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BULLETIN (New Series) OF THE
 AMERICAN MATHEMATICAL SOCIETY
 Volume 25, Number 1, July 1991
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 0273-0979/91 \$1.00 + \$.25 per page

Characterization of Banach Spaces not containing ℓ^1 , by D. van Dulst. Centrum voor Wiskunde en Informatica, Amsterdam 1989, iv + 163 pp. ISBN 90-6196-366-4

1. INTRODUCTION

Central to Banach Space theory is the study of the classical Banach spaces c_0 , ℓ^p , L^p ($1 \leq p \leq \infty$), $C(K)$, and of their relationship with general Banach spaces.

The space ℓ^1 is of special importance in the theory of general Banach spaces. This is due to a phenomenon of considerable interest, namely that many pathological properties of Banach spaces are closely related to the fact that they have subspaces close to ℓ^1 . This is true in the “local” theory of Banach spaces (see e.g. Pisier’s work [P] for a celebrated example) as well as in infinite dimensional theory, the subject of our present interest.

Among the “elementary” spaces c_0 , ℓ^p , $p < \infty$, ℓ^1 is the only one that has a nonseparable dual. If a Banach space contains ℓ^1 , its dual is nonseparable. (For simplicity, we say that a Banach space contains ℓ^1 if it contains a subspace isomorphic to ℓ^1 .) It was conjectured for a long time that the converse holds. This conjecture was disproved in 1974 by a deep example of R. C. James, the so called James tree space JT. (This space and its variations remain of considerable interest.) Thus it came as a surprise that simple criteria allow one to decide whether or not a Banach space contains a copy of ℓ^1 . The main results in that direction were proven in 1974 by H. P. Rosenthal, and surely constitute one of the most beautiful achievements of Banach space theory. One of

the most important results is

Rosenthal's ℓ^1 theorem. *A bounded sequence in a Banach space has a subsequence equivalent to the ℓ^1 basis or has a weak Cauchy subsequence.*

(A sequence (x_n) is weak Cauchy if for all x^* in X^* , the sequence $x^*(x_n)$ converges; it is equivalent to the ℓ^1 basis if $K\|\sum a_n x_n\| \geq \sum |a_n|$ for all finite sequences a_n and some number K .) The importance of Rosenthal's results is that they provide very useful information on the structure of Banach spaces, according to whether or not they contain ℓ^1 . Those which do are wildly non reflexive; those which do not are much less so. Another appeal of Rosenthal's results is that the methods behind their proofs make connection with questions of topology and measure theory that deserve to be studied in their own right; the various section of this review attempt to give some of the highlights of these related topics. Many important results (in particular in §2) have a rather complicated history, since various critical steps have been proved by different authors. A detailed account of this history would (at best) be of interest only to the specialist, and does not have its place in this review. The results will hence be mentioned only in what appears at present to be their final form. Also, we will give only the references not mentioned in van Dulst's book.

2. FUNCTIONS OF FIRST BAIRE CLASS

Rosenthal's original proof of his results was essentially combinatorial. Later on, he initiated a more topological approach, that turned out to be more fruitful. A basic elementary observation is as follows. Consider a Banach space X , and the unit ball T of its dual. Provided with the weak* topology, T is a compact set, and it is metrizable if (and only if) X is separable. The unit ball Z of X can be seen as a set of continuous functions on T . The unit ball of X^{**} can then be seen as the pointwise closure of Z in the set of all functions on T . (The pointwise convergence topology is the coarsest that makes the evaluation $f \rightarrow f(t)$ continuous for each $t \in T$.) The remarkable fact is that the linear structure plays no role in Rosenthal's theorem, and that this theorem boils down to results about sets of continuous functions.

A sequence of functions (f_n) on T is called independent if there exists $\alpha < \beta$, such that, given any two disjoint finite sets

I, J , the set

$$\bigcap_{n \in J} \{f_n \leq \alpha\} \cap \bigcap_{n \in I} \{f_n \geq \beta\}$$

is not empty. The relevance of this notion is the fact that an independent sequence is an ℓ^1 basis for the supremum norm. The archetype of an independent sequence is the sequence of coordinate functions on $T = \{0, 1\}^{\mathbb{N}}$. The key property of this sequence is that all its cluster points for the pointwise convergence topology (these can be identified to ultrafilters on \mathbb{N}) are extremely non-measurable, either in the topological sense, or with respect to the canonical measure on T (a fact going back to W. Sierpiński). The importance of this seemingly specific example is that it is essentially generic, in the sense that any independent sequence of continuous functions on a (metrizable) compact space will mimick its behavior on a certain compact subset.

