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BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 25, Number 1, July 1991

0273-0979/91 $1.00 + $.25 per page

The volume of convex bodies and Banach space geometry, by Gilles Pisier. Cambridge Tracts in Mathematics, vol. 94, Cambridge University Press, Cambridge, 1989, xv + 250 pp., $49.50. ISBN 0-521-364655

A *convex body* is a compact convex subset of $\mathbf{R}^n$ with a nonempty interior. The study of convex bodies has deep roots in the ancient world, especially in the mathematical work of Archimedes and his predecessors. It now has close ties to number theory, probability theory, harmonic analysis, approximation theory, and a host of other subjects.

Consider a symmetric convex body $B \subset \mathbf{R}^n$, the symmetry being with respect to the origin. In most of the classical work, B plays the leading role, but in more recent work, it is often the dimension n. For example, Fritz John (1948) proved that for every such B there is an ellipsoid D such that

$$D \subset B \subset \sqrt{n}D.$$

In general, the number $\sqrt{n}$ cannot be replaced by a smaller number: consider the n-dimensional cube $[-1, 1]^n$.

Dvoretzky (1961) discovered that much sharper ellipsoidal bounds exist for some k-dimensional sections of B provided that k is small relative to n. In fact, to each positive number ε there corresponds a positive number δ such that if

$$1 \leq k \leq \delta \log n, \tag{1}$$

then there is a k-dimensional subspace F of $\mathbf{R}^n$ and a k-dimensional ellipsoid $D \subset F$ such that

$$D \subset B \cap F \subset (1+\varepsilon)D.$$

Thus, for small ε, the section $B \cap F$ is almost ellipsoidal. The number $\delta = \delta(\varepsilon)$ can be chosen independently of n and B. This beautiful discovery has led to many others.

Dvoretzky's proof yields a factor with a smaller order of magnitude than $\log n$ in (1). Milman (1971) gave a different proof using an isoperimetric inequality on the sphere and related work of Paul Lévy to obtain $\log n$, which has the best possible order of magnitude as $n \to \infty$. See also the fundamental paper of Figiel, Lindenstrauss, and Milman (1977) and the references given there.

Gordon (1985, 1988) used his extension of Slepian's inequality for Gaussian processes, which now has a short proof due to Kahane (1986), to give a proof of Dvoretzky's theorem that yields a positive number β such that the optimal choice of $\delta(\varepsilon)$ satisfies $\delta(\varepsilon) \geq \beta\varepsilon^2$ for all $\varepsilon \in (0, 1]$. If a question of Knaster (1947) has an affirmative answer, then, as Milman (1988) has shown, (1) can be replaced by

$$1 \leq k \leq \alpha(n, \varepsilon) \log n / \log 1/\varepsilon$$

where $\alpha(n, \varepsilon) \to 1$ as $n \to \infty$ and $\varepsilon \to 0$.

Earlier, Mahler (1939) had considered the product $|B| \cdot |B^\circ|$ of the volume of a symmetric convex body B and the volume of its polar

$$B^\circ = \{x \in \mathbf{R}^n : \langle x, y \rangle \leq 1 \text{ for all } y \in B\}$$

where $\langle x, y \rangle = \sum_{i=1}^n x_i y_i$. The volume product $|B| \cdot |B^\circ|$ is a linear invariant. If $u : \mathbf{R}^n \to \mathbf{R}^n$ is linear and invertible, then $|u(B)| \cdot |u(B)^\circ| = |B| \cdot |B^\circ|$. In particular, if

$$D_n = \{x \in \mathbf{R}^n : \langle x, x \rangle \leq 1\}$$

and D is any symmetric ellipsoid, then $D = u(D_n)$ for such a map u so

$$|D| \cdot |D^\circ| = |D_n| \cdot |D_n^\circ| = |D_n|^2.$$

Mahler proved that the volume product satisfies

$$4^n (n!)^{-2} \leq |B| \cdot |B^\circ| \leq 4^n \tag{2}$$

and gave applications of this inequality to number theory. Except for $n = 1$, neither side of (2) is sharp. Santaló (1949) found the sharp bound for the right-hand side:

$$|B| \cdot |B^\circ| \leq |D_n|^2. \tag{3}$$

Recent proofs of (3) are contained in the papers of Saint-Raymond (1981) and of Meyer and Pajor (1989).

