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Making Markov martingales meet marginals: with explicit constructions

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We present three generic constructions of martingales that all have the Markov property with known and prespecified marginal densities. These constructions are further investigated for the special case when the prespecified marginals satisfy the scaling property and hence the only datum needed for the construction is the density at unit time. Interesting relations with stochastic orders are presented, along with numerous examples of the resulting martingales.

Keywords: Azéma martingale; convex order; self-similar process; Skorohod embedding

1. Introduction

The role of martingales in the study of stochastic processes, and more generally, probability theory, cannot be overemphasized (see Williams 1991). Mathematical finance, in particular, recognizes martingales as central to the description of economic uncertainty. This paper studies the construction of martingales from a novel perspective motivated by questions arising in the markets for financial derivatives. The more traditional perspective, taken for example in the structure of martingale representation theorems, is to describe all the martingales on a certain underlying stochastic basis. Financial markets trading derivatives, however, identify through option prices the marginal densities of the stochastic process at various – and in principle all future – time points. The underlying stochastic basis is unknown. Conditions of no arbitrage in markets lead us to enquire into the structure of martingales consistent with a prespecified set of marginal densities.

It is useful in the first instance, from both an analytical and a practical perspective, to restrict attention to martingales with the Markov property. Hence, we describe the construction of Markov martingales with fixed marginals. Three distinct solutions are proposed at a general level. This is followed by a discussion of further examples and results when the fixed marginals also satisfy the scaling property. For abstract results on the existence of general and Markov solutions to the problem of matching prespecified marginals, we refer to Strassen (1965), Doob (1968) and Kellerer (1972).

The three constructions developed here are as follows. First, we consider the possibility of obtaining the martingales from a solution of the Skorohod embedding problem as

1350–7265

proposed by Azéma and Yor (1979). The second solution exploits the representation of self-decomposable laws as the unit time densities of self-similar processes by Sato (1991) to construct inhomogeneous Markov martingale processes with independent increments using subordinated Brownian motion. The third solution constructs continuous martingales following methods related to Dupire (1994).

The outline of the paper is as follows. Section 2 presents all constructions at a general level, along with conditions under which they may be implemented. Section 3 considers the constructions under the further hypothesis of fixed marginals that satisfy the scaling property. Section 4 presents a number of examples that have some theoretical or financial importance. Section 5 gives a brief summary.

2. Marginals and Markov martingales

We begin with the marginal densities for a stochastic process that we wish to construct. Suppose the density at time t is given by $g(y, t)$, for $y \in \mathbb{R}$, where $\mathbb{R}$ denotes the real line. We suppose throughout this section that

$$\int |y| g(y, t) \mathrm{d}y < \infty,$$

$$\int y g(y, t) \mathrm{d}y = 0,$$

and hence that the densities are candidates for a martingale begining at zero. From the perspective of the applications we have in mind, we shall primarily be concerned with cases where the densities $g(y, t)$ are strictly positive on the real line for all t. We term our three constructions the Skorohod embedding, the inhomogeneous process of independent increments and the continuous martingale. Three subsections present the details for these cases.

We first note the relation between the existence of a Markov martingale $X(t)$ matching the marginal densities $g(y, t)$ and the convexity order of the marginal densities across maturity. From the martingale property we may infer that for each convex function $\phi(y)$ we must have that, for $s < t$,

$$\mathrm{E}_s[\phi(X(t))] \geqslant \phi(X(s)),$$

where we denote the conditional expectation of $\phi(X(t))$ given information at time s, $\mathcal{T}_s$, by $\mathrm{E}_s[\phi(X(t)]$. Hence, it follows that

$$\mathrm{E}[\phi(X(t))] \geqslant \mathrm{E}[\phi(X(s))],$$

which is equivalent to the density at time s being less than or equal to the density at time t in the convex order (Shaked and Shantikumar (1994), page 55).

Conversely, the following result summarizes studies on this subject prior to 1972.

Theorem 1. *Let $p(y, t)$ be a family of marginal densities, with finite first moment, such that for $s < t$ the density at time t dominates the density at time s in the convex order. Then there* *exists a Markov process $X(t)$ with these marginal densities under which $X(t)$ is a submartingale. Furthermore, if the means are independent of t then $X(t)$ is a martingale.*

Proof. See Kellerer (1972, p. 120). □

Our objective here is to present constructive versions of this result for a number of cases. These constructions are made under more stringent conditions than monotonicity in the convex order.

2.1. Skorohod embedding

Our starting point is the solution to the Skorohod embedding problem that was presented in Azéma and Yor (1979). Here one is given a prespecified probability measure on $\mathbb{R}$, say $\mu(\mathrm{d}y)$, such that $\int |y|\mu(\mathrm{d}y) < \infty$ and $\int y\mu(\mathrm{d}y) = 0$, and we seek to construct a stopping time τ for standard Brownian motion $B(t)$, such that the probability distribution of the stopped random variable $X = B(\tau)$ is given by the measure μ on $\mathbb{R}$. Azéma and Yor (1979) show how one may construct such a stopping time. We first construct the barycentre function $\psi(x)$ defined by

$$\psi(x) = \frac{\displaystyle\int_x^\infty y\mu(\mathrm{d}y)}{\displaystyle\int_x^\infty \mu(\mathrm{d}y)}. \tag{1}$$

We observe that $\psi(x)$ is a positive increasing function that tends to zero as x tends to $-\infty$. Furthermore, $\psi(x) \geqslant x$.

To construct the stopping time τ we simultaneously run the Brownian motion $B(t)$ and its maximum to date $M(t)$, where

$$M(t) = \sup_{0 \leqslant s \leqslant t} B(s),$$

and define τ as the first time $M(t)$ climbs up to the level $\psi(B(t))$. Specifically, we have that

$$\tau = \inf\{s | M(s) \geqslant \psi(B(s))\}.$$

It is instructive to see a graph of the determination of τ from the barycentre function $\psi(x)$, and this is presented in Figure 1.

In Figure 1 we show on the horizontal axis the level of the Brownian motion while the vertical axis records the level of the maximum to date. The horizontal lines indicate the level of the maximum to date as it rises through time, with the stopping time defined as the first time the horizontal lines touch the barycentre curve. In Figure 1 the barycentre curve shown corresponds to a standard normal variate.

This solution to the Skorohod embedding problem may be used to construct martingales with prespecified marginals as follows. For a recent paper that matches a finite set of marginals we refer to Brown *et al*. (2001). Let ψ_t be the barycentre function associated