Consider the class $B_r(T)$ of functions on T , that have the property that for each closed subset U of T , the restriction of f to U has at least one point of continuity. When T is metrizable, $B_r(T)$ coincides with the set $B_1(T)$ of first Baire class functions on T i.e., the pointwise limits of sequences of continuous functions. The central part of Rosenthal's result is the equivalence of (i) to (iv) in the following result. (Part of which were obtained in collaboration with T. Odell.)

Theorem 1. *Consider a bounded set Z of continuous functions on a compact space T , and denote by \overline{Z} its pointwise closure (in the set of all functions on T). The following are then equivalent:*

- (i) Z does not contain a subsequence equivalent to the unit basis of ℓ^1 (for the supremum norm).
- (ii) Each sequence of Z has a pointwise convergent subsequence.
- (iii) Z does not contain an independent sequence.
- (iv) Each sequence of Z has a pointwise cluster point in $B_r(T)$.
- (v) $\overline{Z} \subset B_r(T)$.
- (vi) For each Radon measure μ on T , \overline{Z} consists of μ -measurable functions.

When T is metrizable, (ii) to (vi) remain equivalent under the assumption that T is Polish rather than compact.

One of the meanings of this result is a strong dichotomy. Either \overline{Z} is contained in $B_r(T)$, or it contains very pathological functions.

When the separable Banach space X does not contain ℓ^1 , the unit ball of its second dual identifies to a pointwise compact set of $B_1(T)$; this provides motivation for the study of pointwise compactness in $B_1(T)$. Building on previous work of Rosenthal, Bourgain, Fremlin and Talagrand proved that for a Polish space T , $B_1(T)$ is angelic for the pointwise convergence topology (that is, if a subset A of $B_1(T)$ is such that each countable subset has a cluster point, it is relatively compact, and every $t \in \overline{A}$ is the limit of a sequence of A).

Another delicate result of Bourgain-Fremlin-Talagrand is that the convex hull of a uniformly bounded pointwise compact set in $B_1(T)$ is relatively compact. More generally, consider a bounded function on $Z \times T$, where Z is compact (nonmetrizable in general) and T Polish. Assume that f is continuous in the first variable, and of first Baire class in the second variable. Then, for each Radon probability μ on Z , the function $\int f(z, t)d\mu(z)$ is in $B_1(T)$. This does not stay true when $B_1(T)$ is replaced by the space $B_\alpha(T)$ of functions of Baire class at most α . The correct formulation in that case is to assume Z, T Polish, f Borel on $Z \times T$ such that all the sections $f(z, \cdot)$ belong to $B_\alpha(T)$. In a profound study, A. Louveau [L], relying on tools from effective descriptive theory of Borel sets, proved (among much more) that for each probability μ on T , the function $\int f(z, t)d\mu(t)$ belongs to $B_\alpha(T)$.

While studying sequences in spaces of vector valued measurable functions, the need arises to have a "parametrized" version of the equivalence of (i) to (iii). The correct formulation involves considering not only subsequences of a given sequence, but also sequences of convex combinations. Using this idea, combined with those underlying Theorem 1, I could prove in particular that $L^1(X)$ is weakly sequentially complete if and only if this is the case for X [T1].

Let us now turn to some characterizations, in purely Banach space terms, of the separable Banach spaces not containing ℓ^1 . The equivalence of (i) to (iv) relies on the results presented above. Some extra work, of more Banach space theoretical nature, is needed to obtain the other equivalences.

Theorem 2. *For a separable Banach space X , the following are equivalent*

- (i) X does not contain ℓ^1 .

- (ii) *Every bounded sequence in X has a weak Cauchy subsequence.*
- (iii) *Every bounded subset of X (resp. X^{**}) is weak * sequentially dense in its weak (resp. weak*) closure.*
- (iv) $\text{card } X^{**} = \text{card } \mathbb{R}$.
- (v) *X^* contains no subspace isomorphic to $\ell^1(\Gamma)$ for any uncountable Γ .*
- (vi) *$C([0, 1])$ is not a quotient of X .*
- (vii) *X^* contains no subspace isomorphic to $L^1[0, 1]$ (resp. $C([0, 1]^*)$).*

Finally, let us conclude this section by mentioning that much of Theorems 1 and 2 have been extended (or, rather, “localized”) in [G-G-M-S].