In his 1939 paper, Mahler had conjectured that the sharp lower bound of $|B| \cdot |B^\circ|$ is $4^n/n!$, which is the value of the volume product for $B = [-1, 1]^n$. If this conjecture is correct, then there is a positive number α such that

$$\alpha \leq \left(\frac{|B| \cdot |B^\circ|}{|D_n|^2} \right)^{1/n} \leq 1 \tag{4}$$

for all symmetric convex bodies $B \subset \mathbf{R}^n$ and all $n \geq 1$. Although the correctness of Mahler's conjecture has not yet been decided, Bourgain and Milman (1987) have recently established the existence of such a positive number α. Their methods differ greatly from those used by Mahler and Santaló and rest partly on some of the ideas developed in the extensive exploration of Dvoretzky's theorem and other parts of the local theory of Banach spaces.

The local theory is the study of Banach spaces through their subspaces of finite dimension. Symmetric convex bodies play a key role in this study: if B_X is the closed unit ball of a Banach space X and E is a finite-dimensional subspace of X, then $B_E = B_X \cap E$ can be identified with a symmetric convex body, and of course every symmetric convex body is the closed unit ball of a finite-dimensional Banach space.

If an infinite-dimensional Banach space X has the property that, for some integer $k \geq 2$, all k-dimensional subspaces of X are isometric, is the space X isometric to a Hilbert space? This is one of the many questions posed by Banach in his book (the 1987 edition contains a discussion by Pełczyński and Bessaga of the current status of these questions as well as a survey of some related work). Dvoretzky's theorem yields an affirmative answer. It also confirms the conjecture of Grothendieck that any separable Hilbert space can be finitely represented in every infinite-dimensional Banach space with the same field of scalars. For the real field, see Dvoretzky (1961).

Pisier's book is a valuable guide to some of the recent progress in the local theory of Banach spaces. The presentation is largely self-contained. Most of the material after the first hundred pages of background appears here in a book for the first time. One of the key theorems proved is the Bourgain-Milman inequality (the left-hand side of (4) above). Another is the earlier quotient-of-a-subspace theorem of Milman (1985). Still another is Milman's inverse of the Brunn-Minkowski inequality (1986). Clearly, a complete inverse to the Brunn-Minkowski inequality does not exist but Milman has discovered a partial inverse that has surprising consequences. In fact, it can be used to prove the other two theorems.

In this book, Pisier uses such tools as type, cotype, and K-convexity. Their main source is the paper of Maurey and Pisier (1976) and later papers such as Pisier (1982). He also uses entropy numbers, approximation numbers, Gelfand numbers, Kolmogorov numbers, volume numbers and other analytic, probabilistic, and geometric tools from a variety of sources.

Approximation numbers are used to define the concept of *weak Hilbert space* (cf. Pisier (1988) and the references given there). The Lorentz space $L^{2\infty}(0, 1)$, sometimes denoted by weak-$L^2(0, 1)$, is *not* a weak Hilbert space. The last third of the book is devoted to the exploration of this concept and the related concepts of weak type and weak cotype. Each has a simple geometric characterization in terms of volume ratios. For example, a Banach space X is a weak Hilbert space if and only if there is a real number γ such that if F is an n-dimensional subspace of X, then there are ellipsoids D, $G \subset F$ satisfying

$$D \subset B_X \cap F \subset G \quad \text{and} \quad (|G|/|D|)^{1/n} \leq \gamma.$$

Weak Hilbert spaces have a number of other interesting properties.

For example, consider a complex Banach space X and a nuclear operator $T : X \to X$. If X is a Hilbert space, or is isomorphic to a Hilbert space, then the eigenvalues of T are absolutely summable. See Johnson, König, Maurey, and Retherford (1979). If X is a weak Hilbert space, the sequence (λ_k) of eigenvalues of T, repeated according to their multiplicity and arranged so that $|\lambda_1| \geq |\lambda_2| \geq \cdots$, need not be absolutely summable but does satisfy the weaker condition

$$\sup_k k|\lambda_k| < \infty. \tag{5}$$

More precisely, the left-hand side is majorized by a constant multiple of the nuclear norm of T. In fact, (5) characterizes the weak Hilbert spaces among the complex Banach spaces.

This overview gives only a glimpse of the contents of this substantial and important book.

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BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 25, Number 1, July 1991

0273-0979/91 $1.00 + $.25 per page

Exponential Diophantine equations, by T. N. Shorey and R. Tijdeman. Cambridge Tracts in Mathematics, vol. 87, Cambridge University Press, Cambridge, 1986, x + 240 pp. ISBN 0-521-26826-5

Algorithms for Diophantine equations, by B. M. M. de Weger. CWI Tract 65, Center for Mathematics and Computer Science, POB 4079, 1009 AB Amsterdam, The Netherlands, 1989, viii + 212 pp. ISBN 90-6196-375-3

Diophantine equations (named after Diophantus of Alexandria) were studied much earlier than his time, especially in China. I