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Holomorphic Hilbert modular forms, by Paul B. Garrett. Wadsworth & Brooks/Cole, 1990, 304 pp., ISBN 0-534-10344-8

In the early 1970s many mathematicians, especially number theorists, learned that they were secretly in love with $GL(2)$. This circumstance was brought to light in large part by the publication in 1970 of the book *Automorphic forms on* $GL(2)$ by H. Jacquet and R. P. Langlands [JL]. The last year has seen the publication of no fewer than three introductory books, with completely different tables of contents, on the subject of Jacquet-Langlands' formidable monograph: *Modular forms* by T. Miyake, *Hilbert modular forms* by E. Freitag, and the book under review. Of the three, Miyake's book, which treats only the case of $GL(2, \mathbf{Q})$, is closest in content, if not in spirit, to Jacquet-Langlands; Freitag and Garrett cover much of the same ground but have quite different destinations in mind.

The stated purpose of Garrett's very readable book is to review older, mostly German material in the light of "modern methods," and to provide the technical background for some more recent work, of special interest to number theorists, on the arithmetic of L-functions. Garrett's "modernism" views automorphic forms as functions on adéle groups, and in this his framework most conspicuously diverges from those of Miyake and Freitag, as well as from Shimura's classic *Introduction to the arithmetic theory of automorphic functions* [Sh], the obvious point of comparison for any introductory text on the subject.

The adéles are now an object in their own right, without which modern number theory would be inconceivable. But in the beginning they were a technical device for encoding the observation that, in the arithmetic of global fields (i.e., number fields or finite extensions of the field $k(T)$, where k is a finite field), all absolute values, whether archimedean or p-adic, should be considered equally important. If F is a global field, then the adéle ring $\mathbf{A}_F$ is the ring whose elements are infinite vectors (a_v), where v runs through the set of inequivalent absolute values $|\cdot|_v$ on F, a_v belongs to the completion F_v of v with respect to v, and $|a_v|_v \leq 1$ for all but finitely many v. Then F imbeds naturally as a subring of $\mathbf{A}_F$, so if G is an algebraic group over F, then the group $G(\mathbf{A}_F)$ of $\mathbf{A}_F$-valued points of G is defined. In particular, the multiplicative group $\mathbf{A}_F^\times$ of $\mathbf{A}_F$ is called the group of idéles; $\mathbf{A}_F^\times$ consists of (a_v) such that $a_v \neq 0$ for all v, and such that $|a_v|_v = 1$ for all but finitely many v. Thus the idéle norm $\|(a_v)\| = \prod_v |a_v|_v$ is well defined.

An (adélic) automorphic form on $GL(2)$ over a global field F is a complex-valued function f on the adéle group $GL(2, \mathbf{A}_F)$ which is locally constant in the nonarchimedean variables, C^∞ in the archimedean variables, and which satisfies the following additional conditions, introduced by Harish-Chandra in the context of automorphic forms on real Lie groups:

(i) $f(\gamma g) = f(g)$ for all $\gamma \in GL(2, F)$, $g \in GL(2, \mathbf{A}_F)$,

(ii) Let $G_\infty = GL(2, F\otimes_{\mathbf{Q}}\mathbf{R}) \subset GL(2, \mathbf{A}_F)$, $\mathfrak{g}$ the complexified Lie algebra of G_∞, and $Z(\mathfrak{g})$ the center of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$. Let K_∞ be a maximal compact subgroup of G_∞. Then $Z(\mathfrak{g})$ (resp. K_∞) acts by left-differentiation (resp. left-translation) on the space of smooth functions on $GL(2, \mathbf{A}_F)$, and the $Z(\mathfrak{g})$-module (resp. K_∞-module) generated by f is finite-dimensional.

(iii) f is *slowly increasing* with respect to a certain natural norm on $GL(2, \mathbf{A}_F)$.