3. SETS OF MEASURABLE FUNCTIONS

Much of §2 is concerned with sets of functions that are regular with respect to topology. It is natural to consider similar questions, when topological regularity is replaced by measurability. The landmark result in that direction establishes a close parallel (although only at a rather formal level) with the equivalence of (ii) and (iv) in Theorem 1.

Fremlin’s subsequence theorem. *Consider a bounded sequence of measurable functions on $[0, 1]$, provided with Lebesgue measure. Then either it has a subsequence which converges a.s., or it has a subsequence with no measurable cluster points.*

A function φ from a measure space (Ω, Σ, μ) to a Banach space X is called scalarly measurable if $x^* \circ \varphi$ is measurable whenever $x^* \in X^*$. In that case, $Z_\varphi = \{x^* \circ \varphi; x^* \in X^*, \|x^*\| \leq 1\}$ is a pointwise compact set of measurable functions. This fact and the parallel with the pointwise compact sets of $B_1(T)$, provide motivation for the study of these objects. The main problem is as follows. Consider a compact metric space T , a probability μ on T , and a set Z of continuous functions on T . When is it true that all the pointwise cluster points of Z are μ -measurable? The difference with condition (vi) of Theorem 1 is that we required there that the cluster points be measurable for *all* measures ν on T . It turns out that the techniques needed to study this question are unrelated to those needed for Theorem 1 (and significantly harder). The fundamental concept has been invented by

D. H. Fremlin. Let us say that a set Z of measurable functions on a probability space (Ω, Σ, μ) is stable if the following occurs. Given $A \in \Sigma$, with $\mu(A) > 0$, and $\alpha < \beta$, then for some $n > 0$, we have

$$(\mu^{2n})^* (\{(x_1, \dots, x_n, y_1, \dots, y_n); \\ \exists f \in Z; \forall i \leq n, f(x_i) < \alpha, f(y_i) > \beta\}) < (\mu(A))^{2n}.$$

While grasping the meaning of this condition certainly requires some effort it should nonetheless be immediately apparent that it is of a combinatorial nature. A bounded stable set of measurable functions is pointwise relatively compact in the set of measurable functions. Let us go back to sets Z of continuous functions on T , that have only μ -measurable cluster points. The difficulty is that in order to use the hypothesis, one needs tools to construct nonmeasurable cluster points, and that the tools available require some special axiom (Continuum Hypothesis (CH), or even much weaker consequences, does the job). Using such axioms, Fremlin showed that Z must be μ -stable. A recent forthcoming paper of Fremlin and Shelah [F-S], however, constructs a model of ZFC where this fails. This intervention of special axioms is certainly a nuisance. At second thought, the situation is not so bad. The interesting objects are simply not the pointwise compacts of measurable functions—that could conceivably be very pathological in some unusual models of ZFC—but the stable sets of measurable functions. These sets, that are studied in detail in my AMS memoir (no. 307), certainly have remarkable properties. For example, Fremlin showed that on such a set, the identity map is continuous from the pointwise convergence topology to the topology of convergence in measure. The nicest surprise concerning stable sets (which establishes beyond doubt the importance of the notion) is that, in the uniformly bounded case, they are exactly the sets on which the law of large numbers holds uniformly (in a variety of equivalent forms). This solves the so called Glivenko-Cantelli problem [T2].

A scalarly measurable function φ from a measure space to a Banach space X is called Pettis integrable if, for each measurable set A , there exists $\mu(A) \in X$ such that $x^*(\mu(A)) = \int_A x^* \circ \varphi d\mu$ for all x^* in X^* . The map $A \rightarrow \mu(A)$ is the vector measure, of which φ is called a Pettis derivative. Not surprisingly, the theory of Pettis integration is closely connected to that of pointwise compact sets

of measurable functions. Say now that X has the Weak Radon-Nikodym property (WRNP) if each bounded variation X -valued vector measure has a Pettis derivative. This definition attempts to mimic that of the RNP (where the notion of Pettis integral is replaced by the ordinary Bochner integral). It is however much less successful, and actually little can be proved unless X is a dual, $X = Y^*$. In that case, X has the WRNP if and only if Y does not contain ℓ^1 (L. Janika); this, of course, is related to condition (vi) of Theorem 1, although extra difficulty arise when X is not separable. Also closely related to this is the fact that X does not contain ℓ^1 if and only if each *weak** compact set of X^* is the *norm* closed convex hull of its extreme points (Odell-Rosenthal in the separable case; Haydon in general). This should be compared to the much older result of Huff and Morris that X^* has the Krein-Milman property (KMP), that is, each norm closed convex set is the norm closed convex hull of its extreme points, if and only if it has the RNP (which, when X is separable, means that X^* is separable). In the separable case, the use of the techniques of R. C. James allows one to refine this result considerably (G. Godefroy [G2], S. Stegall [S]).