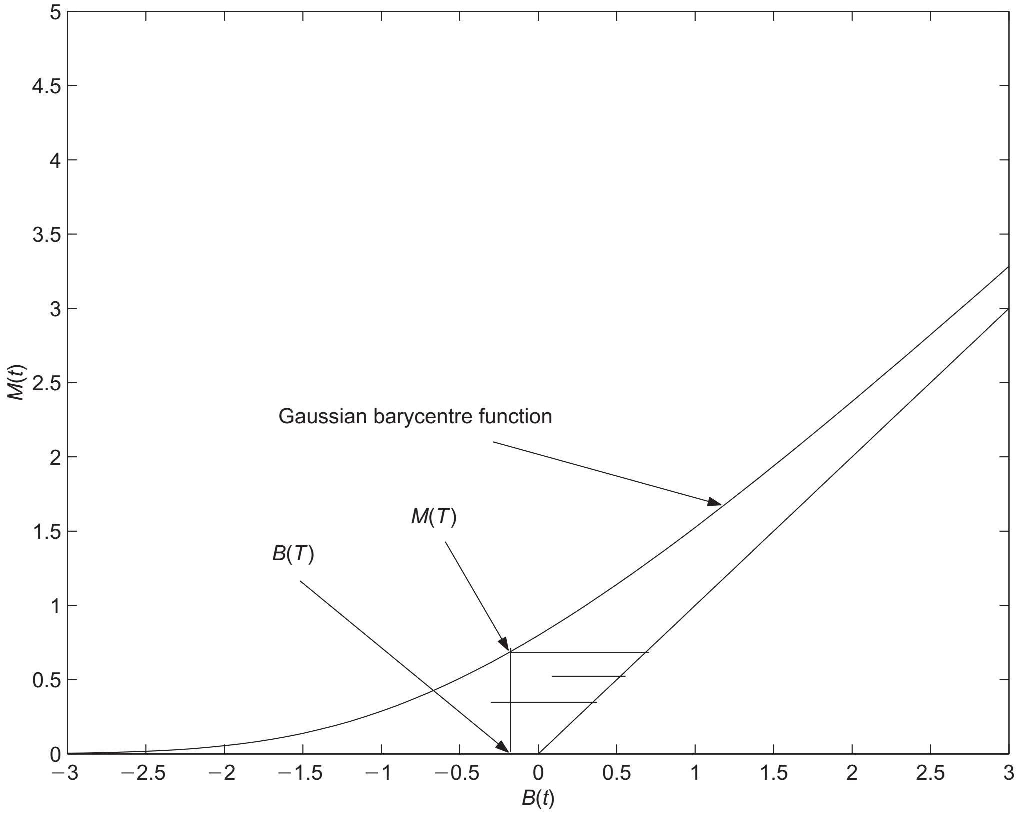

Figure 1. The Azéma–Yor solution to Skorohod's problem for the standard normal distribution.

with the probability $\mu_t(\mathrm{d}y) = g(y, t)\mathrm{d}y$ at time t. Specifically, we define the family of barycentre functions

$$\psi(x, t) = \frac{\int_x^\infty yg(y, t)\mathrm{d}y}{\int_x^\infty g(y, t)\mathrm{d}y}, \tag{2}$$

which we shall sometimes write as $\psi_t(x)$.

Our Skorohod embedding solution constructs a martingale with the specified marginals under the further assumption that $\psi(x, t)$ is increasing in t for each x. This is equivalent to the statement that the random variables through time are ordered by the mean residual life order so that they are increasing in t (Shaked and Shantikumar 1994, p. 43). We say that a family of zero expectation densities has the property of *increasing mean residual value* (IMRV) precisely if the barycentre functions are increasing in t. The IMRV property is stronger than convexity order, as shown in Theorems 3.A.13 and 3.A.16(a) of Shaked and Shantikumar (1994, pp. 91–93).

Theorem 2. *Under the IMRV property for a family of zero mean densities $g(y, t)$ on the real line, let $(B(u), u \geqslant 0)$ be a standard Brownian motion. Then there exists an increasing family of Brownian stopping times $(T_t, t \geqslant 0)$ such that:*

(i) $\beta_t \stackrel{\text{def}}{=} B(T_t)$ *is a martingale.*
(ii) $(\beta_t, t \geqslant 0)$ *is an inhomogeneous Markov process.*
(iii) For each t, the density of β_t *is given by* $g(y, t)$.

Proof. Define T_t by

$$T_t = \inf\{s | M(s) \geqslant \psi(B(s), t)\},$$

where $M(t) = \sup_{0 \leqslant s \leqslant t} B(s)$. It follows from the Azéma and Yor (1979) solution to the Skorohod embedding problem that the law of β_t is $g(y, t)\mathrm{d}y$ for each t. Hence, property (iii) holds.

From the IMRV property we observe from Figure 1 that $T_t \leqslant T_s$ for $t < s$. It follows that β_t is a martingale, i.e.

$$B(T_t) = \mathrm{E}[B(T_s)|\mathcal{T}_{T_t}].$$

Hence, property (i) holds.

For the Markov property we note that, for $t < s$,

$$\begin{aligned} T_s &= \inf\{u | M(u) \geqslant \psi(B(u), s)\} \\ &= T_t + \inf\{v | M(T_t + v) \geqslant \psi(B(T_t + v), s)\} \\ &= T_t + \inf\left\{v \middle| M(T_t) \vee \left(\sup_{T_t \leqslant h \leqslant T_t + v} B(h)\right) \geqslant \psi(B(v), s)\right\} \\ &= T_t + \inf\left\{v \middle| M(T_t) \vee \left(B(T_t) + \sup_{0 \leqslant u \leqslant v} \tilde{B}(u)\right) \geqslant \psi(B(T_t) + \tilde{B}(v), s)\right\}, \end{aligned}$$

where $\tilde{B}(u) = B(T_t + u) - B(T_t)$.

Now define

$$\tilde{T}(\sigma, b) = \inf\{v | \sigma \vee (b + \widetilde{M}(v)) \geqslant \psi(b + \tilde{B}(v), s)\}$$

and observe that, for any test function $f(x)$, we have that

$$\mathrm{E}[f(B(T_s))|\mathcal{T}_{T_t}, B(T_t) = b] = \mathrm{E}[f(b + \tilde{B}(\tilde{T}(\psi(b, t), b)))], \tag{3}$$

and hence, that β_t is an inhomogeneous Markov process and (ii) holds. □

The martingale constructed by this procedure is a one-dimensional Markov process, and it is instructive to develop its infinitesimal generator. To identify the infinitesimal generator of the inhomogeneous Markov process β_t we evaluate, for a test function $f(x)$,

$$\mathcal{A}_t(f)(b) = \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=t^+} \mathrm{E}[f(\beta_s)|\beta_t = b].$$

First, we develop further the expression for the conditional expectation on the right-hand side of equation (3). It is useful to consult Figure 2 in this respect, where we graph the possibilities for $t = 2$ and $s = 6$ when $\psi(x, u)$ is derived from the Gaussian density of mean zero and variance u. As shown in Figure 2, there are two cases to be distinguished. We define

$$d_{T(t)} = \inf\{v > T(t)|M(v) = B(v)\}.$$

In the first case $T(s) < d_{T(t)}$ and Brownian motion has not yet returned to the maximum to date at time t. In this case $M(T(s)) = M(T(t)) = c$, as shown in Figure 2. In the second

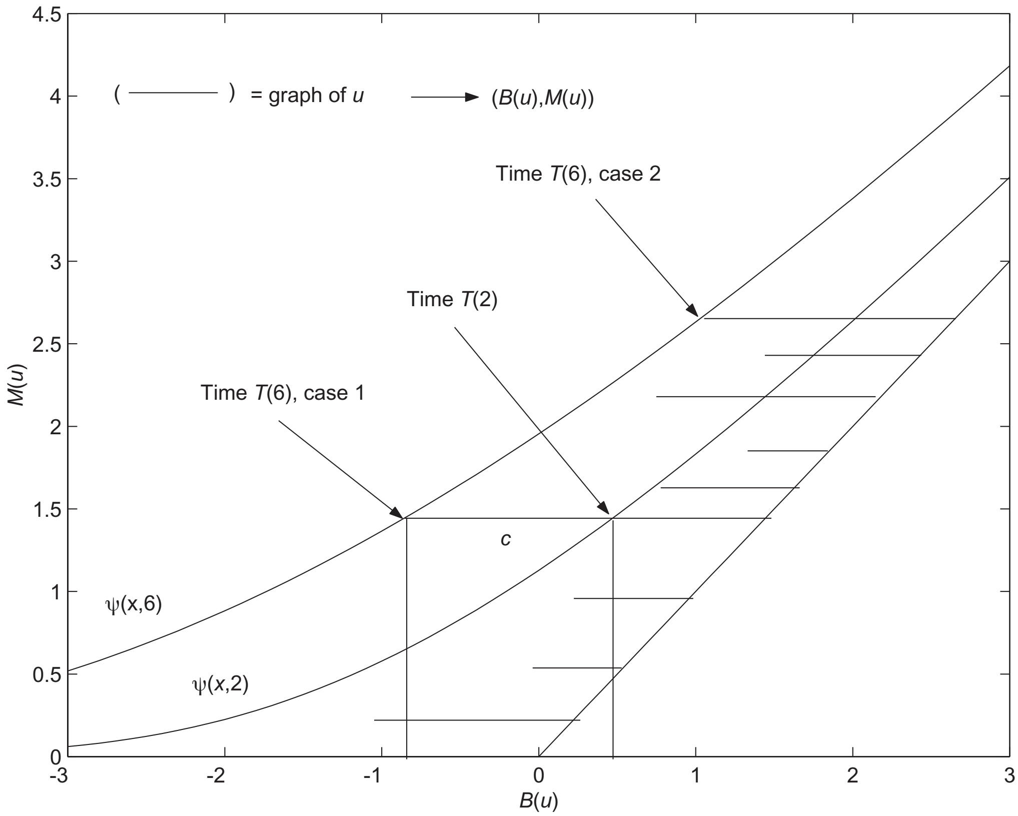

Figure 2. Graph showing the two cases for a later stopping time relative to an earlier stopping time.