Ask a specialist to point to an automorphic form and you will probably be shown one of the two basic examples treated in Garrett's book:

(1)
$$E(h, s, g) = \sum_{\gamma \in B(F)\backslash GL(2, F)} h(\gamma g) \cdot \|a(\gamma g)\|^s \quad \text{(an \textit{Eisenstein series})}$$

or

(2)
$$\Theta(\varphi, g) = \sum_{v \in V(\mathbf{Q})} (r(g)\varphi)(v) \quad \text{(a \textit{theta series}).}$$

This notation requires explanation. Let $B = \{\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)\}$ be the upper triangular Borel subgroup of $GL(2)$, $N = \{\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)\} \subset B$; any $g \in GL(2, \mathbf{A}_F)$ has an Iwasawa decomposition

$$g = n(g) \cdot \begin{bmatrix} a(g) & 0 \\ 0 & 1 \end{bmatrix} \cdot k(g) \cdot z(g)$$

with $n(g) \in N(\mathbf{A}_F)$, $a(g) \in \mathbf{A}_F^\times$, $k(g) \in K_\infty \cdot \prod GL(2, \mathscr{O}_v)$ (here the product runs over all nonarchimedean places v of F, and $\mathscr{O}_v$ is the maximal order at v), and $z(g)$ a diagonal matrix. The idéle $a(g)$ is not uniquely determined, but its idéle norm $\|a(g)\|$ is, and the formula for the Eisenstein series makes sense and converges absolutely if h is a smooth function on $B(F) \cdot N(\mathbf{A}_F)\backslash GL(2, \mathbf{A}_F)$ which transforms on the left by a character of $B(\mathbf{A}_F)/N(\mathbf{A}_F)$, and s is a complex number whose real part is sufficiently large.

The theta series is written as in Weil's papers on the metaplectic group and Siegel's formula. Here for simplicity $F = \mathbf{Q}$, and V is a $\mathbf{Q}$-vector space of even dimension, endowed with a positive-definite quadratic form. To any nontrivial character ψ of the compact group $\mathbf{A}_\mathbb{Q}/\mathbb{Q}$, Weil associates an action r_ψ of $SL(2, \mathbf{A}_\mathbb{Q})$ on the space $\mathscr{S}(V_\mathbf{A})$ of Schwartz-Bruhat functions on $V_\mathbf{A} = V \otimes_\mathbb{Q} \mathbf{A}_\mathbb{Q}$. This action can be extended naturally to a subgroup G^0 of finite index in $GL(2, \mathbf{A}_\mathbb{Q})$. If $\varphi \in \mathscr{S}(V_\mathbf{A})$, then the formula (2) converges absolutely and defines a function $\Theta(\varphi, g)$ on G^0. An application of the *Poisson summation formula* shows that $\Theta(\varphi, g)$ is left-invariant under $G^0 \cap GL(2, \mathbf{Q})$; it is then easy to extend $\Theta(\varphi, g)$ to an automorphic form on $GL(2)$ over $\mathbf{Q}$. In Garrett's book theta series serve as nontrivial examples of cusp forms; when $\dim(V) = 2$, Garrett constructs the theta functions attached to Hecke characters of totally imaginary quadratic extensions of F.

The algebra $U(\mathfrak{g})$, the group K_∞, and the group $GL(2, \mathbf{A}_F^f)$ of nonarchimedean adéles in $GL(2, \mathbf{A}_F)$ act by left-differentiation and left-translation, respectively, on the space $\mathfrak{A}_F$ of automorphic forms on $GL(2)$ over F. Since Jacquet-Langlands, the basic object of study has been the *automorphic representation* of $GL(2, F)$: a subspace π of $\mathfrak{A}_F$ (more generally, a subquotient of $\mathfrak{A}_F$) which is invariant and irreducible with respect to the combined action of $U(\mathfrak{g})$, K_∞, and $GL(2, \mathbf{A}_F^f)$. Most important are the *cuspidal* automorphic representations, those which cannot be embedded as subquotients of the representations constructed from Eisenstein series. The vectors in π are, of course, automorphic forms, and are *cusp forms* when π is cuspidal; they are *Hilbert modular forms* when F is a totally real number field. The name "Hilbert modular form" is often reserved for holomorphic elements of π, when such exist, but following the title of Garrett's book, we make this into an extra condition. Thus, if F is totally real of degree d over $\mathbf{Q}$, the identity component K_∞^+ of K_∞ is isomorphic to $SO(2)^d$. The representations of K_∞^+ are thus parametrized by $\mathbf{Z}^d$. If π contains an irreducible $U(\mathfrak{g}) \times GL(2, \mathbf{A}_F^f)$-submodule π^{hol} whose restriction to K_∞^+ lies in the octant $(\mathbf{Z}_{\geq 0})^d$, then π is said to be of holomorphic type. Then π^{hol} contains a representation K_∞^+ of smallest parameter $\mathbf{k} = (k_1, k_2, \ldots, k_d)$, with each $k_i \geq 0$, and the elements of π^{hol} with parameter $\mathbf{k}$ are called *holomorphic Hilbert modular forms of weight* $\mathbf{k}$.

