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FINITE SECTIONS OF SEGAL-BARGMANN SPACE TOEPLITZ OPERATORS WITH POLYRADIALLY CONTINUOUS SYMBOLS

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ABSTRACT. We establish a criterion for the asymptotic invertibility of Toeplitz operators on the Segal-Bargmann space on $\mathbb{C}^N$ whose symbols have the property that the polyradial limits

$$\lim_{r_1, \dots, r_N \to \infty} a(r_1 t_1, \dots, r_N t_N)$$

exist for all $(t_1, \dots, t_N) \in \mathbb{T}^N$ and represent a continuous function on $\mathbb{T}^N$.

1. INTRODUCTION

The following problem emerges in connection with several questions on Toeplitz operators. Given a Toeplitz operator on some Hilbert space of analytic functions, consider the compressions of the operator to the subspaces of polynomials of degree at most n. If the given operator is invertible, are its compressions invertible for all sufficiently large n and do the inverses of these compressions strongly converge to the inverse of the operator as n goes to infinity? In case the answer to this question is affirmative, one says that the given Toeplitz operator is asymptotically invertible or that the finite section method is applicable. The finite section method has been studied for a long time for Toeplitz operators on Hardy

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spaces (see e.g. the books [8] and [6]) and was also recently tackled by one of the authors for Bergman space Toeplitz operators [4].

The increasing interest in Toeplitz operators on the Segal-Bargmann space (see e.g. [2, 3]) motivates the investigation of their asymptotic invertibility, too. In the present note we announce a criterion for the asymptotic invertibility of Toeplitz operators with continuous symbols on some scales of Hilbert spaces of entire functions on $\mathbb{C}^N$, including the Segal-Bargmann space.

2. Spaces of Segal-Bargmann type on $\mathbb{C}$

Let $d\mu(z)$ be a rotation invariant measure on $\mathbb{C}$, i.e. suppose in polar coordinates we have $d\mu(z) = r d\nu(r) d\theta$, where $d\nu(r)$ is a nonnegative measure on $(0, \infty)$. We assume that

$$\hat{\mu}(n) := \int_{\mathbb{C}} |z|^n \, d\mu(z) = 2\pi \int_0^\infty r^{n+1} \, d\nu(r) < \infty \tag{1}$$

for $n = 0, 1, 2, \ldots$ and we also require that $d\mu(z)$ does not have finite support, i.e. that

$$\sup\{r \colon r \in \operatorname{supp} \nu\} = \infty . \tag{2}$$

Finally, dictated by a proof that will follow below, we must demand that

$$\lim_{n\to\infty} (\hat{\mu}(n)\hat{\mu}(n))/(\hat{\mu}(n-1)\hat{\mu}(n+1)) = 1 . \tag{3}$$

The space $H^2(\mathbb{C}, d\mu)$ is defined as the normed space of all entire functions f for which

$$\|f\|^2 := \int_{\mathbb{C}} |f(z)|^2 \, d\mu(z) < \infty .$$

The Segal-Bargmann space considered by Berger and Coburn in [2] and [3] arises from specifying $d\mu(z)$ to

$$d\mu_\beta(z) = (\beta/\pi) e^{-\beta r^2} r dr d\theta \qquad (\beta > 0) . \tag{4}$$

It is clear that $d\mu_\beta(z)$ satisfies (1) and (2), and since $\hat{\mu}_\beta(n) = \Gamma(n/2 + 1)\beta^{-n/2}$, the equality (3) can be easily checked using Stirling's formula. To have another example, we note that if $d\mu(z) = (\gamma^2/2\pi) e^{-\gamma r} r dr d\theta$, then the requirements (1), (2), and (3) are also met.

Consider the power series

$$p(\lambda) = \sum_{n=0}^{\infty} \lambda^n / \hat{\mu}(2n) . \tag{5}$$

Hölder's inequality with the conjugate exponents $(n+1)/n$ and $n+1$ gives

$$\int_{\mathbb{C}} |z|^{2n}\, d\mu(z) \le \left(\int_{\mathbb{C}} |z|^{2n+2}\, d\mu(z)\right)^{n/(n+1)} \left(\int_{\mathbb{C}} d\mu(z)\right)^{1/(n+1)}$$

and thus

$$\hat{\mu}(2n+2)/\hat{\mu}(2n) \ge \left(\int_{\mathbb{C}} d\mu(z)\right)^{-1/(n+1)} \left(\int_{\mathbb{C}} |z^2|^{n+1}\, d\mu(z)\right)^{1/(n+1)}.$$