case $M(T(s))$ exceeds $c = M(T(t))$ and the maximum to date has climbed higher than it was at time $T(t)$. For the first case we have that

$$\psi(B(T(s), s) = c$$

or

$$B(T(s)) = \psi_s^{-1}(c).$$

Therefore we may write

$$\begin{aligned}
&\mathrm{E}[f(B(T(s)))|\mathcal{T}_{T(t)};\ B(T(t)) = b] \\
&= \mathrm{E}[f(B(T(s)))\mathbf{1}_{T(s)<d_{T(t)}}|\mathcal{T}_{T(t)};\ B(T(t)) = b] + \mathrm{E}[f(B(T(s)))\mathbf{1}_{T(s)>d_{T(t)}}|\mathcal{T}_{T(t)};\ B(T(t)) = b] \\
&= f(\psi_s^{-1}(c))P[T(s) < d_{T(t)}|\mathcal{T}_{T(t)};\ B(T(t)) = b] + \mathrm{E}[f(B(T(s)))\mathbf{1}_{T(s)>d_{T(t)}}|\mathcal{T}_{T(t)};\ B(T(t)) = b]
\end{aligned}$$

We now observe that

$$\begin{aligned}
(T(s) < d_{T(t)}) &= \Big(\inf_{T(t)<u<d_{T(t)}} B(u) \leqslant \psi_s^{-1}(c)\Big) \\
&\equiv (\theta_{b,\psi_s^{-1}(c)} < \theta_{b,c}),
\end{aligned}$$

where we define

$$\theta_{b,y} = \inf\{t|B^b(t) = y\}$$

and $B^b(t)$ is Brownian motion starting at b. Thus we have that

$$\begin{aligned}
&\mathrm{E}[f(B(T(s)))|\mathcal{T}_{T(t)};\ B(T(t)) = b] \\
&\qquad = f(\psi_s^{-1}(c))P\big(\theta_{b,\psi_s^{-1}(c)} < \theta_{b,c}) + \mathrm{E}[f(B(T(s)))\mathbf{1}_{T(s)>d_{T(t)}}|\mathcal{T}_{T(t)};\ B(T(t)) = b].
\end{aligned}$$

From classical results on Brownian motion we know that

$$P(\theta_{b,\psi_s^{-1}(c)} < \theta_{b,c}) = \frac{c-b}{c-\psi_s^{-1}(c)}.$$

We now define

$$\Phi_{t,s}(b) = \psi_s^{-1}(\psi_t(b))$$

and observe that

$$\begin{aligned}
&\mathrm{E}[f(B(T(s)))|\mathcal{T}_{T(t)};\ B(T(t)) = b] \\
&= f(\Phi_{t,s}(b))\frac{\psi_t(b) - b}{\psi_t(b) - \Phi_{t,s}(b)} + \frac{b - \Phi_{t,s}(b)}{\psi_t(b) - \Phi_{t,s}(b)}\mathrm{E}\big[f(B(T(s)))|\mathcal{T}_{T(t)};\ B(T(t)) = b;\ T(s) > d_{T(t)}].
\end{aligned}$$

Now, conditional on $T(s) > d_{T(t)}$, we may wait till time $d_{T(t)}$ when Brownian motion is at the level c. We now consider $\hat{B}(u) = B(u) - c$, starting at 0 at $d_{T(t)}$, and we are interested in the law of when its maximum to date $\hat{M}(u)$ will reach the level of a new ψ function defined by

$$\psi(x) = (\psi(x+c,\, s) - c)^{+}. \tag{4}$$

This is the ψ function of a measure $\rho(x)\mathrm{d}x$ defined by

$$\bar{\rho}(x) = \exp\left(-\int_{-\infty}^{x} \frac{\mathrm{d}\psi(u)}{\psi(u) - u}\right),$$

where $\bar{\rho}(x) = \int_x^\infty \rho(y)\mathrm{d}y$. In our case this new ψ function is given by equation (4) and we have that $\mathrm{d}\psi$ is zero for $(x+c) < \psi_s^{-1}(c)$. Hence, for $x + c > \psi_s^{-1}(c)$, we have that

$$\bar{\rho}(x) = \exp\left(-\int_{\psi_s^{-1}(c)}^{x+c} \frac{\psi_s'(u)}{\psi_s(u) - u}\,\mathrm{d}u\right).$$

But we know that the original density at time s satisfies

$$\bar{\rho}_s(x) = \exp\left(-\int_{-\infty}^{x} \frac{\psi_s'(u)}{\psi_s(u) - u}\,\mathrm{d}u\right).$$

It follows that

$$\bar{\rho}(x) = \frac{\bar{\rho}_s(x+c)}{\bar{\rho}_s(\psi_s^{-1}(c))}$$

and the probability measure of the $B(T(s))$ conditioned on $T(s) > d_{T(t)}$ in the second case is just the original measure conditioned to be above $\psi_s^{-1}(c)$.

We may then write, in summary, that

$$\mathrm{E}[f(B(T(s)))|\mathcal{T}_{T(t)};\, B(T(t)) = b]$$

$$= \alpha f(\psi_s^{-1}(c)) + (1-\alpha)\frac{\displaystyle\int_{\psi_s^{-1}(c)}^{\infty} f(y)g(y,\, s)\mathrm{d}y}{\displaystyle\int_{\psi_s^{-1}(c)}^{\infty} g(y,\, s)\mathrm{d}y},$$

where $\alpha = (c-b)/(c - \psi_s^{-1}(c))$. As a particular case we note that

$$\begin{aligned}\mathrm{E}[B(T(s))|B(T(t)) = b] &= \alpha\psi_s^{-1}(c) + (1-\alpha)c \\ &= \frac{c-b}{c-\psi_s^{-1}(c)}\psi_s^{-1}(c) + \frac{b-\psi_s^{-1}(c)}{c-\psi_s^{-1}(c)}c \\ &= b = B(T(t)),\end{aligned}$$

as expected.