A standard change of variables identifies a holomorphic Hilbert modular form as above with a more familiar object. Let $\mathfrak{h}$ denote the upper half-plane in $\mathbf{C}$, and let σ_i, $i = 1, \ldots, d$, denote the real embeddings of F. The identity component $GL(2, \mathbf{R})^+$ of $GL(2, \mathbf{R})$ acts on $\mathfrak{h}$ by linear fractional transformations. If we denote by $GL(2, F)^+$ the subgroup of $\gamma \in GL(2, F)$ such that $\sigma_i(\gamma) \in GL(2, \mathbf{R})^+$, $i = 1, \ldots, d$, $GL(2, F)^+$ then acts on $\mathfrak{h}^d$ by the formula

$$\gamma(\mathbf{z}) = (\sigma_1(\gamma)(z_1), \ldots, \sigma_d(\gamma)(z_d)) \quad \text{if } \mathbf{z} = (z_1, \ldots, z_d).$$

Then we may identify holomorphic Hilbert modular forms with functions $f = f(\mathbf{z}, g)$, $\mathbf{z} \in \mathfrak{h}^d$, $g \in GL(2, \mathbf{A}_F^f)$, holomorphic in

$\mathbf{z}$, locally constant in g, and satisfying the functional equation

$$(3)\qquad f(\gamma(\mathbf{z}), \gamma g) = \left(\prod_{j=1}^{d}(c_j z_j + d_j)^{k_j}\right) \cdot f(\mathbf{z}, g),$$

$$\text{where } \sigma_j(\gamma) = \begin{bmatrix} a_j & b_j \\ c_j & d_j \end{bmatrix} \qquad j = 1, \ldots, d.$$

Then f has the following Fourier expansion:

$$(4)\qquad f(\mathbf{z}, g) = \sum_{\alpha \in F} W_\alpha(f, g) e^{2\pi i \alpha \cdot \mathbf{z}}$$

where $\alpha \cdot \mathbf{z} = \sum_{j=1}^{d} \sigma_j(\alpha) z_j$ and the *Whittaker function* $W_\alpha(f, g)$ has a factorization over the finite primes

$$(5)\qquad W_\alpha(f, (g_v)) = \prod_v W_{v,\alpha}(f, g_v).$$

Then f is a cusp form iff $W_\alpha(f, g) \equiv 0$ unless $\sigma_j(\alpha) > 0$ for all α. Formula (3) is the definition which best lends itself to arithmetic applications, whereas the group theoretic formalism is more convenient for nearly everything else. Much of Garrett's book is taken up with exhibiting the relative advantages of the two approaches.

The subordination of the classical theory of Hilbert modular forms to group theory did not begin with Jacquet-Langlands. The equivalence of the theory of Hecke operators with that of spherical representations of $GL(2, \mathbf{Q}_p)$ was first observed in the late 1950s. At the Boulder conference of 1966, Satake explained how the Ramanujan-Petersson conjecture on the Fourier coefficients of modular forms could be formulated naturally in terms of the classification of unitary representations of $GL(2, \mathbf{Q}_p)$ [Sa]. By the time Jacquet-Langlands appeared, the influential book [GGP] of Gelfand, Graev, and Piatetski-Shapiro, as well as the papers of Weil mentioned above, had already argued persuasively in favor of the adélic approach.