As z^2 is unbounded, the right-hand side of the latter inequality goes to infinity as $n \to \infty$, which implies that the convergence radius of (5) is infinite. Hence $p(\lambda)$ is an entire function. In the special case where $d\mu(z)$ is the measure (4) we have $p(\lambda) = e^{\beta\lambda}$. Standard computations (using the series representation (5)) yield that the function K_w defined for $w \in \mathbb{C}$ by $K_w(z) = p(w\overline{z})$ belongs to $H^2(\mathbb{C}, d\mu)$, that $\|K_w\|^2 = p(|w|^2)$ and that $p(w\overline{z})$ is the reproducing kernel for $H^2(\mathbb{C}, d\mu)$: if $f \in H^2(\mathbb{C}, d\mu)$ and $w \in \mathbb{C}$, then

$$f(w) = (f, K_w) = \int_{\mathbb{C}} f(z) p(w\overline{z})\, d\mu(z).$$

We have in particular $|f(w)| \le \|f\|(p(|w^2|))^{1/2}$, which shows that point evaluation at each $w \in \mathbb{C}$ is a bounded functional on $H^2(\mathbb{C}, d\mu)$ and that $H^2(\mathbb{C}, d\mu)$ is complete and thus a Hilbert space. The orthogonal projection of $L^2(\mathbb{C}, d\mu)$ onto $H^2(\mathbb{C}, d\mu)$ is given by

$$(Pf)(w) = \int_{\mathbb{C}} f(z) p(w\overline{z})\, d\mu(z),$$

and the functions $\{e_n\}_{n=0}^{\infty}$ defined by

$$e_n(z) = z^n/(\hat{\mu}(2n))^{1/2} \tag{6}$$

form an orthonormal basis in $H^2(\mathbb{C}, d\mu)$.

3. Toeplitz operators on $H^2(\mathbb{C}, d\mu)$

The Toeplitz operator $T^1(a)$ generated by a function a in $L^\infty(\mathbb{C}, d\mu)$ (its so-called symbol) is the operator on $H^2(\mathbb{C}, d\mu)$ that sends a function f to the function $P(af)$. Here we restrict ourselves to Toeplitz operators whose symbol a belongs to $C(\overline{\mathbb{C}})$, which means that a is continuous on $\mathbb{C}$, that the limit

$$a^{\#}(e^{i\theta}) := \lim_{r\to\infty} a(re^{i\theta})$$

exists for each $e^{i\theta}$ on the complex unit circle $\mathbb{T}$ and that $a^{\#}$ is continuous on $\mathbb{T}$. With every function $a \in C(\overline{\mathbb{C}})$ we associate a function $\hat{a} \in L^{\infty}(\mathbb{C}, d\mu)$ by $\hat{a}(re^{i\theta}) := a^{\#}(e^{i\theta})$ $(r > 0)$.

Taking into account that $|f(w)| \le \|f\|(p(|w|^2))^{1/2}$ for f in $H^2(\mathbb{C}, d\mu)$ one can show (e.g. as in [1, Lemmas 4.1 and 4.2]) that $T^1(a) - T^1(\hat{a})$ is compact for $a \in C(\overline{\mathbb{C}})$. The matrix representation of $T^1(\hat{a})$ with respect to the orthonormal basis $\{e_n\}_{n=0}^{\infty}$ given by (6) is

$$(7) \qquad (a_{j-k}\hat{\mu}(k+j)/(\hat{\mu}(2k)\hat{\mu}(2j))^{1/2})_{j,k=0}^{\infty},$$

where $\{a_n\}_{n=-\infty}^{\infty}$ stands for the sequence of the Fourier coefficients of $a^{\#}$.

For $a \in C(\overline{\mathbb{C}})$, we denote by $T^0(a)$ the operator on $H^2(\mathbb{C}, d\mu)$ whose matrix representation with respect to the basis $\{e_n\}_{n=0}^{\infty}$ equals $(a_{j-k})_{j,k=\infty}^{\infty}$. Since $T^0(a)$ is unitarily equivalent to the Toeplitz operator with the symbol $a^{\#}$ on the Hardy space $H^2(\mathbb{T})$, the operator $T^0(a)$ is bounded on $H^2(\mathbb{C}, d\mu)$.