For the infinitesimal generator we seek to evaluate

$$
\begin{aligned}
\mathcal{A}_t(f)(b) &= \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=t^+} (\mathrm{E}[f(\beta_s)|\beta_t = b]) \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=t^+} \left\{ f(\psi_s^{-1}(\psi_t(b))) \frac{\psi_t(b) - b}{\psi_t(b) - \psi_s^{-1}(\psi_t(b))} \right\} \\
&\quad + \left.\frac{d}{\mathrm{d}s}\right|_{s=t^+} \left\{ \frac{b - \psi_s^{-1}(\psi_t(b))}{\psi_t(b) - \psi_s^{-1}(\psi_t(b))} \frac{\int_{\psi_s^{-1}(\psi_t(b))}^{\infty} g(y, s) f(y) \mathrm{d}y}{\int_{\psi_s^{-1}(\psi_t(b))}^{\infty} g(y, s) \mathrm{d}y} \right\}.
\end{aligned}
$$

Computing the derivatives, we obtain that

$$
\mathcal{A}_t(f)(b) = \left\{ f'(b) - \frac{1}{\psi_t(b) - b} \frac{\int_b^{\infty} g(y, t)(f(y) - f(b)) \mathrm{d}y}{\int_b^{\infty} g(y, t) \mathrm{d}y} \right\} \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=t^+} \psi_s^{-1}(\psi_t(b)).
$$

We may further evaluate the final derivative and write

$$
-\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=t^+} \psi_s^{-1}(\psi_t(b)) = \frac{\frac{\partial}{\partial t}\psi(b, t)}{\frac{\partial}{\partial x}\psi(b, t)} = a_t(b), \tag{5}
$$

and then we can write that

$$
\mathcal{A}_t(f)(b) = a_t(b) \left\{ \frac{1}{\psi_t(b) - b} \frac{\int_b^{\infty} g(y, t)(f(y) - f(b)) \mathrm{d}y}{\int_b^{\infty} g(y, t) \mathrm{d}y} - f'(b) \right\}. \tag{6}
$$

By writing $f(y) - f(b)$ in terms of $f''(y)$ and performing the integrations involved, we obtain the simple expression

$$
\mathcal{A}_t(f)(b) = \frac{\frac{\partial}{\partial t}\psi(b, t)}{\frac{\partial}{\partial x}\psi(b, t)} \frac{\int_b^{\infty} f''(v) \mathrm{E}(X_t - v)^+ \mathrm{d}v}{\mathrm{E}(X_t - b)^+}. \tag{7}
$$

From expression (6) we see that the process is a one-sided jump process with jump intensities given by

$$
\frac{a_t(b)}{\psi_t(b) - b} \frac{g(b + x, t)}{\int_0^{\infty} g(b + x, t) \mathrm{d}x}, \qquad \text{for } x > 0
$$

and a drift factor of $-a_t(b)$.

Other more symmetric solutions to the Skorohod embedding problem may also be used along similar lines to form martingales with fixed marginals. We refer the reader to Vallois (1982) and Perkins (1986) for further details on these constructions.

2.2. Inhomogeneous independent increments

A large class of processes with independent increments may be associated with the self-decomposable laws or the distributions of class L (see Khinchine 1938; Gnedenko and Kolmogorov 1968). Sato (1991) has linked these probability laws to the time 1 distributions of self-similar processes with independent increments. See also Jeanblanc *et al*. (2001) for further discussions of various processes attached to self-decomposable laws. These laws may be recognized by the Lévy-Khinchine decomposition of their characteristic functions. In particular, their Lévy densities $k(x)$, when multiplied by the absolute value of x, must be increasing for negative x and decreasing for positive x (Sato 1999, Corollary 15.1, p. 95). For the construction of martingales we focus attention on subordinating Brownian motion by an independent increasing Markov process with independent increments. Specifically, in this approach we seek an increasing Markov process with inhomogeneous independent increments, say $L(t)$, such that the process

$$X(t) = B(L(t))$$

has the requisite marginals, where $B(u)$ is a Brownian motion independent of $(L(t),\ t \geqslant 0)$. For this approach to be successful we may identify the Laplace transform of $L(t)$ by noting that

$$\mathrm{E}[\mathrm{e}^{\mathrm{i}uX(t)}] = \mathrm{E}\left[\exp\left(\frac{-u^2}{2}L(t)\right)\right] = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}uy} g(y,\ t)\mathrm{d}y.$$

The infinitesimal generator for the resulting process is easily identified from the Laplace transform for $L(t)$ written in its infinitely divisible form as

$$\mathrm{E}[\exp(-\lambda L(t))] = \exp\left(\int_0^t\int_0^\infty (\mathrm{e}^{-\lambda x} - 1)k_L(x,\ u)\mathrm{d}x\,\mathrm{d}u\right).$$

In particular, we have from Sato's (1999) Theorems 30.1 and 31.5 that

$$\mathcal{A}_t(f)(x) = \int_{-\infty}^{\infty} (f(x+y) - f(x) - \mathbf{1}_{|y|\leqslant 1} y f'(x))k_X(y,\ t)\mathrm{d}y,$$

where

$$k_X(x,\ t) = \int_0^\infty \frac{1}{\sqrt{2\pi s}} \exp\left(-\frac{x^2}{2s}\right) k_L(s,\ t)\mathrm{d}s.$$

2.3. Continuous martingale constructions

This is the approach of Dupire (1994) and follows on noting that, for a continuous Markov martingale defined by

$$X(t) = \int_0^t \sigma(X(s), s)\mathrm{d}W(s),$$

the forward transition densities $g(y, t)$ satisfy the forward equation

$$\frac{\partial}{\partial t} g(y, t) = \frac{1}{2} \frac{\partial^2}{\partial y^2} (\sigma^2(y, t) g(y, t)).$$

Defining the function

$$C(k, t) = \int_k^\infty (y - k) g(y, t)\mathrm{d}y,$$

one may show by some elementary calculations that

$$\sigma^2(k, t) = \frac{2C_t}{C_{kk}}. \tag{8}$$

Hence, provided the resulting diffusion coefficients computed in accordance with (8) are Lipschitz, we have a continuous martingale representation with the prespecified marginals.

3. Markov martingales for scaling marginals

We now suppose that the prespecified densities scale in the following way. Specifically, we assume that

$$X(t) \overset{\text{law}}{=} \sqrt{t} X(1).$$

Note that other scaling laws may be easily accomodated by deterministic time changes.

It follows that the densities are explicitly given in terms of the density $h(y)$ at unit time by

$$g(y, t) = \frac{1}{\sqrt{t}} h\left(\frac{y}{\sqrt{t}}\right), \tag{9}$$

and hypotheses for the success of the constructions reduce to assumptions on the single density $h(y)$.

3.1. Azéma–Yor under scaling

For the densities to satisfy the IMRV property, we now need that

$$\psi_t(x) = \frac{\int_x^\infty y h(y/\sqrt{t})\mathrm{d}y}{\int_x^\infty h(y/\sqrt{t})\mathrm{d}y} = \frac{\sqrt{t}\int_{x/\sqrt{t}}^\infty u h(u)\mathrm{d}u}{\int_{x/\sqrt{t}} h(u)\mathrm{d}u}$$

be increasing in t. The following lemma gives a useful criterion.

Lemma 3. *The functions ψ_t are increasing with respect to t if and only if*

$$\text{the function } \frac{a}{\psi_1(a)} \text{ is increasing in } a \in \mathbb{R}^+. \tag{10}$$

Proof. Writing

$$\frac{1}{x}\psi_t(x) = \frac{\psi_1(a)}{a},$$

where $a = x/\sqrt{t}$, it follows that the functions ψ_t are increasing in t if and only if the function

$$a \to \frac{a}{\psi_1(a)}$$

is increasing over the entire real line. But it is elementary that $a/\psi_1(a)$ is increasing for $a < 0$, since ψ_1 is increasing and non-negative. □

The IMRV condition is satisfied by an important class of densities of which there are numerous examples.