The principal innovation of Jacquet-Langlands was the systematic use of representation theory to attach L-functions to automorphic representations, and to derive their analytic properties. An automorphic representation π of $GL(2, F)$ is isomorphic to a (restricted) tensor product $\bigotimes_v \pi_v$, where v runs through the places of F and π_v is an irreducible representation of $GL(2, F_v)$ (v nonarchimedean) or the enveloping algebra of the Lie algebra of $GL(2, F_v)$ (v archimedean). Each π_v is admissible, which means in the nonarchimedean case that every vector in π_v is contained in a finite-dimensional $GL(2, \mathscr{O}_v)$-module and the $GL(2, \mathscr{O}_v)$-spectrum in π_v has finite multiplicities; for archime dean v the admissible representations are those studied by Harish-Chandra. Imitating the pattern established in Tate's thesis for the group $GL(1)$, Jacquet-Langlands attaches to each infinite-dimensional admissible π_v the following data: a family of zeta integrals $Z(s, g, W_v)$, $s \in \mathbf{C}$, $g \in GL(2, F_v)$, W_v an element of the Whittaker model of π_v—for example, one of the functions $W_{v,\alpha}(f, \cdot)$, as in (5); a local Euler factor $L(s, \pi_v)$ which is the inverse of a polynomial of degree ≤ 2 in Nv^{-s} (v finite) or a product of an exponential and at most two shifted Γ-functions (v archimedean), and such that

$$\Phi(g, s, W_v) = L(s, \pi_v)^{-1} Z(s, g, W_v)$$

is entire for all g, W_v; and an entire function $\varepsilon(s, \pi_v, \psi_v)$, which depends on an auxiliary unitary character $\psi_v : F_v \to \mathbf{C}^\times$. These functions are related by the *local functional equation*:

$$\text{(6)} \qquad \Phi(wg, 1-s, W_v) = \varepsilon(s, \pi_v, \psi_v) \cdot \Phi(g, s, W_v), \qquad w = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \in GL(2, F_v).$$

If the ψ_v 's are chosen to be local components of an adélic character $\psi : \mathbf{A}_F/F \to \mathbf{C}^\times$, then the product $\varepsilon(s, \pi) = \prod_v \varepsilon(s, \pi_v, \psi_v)$ does not depend on ψ. Then, defining $L(s, \pi) = \prod_v L(s, \pi_v)$, the global functional equation

$$\text{(7)} \qquad L(s, \pi) = \varepsilon(s, \pi) L(1-s, \tilde{\pi}),$$

where $\tilde{\pi}$ is the (admissible) contragredient of π, is a fairly straightforward consequence of the local functional equations and the global invariance properties of automorphic forms.

The last two decades have seen the discovery of several dozen new integral representations for Euler products, with at least one L-function attached to automorphic representations of all but a few exceptional groups. The Jacquet-Langlands approach to the analytic properties of L-functions through a combination of local and global harmonic analysis has served as a model for all subsequent work. Euler products are a principal theme of Garrett's book. They first appear in a convincing account—with all questions of convergence omitted—of the adélic approach to the Euler product for the Mellin transform of a Hecke eigenform, essentially following Jacquet-Langlands in their use of Whittaker functions.

A more substantial section is devoted to Rankin's method for proving the analytic continuation of the L-function attached to two Hilbert modular forms. Nowadays, the terms "Rankin's method" and "Rankin-Selberg convolution" are used to describe any procedure to obtain an Euler product by integrating an automorphic form on one group against an Eisenstein series on a smaller (or larger) group. The functional equation of the Euler product is then a direct consequence of the functional equation of the Eisenstein series. Rankin's original product expresses an L-function attached to two elliptic modular forms f and f' as an integral against an Eisenstein series:

$$\Lambda(s) \cdot L(s, f, f') = \int_{G(\mathbf{Q})Z(\mathbf{A})\backslash G(\mathbf{A})} f(g)\overline{f}'(g)E(h, s, g)\, dg$$

where $G = GL(2, \mathbf{Q})$, Z the subgroup of diagonal matrices, $E(h, s, g)$ is an appropriately chosen Eisenstein series as in (1), and $\Lambda(s)$ is a product of elementary factors. Rankin worked with holomorphic functions, of course; the adélic generalization was carried out by Jacquet [J], and Garrett's account is based on the simpler parts of Jacquet's theory.