4. MORE GEOMETRY OF BANACH SPACES

In a Banach space with the RNP (and in particular in a separable dual) each convex set has a slice (i.e. a non empty intersection with a half space) of arbitrarily small diameter. J. Bourgain proved that in the dual of a space not containing ℓ^1 , for each bounded convex set one can find a finite convex combination of slices of arbitrarily small diameter. Banach spaces with this property are now called strongly regular. This is an interesting property, in particular in view of the fact, proved by W. Schachermayer, that it can be characterized by the fact that all bounded operators T from L^1 to X are such that $T^*(B^\infty)$ is suitably small where B^∞ denotes the unit ball of L^∞ . (This provides a nice analogy with the fact that X has the RNP if and only if $T^*(B^\infty)$ is an equimeasurable set—in the sense of Grothendieck—for all bounded operators from L^1 to X .) We refer the reader to the memoir [G-G-M-S] for this and other notions of regularity.

It is an old problem whether RNP and KMP are equivalent for Banach spaces. W. Schachmayer proved that this is the case for strongly regular Banach spaces.

We have detailed only topics covered by the book under review, but there is a sizable body of literature in geometry of Banach spaces that builds upon these ideas; as a particularly elegant example, one can refer e.g. to Godefroy's work on uniqueness of preduals [G1].

5. THE BOOK

The topics explored in the book extend quite beyond the characterizations of Banach spaces that do not contain ℓ^1 , and the book contains more measure theory than would be needed to describe that topic from, say, the pure geometry of Banach spaces point of view. This is certainly the right way to bring out the subject's real flavor. The book provides a self contained account of a variety of rather far reaching results. It is designed to be accessible with only a general background in functional analysis, but no previous knowledge of the subject. Accordingly, it provides very readable and detailed proofs, and it focuses on the most important results, (thereby staying pleasantly short.) These include Fremlin's subsequence theorem, some material on stable sets of measurable functions, the proofs of Theorems 1,2, the proofs of Bourgain's and Schachermayer's results on strong regularity and a chapter on the ever fascinating James's tree space JT. Seven appendices cover the basis of the necessary "advanced" tools (Radon measures, lifting theorem, analytic sets etc.) and greatly enhance readability for the nonexpert.

My only regret concerns the part of Chapter 2 where it is proved that, under certain conditions, pointwise compact sets of measurable functions are stable. As I mentioned earlier this does require some special axiom. The paper [F-S] demonstrating that this is indeed the case was written after the book, but in any case the proof given in the book does use CH. This is unfortunately mentioned neither in the introduction of the chapter, nor in the statement of the results, and only in the course of one proof does one find stated that CH is used at this place, although it must be said that the notes at the end of the chapter do bring attention to this point. (Clearly, the author considers CH natural, and wishes it were true. So do I, although I would never admit it publicly.)

Overall, however, van Dulst's book is very well written, and should become a standard introduction to a very attractive chapter of functional analysis.

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BULLETIN (New Series) OF THE
 AMERICAN MATHEMATICAL SOCIETY
 Volume 25, Number 1, July 1991
 ©1991 American Mathematical Society
 0273-0979/91 \$1.00 + \$.25 per page

Accessible categories: The foundations of categorical model theory,
 by Michael Makkai and Robert Paré. Contemporary Mathematics, vol. 104, American Mathematical Society, Providence, RI, 1989, 176 pp., \$31.00. ISBN 0-8218-5111-X

1. INTRODUCTION

It is some years since a research level book on “pure” category theory has appeared, and perhaps that is sufficient reason to review it here. Category theory was invented in the 1940s by S. Eilenberg and S. Mac Lane and has gone through a number of transformations since then. At one point, it appeared to be part of homological algebra, while at another point, topos theory swept away all other concerns. It may be that the intensity with which