Theorem 4. *If $h(y) = \exp(-V(y))$ and $yV'(y)$ is increasing in $y > 0$, then $h(y)$ admits IMRV under scaling.*

Proof. We write for, $a > 0$ and $\overline{H}(y) = \int_y^\infty h(u)\mathrm{d}u$, that

$$\begin{aligned}
\frac{\psi_1(a)}{a} &= \frac{1}{a\overline{H}(a)}\int_a^\infty y(-\mathrm{d}\overline{H}(y)) \\
&= 1 + \frac{1}{a\overline{H}(a)}\int_a^\infty \overline{H}(y)\mathrm{d}y \\
&= 1 + \int_1^\infty \frac{\overline{H}(au)}{\overline{H}(a)}\mathrm{d}u.
\end{aligned}$$

Hence, it suffices that

$$\gamma_u(y) \stackrel{\text{def}}{=} \frac{\overline{H}(yu)}{\overline{H}(y)}$$

is decreasing in $y > 0$, for fixed $u \geqslant 1$. We evaluate

$$\frac{\partial}{\partial y}\gamma_u(y) = \frac{u\displaystyle\int_y^\infty (h(y)h(xu) - h(yu)h(x))\mathrm{d}x}{(\overline{H}(y))^2}.$$

For this derivative to be negative for $y > 0$, it suffices that the integrand in the numerator is negative or, in terms of V, that

$$V(xu) - V(x) > V(yu) - V(y).$$

We now write this inequality as

$$\int_x^{xu} V'(\alpha)\mathrm{d}\alpha > \int_y^{yu} V'(\alpha)\mathrm{d}\alpha$$

and, making the change of variables $\alpha = xk$ on the left and $\alpha = yk$ on the right, we have that

$$\int_1^u V'(xk)xk\frac{\mathrm{d}k}{k} > \int_1^u V'(yk)yk\frac{\mathrm{d}k}{k}.$$

The result follows from the fact that $yV'(y)$ is increasing for $y > 0$. □

The class of densities permitting Skorohod embedding using the Azéma–Yor construction after scaling is quite broad and includes the collection of log-concave densities with the additional sufficient provision that $V'(y)$ be positive for positive y. The log-concave densities are an important class of densities constituting the class of strongly unimodal densities (Sato 1999, p. 395).

A further large class of possibilities for this construction is made available by the following result. Suppose that $h(y)$ is written as the Laplace transform in $y^2/2$ of a probability on $\mathbb{R}^+$ for a random variable we shall call T, i.e.

$$h(y) = CE\left[\exp\left(-\frac{y^2}{2}T\right)\right]. \tag{11}$$

The following result gives us a large class of densities that admit the scaled Skorohod embedding constructions.

Theorem 5. *Suppose that* (11) *holds for T infinitely divisible, with*

$$\mathrm{E}\left[\exp\left(-\frac{x^2 T}{2}\right)\right] = \exp\left(-\int_0^\infty \mu(\mathrm{d}t)(1 - \mathrm{e}^{-x^2 t/2}\right).$$

Assume, further, that T is self-decomposable or, equivalently, that

$$\mu(\mathrm{d}t) = \frac{m(t)}{t}\,\mathrm{d}t$$

with $m(t)$ decreasing. Then $yV'(y)$ is increasing in $y > 0$, where

$$V(y) = \int_0^\infty \frac{\mathrm{d}t}{t}\,m(t)(1 - \mathrm{e}^{-y^2 t/2}).$$

Proof. Note that

$$
\begin{aligned}
yV'(y) &= \int_0^\infty \mathrm{d}t m(t) y^2 \,\mathrm{e}^{-y^2 t/2} \\
&= \int_0^\infty \mathrm{d}u m\left(\frac{u}{y^2}\right) \mathrm{e}^{-u/2}.
\end{aligned}
$$

Hence $yV'(y)$ is increasing in $y > 0$. □

3.2. Inhomogeneous process of independent increments under scaling

We now seek to construct martingales using subordination as

$$X(t) = B(L(t)) \tag{12}$$

where $(B(u))$ is a Brownian motion, $L(t)$ is an independent increasing process with independent but generally inhomogeneous increments that has the scaling property

$$L(ct) \stackrel{\text{law}}{=} cL(t), \qquad t > 0. \tag{13}$$

It follows immediately from (12) that

$$h(y) = \mathrm{E}\left[\frac{1}{\sqrt{2\pi L_1}} \exp\left(-\frac{y^2}{2L_1}\right)\right]. \tag{14}$$

Theorem 6. *Let $L_1 \geqslant 0$. Then the following three properties are equivalent:*

(*i*) *There exists an increasing process with independent increments $(L_t, t \geqslant 0)$ which satisfies* (13).
(*ii*) *L_1 is self-decomposable.*
(*iii*) *The Laplace transform of L_1 is given by*

$$\mathrm{E}[\exp(-\lambda L_1)] = \exp\left(-\int_0^\infty \left(1 - \mathrm{e}^{-\lambda l}\right) \nu(\mathrm{d}l)\right)$$

with $\nu(\mathrm{d}l) = \mathrm{d}l(k(l)/l)$ and k is decreasing.

Consequently, there exists a martingale $X(t)$, defined by (12)*, which satisfies* (14).

Proof. See Sato (1991). Here we provide some details for using Kolmogorov's extension theorem and show that, for $s < t$, there exists a non-negative random variable, $L_{s,t}$, such that

$$sL_1 + L_{s,t} \stackrel{\text{law}}{=} tL_1,$$

where on the left-hand side L_1 and $L_{s,t}$ are independent. The Laplace transform for $L_{s,t}$ must then be

$$\begin{aligned}
\mathrm{E}[\exp(-\lambda L_{s,t})] &= \exp\left(-\int_0^\infty \nu(\mathrm{d}l)((1-\mathrm{e}^{-\lambda tl})-(1-\mathrm{e}^{-\lambda sl}))\right)\\
&= \exp\left(-\int_0^\infty \frac{\mathrm{d}l}{l}k(l)((1-\mathrm{e}^{-\lambda tl})-(1-\mathrm{e}^{-\lambda sl}))\right)\\
&= \exp\left(-\int_0^\infty \frac{\mathrm{d}l}{l}\left(k\left(\frac{l}{t}\right)-k\left(\frac{l}{s}\right)\right)(1-\mathrm{e}^{-\lambda l})\right),
\end{aligned}$$

and the hypothesis that k is decreasing implies that the last writen expression is the Laplace transform of a positive (infinitely divisible) random variable. □

3.3. Continuous martingales under scaling

For this purpose we consider the representation

$$X(t) = \int_0^t \sigma(s, X_s)\mathrm{d}W(s). \tag{15}$$

We know by hypothesis that the laws of $X(t)$ satisfy the scaling property in that, for any fixed $c > 0$,

$$(X_{ct},\ t \geqslant 0) \overset{\text{law}}{=} (\sqrt{c}X_t,\ t \geqslant 0); \tag{16}$$

furthermore, we know that equation (9) holds.

We now establish a general result that helps us identify the continuous martingale representation.

Theorem 7. *Assume that a process satisfies the scaling property* (16) *and simultaneously has the representation* (15). *Then we must have that*

$$\sigma^2(s, x) = a\left(\frac{x}{\sqrt{s}}\right), \tag{17}$$

$$a(y) = \frac{1}{h(y)}\int_y^\infty zh(z)\mathrm{d}z. \tag{18}$$

Furthermore, if we have a density h and an associated function $a(y)$ that is Lipschitz, then there exists a continuous martingale satisfying the scaling property (16) *and the Markov property, and for which $\langle X\rangle_t = \int_0^t \mathrm{d}s a(X_s/\sqrt{s})$.*

Proof. Consider first a martingale $X(t)$ satisfying (15) and define the function a by

$$a\left(s, \frac{x}{\sqrt{s}}\right) = \sigma^2(s, x).$$

It follows that

$$\langle X \rangle_t = \int_0^t \mathrm{d}s \ \ a\left(s, \frac{X_s}{\sqrt{s}}\right).$$

Now, for any function $f \in C^2(\mathbb{R})$, with compact support, we have

$$\mathrm{E}[f(X_t)] = f(0) + \int_0^t \mathrm{d}s \frac{1}{2} \mathrm{E}\left[a\left(s, \frac{X_s}{\sqrt{s}}\right) f''(X_s)\right]. \tag{19}$$

Combining (9) and (19), we obtain

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}y}{\sqrt{t}} h\left(\frac{y}{\sqrt{t}}\right) f(y) = f(0) + \int_0^t \mathrm{d}s \int_{-\infty}^{\infty} \frac{\mathrm{d}y}{\sqrt{s}} h\left(\frac{y}{\sqrt{s}}\right) \frac{1}{2} a\left(s, \frac{y}{\sqrt{s}}\right) f''(y).$$