Number theorists' excitement about $GL(2)$ was not, however, primarily a response to the new way to construct L-functions. In fact, an explicit purpose of Jacquet-Langlands was to present evidence that automorphic representations were the natural setting for a nonabelian generalization of class field theory. Langlands' *reciprocity conjectures* predict a one-to-one correspondence between cuspidal automorphic representations of $GL(2, F)$ and irreducible compatible systems of two-dimensional λ-adic representations of $\mathrm{Gal}(\overline{F}/F)$ (more generally, of the Weil group of F). Some examples of the latter:

(i) The Galois action on the points of finite order of an elliptic curve A over F defines a compatible system of two-dimensional l-adic representations ρ_l: $\mathrm{Gal}(\overline{F}/F) \to GL(T_l(A) \otimes \mathbf{Q})$, where $T_l(A)$ is the *Tate module* of A. More generally, if X is any algebraic variety over F, then $\mathrm{Gal}(\overline{F}/F)$ operates on the l-adic cohomology groups $H^i(X_{\overline{F}}, \mathbf{Q}_l)$, which define compatible systems of l-adic representations.

(ii) Suppose F' is a finite Galois extension of F, and let ρ: $\mathrm{Gal}(F'/F) \to GL(2, E)$ be a homomorphism, where E is a number field. Then ρ defines continuous homomorphisms ρ_λ: $\mathrm{Gal}(\overline{F}/F) \to GL(2, E_\lambda)$, where E_λ is the completion of E at the nonarchimedean valuation λ, and the ρ_λ form a compatible system.

The characteristic polynomials of the images under ρ_λ of Frobenius elements determine local Euler factors $L_v(s, \{\rho_\lambda\})$ for almost all v, and with a bit more work for all v. In case (ii) above—more generally, when the ρ_λ are λ-adic realizations of *complex* representations of the Weil group—the Euler product $L(s, \{\rho_\lambda\}) = \prod_v L_v(s, \{\rho_\lambda\})$ is an Artin-Hecke L-function, and as such is known to have a meromorphic continuation and satisfy a functional equation of the form

$$L(s, \{\rho_\lambda\}) = \varepsilon(s, \{\rho_\lambda\})L(1-s, \{\tilde{\rho}_\lambda\}) \tag{8}$$

analogous to (7). Langlands, completing earlier work of Dwork, demonstrated the existence of local functions $\varepsilon_v(s, \{\rho_\lambda\}, \psi_v)$, with ψ_v an additive character as above, which agree with the local factors of Tate when $n = 1$ and have a number of other nice properties, and which satisfy

$$\varepsilon(s, \{\rho_\lambda\}) = \prod_v \varepsilon_v(s, \{\rho_\lambda\}, \psi_v) \tag{9}$$

when the ψ_v come from a global ψ as above. A basic conjecture in number theory is that the Euler product $L(s, \{\rho_\lambda\})$ always has an analytic continuation, and that local constants $\varepsilon_v(s, \{\rho_\lambda\}, \psi_v)$ with certain functorial properties can be defined for any $\{\rho_\lambda\}$ in such a way that the functional equation (8) is satisfied. If $\{\rho_\lambda\}$ is an irreducible 2-dimensional system associated to the cuspidal automorphic representation $\pi = \pi\{\rho_\lambda\}$ by Langlands' reciprocity conjecture, then it is expected that

$$\begin{aligned} &L_v(s, \{\rho_\lambda\}) = L(s, \pi_v), \\ &\varepsilon_v(s, \{\rho_\lambda\}, \psi_v) = \varepsilon(s, \pi_v, \psi_v) \quad \text{for all } v. \end{aligned} \tag{10}$$