The point is that if $a \in C(\overline{\mathbb{C}})$, then $T^1(a) - T^0(a)$ is a compact operator on $H^2(\mathbb{C}, d\mu)$. Indeed, due to a standard approximation argument, it suffices to show that the operators $T^1(z^m/|z|^m) - T^0(z^m/|z|^m)$ are compact for all integers m. From the matrix representation (7) we infer that the compactness of the latter operators is equivalent to the equalities

$$(8) \qquad \lim_{k\to\infty} \hat{\mu}(2k+m)/(\hat{\mu}(2k)\hat{\mu}(2k+2m))^{1/2} = 1.$$

The equalities (8) in turn are an immediate consequence of our hypothesis (3).

Denote by P_n $(n = 0, 1, 2, \dots)$ the orthogonal projection of $H^2(\mathbb{C}, d\mu)$ onto the linear hull of the monomials $1, z, \dots, z^n$. An invertible operator T on $H^2(\mathbb{C}, d\mu)$ is said to be asymptotically invertible if the compressions (finite sections) $P_nTP_n|\operatorname{Im} P_n$ are invertible for all sufficiently large n and the operators $(P_nTP_n)^{-1}P_n$ converge strongly to T^{-1} as $n \to \infty$.

Theorem 1. *Every invertible Toeplitz operator $T^1(a)$ with symbol a in $C(\overline{\mathbb{C}})$ is asymptotically invertible on $H^2(\mathbb{C}, d\mu)$.*

Proof. We have $T^1(a) = T^0(a) + K$ with some compact operator K, and a theorem by Silbermann (explicitly stated as Theorem

7.20 in [6]) says that operators of the form $T^0(a)+K$ are asymptotically invertible whenever they are invertible. □

4. Higher dimensions

Given N measures $d\mu_j(z)$ on $\mathbb{C}$ satisfying (1), (2), and (3), we put $d\mu(z)=d\mu_1(z_1)\cdots d\mu_N(z_N)$ and denote by $H^2(\mathbb{C}^N, d\mu)$ the Hilbert space of all entire functions on $\mathbb{C}^N$ for which $\|f\|^2 := \int_{\mathbb{C}^N}|f(z)|^2\,d\mu(z)<\infty$. Notice that $H^2(\mathbb{C}^N, d\mu)$ decomposes into the Hilbert space tensor product

$$H^2(\mathbb{C}, d\mu_1)\otimes\cdots\otimes H^2(\mathbb{C}, d\mu_N).$$

The orthogonal projection of $L^2(\mathbb{C}^N, d\mu)$ onto $H^2(\mathbb{C}^N, d\mu)$ is denoted by P. We have

$$(Pf)(w)=\int_{\mathbb{C}^N} f(z)p_1(w_1\overline{z}_1)\cdots p_N(w_N\overline{z}_N)\,d\mu(z)$$

for $f\in L^2(\mathbb{C}^N, d\mu)$ and $w\in\mathbb{C}^N$; here $p_j(w\overline{z})$ is the reproducing kernel for $H^2(\mathbb{C}, d\mu_j)$. The Toeplitz operator on $H^2(\mathbb{C}^N, d\mu)$ induced by a function $a\in L^\infty(\mathbb{C}^N, d\mu)$ (its symbol) acts by the rule $f\mapsto P(af)$, and it will be denoted by $T^{1,\ldots,1}(a)$ (N units).

We limit ourselves to Toeplitz operators with symbols in $C(\overline{\mathbb{C}}^N)$. The latter space is defined as the set of all functions a which are continuous on $\mathbb{C}^N$ and for which the limits

$$a^\#(e^{i\theta_1}, \ldots, e^{i\theta_N}) := \lim_{r_1,\ldots,r_N\to\infty} a(r_1e^{i\theta_1}, \ldots, r_Ne^{i\theta_N})$$

exist for all $(e^{i\theta_1}, \ldots, e^{i\theta_N})\in\mathbb{T}^N$ and represent a continuous function on $\mathbb{T}^N$.