Noting that this equation holds for all f, after integrating by parts twice on the right we learn, on taking partials with respect to t, that

$$\frac{\partial}{\partial t}\left(\frac{1}{\sqrt{t}} h\left(\frac{y}{\sqrt{t}}\right)\right) = \frac{\partial^2}{\partial y^2}\left[\frac{1}{\sqrt{t}} h\left(\frac{y}{\sqrt{t}}\right) \frac{1}{2} a\left(t, \frac{y}{\sqrt{t}}\right)\right];$$

equivalently, we have that

$$-\frac{1}{2t^{3/2}} h\left(\frac{y}{\sqrt{t}}\right) + \frac{1}{\sqrt{t}} h'\left(\frac{y}{\sqrt{t}}\right) y \left(-\frac{1}{2t^{3/2}}\right) = \frac{1}{2t^{3/2}} \frac{\partial^2}{\partial x^2}[h(x)a(t, x)]\bigg|_{x=y/\sqrt{t}}. \tag{20}$$

Then we may write (20) as

$$-\frac{d}{\mathrm{d}x}(xh(x)) = \frac{\partial^2}{\partial x^2}[h(x)a(t, x)].$$

Integrating twice, we obtain

$$-\int_{-\infty}^{x} yh(y)\mathrm{d}y = h(x)a(t, x) + u(t)x + v(t).$$

Since the random variable is centred, we may write

$$\int_{x}^{\infty} yh(y)\mathrm{d}y = h(x)a(t, x) + u(t)x + v(t).$$

We then have the candidate solution $a(t, x)$ independent of t,

$$a(x) = \frac{\displaystyle\int_x^{\infty} yh(y)\mathrm{d}y}{h(x)},$$

as was to be shown.

For the converse, suppose we have h or equivalently a satisfying (18), and define

$$X(t) = \int_0^t \left(a\left(\frac{X_s}{\sqrt{s}}\right)\right)^{1/2} \mathrm{d}W(s), \tag{21}$$

where we suppose that $\sigma^2 = a$ is Lipschitz. Now, following Lamperti (1962), define $Y(u)$ such that

$$X(t) = \sqrt{t}Y(\log(t)) \tag{22}$$

or that

$$Y(u) = \exp\left(-\frac{u}{2}\right)X(\exp(u)).$$

By Itô's lemma one easily observes that if $X(t)$ satisfies scaling and the representation (21) then $Y(t)$ satisfies the following conditions:

(i) For every $u \in \mathbb{R}$, there is a Brownian motion $\{\beta_s^{(u)}, s \geqslant 0\}$ such that

$$Y(u+t) = Y(u) + \int_0^t \sigma(Y(u+s))\mathrm{d}\beta_s^{(u)} - \frac{1}{2}\int_0^t Y(u+s)\mathrm{d}s. \tag{23}$$

(ii) $\{Y(w), w \in \mathbb{R}\}$ is a stationary process, i.e. for every $u \in \mathbb{R}$,

$$(Y(u+t), t \geqslant 0) \overset{\text{law}}{=} (Y(t), t \geqslant 0).$$

(iii) For every $u \in \mathbb{R}$,

$$P(Y(u) \in \mathrm{d}y) = h(y)\mathrm{d}y.$$

Hence the required solution for $X(t)$ is constructed by solving equation (23), for which existence and uniqueness hold for given $Y(u)$, since $\sqrt{a}$ is Hölder of order $\frac{1}{2}$ (see Revuz and Yor 1999, Chapter IX), and using equation (22). □

4. Examples of Markov martingales and their marginals

For many families of marginals it is possible to obtain Markov martingales from all three constructions meeting the specified set of marginals. In this section we provide classes of examples for each of our three constructions. We then consider matching the marginals for the bounded and discontinuous Azéma martingales using continuous martingales.

4.1. Skorohod embedding

We construct a sequence of martingale marginal processes by the Skorohod embedding method for various families of distributions, beginning with some simple cases. Our first example considers the very simple family of distributions that are uniform in the interval $[-t, t]$, and we call this the uniform case. We then take up the shifted exponential distributions of mean t and shifted to start at $-t$ to fulfil the zero mean condition for the family. We call this the exponential case. Our third example is the Gaussian family of mean zero and variance t that has a density satisfying the condition that $yV'(y)$ is increasing for $y > 0$ from Theorem 4. This is followed by other examples in this category.

4.1.1. The uniform case

Suppose the family of densities $g(y, t)$ is given by the uniform density in the interval $[-t, t]$. For these densities the ψ function is easily evaluated as

$$\psi(x, t) = \frac{\int_x^t y \, \mathrm{d}y}{\int_x^t \mathrm{d}y}, \qquad -t < x < t,$$

$$= \frac{t^2 - x^2}{2(t - x)}, \qquad -t < x < t,$$

$$= \frac{t + x}{2}, \qquad -t < x < t.$$

The associated stopping time is given by

$$T(t) = \inf\{s|2M(s) - B(s) = t\}.$$

We recall that $\{2M(s) - B(s)\}$ is Pitman's (1975) representation of the norm of a three-dimensional Brownian motion. Then we write the martingale marginal consistent with these densities as

$$X(t) = B(T(t)).$$

4.1.2. The exponential case

Suppose the family of densities is given by the exponential of mean t, shifted to start at $-t$. We then have that

$$g(y, t) = \frac{1}{t} \exp\left(-\frac{y + t}{t}\right), \qquad -t < y < \infty.$$

Once again we may explicitly determine the ψ functions as

$$\psi(x, t) = \frac{\int_x^\infty y \exp(-y/t) \mathrm{d}y}{\int_x^\infty \exp(-y/t) \mathrm{d}y}$$

$$= \frac{xt \exp(-x/t) + t^2 \exp(-x/t)}{t \exp(-x/t)}$$

$$= x + t,$$

for $x \geqslant t$. Hence, $\psi(x, t) = (x + t)^+$.

The time change is then given by

$$T(t) = \inf\{s|M(s) - B(s) = t\},$$

which features Lévy's representation $\{M(s) - B(s), s \geqslant 0\}$ of the reflecting Brownian motion. Since, by Lévy's equivalence theorem,

$$((M(u) - B(u), M(u)), u \geqslant 0) \overset{\text{law}}{=} ((|B(u)|, L(u)), u \geqslant 0),$$

where $L(u)$ is the local time of B at zero, we observe that

$$(B(T(t)), t \geqslant 0) \overset{\text{law}}{=} (L(T^*(t)) - t), \tag{24}$$

where

$$T^*(t) = \inf\{s||B(s)| = t\}.$$

Moreover, as

$$B(t)^+ = |\beta|_{\int_0^t \mathrm{d}s \mathbf{1}_{B(s)>0}}$$

where β is a real-valued Brownian motion, we deduce that

$$(B(T(t)), t \geqslant 0) \overset{\text{law}}{=} \left(\frac{1}{2} L_{S(t)}(B) - t\right),$$

where $S(t) = \inf\{s|B(s) = t\}$.

The process $(L_{S(t)}(B), t \geqslant 0)$ is an inhomogeneous Lévy process and, for fixed t, $\frac{1}{2}L_{S(t)}(B)$ is exponential with mean t due to our embedding of exponential densities at the outset.

4.1.3. The Gaussian case

Here we consider the case where the density $g(y, t)$ is Gaussian with mean zero and variance t. This is the case relevant for the Bachelier option pricing model when the underlying follows the arithmetic Brownian motion model. In this case we have both the scaling and the log-concavity property for the marginals. The marginals are

$$g(y, t) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{y^2}{2t}\right), \qquad -\infty < y < \infty. \tag{25}$$

Theorem 4 applies trivially with $yV'(y) = y^2$.