All through the '60s number theorists had been coming to realize that L-functions of the form $L(s, \{\rho_{l,V}\})$, where $\rho_{l,V}$ is the Galois representation on l-adic cohomology of a smooth projective variety V, were the key to the diophantine geometry of V. The zeroes of $L(s, \{\rho_{l,V}\})$, according to Birch and Swinnerton-Dyer, should control the existence of points of infinite order on V when V is an abelian variety; the poles, according to Tate, should control the existence of algebraic cycles modulo homological equivalence. More recent conjectures of Beilinson, Bloch, and

Bloch-Kato predict that still more of the geometry of V can be read off from its L-functions. The Langlands conjectures promise that all such L-functions are attached to automorphic forms.

Little progress has been made in proving the existence of $\pi\{\rho_\lambda\}$ for general 2-dimensional systems $\{\rho_\lambda\}$ over number fields (over function fields the existence of $\pi(\rho_\lambda)$ was proved by Deligne). The case of finite Galois representations was shown by Jacquet-Langlands to be essentially equivalent to the Artin conjecture for such representations. In the case of the l-adic representations of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ on the Tate modules of elliptic curves, the automorphic forms in question are supposed to be elliptic modular forms of weight 2, and the Langlands conjecture in this case reduces to a conjecture considered earlier by Taniyama, Shimura, and Weil, and which was recently shown by Frey and Ribet to have the Fermat conjecture as an unexpected consequence.

For general number fields F, the Langlands correspondence in the reverse direction—attaching 2-dimensional systems $\{\rho_\lambda(\pi)\}$ to cuspidal automorphic representations of $GL(2, F)$—is equally mysterious (for function fields this is again known, in this case thanks to Drinfeld). But when F is a totally real field and π contains a *holomorphic* Hilbert modular form, the problem of constructing $\{\rho_\lambda(\pi)\}$ was recently completely solved, following more than 30 years of work by many other authors, by R. Taylor [T]; an altogether different solution was subsequently found by Blasius and Rogawski [BR].

Langlands' conjecture that every 2-dimensional compatible λ-adic system is a $\{\rho_\lambda(\pi)\}$ has naturally encouraged the intensive study of λ-adic representations attached to Hilbert modular forms. The interplay between group theory and geometry has been exploited to startling effect to solve classical and not-so-classical problems in arithmetic over $\mathbf{Q}$—for example, the work of Ribet, Mazur, and Wiles on class fields of cyclotomic fields, and partial solutions by Gross-Zagier and Kolyvagin of the Birch-Swinnerton-Dyer and Tate-Shafarevich conjectures.

Applications to arithmetic have been successful largely for a reason known since work of Eichler and Shimura in the '50s, and emphasized in Shimura's book: the relations between automorphic forms, L-functions, and Galois representations can all be interpreted—this is a peculiarity of $GL(2)$—in terms of the *Fourier expansions* (4). On the one hand, $L(s, \pi)$ is completely expressible in terms of the Fourier coefficients of a well chosen form $f_\pi \in \pi$, called a *new form*. On the other hand, the formula (3) exhibits Hilbert modular forms of weight $\mathbf{k}$ as sections of certain line bundles over the *Hilbert modular varieties* $GL(2, F)^+\backslash\mathfrak{h}^d \times GL(2, \mathbf{A}_F^f)$. These varieties parametrize universal families of abelian varieties with additional structure—over $\mathbf{Z}$!—and the Fourier coefficients of holomorphic modular forms have exact interpretations in terms of this modular geometry. In this way, the Fourier coefficients translate statements about the geometry of the Hilbert modular variety into statements about $L(s, \pi)$, and from there to conclusions about the $\{\rho_\lambda(\pi)\}$.