Our next objective is to associate with each function $a\in C(\overline{\mathbb{C}}^N)$ a collection of 2^N "mixed" Toeplitz operators $T^{\varepsilon_1,\ldots,\varepsilon_N}(a)$ ($(\varepsilon_1, \ldots, \varepsilon_N)\in\{0, 1\}^N$) on $H^2(\mathbb{C}^N, d\mu)$. For $N=1$ these two operators are just the operators $T^1(a)$ and $T^0(a)$ introduced in §3. If $N>1$, then $T^{1,\ldots,1}(a)$ stands for the Toeplitz operator defined above. There is no problem in defining the remaining 2^N-1 operators $T^{\varepsilon_1,\ldots,\varepsilon_N}(a)$ is case a is a finite sum of the form

$$a=\sum_j a_1^{j_1}\otimes\cdots\otimes a_N^{j_N}\qquad (a_k^j\in C(\overline{\mathbb{C}}));$$

we then put

$$T^{\varepsilon_1,\ldots,\varepsilon_N}(a) := \sum_j T^{\varepsilon_1}(a_1^{j_1})\otimes\cdots\otimes T^{\varepsilon_N}(a_N^{j_N}).$$

To define $T^{\varepsilon_1, \dots, \varepsilon_N}(a)$ for general $a \in C(\overline{\mathbb{C}}^N)$ we proceed as follows. Let W_n $(n = 0, 1, 2, \dots)$ be the operator given on $H^2(\mathbb{C}, d\mu)$ by

$$W_n: \sum_{k=0}^{\infty} f_k e_k \mapsto f_n e_0 + f_{n-1} e_1 + \cdots + f_0 e_n .$$

We clearly have $W_n T^0(a) W_n = P_n T^0(a^0) P_n$, where $a^0(z) := a(\overline{z})$. If $a \in C(\overline{\mathbb{C}})$, then $T^1(a) = T^0(a) + K$ is compact and hence,

$$W_n T^1(a) W_n = P_n T^0(a^0) P_n + W_n K W_n \to T^0(a^0)$$

strongly as $n \to \infty$, because $W_n \to 0$ weakly as $n \to \infty$. Thus, $T^0(a)$ may also be defined as the strong limit $\lim_{n\to\infty} W_n T^1(a^0) W_n$, and this idea works in the multidimensional case as well.

Namely, write $R_n^0 := W_n$ and $R_n^1 := P_n$, and for $\varepsilon = (\varepsilon_1, \dots, \varepsilon_N)$ in $\{0, 1\}^N$ let $R_n^\varepsilon := R_n^{\varepsilon_1} \otimes \cdots \otimes R_n^{\varepsilon_N}$. For $a \in C(\overline{\mathbb{C}}^N)$ define $a^\varepsilon \in C(\overline{\mathbb{C}}^N)$ by $a^\varepsilon(z_1, \dots, z_N) = a(w_1, \dots, w_N)$, where $w_j := z_j$ if $\varepsilon_j = 1$ and $w_j := \overline{z}_j$ if $\varepsilon_j = 0$. One can show (see [4] for the Bergman space case) that the strong limit

$$\lim_{n\to\infty} R_n^\varepsilon T^{1, \dots, 1}(a^\varepsilon) R_n^\varepsilon$$

exists for every $a \in C(\overline{\mathbb{C}}^N)$, and this limit is, by definition, the mixed Toeplitz operator $T^\varepsilon(a)$.

The following theorem provides a Fredholm criterion for the operators $T^\varepsilon(a)$.

Theorem 2. *Let $a \in C(\overline{\mathbb{C}}^N)$ and $\varepsilon \in \{0, 1\}^N$.*

(a) *If $N \geq 2$, then $T^\varepsilon(a)$ is Fredholm on $H^2(\mathbb{C}^N, d\mu)$ if and only if for each $\tau \in \mathbb{T}$ the N operators*

$$T^{\varepsilon_1, \dots, \varepsilon_{j-1}, \varepsilon_{j+1}, \dots, \varepsilon_N}(a_\tau^j) \qquad (j = 1, \dots, N)$$

are invertible on $H^2(\mathbb{C}^{N-1}, d\mu/d\mu_j)$; here $a_\tau^1, \dots, a_\tau^N \in C(\overline{\mathbb{C}}^{N-1})$ are defined by

$$a_\tau^j(z_1, \dots, z_{j-1}, z_{j+1}, \dots, z_N) := \lim_{r\to\infty} a(z_1, \dots, z_{j-1}, r\tau, z_{j+1}, \dots, z_N).$$