The associated ψ function may once again be analytically derived, and is

$$\psi(x, t) = \frac{\sqrt{t}\exp(-x^2/2t)}{\sqrt{2\pi}(1 - N(x/\sqrt{t}))}. \tag{26}$$

We note that $\psi(x, t) = \sqrt{t}/R(x/\sqrt{t})$, where R is the celebrated Mills ratio (see Johnson and Kotz 1970, Chapter 33, Section 7.1), so called on account of its tabulation by Mills (1926). Instead of using Theorem 3 to check that the ψ_t are increasing in t, we may also use Lemma 2, which requires that

$$\frac{a}{\psi_1(a)} = aR(a)$$

be increasing in $a \geqslant 0$, which is demonstrated by the following:

$$\begin{aligned} aR(a) &= a\int_0^\infty \exp\left(-\frac{(a+w)^2}{2}\right)\mathrm{d}w \exp\left(\frac{a^2}{2}\right) \\ &= a\int_0^\infty \exp\left(-\frac{w^2}{2} - wa\right)\mathrm{d}w \\ &= \int_0^\infty \exp\left(-\frac{v^2}{2a^2} - v\right)\mathrm{d}v. \end{aligned}$$

It may be worth noticing that in fact

$$aR(a) = P\left(\frac{e}{\sqrt{2e'}} \leqslant a\right),$$

where e, e' are two independent exponential variables with mean 1.

We may therefore construct the martingale $B(T(t))$, where

$$T(t) = \inf\{s | M(s) = \psi(B(s), t)\}.$$

The inhomogeneous and discontinuous Markov process here has one-sided jumps given by the normal density truncated at the level of the process with a drift that may be computed in closed form using expression (5).

4.1.4. Other examples of scaled densities permitting the Azéma–Yor Skorohod embedding

There are many other examples of densities meeting the conditions of Theorem 3 that may be scaled and then matched using the Skorohod embedding method.

Double negative exponential. We may consider, for example, the important class of double negative exponential densities defined by

$$g(x, t) = \frac{1}{\sigma\sqrt{2t}} \exp\left(-\frac{\sqrt{2}|x|}{\sigma\sqrt{t}}\right), \qquad -\infty < x < \infty. \tag{27}$$

We may easily compute the variance of x and observe that it is $\sigma^2 t$. We also verify by the change of variables $y = x/\sqrt{t}$ that

$$h(y; \sigma) = \frac{1}{\sigma\sqrt{2}} \exp\left(-\frac{\sqrt{2}|y|}{\sigma}\right) = g(y, 1). \tag{28}$$

It is clear that $yV'(y)$ is increasing in $y > 0$. Hence, the densities admit the Skorohod embedding construction by Theorem 4.

Symmetric powers. An interesting generalization that nests the double negative exponential and the Gaussian models is provided by the class

$$h(y) = \kappa \exp(-c|y|^{\alpha}), \qquad \text{for } \alpha > 0.$$

We observe that for positive y the hypothesis of Theorem 4 holds.

Reciprocal hyperbolic cosine. The density

$$h(y) = \frac{1}{\cosh(y)}$$

may be expressed in the form (11) as shown, for example, in Pitman and Yor (2000), where we find that

$$\frac{1}{\cosh(y)} = \mathrm{E}\left[\exp\left(-\frac{y^2}{2}C_1\right)\right]$$

and C_1 may be realized as the first hitting time of ± 1 by Brownian motion. The Lévy density is given by

$$k_C(x) = \frac{\sum_{n=1}^{\infty} \mathrm{e}^{-\pi^2}(n-1/2)^2 x/2}{x},$$

which has the form required in Theorem 5.

Scaled reciprocal hyperbolic sine. This is the density

$$h(y) = \frac{y}{\sinh(y)},$$

which has the form

$$\frac{y}{\sinh(y)} = \mathrm{E}\left[\exp\left(-\frac{y^2}{2}S_1\right)\right],$$

where S_1 may be realized as the hitting time of the unit sphere by three-dimensional Brownian motion. The Lévy density is given by

$$k_S(x) = \frac{\sum_{n=1}^{\infty} \mathrm{e}^{-\pi^2 n^2 x/2}}{x}$$

and has the requisite form of Theorem 5.

4.2. Independent increments

We begin with the case of the double negative exponential. For this purpose consider the process

$$L(a) = \ell_{T(a)},$$

where $\ell(s)$ is the local time at zero of a Brownian motion and $T(a)$ is the first passage time of this Brownian motion to the level a. Hence $L(a)$ is the local time at zero of a Brownian motion up to the first passage time of this Brownian motion to the level a.

It is well known that $L(a)$ is an exponential random variable with mean $2a$ (see Revuz and Yor 1999, Chapter XIII). It follows that

$$\mathrm{E}[\exp(-\lambda L(a))] = \frac{1}{1+2a\lambda}.$$

We may now compute the characteristic function for an independent Brownian motion evaluated at $L(a)$, $Y(a) = B(L(a))$, and observe that

$$\mathrm{E}[\exp(\mathrm{i}uY(a))] = \frac{1}{1+au^2}.$$

Computing the characteristic function of the double negative exponential density (27), we see that

$$\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}ux} g(x,\, 1)\mathrm{d}x = \frac{1}{1+\sigma^2 u^2}.$$

It follows that the double negative exponential laws are matched by the process

$$X(t) = B(L(\sigma^2 t)),$$

and we have the representation by an inhomogeneous Markov martingale with independent increments.

4.2.1. The Student distributions

The Student distributions (see Johnson and Kotz 1970, Chapter 27) have densities defined by

$$h_m(y) = \frac{C_m}{(1+y^2)^m} \tag{29}$$

for $m > 1$ (which implies that $\int |y| h_m(y)\mathrm{d}y < \infty$). These densities may be associated with the independent increments solution as follows. Consider the random variable

$$L_1 = \frac{1}{2\Gamma_\nu},$$

where Γ_ν denotes a gamma variable with parameter ν.

Evaluating a standard Brownian motion at the independent time L_1, we obtain the density of the random variable

$$\begin{aligned} g(y) &= \mathrm{E}\left[\frac{1}{\sqrt{2\pi L_1}}\exp\left(-\frac{y^2}{2L_1}\right)\right] \\ &= \int_0^\infty \frac{1}{\sqrt{\pi}\Gamma(\nu)}\, t^{\nu-1/2}\mathrm{e}^{-(1+y^2)t}\mathrm{d}t \\ &= \frac{\Gamma(\nu+1/2)}{\sqrt{\pi}\Gamma(\nu)}\frac{1}{(1+y^2)^{\nu+1/2}}, \end{aligned}$$

which agrees with (29) for $m=\nu+\frac{1}{2}$ and $C_m=\Gamma(m)/\sqrt{\pi}\Gamma(m-\frac{1}{2})$.

The representation of L_1 as the value at time 1 of a process of inhomogeneous and independent increments is accomplished as follows. We recall that if $(R_t^\nu,\ t\geqslant 0)$ denotes a Bessel process with index $\nu>0$, starting at 0, and if

$$\Lambda_a^{(\nu)}=\sup\{t>0 : R_t^{(\nu)}=a\},$$

then the process $(\Lambda_a^{(\nu)},\ a\geqslant 0)$ (see Getoor 1979, Pitman and Yor 1981; Revuz and Yor 1999) has independent increments, and

$$\Lambda_a^{(\nu)} \overset{\text{law}}{=} \frac{a^2}{2\Gamma_\nu},$$

where Γ_ν is a gamma variable with parameter ν. As a consequence, the process $(B(\Lambda_a^{(\nu)}),\ a\geqslant 0)$ has independent increments and enjoys the scaling property of order 1, and the density at unit time is given by (29).

4.3. Continuous martingales

We present three sets of examples that generalize in sequence. We begin with the double negative exponential densities, then take up the Student distributions and close with a parametric subclass connected with the Pearson families.