Underlying all arithmetic applications is the classical theorem that the space of modular forms for $GL(2, \mathbf{Q})$ of a given weight and level is spanned by forms with cyclotomic Fourier coefficients. Shimura generalized this theorem to Hilbert modular forms in 1975 (related results were proved earlier by Klingen and others). Garrett calls this generalization the Arithmetic Structure Theorem, and his new proof of this result serves as the focal point of his book, tying together his discussions of L-functions, Eisenstein series, and theta series.

Hidden in Garrett's proof of the Arithmetic Structure Theorem is yet another Euler product, the symmetric square L-function, which can be represented as an integral of a Siegel modular Eisenstein series $E^{(2)}$ on $\mathrm{Sp}(2, \mathbf{A}_F)$ against forms on an embedded $SL(2, \mathbf{A}_F) \times SL(2, \mathbf{A}_F)$. This identity, which Garrett was apparently the first to notice [G], reduces the proof of the Arithmetic Structure Theorem to a calculation, following Siegel, of the Fourier coefficients of $E^{(2)}$, and thence, via Klingen's theorem on the special values of Dirichlet L-functions of totally real fields, to the classical formulas for the special values of Dirichlet L-functions of $\mathbf{Q}$.

Garrett's introduction presents a diagram of the tortuous sequence of implications leading to the Arithmetic Structure Theorem, and one realizes with some astonishment that his derivation of this result uses little more than elementary complex analysis, measure theory, and elementary algebraic number theory. One also realizes that these simple ingredients have been combined to present exemplary applications of most of the standard techniques of the analytic theory of automorphic forms. Better still, starting from scratch, Garrett has succeeded in developing his subject matter to the point of presenting results which, if not exactly the cutting edge of the field, certainly come close. Such results include not only the Arithmetic Structure Theorem but also simplified versions of a series of theorems of Shimura, dating from the '70s, which relate special values of L-functions to periods of integrals. Apart from their intrinsic interest, Shimura's theorems are the starting point for the kind of arithmetic applications discussed above, and Garrett's exposition makes Shimura's difficult theorems seem completely natural.

At times, I was frustrated by Garrett's tendency to drop a subject at a point where a little additional effort would have greatly clarified matters. For example, his comprehensive section on "classical" Hilbert modular forms proves the analytic continuation and functional equation of Dirichlet series attached to cusp forms, but not the Euler product decomposition; later, an adélic version of the same integral representation obtains the Euler product, but not the analytic continuation. Hardly a word is offered to explain the relation between the two approaches. Again, Garrett's treatment of p-adic representation theory is largely limited to a construction of the spherical Hecke algebra and occasional remarks suggesting that the general theory—in particular, the theory of new forms—is too difficult to handle. The reader may well miss the point that the adélic approach simplifies and unifies the theory of L-functions. But one can't object too strenuously to Garrett's omissions, since they are mostly easy to retrieve in the expository literature—for example, in Gelbart's book [Ge] which, although it contains few proofs, is still the most efficient introduction to the material in Jacquet-Langlands. Much of what Garrett covers, on the other hand, was previously available only in the original journal articles.

The enterprise initiated by Langlands' conjectures and by the publication of Jacquet-Langlands has been wildly successful, and progress has come much more rapidly than anyone anticipated 20 years ago. Many conjectures which seemed hopelessly difficult in the mid '70s have now been reduced to a series of problems in harmonic analysis—primarily connected with Arthur's generalization of the Selberg trace formula and its conjectural "stable" variant—whose resolution is expected within a few years. In part for this reason, and in part because the technical baggage of the subject rapidly grows cumbersome as one moves away from GL(2), the number theorists who flocked to the conferences in Antwerp in 1972 and Corvallis in 1977 have largely abandoned automorphic forms to the specialists. Nevertheless, anyone interested in the arithmetic of number fields will eventually have to learn something about Hilbert modular forms. As an introduction to the analytic and arithmetic aspects of the subject, Garrett's book may be the best place to start.

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Coxeter graphs and towers of algebras could not appear at a more auspicious moment. One of the authors, Vaughan F. R. Jones, has just been awarded the Fields medal, and (if we may presume to infer the reasons for the choice of Jones by the Fields Medal Committee), the decision had much to do with the subject of this monograph.