(b) *If $T^\varepsilon(a)$ is not invertible on $H^2(\mathbb{C}^N, d\mu)$ $(N \geq 1)$, then there exists a sequence $\{U_n\}_{n=0}^\infty$ of operators on $H^2(\mathbb{C}^N, d\mu)$ such that $\|U_n\| = 1$ for all n and $\|U_n T^\varepsilon(a)\| \to 0$ or $\|T^\varepsilon(a) U_n\| \to 0$ as $n \to \infty$.*

In various special settings this theorem is well known. In the Wiener-Hopf case, that is for $\varepsilon = (0, \dots, 0)$, it was established by Simonenko [11] and Douglas and Howe [7], for pure Segal-Bargmann space Toeplitz operators, i.e. for $\varepsilon = (1, \dots, 1)$, it follows from the results of Berger and Coburn [2, 3].

The theorem can be proved by arguments similar to the ones employed by Douglas and Howe [7] and Pilidi [10] in the Wiener-Hopf case (also see [6, 8.38 and 8.77]). We remark that the "only if" part of (a) may be shown by induction on N using part (b) for $N-1$.

We now come to asymptotic invertibility. An invertible operator T on $H^2(\mathbb{C}^N, d\mu)$ is said to be asymptotically invertible if the compressions (finite sections)

$$T_n := (P_n \otimes \cdots \otimes P_n)T(P_n \otimes \cdots \otimes P_n)|\operatorname{Im}(P_n \otimes \cdots \otimes P_n)$$

are invertible for all sufficiently large n and the operators $T_n^{-1}(P_n \otimes \cdots \otimes P_n)$ converge strongly to T^{-1} as $n \to \infty$.

Theorem 3. *Let $a \in C(\overline{\mathbb{C}}^N)$ and $\varepsilon \in \{0, 1\}^N$. Then the operator $T^\varepsilon(a)$ is asymptotically invertible on $H^2(\mathbb{C}^N, d\mu)$ if and only if the 2^N operators $T^{\varepsilon\cdot\delta}(a^\delta)$ $(\delta \in \{0, 1\}^N)$ are invertible on $H^2(\mathbb{C}^N, d\mu)$; here $\varepsilon \cdot \delta := (\varepsilon_1\delta_1, \dots, \varepsilon_N\delta_N)$.*

This is our main result. For pure Segal-Bargmann space Toeplitz operators, the theorem says that $T^{1,\dots,1}(a)$ is asymptotically invertible if and only if the 2^N operators $T^\delta(a^\delta)$ $(\delta \in \{0, 1\}^N)$ are invertible. In the case $N = 1$ this theorem really goes over into Theorem 1, because then the invertibility of $T^0(a^0)$ automatically results from the invertibility of $T^1(a)$. In the Wiener-Hopf (Hardy space) case the theorem is Kozak's pioneering result [9]: $T^{0,\dots,0}(a)$ is asymptotically invertible if and only if the 2^N operators $T^{0,\dots,0}(a^\delta)(\delta \in \{0, 1\}^N)$ are invertible. Note that all the operators $T^{0,\dots,0}(a^\delta)$ are Wiener-Hopf, so that no mixed operators appear in the Wiener-Hopf situation.

The proof of Theorem 3 is rather long. It proceeds along the lines of §§8.60–8.69 in [6] (also see [4] and [5]). Two important ingredients of the proof are the Fredholm criterion contained in Theorem 2 and the following Segal-Bargmann analogue of a formula established by Widom [12] in the Wiener-Hopf case: if $a, b \in C(\overline{\mathbb{C}})$ then

$$P_nT^1(ab)P_n = P_nT^1(a)P_nT^1(b)P_n + P_nKP_n + W_nLW_n + C_n,$$

where K and L are compact on $H^2(\mathbb{C}, d\mu)$ and $\|C_n\| \to 0$ as $n \to \infty$.

We finally remark that it is certainly nice to have a single theorem comprising both Kozak's criterion for Hardy space Toeplitz operators and a criterion for the asymptotic invertibility of Segal-Bargmann space Toeplitz operators. However, we wish to emphasize that our theorem as it is stated does not result from the sole endeavor to aesthetics; it is rather determined by our proof: we prove it by induction on the dimension N, and the proof we have does not work when we bound ourselves to pure Segal-Bargmann operators only, whereas it goes smoothly when considering all mixed Toeplitz operators simultaneously.

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