = 6\left(\frac{1}{\Psi(\alpha)^3} - \frac{2}{\Psi(\alpha)^2\Psi(2\alpha)} + \frac{1}{\Psi(\alpha)\Psi(2\alpha)\Psi(3\alpha)}\right),$$

where Ψ is the Laplace exponent of ξ. The formulae for higher moments become increasingly involved. Plainly, these calculations are relevant to the law of the supremum $S_t = \sup\{X_s, 0 \leqslant s \leqslant t\}$ under $\mathbb{P}_{0+}$; however, it is not clear whether explicit results can be derived for the expression of the entrance law.

Acknowledgement

This work was done during a visit of J.B. to the Instituto de Matemáticas at UNAM, with the partial support of the project PAPIIT-TN115799, which is gratefully acknowledged. We are also grateful to Frédérique Petit for discussions and references about the question that motivated this paper. Finally, we should like to thank an anonymous referee for very careful reading and helpful comments.

References

Bertoin, J. (1996) *Lévy Processes*. Cambridge: Cambridge University Press.

Bertoin, J. (2001) Self-similar fragmentations. To appear in *Ann. Inst. H. Poincaré*.

Bertoin, J., van Harn, K. and Steutel, F.W. (1999) Renewal theory and level passage by subordinators. *Statist. Probab. Lett.*, **45**, 65–69.

Bertoin, J. and Yor, M. (2001) The entrance laws of self-similar Markov processes and exponential functionals of Lévy processes. Preprint.

Brennan, M.D. and Durrett, R. (1987) Splitting intervals. II. Limit laws for lengths. *Probab. Theory Related Fields*, **75**, 109–127.

Carmona, P., Petit, F. and Yor, M. (1994) Sur les fonctionnelles exponentielles de certains processus de Lévy. *Stochastics Stochastics Rep.*, **47**, 71–101.

Carmona, P., Petit, F. and Yor, M. (1997) On the distribution and asymptotic results for exponential functionals of Lévy processes. In M. Yor (ed.), *Exponential Functionals and Prinicipal Values Related to Brownian Motion*, Bibl. Rev. Mat. Iber. Madrid: Biblioteca de la Revista Matemática Iberoamericana.

Fitzsimmons, P.J. and Getoor, R.K. (1992) On the distribution of the Hilbert transform of the local time of a symmetric Lévy process. *Ann. Probab.*, **20**, 1484–1497.

Lamperti, J.W. (1962) Semi-stable stochastic processes. *Trans. Amer. Math. Soc.*, **104**, 62–78.

Lamperti, J.W. (1972) Semi-stable Markov processes. *Z. Wahrscheinlichkeitstheorie Verw. Geb.*, **22**, 205–225.

Sato, K.I. (1999) *Lévy Processes and Infinitely Divisible Distributions*. Cambridge: Cambridge University Press.

Vuolle-Apiala, J. (1994) Itô excursion theory for self-similar Markov processes. *Ann. Probab.*, **22**, 546–565.

Received December 2000 and revised September 2001