4.3.1. Double negative exponential

By Theorem 6 we may determine the function $a(y)$ required in the continuous martingale representation by performing an integration with respect to the function h. The result is given by

$$a(y)=\sigma^2+\sigma|y|.$$

It follows that the continuous martingale which solves

$$X(t)=\int_0^t \sqrt{\sigma^2+\sigma\left|\frac{X_s}{\sqrt{s}}\right|}\,\mathrm{d}W(s)$$

has densities that match the marginals (27).

4.3.2. The Student distributions

In this case we have that

$$a_m(y) = \frac{1}{h_m(y)} \int_y^\infty x h_m(x) \mathrm{d}x,$$

and it follows that

$$a_m(y) = \frac{1}{2(m-1)}(1+y^2)$$

so that the associated continuous martingales solve

$$X(t) = \int_0^t \sqrt{\frac{1}{2(m-1)}\left(1+\frac{X(s)^2}{s}\right)} \mathrm{d}W(s).$$

4.3.3. A general class associated with Pearson densities

More generally, we may solve back for $h(y)$ from Lipschitz candidates for $a(y)$. Specifically, we have from equation (18) that

$$\frac{yh(y)}{\displaystyle\int_y^\infty xh(x)\mathrm{d}x} = \frac{y}{a(y)}, \tag{30}$$

which implies that

$$\int_y^\infty xh(x)\mathrm{d}x = C\exp\left(-\int_0^y \frac{z}{a(z)}\mathrm{d}z\right)$$

or that

$$h(y) = \frac{C}{a(y)} \exp\left(-\int_0^y \frac{z}{a(z)}\mathrm{d}z\right).$$

Consider now the general model for the diffusion coefficient given by

$$a(y) = \alpha + \beta|y| + \gamma y^2.$$

It follows, on noting that

$$\begin{aligned}
\int_0^y \frac{z}{\alpha+\beta z+\gamma z^2}\mathrm{d}z &= \frac{1}{2\gamma}\int_0^y \frac{2\gamma z+\beta-\beta}{\alpha+\beta z+\gamma z^2}\mathrm{d}z \\
&= \frac{1}{2\gamma}\log\left(\frac{\alpha+\beta y+\gamma y^2}{\alpha}\right) - \frac{\beta}{2\gamma}\int_0^y \frac{\mathrm{d}z}{\alpha+\beta z+\gamma z^2} \\
&= \frac{1}{2\gamma}\log\left(\frac{\alpha+\beta y+\gamma y^2}{\alpha}\right) - \frac{\beta}{2\gamma}\,\frac{2\arctan[(\beta+2\gamma y)/\sqrt{4\alpha\gamma-\beta^2}]}{\sqrt{4\alpha\gamma-\beta^2}},
\end{aligned}$$

that one may write

$$h(y) = \frac{C}{\alpha + \beta y + \gamma y^2}\left(\frac{\alpha + \beta y + \gamma y^2}{\alpha}\right)^{1/2\gamma} \exp\left(\frac{\beta}{2\gamma}\, \frac{2\arctan[(\beta + 2\gamma y)/\sqrt{4\alpha\gamma - \beta^2}]}{\sqrt{4\alpha\gamma - \beta^2}}\right).$$

Wong (1964; see also 1971, p. 174) shows that these densities are associated with the Pearson family. We thus have a parametric subfamily with three parameters of the Pearson densities that are matched on scaling by the continuous martingale

$$X(t) = \int_0^t \sqrt{\alpha + \beta\left|\frac{X(s)}{\sqrt{s}}\right| + \gamma\frac{X(s)^2}{s}}\,\mathrm{d}W(s).$$

4.3.4. The scaled generalized hyperbolic laws

The martingale component of the generalized hyperbolic law at time 1 may be written as

$$X(1) = B(L_1(1)),$$

where $L_1(1)$ has the generalized inverse Gaussian distribution. To represent the scaled marginals using a continuous martingale, we determine that

$$a(y) = \frac{\mathrm{E}[\sqrt{L_1}\exp(-y^2/2L_1)]}{\mathrm{E}[1/\sqrt{L_1}\exp(-y^2/2L_1)]}.$$

When L_1 has the $GIG(\lambda, \delta, \gamma)$ distribution (Barndorff-Nielsen 1977) we observe that

$$a(y) = \frac{\sqrt{\delta^2 + y^2}}{\gamma}\, \frac{K_{\lambda+1/2}(\gamma\sqrt{\delta^2 + y^2})}{K_{\lambda+3/2}(\gamma\sqrt{\delta^2 + y^2})}.$$

For a discussion of the applications and special cases of this distribution in a financial context, we refer to Bibby and Sørensen (2001).

4.4. Continuous martingales and the Azéma martingale marginals

The Azéma martingale is an interesting case that in the financial context allows for discontinuities and yet maintains the property of a complete market, as shown by Dritschel and Protter (1999). Furthermore, it is shown in Azéma and Yor (1989) – see also Yor (1997) – that this martingale also satisfies the Wiener chaos representation property for all square-integrable random variables. We consider in this section the task of matching the scaled marginals of this martingale using continuous martingales. This martingale is defined by projecting a Brownian motion onto the filtration $(\mathcal{G}_t)$ generated by the sign of the Brownian motion; note that the process of last zeros to date $g_t = \sup\{s \leqslant t : B(s) = 0\}$ is adapted to $(\mathcal{G}_t)$ (Azéma and Yor 1989; Yor 1997). Explicitly, we may write

$$X(t) = \mathrm{sign}(B(t))\sqrt{t - g(t)},$$

where $B(t)$ is a Brownian motion and

$$g(t) = \sup\{s \leqslant t | B(s) = 0\}.$$

The process $X(t)$ is a Markov martingale that satisfies the scaling property, and the law of $X(1)$ may be deduced from the result that $g(1)$ has the arcsine distribution (see Yor 1992, p. 101, for a short proof). Specifically, we have that the density of $X(1)$ is given by

$$h(x) = \frac{1}{\pi} \frac{1}{\sqrt{1-x^2}}, \qquad -1 < x < 1.$$

From Theorem 7 it follows that

$$a(y) = \sqrt{1-y^2} \int_y^1 \frac{u}{\sqrt{1-u^2}} \mathrm{d}u = 1 - y^2.$$

Hence the continuous martingale

$$Z(t) = \int_0^t \sqrt{\left(1 - \frac{Z(s)^2}{s}\right)^+} \mathrm{d}W(s)$$

for a standard Brownian motion $W(t)$ matches the marginals of the Azéma martingale.

On the other hand, consider circular Brownian motion

$$\{C_u = \exp(\mathrm{i}\gamma_u);\ -\infty < u < \infty\}$$

in equilibrium, where the following conditions hold:

(i) For each u, (C_u) is uniformly distributed on the circle.
(ii) (C_u) satisfies

$$C_{t+s} = C_t + \int_t^{t+s} \mathrm{i}C_u \mathrm{d}\gamma_u - \frac{1}{2}\int_t^{t+s} C_u \mathrm{d}u.$$

Then

$$(Y(u),\ u \in \mathbb{R}) \stackrel{\text{law}}{=} (\mathrm{Im}(C_u),\ u \in \mathbb{R})$$

and finally, our continuous martingale matching Azéma's martingale is

$$Z(t) = \sqrt{t} \sin(\gamma(\log(t))).$$

5. Conclusion

We consider three classes of solutions to the problem of finding one-dimensional Markov martingales that have prespecified marginal densities at all time points. The first method exploits the Azéma–Yor solution to the Skorohod embedding problem. The second subordinates Brownian motion to an independent inhomogeneous Lévy process with independent increments. The third constructs a continuous Markov martingale via a suitable choice of a diffusion coefficient meeting the required Lipschitz conditions.

The conditions for the applicability of the methods are related to stochastic orderings of the densities to be matched. For the embedding solution, for example, the densities must be ordered by mean residual value, a modification of mean residual life. The existence of the martingale is related to the convexity order, and self-decomposability also plays an important role. Precise conditions for each construction under scaling are presented, along with numerous examples of theoretical and financial interest.

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